SPECHT MODULES AND BRANCHING RULES FOR ARIKI-KOIKE ALGEBRAS

Jie Du and Hebing Rui

School of Mathematics, University of New South Wales
Sydney, 2052, Australia
Department of Mathematics, University of Shanghai for Science & Technology,
Shanghai, 200093, China

18 January, 1999

Specht modules for an Ariki-Koike algebra $\mathbf{H}_m^r$ have been investigated recently in the context of cellular algebras (see, e.g., [GL] and [DJM]). Thus, these modules are defined as quotient modules of certain "permutation" modules, that is, defined as "cell modules" via cellular bases. So cellular bases play a decisive rôle in these work. However, the classical theory [C] or the work in the case when $m = 1, 2$ (i.e., the case for type A and B Hecke algebras) suggest that a construction as *submodules* without using cellular bases should exist. Following our previous work [DR], we shall introduce in this paper Specht modules for an Ariki-Koike algebra as submodules of those "permutation" modules and investigate their basic properties such as Standard Basis Theorem and the ordinary Branching Theorem, generalizing several classical constructions given in [JK] and [DJ] for type A.

The second part of the paper moves on looking for modular branching rules for Specht and irreducible modules over an Ariki-Koike algebra. These rules for symmetric groups were recently established by Kleshchev [K], and were generalized to Hecke algebras of type A by Brundan [B]. We shall restrict to the case where $\mathbf{H}_m^r$ has a semi-simple bottom in the sense of [DR]. This is because the classification of irreducible modules is known in this case. Our Specht module theory and the Morita equivalence theorem established in [DR] are the main ingredients in this generalization.

We point out that this realization as submodules is actually very important in the Specht/Young module theory for Ariki-Koike algebras and their associated endomorphism algebras. See [CPS] for more details. We emphasise that our method throughout the work is independent of the use of cellular bases. Moreover, in the proofs of the main results (2.2), (3.6), (4.2) and (4.10), one will see how the relevant level 1 results (i.e., the results for type A Hecke algebras) are "lifted" to the corresponding level m results.

Research was supported by the Australian Research Council. The second author was also supported by Sonderforschungsbereich 343 at the University of Bielefeld and the National Natural Science Foundation in China. He wishes to thank the University of Bielefeld for its hospitality during his visit.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

After a first manuscript of the paper was completed, the authors received a preprint entitled “On the classification of simple modules for cyclotomic Hecke algebras of type $G(m,1,n)$ and Kleshchev multipartitions” by S. Ariki. With this latest development, one would expect the branching rules (4.10) holds in general.

1. Specht modules. Let $\mathfrak{S}_r = \mathfrak{S}_{\{1,\cdots,r\}}$ be the symmetric group of all permutations of $1,\cdots,r$ with Coxeter generators $s_i = (i,i+1)$, and let $\mathfrak{S}_\lambda$ denote the Young subgroup corresponding the composition λ of r. (A *composition* λ of r is a finite sequence of non-negative integers $(\lambda_1,\lambda_2,\cdots,\lambda_m)$ such that $|\lambda| = \sum_i \lambda_i = r$.) Thus, we have

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\boldsymbol{a}} = \mathfrak{S}_{\{1,\cdots,a_1\}} \times \mathfrak{S}_{\{a_1+1,\cdots,a_2\}} \times \cdots \times \mathfrak{S}_{\{a_{m-1}+1,\cdots,r\}}, \tag{1.1}$$

where $\boldsymbol{a} = [a_0,a_1,\cdots,a_m]$ with $a_0 = 0$ and $a_i = \lambda_1 + \cdots + \lambda_i$ for all $i = 1,\cdots,m$. We will denote by $\mathcal{D}_\lambda$ the set of distinguished representatives of right $\mathfrak{S}_\lambda$-cosets and write $\mathcal{D}_{\lambda\mu} = \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1}$, which is the set of distinguished representatives of double cosets $\mathfrak{S}_\lambda \backslash \mathfrak{S}_r / \mathfrak{S}_\mu$.

For later use, let $\Lambda(r)$ (resp. $\Lambda^+(r)$) denote the set of all compositions (resp. partitions) of r. For $\lambda \in \Lambda^+(r)$, let λ' be the dual partition of λ: thus $\lambda'_i = \#\{\lambda_j \geqslant i\}$. There is a unique element $w_\lambda \in \mathcal{D}_{\lambda\lambda'}$ with the trivial intersection property:

$$\mathfrak{S}_\lambda^{w_\lambda} \cap \mathfrak{S}_{\lambda'} = w_\lambda^{-1}\mathfrak{S}_\lambda w_\lambda \cap \mathfrak{S}_{\lambda'} = \{1\}. \tag{1.2}$$

If we represent a partition by a Young diagram $\mathcal{Y}(\lambda)$, for example, $\mathcal{Y}(\lambda) = \boxed{\begin{array}{ccc}\square&\square&\square\\ \square&\square&\end{array}}$ represents $\lambda = (32)$, then w_λ is defined by $\mathbf{t}^\lambda w_\lambda = \mathbf{t}_\lambda$, where $\mathbf{t}^\lambda$ (resp. $\mathbf{t}_\lambda$) is the λ-tableau obtained by putting the numbers $1,2,\cdots,r$ in order into the boxes from left to right down successive rows (resp. columns): thus $\mathbf{t}^{(32)} = \boxed{\begin{array}{ccc}1&2&3\\4&5&\end{array}}$, and $\mathbf{t}_{(32)} = \boxed{\begin{array}{ccc}1&3&5\\2&4&\end{array}}$.

