EXPONENTIAL MIXING FOR CLASSICAL CONTINUOUS SYSTEMS

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Abstract. Using the method of cluster expansions, we prove exponential $L_{2+\varepsilon}$-mixing (ε arbitrarily small and positive) for Gibbs measures corresponding to dilute continuous systems of particles interacting via a stable, exponentially decreasing pair potential.

KEY WORDS: Gibbs measure, cluster expansion, exponential mixing.

Introduction

Although the Dobrushin–Lanford–Ruelle approach [4, 5, 6, 18, 30] to investigate equilibrium systems of particles in classical statistical mechanics through the construction and study of corresponding Gibbs measures is widely followed [27, 14, 28, 20, 24, 26, 21, 17], mixing properties for Gibbs measures corresponding to classical continuous systems so far were regarded only by H. Spohn [31]. Using results of G. Del Grosso [7] (see also works of Minlos and Halfina [22, 13]) he proved L_2-exponential mixing for the case of classical continuous systems of particles interacting via a positive potential of finite range.

However, the positivity condition, which seems crucial and unavoidable in [31], imposes too strong restrictions on the class of possible potentials. For example, it excludes a large class of interaction potentials between molecules usually regarded in physics, see Fig. 1.

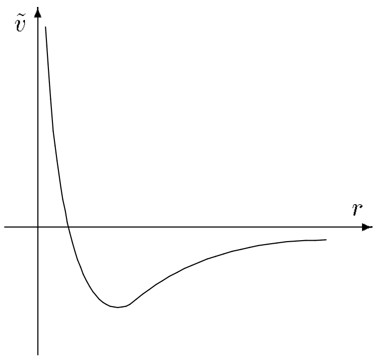

Fig. 1. The most typical interaction potential between two particles situated at points $x, x' \in \mathbb{R}^d$ is a function of $r := |x - x'|$, *ie*, $v(x, x') = \tilde{v}(r)$, with $\tilde{v}$ as above.

Estimates for correlation functions have been previously obtained in [8, 9, 10, 11]. They give mixing properties of the corresponding Gibbs measures on polynomials. It is, however, interesting in itself and important for many applications, such as the study of fluctuations and their stochastic dynamics [31, 3], to have the same estimates for all square integrable functions on the configuration space.

In the present work we prove exponential $L_{2+\varepsilon}$-mixing, where ε is arbitrary small and positive, for Gibbs measures corresponding to dilute continuous systems of particles interacting via a stable, exponentially decreasing pair potential. So, the potential pictured in Fig. 1 is covered by our considerations. Our method relies on the cluster expansion technique developed in [8, 9, 10, 11], see also [19, 23].

1. Poisson measures on configuration space

Let $\mathfrak{B}(\mathbb{R}^d)$ denote the Borel σ-algebra on $\mathbb{R}^d$ and let $\mathcal{L}(\Lambda)$, $\Lambda \in \mathfrak{B}(\mathbb{R}^d)$, be the system of all bounded Borel subsets of Λ. Define the configuration spaces Γ_Λ, $\Gamma_\Lambda^{\text{fin}}$ as the sets of all locally finite and finite configurations in Λ, respectively, *ie*,

$$\Gamma_\Lambda := \{\gamma \subset \Lambda \mid \#(\gamma \cap \Lambda') < \infty \text{ for all } \Lambda' \in \mathcal{L}(\Lambda)\},$$
$$\Gamma_\Lambda^{\text{fin}} := \{\gamma \subset \Lambda \mid \#\gamma < \infty\},$$

$\#\gamma$ being the cardinality of the set γ.

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Let $\mathfrak{B}_{\Lambda'}(\Gamma_\Lambda)$ and $\mathfrak{B}_{\Lambda'}(\Gamma_\Lambda^{\mathrm{fin}})$, $\Lambda, \Lambda' \in \mathfrak{B}(\mathbb{R}^d)$, $\Lambda' \subset \Lambda$, be the σ-algebras on Γ_Λ and $\Gamma_\Lambda^{\mathrm{fin}}$ generated by all mappings of the form

$$\gamma \mapsto \#(\gamma \cap \Lambda'') \in \mathbb{Z}_+ := \{0, 1, 2, \dots\},$$

where $\Lambda'' \in \mathcal{L}(\Lambda')$ and γ is taken from Γ_Λ and $\Gamma_\Lambda^{\mathrm{fin}}$, respectively.

Remark 1. Below, we shall omit the lower indices at Γ, Γ^{fin} and $\mathfrak{B}$ if they are equal to $\mathbb{R}^d$, *eg*, $\mathfrak{B}(\Gamma) := \mathfrak{B}_{\mathbb{R}^d}(\Gamma_{\mathbb{R}^d})$.

We set $\gamma_\Lambda := \gamma \cap \Lambda$.

Let π^z be the Poisson measure over $\mathfrak{B}(\Gamma)$ with intensity $z > 0$. Then for all $\Lambda \in \mathcal{L}(\mathbb{R}^d)$ and any $\mathfrak{B}_\Lambda(\Gamma)$-measurable function F

$$\int_\Gamma F(\gamma)\pi^z(d\gamma) = e^{-z|\Lambda|} \sum_{n=0}^\infty \frac{z^n}{n!} \int_{\Lambda^n} F\left(\{x_i\}_{i=1}^n\right)(dx)_n^1 \tag{1.1}$$

(if the right-hand side of (1.1) exists), where $(dx)_n^1 := dx_1 \dots dx_n$. The existence of π^z is insured by Kolmogorov's theorem.

In a similar way for the Lebesgue–Poisson measure λ_Λ^z over $\mathfrak{B}_\Lambda(\Gamma_\Lambda^{\mathrm{fin}})$, $\Lambda \in \mathfrak{B}(\mathbb{R}^d)$, we have

$$\int_{\Gamma_\Lambda^{\mathrm{fin}}} F(\gamma)\lambda_\Lambda^z(d\gamma) = \sum_{n=0}^\infty \frac{z^n}{n!} \int_{\Lambda^n} F\left(\{x_i\}_{i=1}^n\right)(dx)_n^1 \tag{1.2}$$

for any $\mathfrak{B}_\Lambda(\Gamma_\Lambda^{\mathrm{fin}})$-measurable function F.

Obviously, the Lebesgue–Poisson measure possesses the following 'cluster property'.

Lemma 1. *For all $\Lambda_1, \Lambda_2 \in \mathfrak{B}(\mathbb{R}^d)$ such that $\Lambda_1 \cap \Lambda_2 = \emptyset$ and all $F_i \in L_2\big(\Gamma_{\Lambda_1\cup\Lambda_2}, \mathfrak{B}_{\Lambda_i}(\Gamma_{\Lambda_1\cup\Lambda_2}), \lambda^z_{\Lambda_1\cup\Lambda_2}\big)$, $i = 1, 2$,*

$$\int_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}} F_1(\gamma_{\Lambda_1})F_2(\gamma_{\Lambda_2})\lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma) = \int_{\Gamma^{\mathrm{fin}}_{\Lambda_1}} F_1(\gamma_{\Lambda_1})\lambda^z_{\Lambda_1}(d\gamma) \int_{\Gamma^{\mathrm{fin}}_{\Lambda_2}} F_2(\gamma_{\Lambda_2})\lambda^z_{\Lambda_2}(d\gamma)$$

Below we will also need the following lemma. See also [19, 20].

Lemma 2. *For any $F_1, \dots, F_p \in L_1(\Gamma_\Lambda^{\mathrm{fin}}, \lambda_\Lambda^z)$, $\Lambda \in \mathfrak{B}(\mathbb{R}^d)$,*

$$\int_{\Gamma_\Lambda^{\mathrm{fin}}} \lambda_\Lambda^z(d\gamma) \sum_{\substack{(\gamma^1,\dots,\gamma^p)\\ \gamma^1\cup\dots\cup\gamma^p=\gamma\\ \gamma^i\cap\gamma^j=\emptyset,\ i\neq j}} F_1(\gamma^1)\dots F_p(\gamma^p) = \prod_{i=1}^p \int_{\Gamma_\Lambda^{\mathrm{fin}}} F_i(\gamma^i)\lambda_\Lambda^z(d\gamma^i) \tag{1.3}$$

and

$$\int_{\Gamma_\Lambda^{\mathrm{fin}}\setminus\{\emptyset\}} \lambda_\Lambda^z(d\gamma) \sum_{\substack{(\gamma^1,\dots,\gamma^p)\\ \gamma^1\cup\dots\cup\gamma^p=\gamma\\ \gamma^i\cap\gamma^j=\emptyset,\ i\neq j\\ \gamma^i\neq\emptyset,\ i=1,\dots,p}} F_1(\gamma^1)\dots F_p(\gamma^p) = \prod_{i=1}^p \int_{\Gamma_\Lambda^{\mathrm{fin}}\setminus\{\emptyset\}} F_i(\gamma^i)\lambda_\Lambda^z(d\gamma^i) \tag{1.4}$$

Proof. Concerning formula (1.3), we have:

$$\int_{\Gamma_\Lambda^{\mathrm{fin}}} \lambda_\Lambda^z(d\gamma) \sum_{\substack{(\gamma^1,\dots,\gamma^p)\\ \gamma^1\cup\dots\cup\gamma^p=\gamma\\ \gamma^i\cap\gamma^j=\emptyset,\ i\neq j}} F_1(\gamma^1)\dots F_p(\gamma^p) = \sum_{n=0}^\infty \frac{z^n}{n!} \int_{\Lambda^n} (dx)_n^1 \sum_{\substack{(\gamma^1,\dots,\gamma^p)\\ \gamma^1\cup\dots\cup\gamma^p=\{x_1,\dots,x_n\}\\ \gamma^i\cap\gamma^j=\emptyset,\ i\neq j}} F_1(\gamma^1)\dots F_p(\gamma^p) =$$

$$= \sum_{n=0}^\infty \frac{z^n}{n!} \sum_{\substack{k_1+\dots+k_p=n\\ k_1,\dots,k_p\geq 0}} \int_{\Lambda^n} (dx)_n^1 \sum_{\substack{(\gamma^1,\dots,\gamma^p)\\ \gamma^1\cup\dots\cup\gamma^p=\{x_1,\dots,x_n\}\\ \gamma^i\cap\gamma^j=\emptyset,\ i\neq j\\ \#\gamma^i=k_i,\ i=1,\dots,p}} F_1(\gamma^1)\dots F_p(\gamma^p) =$$

$$= \sum_{n=0}^\infty \sum_{\substack{k_1+\dots+k_p=n\\ k_1,\dots,k_p\geq 0}} \frac{z^{k_1}\dots z^{k_p}}{n!} C_n^{k_1,\dots,k_p} \prod_{i=1}^p \int_{\Lambda^{k_i}} F_i\big(\{x_1^i,\dots,x_{k_i}^i\}\big)(dx^i)_{k_i}^1 =$$

$$= \sum_{k_1=0}^{\infty} \cdots \sum_{k_p=0}^{\infty} \frac{z^{k_1} \dots z^{k_p}}{k_1! \dots k_p!} \prod_{i=1}^{p} \int_{\Lambda^{k_i}} F_i(\{x_1^i, \dots, x_{k_i}^i\})(dx^i)_{k_i}^1 =$$

$$= \prod_{i=1}^{p} \Big(\sum_{k_i=0}^{\infty} \frac{z^{k_i}}{k_i!} \int_{\Lambda^{k_i}} F_i(\{x_1^i, \dots, x_{k_i}^i\})(dx^i)_{k_i}^1 \Big) = \prod_{i=1}^{p} \int_{\Gamma_\Lambda^{\mathrm{fin}}} F_i(\gamma^i)\lambda_\Lambda^z(d\gamma^i).$$

The proof of (1.4) is essentially the same with the only difference that the summation over n, $k_1, \dots, k_p$ should start at 1. □

2. Gibbs perturbations of Poisson measures

In what follows we consider a classical continuous system of particles interacting via a pair potential v, which is supposed to be symmetric: $v(x,x') = v(x',x)$, for all $x, x' \in \mathbb{R}^d$, and translationally invariant: $v(x,x') = \hat{v}(x-x')$, $x, x' \in \mathbb{R}^d$. In addition, we suppose it to be stable:

$$\exists B > 0 : \quad \forall n \geq 2, \quad \forall x_1, \dots, x_n \in \mathbb{R}^d \quad \sum_{1 \leq i < j \leq n} v(x_i, x_j) \geq -Bn, \tag{2.1}$$

and to be exponentially decaying at infinity, *ie*, there exist positive constants v_0, r_0 and $\varkappa$ such that

$$|\hat{v}(x)| \leq v_0 e^{-\varkappa|x|} \quad \forall x : |x| \geq r_0. \tag{2.2}$$

Remark 2. From a physical point of view it is natural to assume even more, namely, that the potential v is not only symmetric and translationally invariant but also rotationally invariant, *ie*, $v(x,x') = \tilde{v}(|x-x'|)$ (*cf* Fig. 1). Having in mind this example of a pair potential we, nevertheless, do not use the assumption of rotational invariance.

To any finite configuration $\gamma_\Lambda \in \Gamma_\Lambda$, in a bounded region $\Lambda \in \mathcal{L}(\mathbb{R}^d)$, and fixed boundary conditions $\gamma_{\Lambda^c} \in \Gamma_{\Lambda^c}$ there corresponds the following potential energy

$$U(\gamma_\Lambda, \gamma_{\Lambda^c}) := \sum_{\{x,x'\} \subset \gamma_\Lambda} v(x,x') + \sum_{x \in \gamma_\Lambda,\, x' \in \gamma_{\Lambda^c}}^{*} v(x,x'), \tag{2.3}$$

where the star attached to the second sum means that, by definition, we put the series equal to infinity if it does not converge absolutely.

Let us introduce Gibbs states $G_\Lambda^{z,\beta}(\cdot \mid \gamma_{\Lambda^c})$ in a finite volume $\Lambda \in \mathcal{L}(\mathbb{R}^d)$ with boundary conditions $\gamma \in \Gamma$ corresponding to a continuous system with inverse temperature $\beta > 0$, activity $z > 0$ and a stable potential v as a probability measures over $\mathfrak{B}(\Gamma)$ as follows

$$G_\Lambda^{z,\beta}(\Delta \mid \gamma_{\Lambda^c}) = \begin{cases} \Xi_\Lambda^{z,\beta}(\gamma_{\Lambda^c})^{-1} \int_\Gamma \chi_\Delta(\gamma'_\Lambda \cup \gamma_{\Lambda^c}) \exp[-\beta U(\gamma'_\Lambda, \gamma_{\Lambda^c})]\pi^z(d\gamma'), & \text{if} \quad \Xi_\Lambda^{z,\beta}(\gamma_{\Lambda^c}) < \infty, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

where $\chi_A(\cdot)$ is the charachteristic function of a set A and

$$\Xi_\Lambda^{z,\beta}(\gamma) := \int_\Gamma \exp[-\beta U(\gamma'_\Lambda, \gamma_{\Lambda^c})]\, \pi^z(d\gamma'). \tag{2.5}$$

Definition 1. A Gibbs state $G^{z,\beta}(\cdot)$ corresponding to a continuous system with inverse temperature $\beta > 0$, activity $z > 0$ and a stable potential v is a probability measure on $(\Gamma, \mathfrak{B}(\Gamma))$ such that for any $\Lambda \in \mathcal{L}(\mathbb{R}^d)$ and $\Delta \in \mathfrak{B}_\Lambda(\Gamma)$

$$\int_\Gamma G_\Lambda^{z,\beta}(\Delta \mid \gamma_{\Lambda^c}) G^{z,\beta}(d\gamma) = G^{z,\beta}(\Delta). \tag{2.6}$$

Remark 3. The stability condition on the potential of interaction ensures the existence of finite volume Gibbs states with empty boundary conditions. See *eg* [14, 19, 27, 2, 17] for more details.

It is a rather complicated question whether the set of Gibbs measures corresponding to a classical continuous system interacting via a pair potential v is not empty at least for some z and β. One of the possible ways to show this is to construct a cluster expansion of the Radon-Nikodym derivatives

$$p_\Lambda(\gamma_\Lambda) := \frac{G_\Lambda^{z,\beta}(d\gamma_\Lambda|\{\emptyset\})}{\lambda_\Lambda^z(d\gamma_\Lambda)}, \quad \Lambda \in \mathcal{L}(\mathbb{R}^d), \tag{2.7}$$

and prove its convergence as $\Lambda\nearrow\mathbb{R}^d$. From this it follows that the limit

$$\lim_{\Lambda\nearrow\mathbb{R}^d} G_\Lambda^{z,\beta}(\cdot\,|\emptyset) \tag{2.8}$$

exists. Then, under our conditions on the potential, by one of the main theorems from [15] the limiting measure will be Gibbsian. This method is implemented in Section 3 below (*cf* Theorem 1).

Remark 4. The limit in (2.8) we will always understand in sense of weak local convergence, *ie*, we will say that a measure $G^{z,\beta}(\cdot)$ is a (weak local) limit of finite volume Gibbs measures $G_\Lambda^{z,\beta}(\cdot\mid\gamma_{\Lambda^c})$ with boundary condition γ if and only if for any $\Lambda'\in\mathcal{L}(\mathbb{R}^d)$ and any $\mathfrak{B}_{\Lambda'}(\Gamma)$-measurable function F

$$\int_\Gamma F(\gamma)G^{z,\beta}(d\gamma)=\lim_{\Lambda\nearrow\mathbb{R}^d}\int_\Gamma F(\gamma'_\Lambda)G_\Lambda^{z,\beta}(d\gamma'\mid\gamma_{\Lambda^c}).$$

Remark 5. As it follows from (2.4), (2.5), (1.1) and (1.2), the density $p_\Lambda(\gamma_\Lambda)$, $\Lambda\in\mathcal{L}(\mathbb{R}^d)$, has the following form:

$$p_\Lambda(\gamma_\Lambda)=\big(\widetilde{\Xi}_\Lambda^{z,\beta}\big)^{-1}\exp[-\beta U(\gamma_\Lambda,\emptyset)] \tag{2.9}$$

with

$$\widetilde{\Xi}_\Lambda^{z,\beta}:=e^{z|\Lambda|}\Xi_\Lambda^{z,\beta}(\emptyset)=\int_{\Gamma_\Lambda^{\mathrm{fin}}}\exp\big[-\beta U(\gamma_\Lambda,\emptyset)\big]\lambda_\Lambda^z(d\gamma_\Lambda). \tag{2.10}$$

For any finite configuration γ, let $\sum^{(\gamma)}$ denote the sum over all partitions of γ into non-empty subsets:

$$\sum\nolimits^{(\gamma)}[\ldots]=\sum_{n=1}^{\#\gamma}\sum_{\substack{\{\gamma^1,\ldots,\gamma^n\}\\ \gamma^1\cup\cdots\cup\gamma^n=\gamma\\ \gamma^i\cap\gamma^j=\emptyset,\ i\neq j\\ \gamma^i\neq\emptyset,\ i=1,\ldots,n}}[\ldots]=\sum_{n=1}^{\#\gamma}\frac{1}{n!}\sum_{\substack{(\gamma^1,\ldots,\gamma^n)\\ \gamma^1\cup\cdots\cup\gamma^n=\gamma\\ \gamma^i\cap\gamma^j=\emptyset,\ i\neq j\\ \gamma^i\neq\emptyset,\ i=1,\ldots,n}}[\ldots], \tag{2.11}$$

let $\mathfrak{G}(\gamma)$, $\mathfrak{G}_c(\gamma)$, $\mathfrak{T}(\gamma)$ be the sets of all graphs, connected graphs, trees[1] respectively whose vertices coincide with elements, or points, of γ, and let

$$\delta_\emptyset(\gamma)=\begin{cases}1, & \gamma=\emptyset,\\ 0, & \gamma\neq\emptyset.\end{cases}$$

In what follows we are going to work with a cluster expansion suggested and further developed in [8, 9, 10, 11, 19], namely, we are going to represent $p_\Lambda(\gamma_\Lambda)$, $\Lambda\in\mathcal{L}(\mathbb{R}^d)$, in the form

$$p_\Lambda(\gamma_\Lambda)=\big(\widetilde{\Xi}_\Lambda^{z,\beta}\big)^{-1}\Big\{\delta_\emptyset(\gamma_\Lambda)+\sum\nolimits^{(\gamma_\Lambda)}k(\gamma_\Lambda^1)\ldots k(\gamma_\Lambda^n)\Big\}, \tag{2.12}$$

with some function $k(\cdot)$ over $\Gamma^{\mathrm{fin}}\setminus\{\emptyset\}$ satisfying the estimate

$$|k(\gamma)|\le(e^{2\beta B})^{\#\gamma}\sum_{\mathcal{T}\in\mathfrak{T}(\gamma)}\prod_{\{x,x'\}\in\mathcal{T}}\big|e^{-\beta v(x,x')}-1\big|. \tag{2.13}$$

To construct this cluster expansion it suffices to show that for any non-empty finite configuration γ

$$\exp\Big[-\beta\sum_{\{x,x'\}\subset\gamma}v(x,x')\Big]=\sum\nolimits^{(\gamma)}k(\gamma_1)\ldots k(\gamma_n), \tag{2.14}$$

but

$$\exp\Big[-\beta\sum_{\{x,x'\}\subset\gamma}v(x,x')\Big]=\prod_{\{x,x'\}\subset\gamma}\exp\big[-\beta v(x,x')\big]=$$
$$=\prod_{\{x,x'\}\subset\gamma}\Big(\big(\exp\big[-\beta v(x,x')\big]-1\big)+1\Big)=\sum_{G\in\mathfrak{G}(\gamma)}\prod_{\{x,x'\}\in G}\big(\exp\big[-\beta v(x,x')\big]-1\big),$$

and so (2.14) is satisfied with

$$k(\gamma):=\sum_{G_c\in\mathfrak{G}_c(\gamma)}\prod_{\{x,x'\}\in G_c}\Big(\exp\big[-\beta v(x,x')\big]-1\Big),\quad\gamma\in\Gamma^{\mathrm{fin}}\setminus\{\emptyset\}. \tag{2.15}$$

[1]For definition of a tree, or tree graph, see *eg* [25].

Less trivial is the proof of the bound (2.13) on the function $k(\gamma)$ defined by (2.15). For the proof see Lemma 2 from § 4, Chapter 4 in [19], or [16] for more details.

Remark 6. It is useful to put $k(\emptyset) := 0$ and regard $k(\cdot)$ as a function over Γ^{fin}.

Proposition 1. Let $f : \mathbb{R}_+ \mapsto \mathbb{R}$ be such that $\big|f(x)\big| \le Ce^{-x}$, then

$$|e^{f(x)} - 1| \le \psi(C)e^{-x}, \quad x \in \mathbb{R}_+,$$

where $\psi(C) := Ce^C$.

Proof.

$$|e^{f(x)} - 1| = \left|\sum_{n=1}^{\infty} \frac{\big(f(x)\big)^n}{n!}\right| \le \sum_{n=1}^{\infty} \frac{\big|f(x)\big|^n}{n!} = \big|f(x)\big| \sum_{n=1}^{\infty} \frac{\big|f(x)\big|^{n-1}}{n!} \le \big|f(x)\big| \sum_{n=1}^{\infty} \frac{\big|f(x)\big|^{n-1}}{(n-1)!} =$$

$$= \big|f(x)\big| \sum_{n=0}^{\infty} \frac{\big|f(x)\big|^n}{n!} = \big|f(x)\big| e^{|f(x)|} \le Ce^{-x}e^C = \psi(C)e^{-x}. \qquad \square$$

Proposition 2. If the interaction potential v exponentially decays at infinity, then

$$|e^{-\beta\hat{v}(x)} - 1| < K(\beta)e^{-\varkappa|x|}, \tag{2.16}$$

where

$$K(\beta) = \max\big\{(e^{2\beta B} + 1)e^{\varkappa r_0}, \psi(\beta v_0)\big\}.$$

Proof. For all $x \in \mathbb{R}^d$ such that $|x| \ge r_0$, Proposition 1 and (2.2) give that

$$\big|e^{-\beta\hat{v}(x)} - 1\big| \le \psi(\beta v_0)e^{-\varkappa|x|} \le K(\beta)e^{-\varkappa|x|},$$

whereas for all $x \in \mathbb{R}^d$ such that $|x| < r_0$, the stability condition (2.1) and the assumption that $v(x, x') = \hat{v}(x-x')$ imply that

$$\big|e^{-\beta\hat{v}(x)} - 1\big| \le e^{2\beta B} + 1 < (e^{2\beta B} + 1)e^{\varkappa(r_0 - |x|)} \le K(\beta)e^{-\varkappa|x|}. \qquad \square$$

Remark 7. Estimate (2.13) on the absolute value of the function k, inequality

$$n^{n-2} \le e^n n!, \tag{2.17}$$

the fact that [25]

$$\sum_{\mathcal{T} \in \mathfrak{T}(\{1,\dots,n\})} 1 = n^{n-2}, \tag{2.18}$$

Remark 6 and the assumptions of stability and exponential decay at infinity on the interaction potential v imply that $k \in L_1(\Gamma_\Lambda, \lambda_\Lambda^z)$ for any $\Lambda \in \mathcal{L}(\mathbb{R}^d)$. Indeed, using the definition of Lebesgue–Poisson measure (1.2) and subsequently integrating with respect to the ending points of a tree graph $\mathcal{T} \in \mathfrak{T}\big(\{1, \dots, n\}\big)$, we get that

$$\int_{\Gamma_\Lambda^{\mathrm{fin}}} \big|k(\gamma_\Lambda)\big| \lambda_\Lambda^z(d\gamma_\Lambda) = \sum_{n=1}^{\infty} \frac{(ze^{2\beta B})^n}{n!} \int_{\Lambda^n} (dx)_n^1 \sum_{\mathcal{T} \in \mathfrak{T}(\{x_1,\dots,x_n\})} \prod_{\{x,x'\} \in \mathcal{T}} \big|e^{-\beta\hat{v}(x-x')} - 1\big| \le$$

$$\le |\Lambda| C(\beta)^{-1} \sum_{n=1}^{\infty} \frac{\big(ze^{2\beta B}C(\beta)\big)^n}{n!} \sum_{\mathcal{T} \in \mathfrak{T}(\{1,\dots,n\})} 1 = |\Lambda| C(\beta)^{-1} \sum_{n=1}^{\infty} \frac{\big(ze^{2\beta B}C(\beta)\big)^n}{n!} n^{n-2} \le$$

$$\le |\Lambda| C(\beta)^{-1} \sum_{n=1}^{\infty} \big(ze^{2\beta B+1}C(\beta)\big)^n = \frac{z|\Lambda|e^{2\beta B+1}}{1 - ze^{2\beta B+1}C(\beta)},$$

where

$$C(\beta) := \int_{\mathbb{R}^d} |e^{-\beta\hat{v}(x)} - 1| dx < \infty,$$

and hence by (2.16)

$$C(\beta) < K(\beta) \int_{\mathbb{R}^d} e^{-\varkappa|x|} dx = \Omega(d)K(\beta)\frac{(d-1)!}{\varkappa^d}, \tag{2.19}$$

where $\Omega(d)$ is a volume of a d-dimensional unit sphere.

Proposition 3. If formula (2.12) for the cluster expansion of $p_\Lambda(\gamma_\Lambda)$, $\Lambda\in\mathcal{L}(\mathbb{R}^d)$, defined by (2.7) holds with $k\in L_1(\Gamma_\Lambda^{\mathrm{fin}},\lambda_\Lambda^z)$, then

$$\widetilde{\Xi}_\Lambda^{z,\beta}=\int\limits_{\Gamma_\Lambda^{\mathrm{fin}}}\lambda_\Lambda^z(d\gamma_\Lambda)\Big\{\delta_\emptyset(\gamma_\Lambda)+\sum\nolimits^{(\gamma_\Lambda)}k(\gamma_\Lambda^1)\dots k(\gamma_\Lambda^n)\Big\}=\exp\Big[\int\limits_{\Gamma_\Lambda^{\mathrm{fin}}}k(\gamma_\Lambda)\lambda_\Lambda^z(d\gamma_\Lambda)\Big]. \tag{2.20}$$

Proof. As it follows from (2.9), (2.10) and (2.12)

$$1=\int\limits_{\Gamma_\Lambda^{\mathrm{fin}}}p_\Lambda(\gamma_\Lambda)\lambda_\Lambda^z(d\gamma_\Lambda)=\big(\widetilde{\Xi}_\Lambda^{z,\beta}\big)^{-1}\int\limits_{\Gamma_\Lambda^{\mathrm{fin}}}\lambda_\Lambda^z(d\gamma_\Lambda)\Big\{\delta_\emptyset(\gamma_\Lambda)+\sum\nolimits^{(\gamma_\Lambda)}k(\gamma_\Lambda^1)\dots k(\gamma_\Lambda^n)\Big\}.$$

Whence we get the first equality in (2.20), whereas the second one is an easy consequence of Lemma 2, Remark 6, definition (2.11) and the fact that $\lambda_\Lambda^z(\emptyset)=1$ for all $\Lambda\subseteq\mathbb{R}^d$. Indeed,

$$\int\limits_{\Gamma_\Lambda^{\mathrm{fin}}}\lambda_\Lambda^z(d\gamma_\Lambda)\Big\{\delta_\emptyset(\gamma_\Lambda)+\sum\nolimits^{(\gamma_\Lambda)}k(\gamma_\Lambda^1)\dots k(\gamma_\Lambda^n)\Big\}=1+\int\limits_{\Gamma_\Lambda^{\mathrm{fin}}}\lambda_\Lambda^z(d\gamma_\Lambda)\sum_{n=1}^{\#\gamma_\Lambda}\frac{1}{n!}\sum_{\substack{(\gamma_\Lambda^1,\dots,\gamma_\Lambda^n)\\ \gamma_\Lambda^1\cup\dots\cup\gamma_\Lambda^n=\gamma_\Lambda\\ \gamma_\Lambda^i\cap\gamma_\Lambda^j=\emptyset,\ i\neq j\\ \gamma_\Lambda^i\neq\emptyset,\ i=1,\dots,n}}k(\gamma_\Lambda^1)\dots k(\gamma_\Lambda^n)=$$

$$=1+\sum_{n=1}^{\infty}\frac{1}{n!}\int\limits_{\Gamma_\Lambda^{\mathrm{fin}}}\lambda_\Lambda^z(d\gamma_\Lambda)\sum_{\substack{(\gamma_\Lambda^1,\dots,\gamma_\Lambda^n)\\ \gamma_\Lambda^1\cup\dots\cup\gamma_\Lambda^n=\gamma_\Lambda\\ \gamma_\Lambda^i\cap\gamma_\Lambda^j=\emptyset,\ i\neq j}}k(\gamma_\Lambda^1)\dots k(\gamma_\Lambda^n)=$$

$$=1+\sum_{n=1}^{\infty}\frac{1}{n!}\Big(\int\limits_{\Gamma_\Lambda^{\mathrm{fin}}}k(\gamma'_\Lambda)\lambda_\Lambda^z(d\gamma'_\Lambda)\Big)^n=\exp\Big[\int\limits_{\Gamma_\Lambda^{\mathrm{fin}}}k(\gamma'_\Lambda)\lambda_\Lambda^z(d\gamma'_\Lambda)\Big].\qquad\square$$

3. Convergence of cluster expansion and existence of limiting Gibbs measure

Estimate (2.13) is crucial to prove the existence of the limit in (2.8) (*cf* [19] and the proof below). Moreover, by one of the main theorems from [15] we then obtain existence of Gibbs measure. We summarize all this in the following

Theorem 1. *For a stable exponentially decreasing potential v, and sufficiently small (but positive) inverse temperature β and activity z, namely, such that*

$$2ze^{2\beta B+1}\Omega(d)K(\beta)\frac{(d-1)!}{\varkappa^d}<1, \tag{3.1}$$

there exists a Gibbs measure $G^{z,\beta}$ corresponding to the limit (2.8) (in sense of the weak local convergence) of finite volume Gibbs states $G_\Lambda^{z,\beta}$ with empty boundary conditions. Moreover, the density of its restriction to $\mathfrak{B}_{\Lambda'}(\Gamma)$, $\Lambda'\in\mathcal{L}(\mathbb{R}^d)$, (with respect to Lebesgue–Poisson measure $\lambda_{\Lambda'}^z$)

$$p^{\Lambda'}(\gamma_{\Lambda'}):=\frac{G^{z,\beta}\restriction_{\mathfrak{B}_{\Lambda'}(\Gamma)}(d\gamma_{\Lambda'})}{\lambda_{\Lambda'}^z(d\gamma_{\Lambda'})} \tag{3.2}$$

is given by the following formula

$$p^{\Lambda'}(\gamma_{\Lambda'})=f(\Lambda')\widetilde{p}^{\Lambda'}(\gamma_{\Lambda'})=f(\Lambda')\Big\{\delta_\emptyset(\gamma_{\Lambda'})+\sum\nolimits^{(\gamma_{\Lambda'})}r^{\Lambda'}(\gamma_{\Lambda'}^1)\dots r^{\Lambda'}(\gamma_{\Lambda'}^n)\Big\}, \tag{3.3}$$

where

$$r^{\Lambda'}(\gamma_{\Lambda'}):=\int\limits_{\Gamma_{\Lambda'^c}^{\mathrm{fin}}}k(\gamma_{\Lambda'}\cup\gamma_{\Lambda'^c})\lambda_{\Lambda'^c}^z(d\gamma_{\Lambda'^c}) \tag{3.4}$$

and

$$f(\Lambda')=\exp\Big[\int\limits_{\Gamma_{\Lambda'^c}^{\mathrm{fin}}}k(\gamma_{\Lambda'^c})\lambda_{\Lambda'^c}^z(d\gamma_{\Lambda'^c})-\int\limits_{\Gamma^{\mathrm{fin}}}k(\gamma)\lambda^z(d\gamma)\Big]. \tag{3.5}$$

Proof. Let us first show that for any $\Lambda' \in \mathfrak{B}(\Lambda)$, $\Lambda \in \mathcal{L}(\mathbb{R}^d)$, the density

$$p_\Lambda^{\Lambda'}(\gamma_{\Lambda'}) := \frac{G_\Lambda^{z,\beta} \restriction_{\mathfrak{B}_{\Lambda'}(\Gamma_\Lambda)} (d\gamma_{\Lambda'})}{\lambda_{\Lambda'}^z(d\gamma_{\Lambda'})}$$

of the restriction of the finite volume Gibbs measure $G_\Lambda^{z,\beta}$ to $\mathfrak{B}_{\Lambda'}(\Gamma_\Lambda)$ (with respect to Lebesgue–Poisson measure $\lambda_{\Lambda'}^z$) is given by the following formula

$$p_\Lambda^{\Lambda'}(\gamma_{\Lambda'}) = f_\Lambda(\Lambda')\Big\{\delta_\emptyset(\gamma_{\Lambda'}) + \sum\nolimits^{(\gamma_{\Lambda'})} r_\Lambda^{\Lambda'}(\gamma_{\Lambda'}^1)\dots r_\Lambda^{\Lambda'}(\gamma_{\Lambda'}^n)\Big\}, \tag{3.6}$$

where

$$r_\Lambda^{\Lambda'}(\gamma_{\Lambda'}) := \int\limits_{\Gamma_{\Lambda\setminus\Lambda'}^{\mathrm{fin}}} k(\gamma_{\Lambda'}\cup\gamma_{\Lambda\setminus\Lambda'})\lambda_{\Lambda\setminus\Lambda'}^z(d\gamma_{\Lambda\setminus\Lambda'}) \tag{3.7}$$

and

$$f_\Lambda(\Lambda') := \widetilde{\Xi}_{\Lambda\setminus\Lambda'}^{z,\beta}/\widetilde{\Xi}_\Lambda^{z,\beta} = \exp\Bigg[\int\limits_{\Gamma_{\Lambda\setminus\Lambda'}^{\mathrm{fin}}} k(\gamma_{\Lambda\setminus\Lambda'})\lambda_{\Lambda\setminus\Lambda'}^z(d\gamma_{\Lambda\setminus\Lambda'}) - \int\limits_{\Gamma_\Lambda^{\mathrm{fin}}} k(\gamma_\Lambda)\lambda_\Lambda^z(d\gamma_\Lambda)\Bigg]. \tag{3.8}$$

Indeed, using formulae (2.12), (1.3), Proposition 3 and the fact that $\lambda_\Lambda^z(\emptyset) = 1$ for all $\Lambda \subseteq \mathbb{R}^d$, we get

$$\begin{aligned}
p_\Lambda^{\Lambda'}(\gamma_{\Lambda'}) = &\int\limits_{\Gamma_{\Lambda\setminus\Lambda'}^{\mathrm{fin}}} p_\Lambda(\gamma_{\Lambda'}\cup\gamma_{\Lambda\setminus\Lambda'})\lambda_{\Lambda\setminus\Lambda'}^z(d\gamma_{\Lambda\setminus\Lambda'}) = \\
&=\big(\widetilde{\Xi}_\Lambda^{z,\beta}\big)^{-1}\int\limits_{\Gamma_{\Lambda\setminus\Lambda'}^{\mathrm{fin}}} \lambda_{\Lambda\setminus\Lambda'}^z(d\gamma_{\Lambda\setminus\Lambda'})\Big\{\delta_\emptyset(\gamma_{\Lambda'}\cup\gamma_{\Lambda\setminus\Lambda'}) + \sum\nolimits^{(\gamma_{\Lambda'}\cup\gamma_{\Lambda\setminus\Lambda'})} k(\gamma_\Lambda^1)\dots k(\gamma_\Lambda^n)\Big\} = \\
&=\big(\widetilde{\Xi}_\Lambda^{z,\beta}\big)^{-1}\int\limits_{\Gamma_{\Lambda\setminus\Lambda'}^{\mathrm{fin}}} \lambda_{\Lambda\setminus\Lambda'}^z(d\gamma_{\Lambda\setminus\Lambda'}) \sum_{\substack{(\gamma_1,\gamma_2)\\ \gamma_1\cup\gamma_2=\gamma_{\Lambda\setminus\Lambda'}\\ \gamma_1\cap\gamma_2=\emptyset}} \Big[\delta_\emptyset(\gamma_{\Lambda'}\cup\gamma_1) + \\
&\quad + \sum\nolimits^{(\gamma_{\Lambda'})} \sum_{\substack{(\gamma_1^1,\dots,\gamma_1^{n'})\\ \gamma_1^1\cup\dots\cup\gamma_1^{n'}=\gamma_1\\ \gamma_1^i\cap\gamma_1^j=\emptyset,\ i\neq j}} k(\gamma_{\Lambda'}^1\cup\gamma_1^1)\dots k(\gamma_{\Lambda'}^{n'}\cup\gamma_1^{n'})\Big]\Big[\delta_\emptyset(\gamma_2) + \sum\nolimits^{(\gamma_2)} k(\gamma_2^1)\dots k(\gamma_2^{n_2})\Big] = \\
&=\big(\widetilde{\Xi}_\Lambda^{z,\beta}\big)^{-1}\int\limits_{\Gamma_{\Lambda\setminus\Lambda'}^{\mathrm{fin}}} \lambda_{\Lambda\setminus\Lambda'}^z(d\gamma_1)\Big[\delta_\emptyset(\gamma_{\Lambda'})\delta_\emptyset(\gamma_1) + \sum\nolimits^{(\gamma_{\Lambda'})} \sum_{\substack{(\gamma_1^1,\dots,\gamma_1^{n'})\\ \gamma_1^1\cup\dots\cup\gamma_1^{n'}=\gamma_1\\ \gamma_1^i\cap\gamma_1^j=\emptyset,\ i\neq j}} k(\gamma_{\Lambda'}^1\cup\gamma_1^1)\dots k(\gamma_{\Lambda'}^{n'}\cup\gamma_1^{n'})\Big] \times \\
&\quad \times \int\limits_{\Gamma_{\Lambda\setminus\Lambda'}^{\mathrm{fin}}} \lambda_{\Lambda\setminus\Lambda'}^z(d\gamma_2)\Big[\delta_\emptyset(\gamma_2) + \sum\nolimits^{(\gamma_2)} k(\gamma_2^1)\dots k(\gamma_2^{n_2})\Big] = \\
&=\big(\widetilde{\Xi}_\Lambda^{z,\beta}\big)^{-1}\Big[\delta_\emptyset(\gamma_{\Lambda'}) + \sum\nolimits^{(\gamma_{\Lambda'})} \int\limits_{\Gamma_{\Lambda\setminus\Lambda'}^{\mathrm{fin}}} \lambda_{\Lambda\setminus\Lambda'}^z(d\gamma_1) \sum_{\substack{(\gamma_1^1,\dots,\gamma_1^{n'})\\ \gamma_1^1\cup\dots\cup\gamma_1^{n'}=\gamma_1\\ \gamma_1^i\cap\gamma_1^j=\emptyset,\ i\neq j}} k(\gamma_{\Lambda'}^1\cup\gamma_1^1)\dots k(\gamma_{\Lambda'}^{n'}\cup\gamma_1^{n'})\Big]\widetilde{\Xi}_{\Lambda\setminus\Lambda'}^{z,\beta} = \\
&=\widetilde{\Xi}_{\Lambda\setminus\Lambda'}^{z,\beta}/\widetilde{\Xi}_\Lambda^{z,\beta}\Big[\delta_\emptyset(\gamma_{\Lambda'}) + \sum\nolimits^{(\gamma_{\Lambda'})} \prod_{i=1}^{n'} \int\limits_{\Gamma_{\Lambda\setminus\Lambda'}^{\mathrm{fin}}} \lambda_{\Lambda\setminus\Lambda'}^z(d\gamma_1^i)k(\gamma_{\Lambda'}^i\cup\gamma_1^i)\Big] = \\
&=f_\Lambda(\Lambda')\Big[\delta_\emptyset(\gamma_{\Lambda'}) + \sum\nolimits^{(\gamma_{\Lambda'})} \prod_{i=1}^{n'} r_\Lambda^{\Lambda'}(\gamma_{\Lambda'}^i)\Big].
\end{aligned}$$

As follows from the 'cluster property' of Lebesgue–Poisson measure

$$\int_{\Gamma^{\rm fin}_{\Lambda}} k(\gamma_\Lambda)\lambda^z_\Lambda(d\gamma_\Lambda) - \int_{\Gamma^{\rm fin}_{\Lambda\setminus\Lambda'}} k(\gamma_{\Lambda\setminus\Lambda'})\lambda^z_{\Lambda\setminus\Lambda'}(d\gamma_{\Lambda\setminus\Lambda'}) =$$

$$= \int_{\Gamma^{\rm fin}_{\Lambda'}} \lambda^z_{\Lambda'}(d\gamma_{\Lambda'}) \int_{\Gamma^{\rm fin}_{\Lambda\setminus\Lambda'}} \lambda^z_{\Lambda\setminus\Lambda'}(d\gamma_{\Lambda\setminus\Lambda'}) k(\gamma_{\Lambda'}\cup\gamma_{\Lambda\setminus\Lambda'}) - \int_{\Gamma^{\rm fin}_{\Lambda\setminus\Lambda'}} k(\gamma_{\Lambda\setminus\Lambda'})\lambda^z_{\Lambda\setminus\Lambda'}(d\gamma_{\Lambda\setminus\Lambda'}) =$$

$$= \int_{\Gamma^{\rm fin}_{\Lambda'}\setminus\{\emptyset\}} \lambda^z_{\Lambda'}(d\gamma_{\Lambda'}) \int_{\Gamma^{\rm fin}_{\Lambda\setminus\Lambda'}} \lambda^z_{\Lambda\setminus\Lambda'}(d\gamma_{\Lambda\setminus\Lambda'}) k(\gamma_{\Lambda'}\cup\gamma_{\Lambda\setminus\Lambda'}),$$

and so $f_\Lambda(\Lambda')$ also can be expressed by the following formula

$$f_\Lambda(\Lambda') = \exp\left[- \int_{\Gamma^{\rm fin}_{\Lambda'}\setminus\{\emptyset\}} \lambda^z_{\Lambda'}(d\gamma_{\Lambda'}) \int_{\Gamma^{\rm fin}_{\Lambda\setminus\Lambda'}} \lambda^z_{\Lambda\setminus\Lambda'}(d\gamma_{\Lambda\setminus\Lambda'}) k(\gamma_{\Lambda'}\cup\gamma_{\Lambda\setminus\Lambda'})\right]. \tag{3.8$'$}$$

In the same way as it was shown in Remark 7, estimate (2.13) on the absolute value of function k, the definition of Lebesgue–Poisson measure (1.2), formula (2.18) and inequality (2.17) imply that

$$\int_{\Gamma^{\rm fin}_{\Lambda'}\setminus\{\emptyset\}} \lambda^z_{\Lambda'}(d\gamma_{\Lambda'}) \int_{\Gamma^{\rm fin}_{\Lambda\setminus\Lambda'}} \lambda^z_{\Lambda\setminus\Lambda'}(d\gamma_{\Lambda\setminus\Lambda'}) \big|k(\gamma_{\Lambda'}\cup\gamma_{\Lambda\setminus\Lambda'})\big| \le$$

$$\le \int_{\Gamma^{\rm fin}_{\Lambda'}\setminus\{\emptyset\}} \lambda^z_{\Lambda'}(d\gamma_{\Lambda'}) \int_{\Gamma^{\rm fin}_{\Lambda\setminus\Lambda'}} \lambda^z_{\Lambda\setminus\Lambda'}(d\gamma_{\Lambda\setminus\Lambda'}) (e^{2\beta B})^{\#(\gamma_{\Lambda'}\cup\gamma_{\Lambda\setminus\Lambda'})} \sum_{\mathcal{T}\in\mathfrak{T}(\gamma_{\Lambda'}\cup\gamma_{\Lambda\setminus\Lambda'})} \prod_{\{x,x'\}\in\mathcal{T}} \big|e^{-\beta v(x,x')}-1\big| =$$

$$\le \sum_{n=1}^\infty \frac{(ze^{2\beta B})^n}{n!} \sum_{m=0}^\infty \frac{(ze^{2\beta B})^m}{m!} \int_{(\Lambda')^n} (dx)^1_n \int_{(\Lambda\setminus\Lambda')^m} (dx)^{n+1}_{n+m} \sum_{\mathcal{T}\in\mathfrak{T}(\{x_1,\dots,x_{n+m}\})} \prod_{\{x,x'\}\in\mathcal{T}} \big|e^{-\beta\hat v(x-x')}-1\big| \le$$

$$\le |\Lambda'| C(\beta)^{-1} \sum_{n=1}^\infty \frac{\big(ze^{2\beta B}C(\beta)\big)^n}{n!} \sum_{m=0}^\infty \frac{\big(ze^{2\beta B}C(\beta)\big)^m}{m!} e^{n+m}(n+m)! \le$$

$$\le |\Lambda'| C(\beta)^{-1} \sum_{p=1}^\infty \big(ze^{2\beta B+1}C(\beta)\big)^p \sum_{\substack{n+m=p\\ n\ge 0,\, m\ge 0}} \frac{(n+m)!}{n!m!} = |\Lambda'| C(\beta)^{-1} \sum_{p=1}^\infty \big(2ze^{2\beta B+1}C(\beta)\big)^p =$$

$$\le \frac{2ze^{2\beta B+1}|\Lambda'|}{1-2ze^{2\beta B+1}C(\beta)} =: D_1\big(z,\beta,|\Lambda'|\big). \tag{3.9}$$

This also means that

$$\int_{\Gamma^{\rm fin}_{\Lambda\setminus\Lambda'}} \lambda^z_{\Lambda\setminus\Lambda'}(d\gamma_{\Lambda\setminus\Lambda'}) \big|k(\gamma_{\Lambda'}\cup\gamma_{\Lambda\setminus\Lambda'})\big|$$

is uniformly (with respect to $\Lambda\subseteq\mathbb{R}^d$) bounded for $\lambda^z_{\Lambda'}$-almost all $\gamma_{\Lambda'}\in\Gamma^{\rm fin}_{\Lambda'}\setminus\{\emptyset\}$.

It is easy to see that for any $\gamma\in\Gamma^{\rm fin}$ there exists a limit of $\chi_{\Gamma_{\Lambda\setminus\Lambda'}}(\gamma)k(\gamma_{\Lambda'}\cup\gamma)$ as $\Lambda\nearrow\mathbb{R}^d$, and moreover,

$$\forall\gamma\in\Gamma^{\rm fin} \qquad \lim_{\Lambda\nearrow\mathbb{R}^d} \chi_{\Gamma_{\Lambda\setminus\Lambda'}}(\gamma)k(\gamma_{\Lambda'}\cup\gamma) = \chi_{\Gamma_{(\Lambda')^c}}(\gamma)k(\gamma_{\Lambda'}\cup\gamma).$$

Therefore, there exist limits of $f_\Lambda(\Lambda')$, $r^{\Lambda'}_\Lambda(\gamma_{\Lambda'})$ and $p^{\Lambda'}_\Lambda(\gamma_{\Lambda'})$ (for $\lambda^z_{\Lambda'}$-almost all $\gamma_{\Lambda'}\in\Gamma^{\rm fin}_{\Lambda'}\setminus\{\emptyset\}$) as $\Lambda\nearrow\mathbb{R}^d$, and moreover, they are equal to $f(\Lambda')$, $r^{\Lambda'}(\gamma_{\Lambda'})$ and $p^{\Lambda'}(\gamma_{\Lambda'})$ respectively, which proves the existence of the limiting measure $G^{z,\beta}$ in (2.8) as well as formulae (3.3)–(3.5). As follows from one of the main theorems from [15] under our conditions on the interaction potential v, this limiting measure satisfies the DLR equation (2.6), and hence is a Gibbs measure. □

Remark 8. Formulae (3.8′) and (3.9) also show that

$$f(\Lambda') \le e^{D_1(|\Lambda'|)}. \tag{3.10}$$

4. Decay of correlations

From the convergence of the cluster expansion (2.12) it is also possible to extract some L_p-exponential mixing properties for the limiting Gibbs measure, also known as exponential decay of correlations in statistical mechanics. In this section we formulate and prove the $L_{2+\varepsilon}$-exponential mixing property, where ε can be arbitrary small but positive.

Theorem 2. *Let the interaction potential v be stable and exponentially decaying at infinity in the sense of (2.2). Then, for any $\varepsilon, \delta > 0$, sufficiently small inverse temperature β and activity z, namely, such that*

$$2ze^{2\beta B+1}\Omega(d)K(\beta)\frac{(d-1)!}{\varkappa^d}\Big(1+\frac{1}{\delta}\Big)^d < 1, \tag{4.1}$$

any $\Lambda_1, \Lambda_2 \in \mathcal{L}(\mathbb{R}^d)$ such that $\Lambda_1 \cap \Lambda_2 = \emptyset$, and all $F_i \in L_{2+\varepsilon}\big(\Gamma, \mathfrak{B}_{\Lambda_i}(\Gamma), G^{z,\beta}\big)$, $i = 1, 2$, there exists a constant

$$D = D\big(z, \beta, \varkappa, \varepsilon, \delta, |\Lambda_1|, |\Lambda_2|, \|F_1\|_{L_{2+\varepsilon}}, \|F_2\|_{L_{2+\varepsilon}}\big) \tag{4.2}$$

such that

$$\big|\langle F_1F_2\rangle - \langle F_1\rangle\langle F_2\rangle\big| \le D\exp\Big[-\frac{\varkappa\varepsilon}{(1+\delta)(2+\varepsilon)}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big], \tag{4.3}$$

where $\langle\,\cdot\,\rangle$ denotes the expectation with respect to the measure $G^{z,\beta}$ constructed in Theorem 1.

By a standard technique (see, *eg*, the proof of Theorem 17.2.2 in [12] or its modification to our case in Appendix A), the statement of Theorem 2 follows from the next L_∞-exponential mixing property of the measure $G^{z,\beta}$.

Theorem 3. *Let the interaction potential v be stable and exponentially decaying at infinity. Then for any $\delta > 0$, $\Lambda_1, \Lambda_2 \in \mathcal{L}(\mathbb{R}^d)$ such that $\Lambda_1 \cap \Lambda_2 = \emptyset$, and all $F_i \in L_\infty\big(\Gamma, \mathfrak{B}_{\Lambda_i}(\Gamma), G^{z,\beta}\big)$, $i = 1, 2$, there exists a constant*

$$D' = D'\big(z, \beta, \varkappa, \delta, |\Lambda_1|, |\Lambda_2|\big) \tag{4.4}$$

such that

$$\big|\langle F_1F_2\rangle - \langle F_1\rangle\langle F_2\rangle\big| \le D'\exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big]\|F_1\|_{L_\infty}\|F_2\|_{L_\infty}, \tag{4.5}$$

for sufficiently small z and β satisfying (4.1).

Remark 9. The proof of this theorem is quite long and essentially based on the analysis of the cluster expansion obtained in Section 2.

Proof. Using formula (3.2) and the first equality from (3.3), we can write

$$\begin{aligned}
\langle F_1F_2\rangle - \langle F_1\rangle\langle F_2\rangle &= \int_\Gamma F_1(\gamma_{\Lambda_1})F_2(\gamma_{\Lambda_2})G^{z,\beta}(d\gamma) - \int_\Gamma F_1(\gamma_{\Lambda_1})G^{z,\beta}(d\gamma)\int_\Gamma F_2(\gamma_{\Lambda_2})G^{z,\beta}(d\gamma) = \\
&= \int_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}} F_1(\gamma_{\Lambda_1})F_2(\gamma_{\Lambda_2})p^{\Lambda_1\cup\Lambda_2}(\gamma_{\Lambda_1\cup\Lambda_2})\lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2}) - \\
&\quad - \int_{\Gamma^{\mathrm{fin}}_{\Lambda_1}} F_1(\gamma_{\Lambda_1})p^{\Lambda_1}(\gamma_{\Lambda_1})\lambda^z_{\Lambda_1}(d\gamma_{\Lambda_1})\int_{\Gamma^{\mathrm{fin}}_{\Lambda_2}} F_2(\gamma_{\Lambda_2})p^{\Lambda_2}(\gamma_{\Lambda_2})\lambda^z_{\Lambda_2}(d\gamma_{\Lambda_2}) = \\
&= f(\Lambda_1\cup\Lambda_2)\int_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}} F_1(\gamma_{\Lambda_1})F_2(\gamma_{\Lambda_2})\widetilde{p}^{\Lambda_1\cup\Lambda_2}(\gamma_{\Lambda_1\cup\Lambda_2})\lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2}) - \\
&\quad - f(\Lambda_1)f(\Lambda_2)\int_{\Gamma^{\mathrm{fin}}_{\Lambda_1}} F_1(\gamma_{\Lambda_1})\widetilde{p}^{\Lambda_1}(\gamma_{\Lambda_1})\lambda^z_{\Lambda_1}(d\gamma_{\Lambda_1})\int_{\Gamma^{\mathrm{fin}}_{\Lambda_2}} F_2(\gamma_{\Lambda_2})\widetilde{p}^{\Lambda_2}(\gamma_{\Lambda_2})\lambda^z_{\Lambda_2}(d\gamma_{\Lambda_2}) =
\end{aligned}$$

$$=\big(f(\Lambda_1\cup\Lambda_2)-f(\Lambda_1)f(\Lambda_2)\big)\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}} F_1(\gamma_{\Lambda_1})F_2(\gamma_{\Lambda_2})\widetilde{p}^{\Lambda_1\cup\Lambda_2}(\gamma_{\Lambda_1\cup\Lambda_2})\lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2})+$$

$$+f(\Lambda_1)f(\Lambda_2)\Bigg[\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}} F_1(\gamma_{\Lambda_1})F_2(\gamma_{\Lambda_2})\widetilde{p}^{\Lambda_1\cup\Lambda_2}(\gamma_{\Lambda_1\cup\Lambda_2})\lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2})-$$

$$-\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1}} F_1(\gamma_{\Lambda_1})\widetilde{p}^{\Lambda_1}(\gamma_{\Lambda_1})\lambda^z_{\Lambda_1}(d\gamma_{\Lambda_1})\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_2}} F_2(\gamma_{\Lambda_2})\widetilde{p}^{\Lambda_2}(\gamma_{\Lambda_2})\lambda^z_{\Lambda_2}(d\gamma_{\Lambda_2})\Bigg]. \tag{4.6}$$

Taking into account the fact that

$$\left|f(\Lambda_1\cup\Lambda_2)\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}} F_1(\gamma_{\Lambda_1})F_2(\gamma_{\Lambda_2})\widetilde{p}^{\Lambda_1\cup\Lambda_2}(\gamma_{\Lambda_1\cup\Lambda_2})\lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2})\right|=$$

$$=\left|\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}} F_1(\gamma_{\Lambda_1})F_2(\gamma_{\Lambda_2})p^{\Lambda_1\cup\Lambda_2}(\gamma_{\Lambda_1\cup\Lambda_2})\lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2})\right|=$$

$$=\left|\int\limits_{\Gamma} F_1(\gamma_{\Lambda_1})F_2(\gamma_{\Lambda_2})G^{z,\beta}(d\gamma)\right|\le\|F_1\|_{L_\infty}\|F_2\|_{L_\infty},$$

we can majorize the absolute value of the first summand in (4.6) by

$$\left|1-\frac{f(\Lambda_1)f(\Lambda_2)}{f(\Lambda_1\cup\Lambda_2)}\right|\|F_1\|_{L_\infty}\|F_2\|_{L_\infty}. \tag{4.7}$$

Next, using formula (3.5), we get

$$\left|\frac{f(\Lambda_1)f(\Lambda_2)}{f(\Lambda_1\cup\Lambda_2)}-1\right|=$$

$$=\left|\frac{\exp\left[\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1^c}} k(\gamma_{\Lambda_1^c})\lambda^z_{\Lambda_1^c}(d\gamma_{\Lambda_1^c})-\int\limits_{\Gamma^{\mathrm{fin}}} k(\gamma)\lambda^z(d\gamma)\right]\exp\left[\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_2^c}} k(\gamma_{\Lambda_2^c})\lambda^z_{\Lambda_2^c}(d\gamma_{\Lambda_1^c})-\int\limits_{\Gamma^{\mathrm{fin}}} k(\gamma)\lambda^z(d\gamma)\right]}{\exp\left[\int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} k(\gamma_{(\Lambda_1\cup\Lambda_2)^c})\lambda^z_{(\Lambda_1\cup\Lambda_2)^c}(d\gamma_{(\Lambda_1\cup\Lambda_2)^c})-\int\limits_{\Gamma^{\mathrm{fin}}} k(\gamma)\lambda^z(d\gamma)\right]}-1\right|=$$

$$=\left|\exp\left[-\int\limits_{\Gamma^{\mathrm{fin}}} k(\gamma)\lambda^z(d\gamma)+\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1^c}} k(\gamma)\lambda^z_{\Lambda_1^c}(d\gamma)+\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_2^c}} k(\gamma)\lambda^z_{\Lambda_2^c}(d\gamma_{\Lambda_2^c})-\int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} k(\gamma)\lambda^z_{(\Lambda_1\cup\Lambda_2)^c}(d\gamma)\right]-1\right|=$$

$$=\left|\exp\left[-\int\limits_{\Gamma^{\mathrm{fin}}} k(\gamma)\lambda^z(d\gamma)+\int\limits_{\{\gamma\in\Gamma^{\mathrm{fin}}|\gamma\cap\Lambda_1=\emptyset\}} k(\gamma)\lambda^z(d\gamma)+\right.\right.$$

$$\left.\left.+\int\limits_{\{\gamma\in\Gamma^{\mathrm{fin}}|\gamma\cap\Lambda_2=\emptyset\}} k(\gamma)\lambda^z(d\gamma)-\int\limits_{\{\gamma\in\Gamma^{\mathrm{fin}}|\gamma\cap\Lambda_1=\emptyset,\gamma\cap\Lambda_2=\emptyset\}} k(\gamma)\lambda^z(d\gamma)\right]-1\right|=$$

$$=\left|\exp\left[-\int\limits_{\{\gamma\in\Gamma^{\mathrm{fin}}|\gamma\cap\Lambda_1\neq\emptyset\}} k(\gamma)\lambda^z(d\gamma)+\int\limits_{\{\gamma\in\Gamma^{\mathrm{fin}}|\gamma\cap\Lambda_1\neq\emptyset,\gamma\cap\Lambda_2=\emptyset\}} k(\gamma)\lambda^z(d\gamma)\right]-1\right|=$$

$$=\left|\exp\left[-\int\limits_{\{\gamma\in\Gamma^{\mathrm{fin}}|\gamma\cap\Lambda_1\neq\emptyset,\gamma\cap\Lambda_2\neq\emptyset\}} k(\gamma)\lambda^z(d\gamma)\right]-1\right|,$$

and so, as follows from Proposition 1, we will prove that the first term in the right-hand side of formula (4.6) satisfies the estimate

$$\left|\big(f(\Lambda_1\cup\Lambda_2)-f(\Lambda_1)f(\Lambda_2)\big)\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}} F_1(\gamma_{\Lambda_1})F_2(\gamma_{\Lambda_2})\widetilde{p}^{\Lambda_1\cup\Lambda_2}(\gamma_{\Lambda_1\cup\Lambda_2})\lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2})\right|\le$$

$$\le\psi(D_0)\exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big]\|F_1\|_{L_\infty}\|F_2\|_{L_\infty}\tag{4.8}$$

with some constant $D_0=D_0\big(z,\beta,\varkappa,\delta,\min\{|\Lambda_1|,|\Lambda_2|\}\big)$, if we show that

$$I:=\int\limits_{\{\gamma\in\Gamma^{\mathrm{fin}}|\gamma\cap\Lambda_1\ne\emptyset,\gamma\cap\Lambda_2\ne\emptyset\}}\big|k(\gamma)\big|\lambda^z(d\gamma)\le D_0\exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big].\tag{4.9}$$

It is easy to see that

$$\int\limits_{\{\gamma\in\Gamma^{\mathrm{fin}}|\gamma\cap\Lambda_1\ne\emptyset,\gamma\cap\Lambda_2\ne\emptyset\}}\big|k(\gamma)\big|\lambda^z(d\gamma)=\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1}\setminus\{\emptyset\}}\lambda^z(d\gamma_{\Lambda_1})\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_2}\setminus\{\emptyset\}}\lambda^z(d\gamma_{\Lambda_2})\int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}}\lambda^z(d\gamma_{(\Lambda_1\cup\Lambda_2)^c})\big|k(\gamma)\big|.$$

and so, taking into account (2.13) and (2.16), we get that

$$\begin{aligned}I\le&\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1}\setminus\{\emptyset\}}\lambda^z(d\gamma_{\Lambda_1})\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_2}\setminus\{\emptyset\}}\lambda^z(d\gamma_{\Lambda_2})\int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}}\lambda^z(d\gamma_{(\Lambda_1\cup\Lambda_2)^c})(e^{2\beta B})^{\#\gamma}\sum_{\mathcal{T}\in\mathfrak{T}(\gamma)}\prod_{\{x,x'\}\in\mathcal{T}}\big|e^{-\beta\hat v(x-x')}-1\big|\le\\
\le&\big(K(\beta)\big)^{-1}\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1}\setminus\{\emptyset\}}\lambda^z(d\gamma_{\Lambda_1})\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_2}\setminus\{\emptyset\}}\lambda^z(d\gamma_{\Lambda_2})\times\\
&\times\int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}}\lambda^z(d\gamma_{(\Lambda_1\cup\Lambda_2)^c})\big(e^{2\beta B}K(\beta)\big)^{\#\gamma}\sum_{\mathcal{T}\in\mathfrak{T}(\gamma)}\prod_{\{x,x'\}\in\mathcal{T}}e^{-\varkappa|x-x'|}.\end{aligned}$$

Following the idea of [23], we can represent $e^{-\varkappa|x-x'|}$ as

$$\exp\Big[-\frac{\varkappa}{1+\delta}|x-x'|\Big]\exp\Big[-\frac{\delta\varkappa}{1+\delta}|x-x'|\Big],\tag{4.10}$$

and taking into account that the sum of lengths of all bonds which constitute a tree connecting two different subsets Λ_1 and Λ_2 of $\mathbb{R}^d$ is always greater than or equal to the distance between these two subsets, we get that

$$\begin{aligned}I\le&\big(K(\beta)\big)^{-1}\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1}\setminus\{\emptyset\}}\lambda^z(d\gamma_{\Lambda_1})\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_2}\setminus\{\emptyset\}}\lambda^z(d\gamma_{\Lambda_2})\int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}}\lambda^z(d\gamma_{(\Lambda_1\cup\Lambda_2)^c})\big(e^{2\beta B}K(\beta)\big)^{\#\gamma}\times\\
&\times\sum_{\mathcal{T}\in\mathfrak{T}(\gamma)}\exp\Big[-\frac{\varkappa}{1+\delta}\sum_{\{x,x'\}\in\mathcal{T}}|x-x'|\Big]\exp\Big[-\frac{\delta\varkappa}{1+\delta}\sum_{\{x,x'\}\in\mathcal{T}}|x-x'|\Big]\le\\
\le&\big(K(\beta)\big)^{-1}\exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big]\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1}\setminus\{\emptyset\}}\lambda^z(d\gamma_{\Lambda_1})\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_2}\setminus\{\emptyset\}}\lambda^z(d\gamma_{\Lambda_2})\times\\
&\times\int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}}\lambda^z(d\gamma_{(\Lambda_1\cup\Lambda_2)^c})\big(e^{2\beta B}K(\beta)\big)^{\#\gamma}\sum_{\mathcal{T}\in\mathfrak{T}(\gamma)}\exp\Big[-\frac{\delta\varkappa}{1+\delta}\sum_{\{x,x'\}\in\mathcal{T}}|x-x'|\Big]\le\\
\le&\big(K(\beta)\big)^{-1}\exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big]\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1}\setminus\{\emptyset\}}\lambda^z(d\gamma_{\Lambda_1})\times\\
&\times\int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1)^c}\setminus\{\emptyset\}}\lambda^z(d\gamma_{(\Lambda_1)^c})\big(e^{2\beta B}K(\beta)\big)^{\#\gamma}\sum_{\mathcal{T}\in\mathfrak{T}(\gamma)}\exp\Big[-\frac{\delta\varkappa}{1+\delta}\sum_{\{x,x'\}\in\mathcal{T}}|x-x'|\Big].\end{aligned}\tag{4.11}$$

Using the definition of Lebesgue–Poisson measure (1.2), formulae (2.17), (2.18), and subsequently integrating with respect to the ending points of a tree graph $\mathcal{T} \in \mathfrak{T}_{m+n}$ we get that

$$\begin{aligned}
I \leq & \big(K(\beta)\big)^{-1} \exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big] \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\big(ze^{2\beta B}K(\beta)\big)^{n+m}}{n!\,m!} \times \\
& \times \sum_{\mathcal{T}\in\mathfrak{T}_{n+m}} \int_{(\Lambda_1)^n} (dx)_n^1 \int_{(\mathbb{R}^d)^m} (dx)_{n+m}^{n+1} \prod_{\{x,x'\}\in\mathcal{T}} \exp\Big[-\frac{\delta\varkappa}{1+\delta}|x-x'|\Big] \leq \\
\leq & |\Lambda_1|\big(A(\varkappa)K(\beta)\big)^{-1} \exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big] \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\big(zA(\varkappa)e^{2\beta B}K(\beta)\big)^{n+m}}{n!\,m!} \sum_{\mathcal{T}\in\mathfrak{T}_{n+m}} 1 \leq \\
\leq & |\Lambda_1|\big(A(\varkappa)K(\beta)\big)^{-1} \exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big] \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(m+n)!}{n!\,m!}\big(2ze^{2\beta B+1}A(\varkappa)K(\beta)\big)^{n+m} \leq \\
\leq & |\Lambda_1|\big(A(\varkappa)K(\beta)\big)^{-1} \exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big] \sum_{k=2}^{\infty} \big(2ze^{2\beta B+1}A(\varkappa)K(\beta)\big)^{k},
\end{aligned} \tag{4.12}$$

where

$$A(\varkappa) := \int_{\mathbb{R}^d} \exp\Big[-\frac{\delta\varkappa}{1+\delta}|x|\Big]dx = \Omega(d)\frac{(d-1)!}{\varkappa^d}\Big(1+\frac{1}{\delta}\Big)^d. \tag{4.13}$$

So, (4.9) is satisfied with the constant

$$D_0 = \frac{(2ze^{2\beta B+1})^2 A(\varkappa)K(\beta)\min\{|\Lambda_1|,|\Lambda_2|\}}{1-2ze^{\beta B+1}A(\varkappa)K(\beta)}. \tag{4.14}$$

Here, we took into account that I is symmetric with respect to Λ_1 and Λ_2.

Similar but a bit more complicated estimates can be used to bound the second term from (4.6).

Using the second equality from (3.3), formula (3.4) and the 'cluster property' of Lebesgue–Poisson measure we obtain that

$$\begin{aligned}
f(\Lambda_1)f(\Lambda_2)\Bigg[& \int_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}} F_1(\gamma_{\Lambda_1})F_2(\gamma_{\Lambda_2})\widetilde{p}^{\Lambda_1\cup\Lambda_2}(\gamma_{\Lambda_1\cup\Lambda_2})\lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2}) - \\
& - \int_{\Gamma^{\mathrm{fin}}_{\Lambda_1}} F_1(\gamma_{\Lambda_1})\widetilde{p}^{\Lambda_1}(\gamma_{\Lambda_1})\lambda^z_{\Lambda_1}(d\gamma_{\Lambda_1}) \int_{\Gamma^{\mathrm{fin}}_{\Lambda_2}} F_2(\gamma_{\Lambda_2})\widetilde{p}^{\Lambda_2}(\gamma_{\Lambda_2})\lambda^z_{\Lambda_2}(d\gamma_{\Lambda_2})\Bigg] = \\
= f(\Lambda_1)f(\Lambda_2)\Bigg[& \int_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}} F_1(\gamma_{\Lambda_1})F_2(\gamma_{\Lambda_2}) \sum\nolimits^{(\gamma_{\Lambda_1\cup\Lambda_2})} \prod_{i=1}^{n} r^{\Lambda_1\cup\Lambda_2}(\gamma^i_{\Lambda_1\cup\Lambda_2})\lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2}) - \\
& - \int_{\Gamma^{\mathrm{fin}}_{\Lambda_1}} F_1(\gamma_{\Lambda_1}) \sum\nolimits^{(\gamma_{\Lambda_1})} \prod_{i_1=1}^{n_1} r^{\Lambda_1}(\gamma^{i_1}_{\Lambda_1})\lambda^z_{\Lambda_1}(d\gamma_{\Lambda_1}) \int_{\Gamma^{\mathrm{fin}}_{\Lambda_2}} F_2(\gamma_{\Lambda_2}) \sum\nolimits^{(\gamma_{\Lambda_2})} \prod_{i_2=1}^{n_2} r^{\Lambda_2}(\gamma^{i_2}_{\Lambda_2})\lambda^z_{\Lambda_2}(d\gamma_{\Lambda_2})\Bigg] = \\
= f(\Lambda_1)f(\Lambda_2) & \int_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}} \lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2})F_1(\gamma_{\Lambda_1})F_2(\gamma_{\Lambda_2})\Bigg[\sum\nolimits^{(\gamma_{\Lambda_1\cup\Lambda_2})} \prod_{i=1}^{n} r^{\Lambda_1\cup\Lambda_2}(\gamma^i_{\Lambda_1\cup\Lambda_2}) - \\
& - \sum\nolimits^{(\gamma_{\Lambda_1})} \prod_{i_1=1}^{n_1} r^{\Lambda_1}(\gamma^{i_1}_{\Lambda_1}) \sum\nolimits^{(\gamma_{\Lambda_2})} \prod_{i_2=1}^{n_2} r^{\Lambda_2}(\gamma^{i_2}_{\Lambda_2})\Bigg] =
\end{aligned}$$

$$
\begin{aligned}
=& f(\Lambda_1)f(\Lambda_2)\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}} \lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2})F_1(\gamma_{\Lambda_1})F_2(\gamma_{\Lambda_2})\times\\
&\times\Bigg[\sum\nolimits^{(\gamma_{\Lambda_1\cup\Lambda_2})}\prod_{i=1}^{n}\int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} k\big(\gamma^i_{\Lambda_1\cup\Lambda_2}\cup\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\big)\lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\big)-\\
&-\sum\nolimits^{(\gamma_{\Lambda_1})}\prod_{i_1=1}^{n_1}\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1^c}} k(\gamma^{i_1}_{\Lambda_1}\cup\gamma^{i_1}_{\Lambda_1^c})\lambda^z_{\Lambda_1{}^c}(d\gamma^{i_1}_{\Lambda_1^c})\sum\nolimits^{(\gamma_{\Lambda_2})}\prod_{i_2=1}^{n_2}\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_2{}^c}} k(\gamma^{i_2}_{\Lambda_2}\cup\gamma^{i_2}_{\Lambda_2^c})\lambda^z_{\Lambda_2^c}(d\gamma^{i_2}_{\Lambda_2^c})\Bigg]=\\
=& f(\Lambda_1)f(\Lambda_2)\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}} \lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2})F_1(\gamma_{\Lambda_1})F_2(\gamma_{\Lambda_2})\times\\
&\times\Bigg[\sum\nolimits^{(\gamma_{\Lambda_1\cup\Lambda_2})}\prod_{i=1}^{n}\int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} k\big(\gamma^i_{\Lambda_1\cup\Lambda_2}\cup\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\big)\lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\big)-\\
&-\sum\nolimits^{(\gamma_{\Lambda_1})}\prod_{i_1=1}^{n_1}\int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} \lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^{i_1}_{(\Lambda_1\cup\Lambda_2)^c}\big)\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_2}} \lambda^z_{\Lambda_2}(d\gamma^{i_1}_{\Lambda_2})k\big(\gamma^{i_1}_{\Lambda_1}\cup\gamma^{i_1}_{\Lambda_2}\cup\gamma^{i_1}_{(\Lambda_1\cup\Lambda_2)^c}\big)\times\\
&\times\sum\nolimits^{(\gamma_{\Lambda_2})}\prod_{i_2=1}^{n_2}\int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} \lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^{i_2}_{(\Lambda_1\cup\Lambda_2)^c}\big)\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1}} \lambda^z_{\Lambda_1}(d\gamma^{i_2}_{\Lambda_1})k\big(\gamma^{i_2}_{\Lambda_1}\cup\gamma^{i_2}_{\Lambda_2}\cup\gamma^{i_2}_{(\Lambda_1\cup\Lambda_2)^c}\big)\Bigg]=\\
=& f(\Lambda_1)f(\Lambda_2)\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}} \lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2})F_1(\gamma_{\Lambda_1})F_2(\gamma_{\Lambda_2})\times\\
&\times\Bigg[\sum\nolimits^{(\gamma_{\Lambda_1\cup\Lambda_2})}\prod_{i=1}^{n}\int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} k\big(\gamma^i_{\Lambda_1\cup\Lambda_2}\cup\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\big)\lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\big)-\\
&-\sum\nolimits^{(\gamma_{\Lambda_1})}\prod_{i_1=1}^{n_1}\int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} \lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^{i_1}_{(\Lambda_1\cup\Lambda_2)^c}\big)k\big(\gamma^{i_1}_{\Lambda_1}\cup\gamma^{i_1}_{(\Lambda_1\cup\Lambda_2)^c}\big)\times\\
&\times\sum\nolimits^{(\gamma_{\Lambda_2})}\prod_{i_2=1}^{n_2}\int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} \lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^{i_2}_{(\Lambda_1\cup\Lambda_2)^c}\big)k\big(\gamma^{i_2}_{\Lambda_2}\cup\gamma^{i_2}_{(\Lambda_1\cup\Lambda_2)^c}\big)-\\
&-\sum\nolimits^{(\gamma_{\Lambda_1})}\sum\nolimits^{(\gamma_{\Lambda_2})}\sum_{\substack{(\Theta^1_{\Lambda_2},\dots,\Theta^{n_1}_{\Lambda_2},\Theta^1_{\Lambda_1},\dots,\Theta^{n_2}_{\Lambda_1})\\ \Theta^{i_1}_{\Lambda_2}\in\{\{\emptyset\},\Gamma^{\mathrm{fin}}_{\Lambda_2}\setminus\{\emptyset\}\},\ i_1=1,\dots,n_1\\ \Theta^{i_2}_{\Lambda_1}\in\{\{\emptyset\},\Gamma^{\mathrm{fin}}_{\Lambda_1}\setminus\{\emptyset\}\},\ i_2=1,\dots,n_2\\ \#(\Theta^1_{\Lambda_2}\cup\dots\cup\Theta^{n_1}_{\Lambda_2}\cup\Theta^1_{\Lambda_1}\cup\dots\cup\Theta^{n_2}_{\Lambda_1})\neq 1}}\prod_{i_1=1}^{n_1}\int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} \lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^{i_1}_{(\Lambda_1\cup\Lambda_2)^c}\big)\times\\
&\times\int\limits_{\Theta^{i_1}_{\Lambda_2}} \lambda^z_{\Lambda_2}(d\gamma^{i_1}_{\Lambda_2})k\big(\gamma^{i_1}_{\Lambda_1}\cup\gamma^{i_1}_{\Lambda_2}\cup\gamma^{i_1}_{(\Lambda_1\cup\Lambda_2)^c}\big)\prod_{i_2=1}^{n_2}\int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} \lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^{i_2}_{(\Lambda_1\cup\Lambda_2)^c}\big)\times\\
&\times\int\limits_{\Theta^{i_2}_{\Lambda_1}} \lambda^z_{\Lambda_1}(d\gamma^{i_2}_{\Lambda_1})k\big(\gamma^{i_2}_{\Lambda_1}\cup\gamma^{i_2}_{\Lambda_2}\cup\gamma^{i_2}_{(\Lambda_1\cup\Lambda_2)^c}\big)\Bigg]=
\end{aligned}
$$

$$
\begin{aligned}
=&f(\Lambda_1)f(\Lambda_2)\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}}\lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2})F_1(\gamma_{\Lambda_1})F_2(\gamma_{\Lambda_2})\times\\
&\times\Bigg[\sum\nolimits^{(\gamma_{\Lambda_1},\gamma_{\Lambda_2})}\prod_{i=1}^{n}\int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}}k\big(\gamma^i_{\Lambda_1\cup\Lambda_2}\cup\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\big)\lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\big)-\\
&-\sum\nolimits^{(\gamma_{\Lambda_1})}\sum\nolimits^{(\gamma_{\Lambda_2})}\sum_{\substack{(\Theta^1_{\Lambda_2},\dots,\Theta^{n_1}_{\Lambda_2},\Theta^1_{\Lambda_1},\dots,\Theta^{n_2}_{\Lambda_1})\\ \Theta^{i_1}_{\Lambda_2}\in\{\{\emptyset\},\Gamma^{\mathrm{fin}}_{\Lambda_2}\setminus\{\emptyset\}\},\ i_1=1,\dots,n_1\\ \Theta^{i_2}_{\Lambda_1}\in\{\{\emptyset\},\Gamma^{\mathrm{fin}}_{\Lambda_1}\setminus\{\emptyset\}\},\ i_2=1,\dots,n_2\\ \#(\Theta^1_{\Lambda_2}\cup\dots\cup\Theta^{n_1}_{\Lambda_2}\cup\Theta^1_{\Lambda_1}\cup\dots\cup\Theta^{n_2}_{\Lambda_1})\neq 1}}\prod_{i_1=1}^{n_1}\int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}}\lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^{i_1}_{(\Lambda_1\cup\Lambda_2)^c}\big)\times\\
&\times\int\limits_{\Theta^{i_1}_{\Lambda_2}}\lambda^z_{\Lambda_2}(d\gamma^{i_1}_{\Lambda_2})k\big(\gamma^{i_1}_{\Lambda_1}\cup\gamma^{i_1}_{\Lambda_2}\cup\gamma^{i_1}_{(\Lambda_1\cup\Lambda_2)^c}\big)\prod_{i_2=1}^{n_2}\int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}}\lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^{i_2}_{(\Lambda_1\cup\Lambda_2)^c}\big)\times\\
&\times\int\limits_{\Theta^{i_2}_{\Lambda_1}}\lambda^z_{\Lambda_1}(d\gamma^{i_2}_{\Lambda_1})k\big(\gamma^{i_2}_{\Lambda_1}\cup\gamma^{i_2}_{\Lambda_2}\cup\gamma^{i_2}_{(\Lambda_1\cup\Lambda_2)^c}\big)\Bigg].
\end{aligned}
\tag{4.15}
$$

Here $\sum^{(\gamma_{\Lambda_1},\gamma_{\Lambda_2})}$ denotes the sum over all partitions of $\gamma_{\Lambda_1}\cup\gamma_{\Lambda_2}$ into non-empty subsets so that at least one subset contains points, or elements, both from γ_{Λ_1} and γ_{Λ_2}, *ie*,

$$
\sum\nolimits^{(\gamma_1,\gamma_2)}[\dots]=\sum_{n=1}^{\#(\gamma_1\cup\gamma_2)}\sum_{\substack{\{\gamma^1,\dots,\gamma^n\}\\ \gamma^1\cup\dots\cup\gamma^n=\gamma_1\cup\gamma_2\\ \gamma^i\cap\gamma^j=\emptyset,\ i\neq j\\ \gamma^i\neq\emptyset,\ i=1,\dots,n\\ \exists k\in\{1,\dots,n\}:\gamma_i\cap\gamma^k\neq\emptyset,\ i=1,2}}[\dots].
\tag{4.16}
$$

Let us estimate the absolute value of the first term in the right-hand side of (4.15). Using estimates (2.13) and (3.10) on the absolute values of $k(\gamma)$ and $f(\Lambda)$, inequality (2.16), and estimating the absolute values of F_1 and F_2 by their supremum norms, we get that

$$
\begin{aligned}
I_1=&f(\Lambda_1)f(\Lambda_2)\Bigg|\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}}\lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2})F_1(\gamma_{\Lambda_1})F_2(\gamma_{\Lambda_2})\times\\
&\times\sum\nolimits^{(\gamma_{\Lambda_1},\gamma_{\Lambda_2})}\prod_{i=1}^{n}\int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}}k\big(\gamma^i_{\Lambda_1\cup\Lambda_2}\cup\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\big)\lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\big)\Bigg|\leq\\
\leq&f(\Lambda_1)f(\Lambda_2)\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}}\lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2})\big|F_1(\gamma_{\Lambda_1})\big|\,\big|F_2(\gamma_{\Lambda_2})\big|\times\\
&\times\sum\nolimits^{(\gamma_{\Lambda_1},\gamma_{\Lambda_2})}\prod_{i=1}^{n}\int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}}\big|k\big(\gamma^i_{\Lambda_1\cup\Lambda_2}\cup\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\big)\big|\lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\big)\leq\\
\leq&e^{D_1(|\Lambda_1|)}e^{D_1(|\Lambda_2|)}\int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}}\lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2})\big|F_1(\gamma_{\Lambda_1})\big|\,\big|F_2(\gamma_{\Lambda_2})\big|\times\\
&\times\sum\nolimits^{(\gamma_{\Lambda_1},\gamma_{\Lambda_2})}\prod_{i=1}^{n}\int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}}(e^{2\beta B})^{\#\left(\gamma^i_{\Lambda_1\cup\Lambda_2}\cup\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\right)}\times\\
&\times\sum_{\mathcal{T}\in\mathfrak{T}\left(\gamma^i_{\Lambda_1\cup\Lambda_2}\cup\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\right)}\prod_{\{x,x\}\in\mathcal{T}}|e^{-\beta v(x,x')}-1|\,\lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\big)\leq
\end{aligned}
$$

$$
\begin{aligned}
&\leq e^{D_1\left(|\Lambda_1|\right)} e^{D_1\left(|\Lambda_2|\right)} \int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}} \lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2}) \left|F_1(\gamma_{\Lambda_1})\right| \left|F_2(\gamma_{\Lambda_2})\right| \times \\
&\quad \times \sum\nolimits^{(\gamma_{\Lambda_1},\gamma_{\Lambda_2})} \prod_{i=1}^{n} \int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} (e^{2\beta B})^{\#\left(\gamma^i_{\Lambda_1\cup\Lambda_2}\cup\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\right)} \times \\
&\qquad \times \sum_{\mathcal{T}\in\mathfrak{T}\left(\gamma^i_{\Lambda_1\cup\Lambda_2}\cup\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\right)} \prod_{\{x,x\}\in\mathcal{T}} \left(K(\beta)e^{-\varkappa|x-x|}\right) \lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\left(d\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\right) \leq \\
&\leq e^{D_1\left(|\Lambda_1|\right)} e^{D_1\left(|\Lambda_2|\right)} \|F_1\|_{L_\infty}\|F_2\|_{L_\infty} \int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}} \lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2}) \times \\
&\quad \times \sum\nolimits^{(\gamma_{\Lambda_1},\gamma_{\Lambda_2})} K(\beta)^{-n} \prod_{i=1}^{n} \int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} \left(e^{2\beta B}K(\beta)\right)^{\#\left(\gamma^i_{\Lambda_1\cup\Lambda_2}\cup\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\right)} \times \\
&\qquad \times \sum_{\mathcal{T}\in\mathfrak{T}\left(\gamma^i_{\Lambda_1\cup\Lambda_2}\cup\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\right)} \exp\Big[-\sum_{\{x,x\}\in\mathcal{T}} \varkappa|x-x|\Big] \lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\left(d\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\right).
\end{aligned}
$$

Because of definition (4.16) the same trick as in (4.10)–(4.11) yields that

$$
\begin{aligned}
I_1 &\leq e^{D_1\left(|\Lambda_1|\right)} e^{D_1\left(|\Lambda_2|\right)} \|F_1\|_{L_\infty}\|F_2\|_{L_\infty} \exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big] \int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}} \lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2}) \times \\
&\quad \times \sum\nolimits^{(\gamma_{\Lambda_1},\gamma_{\Lambda_2})} K(\beta)^{-n} \prod_{i=1}^{n} \int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} \left(e^{2\beta B}K(\beta)\right)^{\#\left(\gamma^i_{\Lambda_1\cup\Lambda_2}\cup\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\right)} \times \\
&\qquad \times \sum_{\mathcal{T}\in\mathfrak{T}\left(\gamma^i_{\Lambda_1\cup\Lambda_2}\cup\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\right)} \exp\Big[-\frac{\delta\varkappa}{1+\delta}\sum_{\{x,x\}\in\mathcal{T}} |x-x|\Big] \lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\left(d\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\right) \leq \\
&\leq e^{D_1\left(|\Lambda_1|\right)} e^{D_1\left(|\Lambda_2|\right)} \|F_1\|_{L_\infty}\|F_2\|_{L_\infty} \exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big] \int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}\setminus\{\emptyset\}} \lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2}) \times \\
&\quad \times \sum\nolimits^{(\gamma_{\Lambda_1}\cup\gamma_{\Lambda_2})} K(\beta)^{-n} \prod_{i=1}^{n} \int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} \left(e^{2\beta B}K(\beta)\right)^{\#\left(\gamma^i_{\Lambda_1\cup\Lambda_2}\cup\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\right)} \times \\
&\qquad \times \sum_{\mathcal{T}\in\mathfrak{T}\left(\gamma^i_{\Lambda_1\cup\Lambda_2}\cup\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\right)} \prod_{\{x,x\}\in\mathcal{T}} \exp\Big[-\frac{\delta\varkappa}{1+\delta}|x-x|\Big] \lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\left(d\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\right),
\end{aligned} \tag{4.17}
$$

where we used the fact that

$$
\sum\nolimits^{(\gamma_1,\gamma_2)}[\ldots] = 0 \quad \text{if} \quad \gamma_1\cup\gamma_2 = \emptyset,
$$

and then, changed the sum $\sum^{(\gamma_{\Lambda_1},\gamma_{\Lambda_2})}$ into $\sum^{(\gamma_{\Lambda_1}\cup\gamma_{\Lambda_2})}$.

Taking into account definition (2.11) and applying formula (1.4), we obtain that

$$
\begin{aligned}
I_1 \leq & e^{D_1\left(|\Lambda_1|\right)} e^{D_1\left(|\Lambda_2|\right)} \|F_1\|_{L_\infty} \|F_2\|_{L_\infty} \exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big] \int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}\setminus\{\emptyset\}} \lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2}) \times \\
& \times \sum_{n=1}^{\#(\gamma_{\Lambda_1}\cup\gamma_{\Lambda_2})} \frac{1}{n!} \sum_{\substack{(\gamma^1_{\Lambda_1\cup\Lambda_2},\dots,\gamma^n_{\Lambda_1\cup\Lambda_2}) \\ \gamma^1_{\Lambda_1\cup\Lambda_2}\cup\dots\cup\gamma^n_{\Lambda_1\cup\Lambda_2}=\gamma \\ \gamma^i_{\Lambda_1\cup\Lambda_2}\cap\gamma^j_{\Lambda_1\cup\Lambda_2}=\emptyset,\ i\neq j \\ \gamma^i_{\Lambda_1\cup\Lambda_2}\neq\emptyset,\ i=1,\dots,n}} K(\beta)^{-n} \prod_{i=1}^n \int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} \left(e^{2\beta B}K(\beta)\right)^{\#\left(\gamma^i_{\Lambda_1\cup\Lambda_2}\cup\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\right)} \times \\
& \qquad \times \sum_{\mathcal{T}\in\mathfrak{T}\left(\gamma^i_{\Lambda_1\cup\Lambda_2}\cup\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\right)} \prod_{\{x,x\}\in\mathcal{T}} \exp\Big[-\frac{\delta\varkappa}{1+\delta}|x-x|\Big] \lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\left(d\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\right) = \\
\leq & e^{D_1\left(|\Lambda_1|\right)} e^{D_1\left(|\Lambda_2|\right)} \|F_1\|_{L_\infty} \|F_2\|_{L_\infty} \exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big] \sum_{n=1}^\infty \frac{1}{n!} \int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}\setminus\{\emptyset\}} \lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2}) \times \\
& \times \sum_{\substack{(\gamma^1_{\Lambda_1\cup\Lambda_2},\dots,\gamma^n_{\Lambda_1\cup\Lambda_2}) \\ \gamma^1_{\Lambda_1\cup\Lambda_2}\cup\dots\cup\gamma^n_{\Lambda_1\cup\Lambda_2}=\gamma \\ \gamma^i_{\Lambda_1\cup\Lambda_2}\cap\gamma^j_{\Lambda_1\cup\Lambda_2}=\emptyset,\ i\neq j \\ \gamma^i_{\Lambda_1\cup\Lambda_2}\neq\emptyset,\ i=1,\dots,n}} K(\beta)^{-n} \prod_{i=1}^n \int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} \left(e^{2\beta B}K(\beta)\right)^{\#\left(\gamma^i_{\Lambda_1\cup\Lambda_2}\cup\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\right)} \times \\
& \qquad \times \sum_{\mathcal{T}\in\mathfrak{T}\left(\gamma^i_{\Lambda_1\cup\Lambda_2}\cup\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\right)} \prod_{\{x,x\}\in\mathcal{T}} \exp\Big[-\frac{\delta\varkappa}{1+\delta}|x-x|\Big] \lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\left(d\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\right) = \\
\leq & e^{D_1\left(|\Lambda_1|\right)} e^{D_1\left(|\Lambda_2|\right)} \|F_1\|_{L_\infty} \|F_2\|_{L_\infty} \exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big] \sum_{n=1}^\infty \frac{1}{n!} \times \\
& \times \Bigg(K(\beta)^{-1} \int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}\setminus\{\emptyset\}} \lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2}) \int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} \lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\left(d\gamma_{(\Lambda_1\cup\Lambda_2)^c}\right) \times \\
& \qquad \times \left(e^{2\beta B}K(\beta)\right)^{\#\left(\gamma_{\Lambda_1\cup\Lambda_2}\cup\gamma_{(\Lambda_1\cup\Lambda_2)^c}\right)} \sum_{\mathcal{T}\in\mathfrak{T}\left(\gamma^i_{\Lambda_1\cup\Lambda_2}\cup\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\right)} \prod_{\{x,x\}\in\mathcal{T}} \exp\Big[-\frac{\delta\varkappa}{1+\delta}|x-x|\Big] \Bigg)^n.
\end{aligned}
$$

But in the same way as it was done in (4.11)–(4.12) we get that

$$
\begin{aligned}
& \int\limits_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}\setminus\{\emptyset\}} \lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2}) \int\limits_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} \lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\left(d\gamma_{(\Lambda_1\cup\Lambda_2)^c}\right) \left(e^{2\beta B}K(\beta)\right)^{\#\left(\gamma_{\Lambda_1\cup\Lambda_2}\cup\gamma_{(\Lambda_1\cup\Lambda_2)^c}\right)} \times \\
& \qquad \times \sum_{\mathcal{T}\in\mathfrak{T}\left(\gamma^i_{\Lambda_1\cup\Lambda_2}\cup\gamma^i_{(\Lambda_1\cup\Lambda_2)^c}\right)} \prod_{\{x,x\}\in\mathcal{T}} \exp\Big[-\frac{\delta\varkappa}{1+\delta}|x-x|\Big] = \\
& = \sum_{n=1}^\infty \sum_{m=0}^\infty \frac{\left(ze^{2\beta B}K(\beta)\right)^{n+m}}{n!m!} \int\limits_{(\Lambda_1\cup\Lambda_2)^n} (dx)^1_n \int\limits_{\left((\Lambda_1\cup\Lambda_2)^c\right)^m} (dx)^{n+1}_{n+m} \times \\
& \qquad \times \sum_{\mathcal{T}\in\mathfrak{T}\left(\{x_1,\dots,x_{n+m}\}\right)} \prod_{\{x,x\}\in\mathcal{T}} \exp\Big[-\frac{\delta\varkappa}{1+\delta}|x-x|\Big] \leq \\
& \leq \left(|\Lambda_1\cup\Lambda_2|\right) A(\varkappa)^{-1} \sum_{n=1}^\infty \sum_{m=0}^\infty \frac{(n+m)!}{n!m!} \left(ze^{2\beta B+1}A(\varkappa)K(\beta)\right)^{n+m} \leq \frac{2ze^{2\beta B+1}K(\beta)|\Lambda_1\cup\Lambda_2|}{1-2zA(\varkappa)e^{2\beta B+1}K(\beta)},
\end{aligned}
$$

which implies that

$$I_1 \le D_2 \exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big]\|F_1\|_{L_\infty}\|F_2\|_{L_\infty} \tag{4.18}$$

with

$$D_2 = D_2(z,\beta,\varkappa,|\Lambda_1|,|\Lambda_2|) := e^{D_1(|\Lambda_1|)}e^{D_1(|\Lambda_2|)}\Big\{\exp\Big[\frac{4ze^{2\beta B+1}\max\{|\Lambda_1|,|\Lambda_2|\}}{1-2ze^{\beta B+1}A(\varkappa)K(\beta)}\Big]-1\Big\}. \tag{4.19}$$

Remark 10. It is possible, starting from formula (4.17) to obtain the same estimate on I_1 in a way which does not use the formula (1.4). The corresponding calculations are given in Appendix B.

Let us now go to the estimate of the absolute value of the second term in the right-hand side of (4.15). Estimating the absolute values of F_1 and F_2 by their supremum norms, taking into account (3.10) and using (1.4) in a way similar to that in estimating I_1, we get

$$\begin{aligned}
I_2 =& f(\Lambda_1)f(\Lambda_2)\Bigg|\int_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}}\lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2})F_1(\gamma_{\Lambda_1})F_2(\gamma_{\Lambda_2})\sum\nolimits^{(\gamma_{\Lambda_1})}\sum\nolimits^{(\gamma_{\Lambda_2})}\sum_{\substack{(\Theta^1_{\Lambda_2},\dots,\Theta^{n_1}_{\Lambda_2},\Theta^1_{\Lambda_1},\dots,\Theta^{n_2}_{\Lambda_1})\\ \Theta^{i_1}_{\Lambda_2}\in\{\{\emptyset\},\Gamma^{\mathrm{fin}}_{\Lambda_2}\setminus\{\emptyset\}\},\ i_1=1,\dots,n_1\\ \Theta^{i_2}_{\Lambda_1}\in\{\{\emptyset\},\Gamma^{\mathrm{fin}}_{\Lambda_1}\setminus\{\emptyset\}\},\ i_2=1,\dots,n_2\\ \#(\Theta^1_{\Lambda_2}\cup\dots\cup\Theta^{n_1}_{\Lambda_2}\cup\Theta^1_{\Lambda_1}\cup\dots\cup\Theta^{n_2}_{\Lambda_1})\neq 1}}\times\\
&\times\prod_{i_1=1}^{n_1}\int_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}}\lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^{i_1}_{(\Lambda_1\cup\Lambda_2)^c}\big)\int_{\Theta^{i_1}_{\Lambda_2}}\lambda^z_{\Lambda_2}(d\gamma^{i_1}_{\Lambda_2})k\big(\gamma^{i_1}_{\Lambda_1}\cup\gamma^{i_1}_{\Lambda_2}\cup\gamma^{i_1}_{(\Lambda_1\cup\Lambda_2)^c}\big)\times\\
&\times\prod_{i_2=1}^{n_2}\int_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}}\lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^{i_2}_{(\Lambda_1\cup\Lambda_2)^c}\big)\int_{\Theta^{i_2}_{\Lambda_1}}\lambda^z_{\Lambda_1}(d\gamma^{i_2}_{\Lambda_1})k\big(\gamma^{i_2}_{\Lambda_1}\cup\gamma^{i_2}_{\Lambda_2}\cup\gamma^{i_2}_{(\Lambda_1\cup\Lambda_2)^c}\big)\Bigg|\le\\
\le& e^{D_1(|\Lambda_1|)}e^{D_1(|\Lambda_2|)}\|F_1\|_{L_\infty}\|F_2\|_{L_\infty}\int_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}}\lambda^z_{\Lambda_1\cup\Lambda_2}(d\gamma_{\Lambda_1\cup\Lambda_2})\sum\nolimits^{(\gamma_{\Lambda_1})}\sum\nolimits^{(\gamma_{\Lambda_2})}\sum_{\substack{(\Theta^1_{\Lambda_2},\dots,\Theta^{n_1}_{\Lambda_2},\Theta^1_{\Lambda_1},\dots,\Theta^{n_2}_{\Lambda_1})\\ \Theta^{i_1}_{\Lambda_2}\in\{\{\emptyset\},\Gamma^{\mathrm{fin}}_{\Lambda_2}\setminus\{\emptyset\}\},\ i_1=1,\dots,n_1\\ \Theta^{i_2}_{\Lambda_1}\in\{\{\emptyset\},\Gamma^{\mathrm{fin}}_{\Lambda_1}\setminus\{\emptyset\}\},\ i_2=1,\dots,n_2\\ \#(\Theta^1_{\Lambda_2}\cup\dots\cup\Theta^{n_1}_{\Lambda_2}\cup\Theta^1_{\Lambda_1}\cup\dots\cup\Theta^{n_2}_{\Lambda_1})\neq 1}}\times\\
&\times\prod_{i_1=1}^{n_1}\int_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}}\lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^{i_1}_{(\Lambda_1\cup\Lambda_2)^c}\big)\int_{\Theta^{i_1}_{\Lambda_2}}\lambda^z_{\Lambda_2}(d\gamma^{i_1}_{\Lambda_2})\big|k\big(\gamma^{i_1}_{\Lambda_1}\cup\gamma^{i_1}_{\Lambda_2}\cup\gamma^{i_1}_{(\Lambda_1\cup\Lambda_2)^c}\big)\big|\times\\
&\times\prod_{i_2=1}^{n_2}\int_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}}\lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^{i_2}_{(\Lambda_1\cup\Lambda_2)^c}\big)\int_{\Theta^{i_2}_{\Lambda_1}}\lambda^z_{\Lambda_1}(d\gamma^{i_2}_{\Lambda_1})\big|k\big(\gamma^{i_2}_{\Lambda_1}\cup\gamma^{i_2}_{\Lambda_2}\cup\gamma^{i_2}_{(\Lambda_1\cup\Lambda_2)^c}\big)\big|=\\
\le& e^{D_1(|\Lambda_1|)}e^{D_1(|\Lambda_2|)}\|F_1\|_{L_\infty}\|F_2\|_{L_\infty}\int_{\Gamma^{\mathrm{fin}}_{\Lambda_1}\setminus\{\emptyset\}}\lambda^z_{\Lambda_1}(d\gamma_{\Lambda_1})\int_{\Gamma^{\mathrm{fin}}_{\Lambda_2}\setminus\{\emptyset\}}\lambda^z_{\Lambda_2}(d\gamma_{\Lambda_2})\times\\
&\times\sum_{n_1=1}^{\#\gamma_{\Lambda_1}}\frac{1}{n_1!}\sum_{\substack{(\gamma^1_{\Lambda_1},\dots,\gamma^{n_1}_{\Lambda_1})\\ \gamma^1_{\Lambda_1}\cup\dots\cup\gamma^{n_1}_{\Lambda_1}=\gamma_{\Lambda_1}\\ \gamma^{i_1}_{\Lambda_1}\cap\gamma^{j_1}_{\Lambda_1}=\emptyset,\ i_1\neq j_1\\ \gamma^{i_1}_{\Lambda_1}\neq\emptyset,\ i_1=1,\dots,n_1}}\sum_{n_2=1}^{\#\gamma_{\Lambda_2}}\frac{1}{n_2!}\sum_{\substack{(\gamma^1_{\Lambda_2},\dots,\gamma^{n_2}_{\Lambda_2})\\ \gamma^1_{\Lambda_2}\cup\dots\cup\gamma^{n_2}_{\Lambda_2}=\gamma_{\Lambda_2}\\ \gamma^{i_2}_{\Lambda_2}\cap\gamma^{j_2}_{\Lambda_2}=\emptyset,\ i_2\neq j_2\\ \gamma^{i_2}_{\Lambda_2}\neq\emptyset,\ i_2=1,\dots,n_2}}\sum_{\substack{(\Theta^1_{\Lambda_2},\dots,\Theta^{n_1}_{\Lambda_2},\Theta^1_{\Lambda_1},\dots,\Theta^{n_2}_{\Lambda_1})\\ \Theta^{i_1}_{\Lambda_2}\in\{\{\emptyset\},\Gamma^{\mathrm{fin}}_{\Lambda_2}\setminus\{\emptyset\}\},\ i_1=1,\dots,n_1\\ \Theta^{i_2}_{\Lambda_1}\in\{\{\emptyset\},\Gamma^{\mathrm{fin}}_{\Lambda_1}\setminus\{\emptyset\}\},\ i_2=1,\dots,n_2\\ \#(\Theta^1_{\Lambda_2}\cup\dots\cup\Theta^{n_1}_{\Lambda_2}\cup\Theta^1_{\Lambda_1}\cup\dots\cup\Theta^{n_2}_{\Lambda_1})\neq 1}}\times\\
&\times\prod_{i_1=1}^{n_1}\int_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}}\lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^{i_1}_{(\Lambda_1\cup\Lambda_2)^c}\big)\int_{\Theta^{i_1}_{\Lambda_2}}\lambda^z_{\Lambda_2}(d\gamma^{i_1}_{\Lambda_2})\big|k\big(\gamma^{i_1}_{\Lambda_1}\cup\gamma^{i_1}_{\Lambda_2}\cup\gamma^{i_1}_{(\Lambda_1\cup\Lambda_2)^c}\big)\big|\times\\
&\times\prod_{i_2=1}^{n_2}\int_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}}\lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^{i_2}_{(\Lambda_1\cup\Lambda_2)^c}\big)\int_{\Theta^{i_2}_{\Lambda_1}}\lambda^z_{\Lambda_1}(d\gamma^{i_2}_{\Lambda_1})\big|k\big(\gamma^{i_2}_{\Lambda_1}\cup\gamma^{i_2}_{\Lambda_2}\cup\gamma^{i_2}_{(\Lambda_1\cup\Lambda_2)^c}\big)\big|=
\end{aligned}$$

$$\leq e^{D_1(|\Lambda_1|)} e^{D_1(|\Lambda_2|)} \|F_1\|_{L_\infty} \|F_2\|_{L_\infty} \sum_{n_1=1}^{\infty} \frac{1}{n_1!} \sum_{n_2=1}^{\infty} \frac{1}{n_2!} \sum_{\substack{(\Theta^1_{\Lambda_2},\dots,\Theta^{n_1}_{\Lambda_2},\Theta^1_{\Lambda_1},\dots,\Theta^{n_2}_{\Lambda_1}) \\ \Theta^{i_1}_{\Lambda_2} \in \{\{\emptyset\},\Gamma^{\mathrm{fin}}_{\Lambda_2}\setminus\{\emptyset\}\},\ i_1=1,\dots,n_1 \\ \Theta^{i_2}_{\Lambda_1} \in \{\{\emptyset\},\Gamma^{\mathrm{fin}}_{\Lambda_1}\setminus\{\emptyset\}\},\ i_2=1,\dots,n_2 \\ \#(\Theta^1_{\Lambda_2}\cup\cdots\cup\Theta^{n_1}_{\Lambda_2}\cup\Theta^1_{\Lambda_1}\cup\cdots\cup\Theta^{n_2}_{\Lambda_1})\neq 1}} \times$$

$$\times \int_{\Gamma^{\mathrm{fin}}_{\Lambda_1}\setminus\{\emptyset\}} \lambda^z_{\Lambda_1}(d\gamma_{\Lambda_1}) \sum_{\substack{(\gamma^1_{\Lambda_1},\dots,\gamma^{n_1}_{\Lambda_1}) \\ \gamma^1_{\Lambda_1}\cup\cdots\cup\gamma^{n_1}_{\Lambda_1}=\gamma_{\Lambda_1} \\ \gamma^{i_1}_{\Lambda_1}\cap\gamma^{j_1}_{\Lambda_1}=\emptyset,\ i_1\neq j_1 \\ \gamma^{i_1}_{\Lambda_1}\neq\emptyset,\ i_1=1,\dots,n_1}} \prod_{i_1=1}^{n_1} \int_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} \lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^{i_1}_{(\Lambda_1\cup\Lambda_2)^c}\big) \times$$

$$\times \int_{\Theta^{i_1}_{\Lambda_2}} \lambda^z_{\Lambda_2}(d\gamma^{i_1}_{\Lambda_2}) \big| k\big(\gamma^{i_1}_{\Lambda_1}\cup\gamma^{i_1}_{\Lambda_2}\cup\gamma^{i_1}_{(\Lambda_1\cup\Lambda_2)^c}\big)\big| \times$$

$$\times \int_{\Gamma^{\mathrm{fin}}_{\Lambda_2}\setminus\{\emptyset\}} \lambda^z_{\Lambda_2}(d\gamma_{\Lambda_2}) \sum_{\substack{(\gamma^1_{\Lambda_2},\dots,\gamma^{n_2}_{\Lambda_2}) \\ \gamma^1_{\Lambda_2}\cup\cdots\cup\gamma^{n_2}_{\Lambda_2}=\gamma_{\Lambda_2} \\ \gamma^{i_2}_{\Lambda_2}\cap\gamma^{j_2}_{\Lambda_2}=\emptyset,\ i_2\neq j_2 \\ \gamma^{i_2}_{\Lambda_2}\neq\emptyset,\ i_2=1,\dots,n_2}} \prod_{i_2=1}^{n_2} \int_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} \lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^{i_2}_{(\Lambda_1\cup\Lambda_2)^c}\big) \times$$

$$\times \int_{\Theta^{i_2}_{\Lambda_1}} \lambda^z_{\Lambda_1}(d\gamma^{i_2}_{\Lambda_1}) \big| k\big(\gamma^{i_2}_{\Lambda_1}\cup\gamma^{i_2}_{\Lambda_2}\cup\gamma^{i_2}_{(\Lambda_1\cup\Lambda_2)^c}\big)\big| =$$

$$\leq e^{D_1(|\Lambda_1|)} e^{D_1(|\Lambda_2|)} \|F_1\|_{L_\infty} \|F_2\|_{L_\infty} \sum_{n_1=1}^{\infty} \frac{1}{n_1!} \sum_{n_2=1}^{\infty} \frac{1}{n_2!} \sum_{\substack{(\Theta^1_{\Lambda_2},\dots,\Theta^{n_1}_{\Lambda_2},\Theta^1_{\Lambda_1},\dots,\Theta^{n_2}_{\Lambda_1}) \\ \Theta^{i_1}_{\Lambda_2} \in \{\{\emptyset\},\Gamma^{\mathrm{fin}}_{\Lambda_2}\setminus\{\emptyset\}\},\ i_1=1,\dots,n_1 \\ \Theta^{i_2}_{\Lambda_1} \in \{\{\emptyset\},\Gamma^{\mathrm{fin}}_{\Lambda_1}\setminus\{\emptyset\}\},\ i_2=1,\dots,n_2 \\ \#(\Theta^1_{\Lambda_2}\cup\cdots\cup\Theta^{n_1}_{\Lambda_2}\cup\Theta^1_{\Lambda_1}\cup\cdots\cup\Theta^{n_2}_{\Lambda_1})\neq 1}} \times$$

$$\times \prod_{i_1=1}^{n_1} \int_{\Gamma^{\mathrm{fin}}_{\Lambda_1}\setminus\{\emptyset\}} \lambda^z_{\Lambda_1}(d\gamma_{\Lambda_1}) \int_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} \lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^{i_1}_{(\Lambda_1\cup\Lambda_2)^c}\big) \int_{\Theta^{i_1}_{\Lambda_2}} \lambda^z_{\Lambda_2}(d\gamma^{i_1}_{\Lambda_2}) \big| k\big(\gamma^{i_1}_{\Lambda_1}\cup\gamma^{i_1}_{\Lambda_2}\cup\gamma^{i_1}_{(\Lambda_1\cup\Lambda_2)^c}\big)\big| \times$$

$$\times \prod_{i_2=1}^{n_2} \int_{\Gamma^{\mathrm{fin}}_{\Lambda_2}\setminus\{\emptyset\}} \lambda^z_{\Lambda_2}(d\gamma_{\Lambda_2}) \int_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} \lambda^z_{(\Lambda_1\cup\Lambda_2)^c}\big(d\gamma^{i_2}_{(\Lambda_1\cup\Lambda_2)^c}\big) \int_{\Theta^{i_2}_{\Lambda_1}} \lambda^z_{\Lambda_1}(d\gamma^{i_2}_{\Lambda_1}) \big| k\big(\gamma^{i_2}_{\Lambda_1}\cup\gamma^{i_2}_{\Lambda_2}\cup\gamma^{i_2}_{(\Lambda_1\cup\Lambda_2)^c}\big)\big|. \tag{4.20}$$

As it follows from the condition $\#(\Theta^1_{\Lambda_2}\cup\cdots\cup\Theta^{n_1}_{\Lambda_2}\cup\Theta^1_{\Lambda_1}\cup\cdots\cup\Theta^{n_2}_{\Lambda_1})\neq 1$ in the sum over $(\Theta^1_{\Lambda_2},\dots,\Theta^{n_1}_{\Lambda_2},\Theta^1_{\Lambda_1},\dots,\dots,\Theta^{n_2}_{\Lambda_1})$, every summand in the right-hand side of (4.20) contains at least one integral of the form

$$\int_{\Gamma^{\mathrm{fin}}_{\Lambda_1}\setminus\{\emptyset\}} \lambda^z(d\gamma_{\Lambda_1}) \int_{\Gamma^{\mathrm{fin}}_{\Lambda_2}\setminus\{\emptyset\}} \lambda^z(d\gamma_{\Lambda_2}) \int_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} \lambda^z(d\gamma_{(\Lambda_1\cup\Lambda_2)^c}) \big| k(\gamma_{\Lambda_1}\cup\gamma_{\Lambda_2}\cup\gamma_{(\Lambda_1\cup\Lambda_2)^c})\big|.$$

But as it was already shown in (4.9)–(4.14), this integral can be estimated by

$$D_0 \exp\Big[-\frac{\varkappa}{1+\delta}\,\mathrm{dist}(\Lambda_1,\Lambda_2)\Big].$$

The other integrals of the type

$$\int_{\Gamma^{\mathrm{fin}}_{\Lambda_i}\setminus\{\emptyset\}} \lambda^z(d\gamma_{\Lambda_i}) \int_{\Gamma^{\mathrm{fin}}_{\Lambda_{\{1,2\}\setminus\{i\}}}\setminus\{\emptyset\}} \lambda^z(d\gamma_{\Lambda_{\{1,2\}\setminus\{i\}}}) \int_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} \lambda^z(d\gamma_{(\Lambda_1\cup\Lambda_2)^c}) \big| k(\gamma_{\Lambda_1}\cup\gamma_{\Lambda_2}\cup\gamma_{(\Lambda_1\cup\Lambda_2)^c})\big|, \qquad i=1,2,$$

can be estimated by

$$\int_{\Gamma^{\mathrm{fin}}_{\Lambda_1\cup\Lambda_2}\setminus\{\emptyset\}} \lambda^z_{\Lambda'}(d\gamma_{\Lambda_1\cup\Lambda_2}) \int_{\Gamma^{\mathrm{fin}}_{(\Lambda_1\cup\Lambda_2)^c}} \lambda^z_{(\Lambda_1\cup\Lambda_2)^c}(d\gamma) \big| k(\gamma_{\Lambda_1\cup\Lambda_2}\cup\gamma_{(\Lambda_1\cup\Lambda_2)^c})\big|,$$

which, as follows from (3.9), is less than or equal to $D_1\big(z,\beta,|\Lambda_1\cup\Lambda_2|\big)$. So, the right-hand side of (4.20) is less than or equal to

$$e^{D_1\left(|\Lambda_1\cup\Lambda_2|\right)}\|F_1\|_{L_\infty}\|F_2\|_{L_\infty}\frac{D_0}{D_1(|\Lambda_1\cup\Lambda_2|)}\sum_{n_1=1}^{\infty}\frac{1}{n_1!}\sum_{n_2=1}^{\infty}\frac{1}{n_2!}D_1\big(|\Lambda_1\cup\Lambda_2|\big)^{n_1+n_2}\times$$
$$\times\exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big]\sum_{\substack{(\Theta^1_{\Lambda_2},\dots,\Theta^{n_1}_{\Lambda_2},\Theta^1_{\Lambda_1},\dots,\Theta^{n_2}_{\Lambda_1})\\ \Theta^{i_1}_{\Lambda_2}\in\{\{\emptyset\},\Gamma^{\mathrm{fin}}_{\Lambda_2}\setminus\{\emptyset\}\},\ i_1=1,\dots,n_1\\ \Theta^{i_2}_{\Lambda_1}\in\{\{\emptyset\},\Gamma^{\mathrm{fin}}_{\Lambda_1}\setminus\{\emptyset\}\},\ i_2=1,\dots,n_2\\ \#(\Theta^1_{\Lambda_2}\cup\dots\cup\Theta^{n_1}_{\Lambda_2}\cup\Theta^1_{\Lambda_1}\cup\dots\cup\Theta^{n_2}_{\Lambda_1})\neq 1}}1=$$
$$=e^{D_1\left(|\Lambda_1\cup\Lambda_2|\right)}\|F_1\|_{L_\infty}\|F_2\|_{L_\infty}\frac{D_0}{D_1(|\Lambda_1\cup\Lambda_2|)}\times$$
$$\times\sum_{n_1=1}^{\infty}\frac{1}{n_1!}\sum_{n_2=1}^{\infty}\frac{1}{n_2!}\big(2D_1(|\Lambda_1\cup\Lambda_2|)\big)^{n_1+n_2}\exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big]\le$$
$$\le\|F_1\|_{L_\infty}\|F_2\|_{L_\infty}\frac{D_0}{D_1(|\Lambda_1\cup\Lambda_2|)}\exp\Big[5D_1\big(|\Lambda_1\cup\Lambda_2|\big)\Big]\exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big].\tag{4.21}$$

Estimates (4.8), (4.18) and (4.21) show that inequality (4.5) holds with

$$D'=\psi(D_0)+D_2+\frac{D_0}{D_1(|\Lambda_1\cup\Lambda_2|)}\exp\Big[5D_1\big(|\Lambda_1\cup\Lambda_2|\big)\Big].\qquad\square$$

Remark 11. It would be interesting to extend the result of Theorem 2, and so that of Theorem 1, to a more general case of many-body potentials by using the technique of the work [29].

Appendix A.

In this appendix we provide a proof of Theorem 2, which — as was already mentioned before — is a slight modification of the proof of Theorem 17.2.2 in [12].

For the given $F_i\in L_{2+\varepsilon}\big(\Gamma,\mathfrak{B}_{\Lambda_i}(\Gamma),G^{z,\beta}\big)$, $i=1,2$, let us introduce $F_{i,N}$, $F'_{i,N}$, $N>0$, as follows

$$F_{i,N}:=\begin{cases}F_i, & |F_i|\le N,\\ 0, & |F_i|>N,\end{cases}\qquad F'_{i,N}:=\begin{cases}0, & |F_i|\le N,\\ F_i, & |F_i|>N.\end{cases}$$

Then $F_i=F_{i,N}+F'_{i,N}$, and so

$$\begin{aligned}\big|\langle F_1F_2\rangle-\langle F_1\rangle\langle F_2\rangle\big|=&\big|\langle(F_{1,N}+F'_{1,N})(F_{2,N}+F'_{2,N})\rangle-\langle F_{1,N}+F'_{1,N}\rangle\langle F_{2,N}+F'_{2,N}\rangle\big|=\\ =&\big|\langle F_{1,N}F_{2,N}\rangle+\langle F'_{1,N}F_{2,N}\rangle+\langle F_{1,N}F'_{2,N}\rangle+\langle F'_{1,N}F'_{2,N}\rangle-\\ &-\langle F_{1,N}\rangle\langle F_{2,N}\rangle-\langle F'_{1,N}\rangle\langle F_{2,N}\rangle-\langle F_{1,N}\rangle\langle F'_{2,N}\rangle)-\langle F'_{1,N}\rangle\langle F'_{2,N}\rangle\big|\le\\ \le&\big|\langle F_{1,N}F_{2,N}\rangle-\langle F_{1,N}\rangle\langle F_{2,N}\rangle\big|+\big|\langle F'_{1,N}F_{2,N}\rangle\big|+\big|\langle F_{1,N}F'_{2,N}\rangle\big|+\\ &+\big|\langle F'_{1,N}F'_{2,N}\rangle\big|+\big|\langle F'_{1,N}\rangle\langle F_{2,N}\rangle\big|+\big|\langle F_{1,N}\rangle\langle F'_{2,N}\rangle)\big|+\big|\langle F'_{1,N}\rangle\langle F'_{2,N}\rangle\big|.\end{aligned}\tag{A.1}$$

By Theorem 3

$$\big|\langle F_{1,N}F_{2,N}\rangle-\langle F_{1,N}\rangle\langle F_{2,N}\rangle\big|\le D'N^2\exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big].\tag{A.2}$$

Next, taking into account that $\max\big\{|F_{i,N}|,|F'_{i,N}|\big\}\le|F_i|$, $i=1,2$, (just because $|F_i|=|F_{i,N}|+|F'_{i,N}|$, $i=1,2$) and using the Schwarz inequality with respect to the probability measure $G^{z,\beta}$, we get

$$\begin{aligned}\big|\langle F'_{1,N}F_{2,N}\rangle\big|\le\big\langle|F'_{1,N}|\,|F_{2,N}|\big\rangle\le\Big\langle\Big(\frac{|F'_{1,N}|}{N}\Big)^{\varepsilon}|F'_{1,N}|\,|F_{2,N}|\Big\rangle=N^{-\varepsilon}\big\langle|F'_{1,N}|^{1+\varepsilon}|F_{2,N}|\big\rangle\le\\ \le N^{-\varepsilon}\big\langle|F_1|^{1+\varepsilon}|F_2|\big\rangle\le N^{-\varepsilon}\Big\langle\big(|F_1|^{1+\varepsilon}\big)^{\frac{2+\varepsilon}{1+\varepsilon}}\Big\rangle^{\frac{1+\varepsilon}{2+\varepsilon}}\langle|F_2|^{2+\varepsilon}\rangle^{\frac{1}{2+\varepsilon}}=N^{-\varepsilon}\|F_1\|^{1+\varepsilon}_{L_{2+\varepsilon}}\|F_2\|_{L_{2+\varepsilon}},\end{aligned}\tag{A.3}$$

$$\begin{aligned}\big|\langle F_{1,N}F'_{2,N}\rangle\big| &\le \big\langle |F_{1,N}|\,|F'_{2,N}|\big\rangle \le \Big\langle |F_{1,N}|\Big(\frac{|F'_{2,N}|}{N}\Big)^{\varepsilon}|F'_{2,N}|\Big\rangle = N^{-\varepsilon}\big\langle |F_{1,N}|\,|F'_{2,N}|^{1+\varepsilon}\big\rangle \le \\ &\le N^{-\varepsilon}\big\langle |F_1|\,|F_2|^{1+\varepsilon}\big\rangle \le N^{-\varepsilon}\langle |F_1|^{2+\varepsilon}\rangle^{\frac{1}{2+\varepsilon}}\Big\langle \big(|F_2|^{1+\varepsilon}\big)^{\frac{2+\varepsilon}{1+\varepsilon}}\Big\rangle^{\frac{1+\varepsilon}{2+\varepsilon}} = N^{-\varepsilon}\|F_1\|_{L_{2+\varepsilon}}\|F_2\|^{1+\varepsilon}_{L_{2+\varepsilon}},\end{aligned}\tag{A.4}$$

$$\begin{aligned}\big|\langle F'_{1,N}F'_{2,N}\rangle\big| &\le \big\langle |F'_{1,N}|\,|F'_{2,N}|\big\rangle \le \Big\langle |F'_{1,N}|\frac{1}{2}\Big[\Big(\frac{|F'_{1,N}|}{N}\Big)^{\varepsilon}+\Big(\frac{|F'_{2,N}|}{N}\Big)^{\varepsilon}\Big]|F'_{2,N}|\Big\rangle = \\ &= \frac{1}{2}N^{-\varepsilon}\Big[\langle |F'_{1,N}|^{1+\varepsilon}|F'_{2,N}|\rangle + \langle |F'_{1,N}|\,|F'_{2,N}|^{1+\varepsilon}\rangle\Big] \le \frac{1}{2}N^{-\varepsilon}\Big[\langle |F_1|^{1+\varepsilon}|F_2|\rangle + \langle |F_1|\,|F_2|^{1+\varepsilon}\rangle\Big] \le \\ &\le \frac{1}{2}N^{-\varepsilon}\Big[\Big\langle \big(|F_1|^{1+\varepsilon}\big)^{\frac{2+\varepsilon}{1+\varepsilon}}\Big\rangle^{\frac{1+\varepsilon}{2+\varepsilon}}\big\langle |F_2|^{2+\varepsilon}\big\rangle^{\frac{1}{2+\varepsilon}} + \big\langle |F_1|^{2+\varepsilon}\big\rangle^{\frac{1}{2+\varepsilon}}\Big\langle \big(|F_2|^{1+\varepsilon}\big)^{\frac{2+\varepsilon}{1+\varepsilon}}\Big\rangle^{\frac{1+\varepsilon}{2+\varepsilon}}\Big] = \\ &= \frac{1}{2}N^{-\varepsilon}\Big[\|F_1\|^{1+\varepsilon}_{L_{2+\varepsilon}}\|F_2\|_{L_{2+\varepsilon}} + \|F_1\|_{L_{2+\varepsilon}}\|F_2\|^{1+\varepsilon}_{L_{2+\varepsilon}}\Big],\end{aligned}\tag{A.5}$$

$$\begin{aligned}\big|\langle F'_{1,N}\rangle\langle F_{2,N}\rangle\big| &\le \big\langle |F'_{1,N}|\big\rangle\big\langle |F_{2,N}|\big\rangle \le \Big\langle \Big(\frac{|F'_{1,N}|}{N}\Big)^{\varepsilon}|F'_{1,N}|\Big\rangle\big\langle |F_{2,N}|\big\rangle = N^{-\varepsilon}\big\langle |F'_{1,N}|^{1+\varepsilon}\big\rangle\big\langle |F_{2,N}|\big\rangle \le \\ &\le N^{-\varepsilon}\big\langle |F_1|^{1+\varepsilon}\big\rangle\big\langle |F_2|\big\rangle \le N^{-\varepsilon}\langle 1^{2+\varepsilon}\rangle^{\frac{1}{2+\varepsilon}}\Big\langle \big(|F_1|^{1+\varepsilon}\big)^{\frac{2+\varepsilon}{1+\varepsilon}}\Big\rangle^{\frac{1+\varepsilon}{2+\varepsilon}}\big\langle |F_2|^{2+\varepsilon}\big\rangle^{\frac{1}{2+\varepsilon}}\big\langle 1^{\frac{2+\varepsilon}{1+\varepsilon}}\big\rangle^{\frac{1+\varepsilon}{2+\varepsilon}} = \\ &= N^{-\varepsilon}\|F_1\|^{1+\varepsilon}_{L_{2+\varepsilon}}\|F_2\|_{L_{2+\varepsilon}},\end{aligned}\tag{A.6}$$

$$\begin{aligned}\big|\langle F_{1,N}\rangle\langle F'_{2,N}\rangle\big| &\le \big\langle |F_{1,N}|\big\rangle\big\langle |F'_{2,N}|\big\rangle \le \big\langle |F_{1,N}|\big\rangle\Big\langle \Big(\frac{|F'_{2,N}|}{N}\Big)^{\varepsilon}|F_{2,N}|\Big\rangle = N^{-\varepsilon}\big\langle |F_{1,N}|\big\rangle\big\langle |F'_{2,N}|^{1+\varepsilon}\big\rangle \le \\ &\le N^{-\varepsilon}\big\langle |F_1|\big\rangle\big\langle |F_2|^{1+\varepsilon}\big\rangle \le N^{-\varepsilon}\big\langle |F_1|^{2+\varepsilon}\big\rangle^{\frac{1}{2+\varepsilon}}\big\langle 1^{\frac{2+\varepsilon}{1+\varepsilon}}\big\rangle^{\frac{1+\varepsilon}{2+\varepsilon}}\langle 1^{2+\varepsilon}\rangle^{\frac{1}{2+\varepsilon}}\Big\langle \big(|F_2|^{1+\varepsilon}\big)^{\frac{2+\varepsilon}{1+\varepsilon}}\Big\rangle^{\frac{1+\varepsilon}{2+\varepsilon}} = \\ &= N^{-\varepsilon}\|F_1\|_{L_{2+\varepsilon}}\|F_2\|^{1+\varepsilon}_{L_{2+\varepsilon}},\end{aligned}\tag{A.7}$$

$$\begin{aligned}\big|\langle F'_{1,N}\rangle\langle F'_{2,N}\rangle\big| &\le \big\langle |F'_{1,N}|\big\rangle\big\langle |F'_{2,N}|\big\rangle = \frac{1}{2}\big\langle |F'_{1,N}|\big\rangle\big\langle |F'_{2,N}|\big\rangle + \frac{1}{2}\big\langle |F'_{1,N}|\big\rangle\big\langle |F'_{2,N}|\big\rangle \le \\ &\le \frac{1}{2}\Big\langle \Big(\frac{|F'_{1,N}|}{N}\Big)^{\varepsilon}|F'_{1,N}|\Big\rangle\big\langle |F'_{2,N}|\big\rangle + \frac{1}{2}\big\langle |F'_{1,N}|\big\rangle\Big\langle \Big(\frac{|F'_{2,N}|}{N}\Big)^{\varepsilon}|F_{2,N}|\Big\rangle = \\ &= \frac{1}{2}N^{-\varepsilon}\Big[\big\langle |F'_{1,N}|^{1+\varepsilon}\big\rangle\big\langle |F'_{2,N}|\big\rangle + \big\langle |F'_{1,N}|\big\rangle\big\langle |F'_{2,N}|^{1+\varepsilon}\big\rangle\Big] \le \frac{1}{2}N^{-\varepsilon}\Big[\big\langle |F_1|^{1+\varepsilon}\big\rangle\big\langle |F_2|\big\rangle + \big\langle |F_1|\big\rangle\big\langle |F_2|^{1+\varepsilon}\big\rangle\Big] \le \\ &\le \frac{1}{2}N^{-\varepsilon}\Big[\langle 1^{2+\varepsilon}\rangle^{\frac{1}{2+\varepsilon}}\Big\langle \big(|F_1|^{1+\varepsilon}\big)^{\frac{2+\varepsilon}{1+\varepsilon}}\Big\rangle^{\frac{1+\varepsilon}{2+\varepsilon}}\big\langle |F_2|^{2+\varepsilon}\big\rangle^{\frac{1}{2+\varepsilon}}\big\langle 1^{\frac{2+\varepsilon}{1+\varepsilon}}\big\rangle^{\frac{1+\varepsilon}{2+\varepsilon}} + \\ &\quad + \big\langle |F_1|^{2+\varepsilon}\big\rangle^{\frac{1}{2+\varepsilon}}\big\langle 1^{\frac{2+\varepsilon}{1+\varepsilon}}\big\rangle^{\frac{1+\varepsilon}{2+\varepsilon}}\langle 1^{2+\varepsilon}\rangle^{\frac{1}{2+\varepsilon}}\Big\langle \big(|F_2|^{1+\varepsilon}\big)^{\frac{2+\varepsilon}{1+\varepsilon}}\Big\rangle^{\frac{1+\varepsilon}{2+\varepsilon}}\Big] = \\ &= \frac{1}{2}N^{-\varepsilon}\Big[\|F_1\|^{1+\varepsilon}_{L_{2+\varepsilon}}\|F_2\|_{L_{2+\varepsilon}} + \|F_1\|_{L_{2+\varepsilon}}\|F_2\|^{1+\varepsilon}_{L_{2+\varepsilon}}\Big]\end{aligned}\tag{A.8}$$

Substituting (A.2)–(A.8) into (A.1), we obtain

$$\big|\langle F_1F_2\rangle - \langle F_1\rangle\langle F_2\rangle\big| \le D'N^2\exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big] + 3N^{-\varepsilon}\Big(\|F_1\|^{1+\varepsilon}_{L_{2+\varepsilon}}\|F_2\|_{L_{2+\varepsilon}} + \|F_1\|_{L_{2+\varepsilon}}\|F_2\|^{1+\varepsilon}_{L_{2+\varepsilon}}\Big),$$

whence (4.3) follows by setting

$$N = \exp\Big[\frac{\varkappa}{(1+\delta)(2+\varepsilon)}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big]$$

with

$$D = \Big[D' + 3\big(\|F_1\|^{1+\varepsilon}_{L_{2+\varepsilon}}\|F_2\|_{L_{2+\varepsilon}} + \|F_1\|_{L_{2+\varepsilon}}\|F_2\|^{1+\varepsilon}_{L_{2+\varepsilon}}\big)\Big]. \qquad \square$$

Appendix B.

As it was already mentioned before, in this appendix we are going to present an alternative way to estimate I_1 starting from formula (4.17). Namely, plugging the definition of Lebesgue–Poisson measure (1.2) into the right-hand side of (4.17), we can get

$$
\begin{aligned}
I_1 \leq & f(\Lambda_1)f(\Lambda_2)\|F_1\|_{L_\infty}\|F_2\|_{L_\infty}\exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big]\sum_{m=1}^{\infty}\frac{z^m}{m!}\sum\nolimits^m K(\beta)^{-n}\times\\
&\times\sum_{k_1=0}^{\infty}\frac{z^{k_1}}{k_1!}\cdots\sum_{k_1=0}^{\infty}\frac{z^{k_n}}{k_n!}\sum_{\mathcal{T}\in\mathfrak{T}_{m^1+k_1}}\cdots\sum_{\mathcal{T}\in\mathfrak{T}_{m^n+k_n}}\big(e^{2\beta B}K(\beta)\big)^{m+k_1+\cdots+k_n}\times\\
&\times\int\limits_{(\Lambda_1\cup\Lambda_2)^m\times\left((\Lambda_1\cup\Lambda_2)^c\right)^{k_1+\cdots+k_n}}(dx)^1_{m+k_1+\cdots+k_n}\prod_{\{x,x\}\in\mathcal{T}}\exp\Big[-\frac{\delta\varkappa}{1+\delta}|x-x|\Big],
\end{aligned}
$$

where

$$
\sum\nolimits^m[\ldots]:=\sum_{n=1}^{m}\frac{1}{n!}\sum_{\substack{m^1+\cdots+m^n=m\\ m^1,\ldots,m^n\geq 1}}\frac{m!}{m^1!\ldots m^n!}[\ldots]. \tag{B.1}
$$

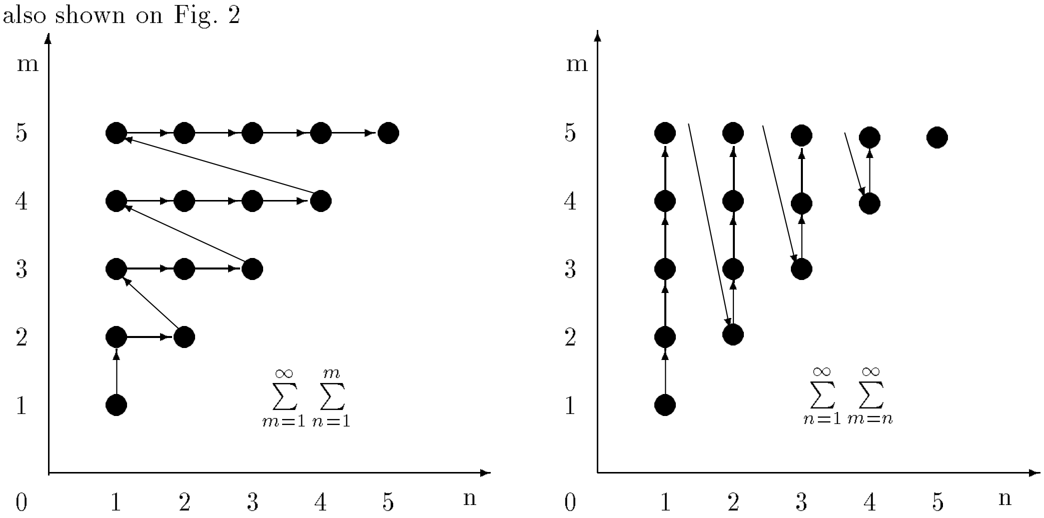
Then, subsequently integrating with respect to the ending points of tree graphs $\mathcal{T}_i\in\mathfrak{T}_{m^i+k_i}$, $i=1,\ldots,n$, just as in (4.12), and using (2.18), (2.17), we get that

$$
\begin{aligned}
I_1 \leq & f(\Lambda_1)f(\Lambda_2)\|F_1\|_{L_\infty}\|F_2\|_{L_\infty}\exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big]\sum_{m=1}^{\infty}\sum\nolimits^m\left(\frac{|\Lambda_1\cup\Lambda_2|}{A(\varkappa)K(\beta)}\right)^n\times\\
&\times\sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}\frac{\big(ze^{2\beta B}K(\beta)A(\varkappa)\big)^{m+k_1+\cdots+k_n}}{m!k_1!\ldots k_n!}\sum_{\mathcal{T}\in\mathfrak{T}_{m^1+k_1}}\cdots\sum_{\mathcal{T}\in\mathfrak{T}_{m^n+k_n}}1\leq\\
\leq & f(\Lambda_1)f(\Lambda_2)K(\beta)^{-1}\|F_1\|_{L_\infty}\|F_2\|_{L_\infty}\exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big]\sum_{m=1}^{\infty}\sum\nolimits^m\left(\frac{2\max\{|\Lambda_1|,|\Lambda_2|\}}{A(\varkappa)K(\beta)}\right)^n\times\\
&\times\sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}\frac{\big(ze^{2\beta B+1}K(\beta)A(\varkappa)\big)^{m+k_1+\cdots+k_n}}{m!k_1!\ldots k_n!}(m^1+k_1)!\ldots(m^n+k_n)!=\\
\leq & f(\Lambda_1)f(\Lambda_2)\|F_1\|_{L_\infty}\|F_2\|_{L_\infty}\exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big]\sum_{m=1}^{\infty}\sum_{n=1}^{m}\frac{1}{n!}\left(\frac{2\max\{|\Lambda_1|,|\Lambda_2|\}}{A(\varkappa)K(\beta)}\right)^n\times\\
&\times\sum_{\substack{m^1+\cdots+m^n=m\\ m^1,\ldots,m^n\geq 1}}\sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}\frac{(m^1+k_1)!\ldots(m^n+k_n)!}{m^1!\ldots m^n!k_1!\ldots k_n!}\big(ze^{2\beta B+1}K(\beta)A(\varkappa)\big)^{m+k_1+\cdots+k_n}.
\end{aligned}
$$

Resummation

$$
\sum_{m=1}^{\infty}\sum_{n=1}^{m}[\ldots]=\sum_{n=1}^{\infty}\sum_{m=n}^{\infty}[\ldots],
$$

as also shown on Fig. 2



and the fact that

$$\sum_{m=n}^{\infty}\sum_{\substack{m^1+\cdots+m^n=m\\ m^1,\dots,m^n\geq 1}}[\dots]=\sum_{m^1=1}^{\infty}\cdots\sum_{m^n=1}^{\infty}[\dots],$$

yields

$$\begin{aligned}
I_1 \leq & f(\Lambda_1)f(\Lambda_2)\|F_1\|_{L_\infty}\|F_2\|_{L_\infty}\exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big]\sum_{n=1}^{\infty}\frac{1}{n!}\Big(\frac{2\max\{|\Lambda_1|,|\Lambda_2|\}}{A(\varkappa)K(\beta)}\Big)^n\times\\
&\times\sum_{m=n}^{\infty}\sum_{\substack{m^1+\cdots+m^n=m\\ m^1,\dots,m^n\geq 1}}\sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}\frac{(m^1+k_1)!\dots(m^n+k_n)!}{m^1!\dots m^n!k_1!\dots k_n!}\big(ze^{2\beta B+1}K(\beta)A(\varkappa)\big)^{m+k_1+\cdots+k_n}=\\
\leq & f(\Lambda_1)f(\Lambda_2)\|F_1\|_{L_\infty}\|F_2\|_{L_\infty}\exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big]\times\\
&\times\sum_{n=1}^{\infty}\frac{1}{n!}\Big(\frac{2\max\{|\Lambda_1|,|\Lambda_2|\}}{A(\varkappa)K(\beta)}\sum_{m'=1}^{\infty}\sum_{k'=0}^{\infty}\frac{(m'+k')!}{m'!k'!}\big(ze^{2\beta B+1}K(\beta)A(\varkappa)\big)^{m'}\big(ze^{2\beta B+1}K(\beta)A(\varkappa)\big)^{k'}\Big)^n=\\
\leq & f(\Lambda_1)f(\Lambda_2)\|F_1\|_{L_\infty}\|F_2\|_{L_\infty}\exp\Big[-\frac{\varkappa}{1+\delta}\operatorname{dist}(\Lambda_1,\Lambda_2)\Big]\times\\
&\times\Big\{\exp\Big[\frac{2\max\{|\Lambda_1|,|\Lambda_2|\}}{A(\varkappa)K(\beta)}\sum_{m'=1}^{\infty}\sum_{k'=0}^{\infty}\frac{(m'+k')!}{m'!k'!}\big(ze^{2\beta B+1}K(\beta)A(\varkappa)\big)^{m'}\big(ze^{2\beta B+1}K(\beta)A(\varkappa)\big)^{k'}\Big]-1\Big\}.
\end{aligned}$$

Finally, taking into account condition (4.1), formula (4.13), and the fact that

$$\begin{aligned}
&\sum_{m'=1}^{\infty}\sum_{k'=0}^{\infty}\frac{(m'+k')!}{m'!k'!}\big(ze^{2\beta B+1}K(\beta)A(\varkappa)\big)^{m'}\big(ze^{2\beta B+1}K(\beta)A(\varkappa)\big)^{k'}=\\
&\quad=\sum_{l=1}^{\infty}\big(ze^{2\beta B+1}K(\beta)A(\varkappa)\big)^{l}\sum_{\substack{m'+k'=l\\ m'\geq 1,\,k'\geq 0}}\frac{l!}{m'!k'!}\leq\\
&\quad\leq\sum_{l=1}^{\infty}\big(ze^{2\beta B+1}K(\beta)A(\varkappa)\big)^{l}\sum_{m'=0}^{l}C_l^{m'}=\sum_{l=1}^{\infty}\big(2ze^{2\beta B+1}K(\beta)A(\varkappa)\big)^{l},
\end{aligned}$$

we obtain exactly the same estimate as in (4.18)–(4.19).

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