An m-tuple $\boldsymbol{\lambda} = (\lambda^{(1)},\cdots,\lambda^{(m)})$ of partitions with $r = \sum_i |\lambda^{(i)}|$ is called a multi-partition or an m-partition of r. Define m-composition similarly. Let $\boldsymbol{\lambda}' = (\lambda^{(m)\prime},\cdots,\lambda^{(1)\prime})$ denote the m-composition *dual* to $\boldsymbol{\lambda}$. By concatenating the components of $\boldsymbol{\lambda}$, the resulting composition of r will be denoted by

$$\bar{\boldsymbol{\lambda}} = \lambda^{(1)} \vee \cdots \vee \lambda^{(m)}.$$

Thus, the bar $^-$ defines a map from the set $\Lambda_m^+(r)$ of all m-partitions of r to $\Lambda(r)$.

For each m-composition $\boldsymbol{\lambda} = (\lambda^{(1)},\cdots,\lambda^{(m)})$, define $[\boldsymbol{\lambda}] = [a_0,a_1,\cdots,a_m]$ such that $a_0 = 0$ and $a_i = \sum_{j=1}^i |\lambda^{(j)}|$. Recall from [DR, §1] that the set of all $[\boldsymbol{\lambda}]$ form a poset $\Lambda[m,r]$ which is isomorphic to the poset $\Lambda(m,r)$ of all compositions of r with m parts. Here the partial ordering on $\Lambda[m,r]$ is given by $\preccurlyeq$: $[a_i] \preccurlyeq [b_i]$ if $a_i \leqslant b_i$, $i = 1,\cdots,m$, while $\Lambda(m,r)$ has the usual dominance order $\trianglelefteq$.

For any $\boldsymbol{a} = [a_i] \in \Lambda[m,r]$, following [DR, (1.6)], let $w_{\boldsymbol{a}} \in \mathfrak{S}_r$ be the element defined by

$$(a_{i-1}+l)w_{\boldsymbol{a}} = r - a_i + l \text{ for all } i \text{ with } a_{i-1} < a_i, 1 \leqslant l \leqslant a_i - a_{i-1}.$$

For example, $w_{\boldsymbol{a}} = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ \underline{6\,7\,8\,9}\;\underline{2\,3\,4\,5}\;\underline{1} \end{pmatrix}$, if $\boldsymbol{a} = [0, 4, 8, 9]$. Note that $w_{\boldsymbol{a}}$ is the unique distinguished double coset representative satisfying

$$w_{\boldsymbol{a}}^{-1} \mathfrak{S}_{\boldsymbol{a}} w_{\boldsymbol{a}} = \mathfrak{S}_{\boldsymbol{a}'}, \tag{1.3}$$

where, for $\boldsymbol{a} = [a_0, a_1, \cdots, a_m] \in \Lambda[m, r]$,

$$\boldsymbol{a}' = [r - a_m, r - a_{m-1}, \cdots, r - a_0]$$

(see [DR, (1.2)]), and $\mathfrak{S}_{\boldsymbol{a}}$ and $\mathfrak{S}_{\boldsymbol{a}'}$ are the Young subgroups defined in (1.1).

We may also identify $\boldsymbol{\lambda}$ with its Young diagram $\mathcal{Y}(\boldsymbol{\lambda})$. For example, $\boldsymbol{\lambda} = ((31), (22), (1))$ is identified with

$$\mathcal{Y}(\boldsymbol{\lambda}) = \boxed{\begin{array}{lll} \Box\Box\Box & \Box\Box & \Box \\ \Box & \Box\Box & \end{array}}.$$

Let $\mathbf{t}^{\boldsymbol{\lambda}}$ be the $\boldsymbol{\lambda}$-tableau obtained by putting the numbers $1, \cdots, r$ in order into the boxes in the Young diagram of $\boldsymbol{\lambda}$ from left to right down successive rows. ¿From the example above, we have

$$\mathbf{t}^{\boldsymbol{\lambda}} = \boxed{\begin{array}{lll} 1\,2\,3 & 5\,6 & 9 \\ 4 & 7\,8 & \end{array}}.$$

We also define the $\boldsymbol{\lambda}$-tableau $\mathbf{t}_{\boldsymbol{\lambda}}$ by putting the numbers from right to left down successive columns as illustrated in the following example

$$\mathbf{t}_{\boldsymbol{\lambda}} = \boxed{\begin{array}{lll} 6\,8\,9 & 2\,4 & 1 \\ 7 & 3\,5 & \end{array}}.$$

Now, associated to an m-partition $\boldsymbol{\lambda} = (\lambda^{(1)}, \cdots, \lambda^{(m)})$ of r, we define the element $w_{\boldsymbol{\lambda}} \in \mathfrak{S}_r$ by $\mathbf{t}^{\boldsymbol{\lambda}} w_{\boldsymbol{\lambda}} = \mathbf{t}_{\boldsymbol{\lambda}}$. More precisely, if $\mathbf{t}^i$ (resp. $\mathbf{t}_i$) denote the i-th subtableau of $\mathbf{t}^{\boldsymbol{\lambda}}$ (resp. $\mathbf{t}_{\boldsymbol{\lambda}} w_{[\boldsymbol{\lambda}]}^{-1}$) and define $w_{(i)}$ by $\mathbf{t}^i w_{(i)} = \mathbf{t}_i$, then $\mathbf{t}^{\boldsymbol{\lambda}} w_{(1)} \cdots w_{(m)} w_{[\boldsymbol{\lambda}]} = \mathbf{t}_{\boldsymbol{\lambda}}$. Likewise, if we define $\tilde{\mathbf{t}}^i$ (resp. $\tilde{\mathbf{t}}_i$) the i-th subtableau of $\mathbf{t}^{\boldsymbol{\lambda}} w_{[\boldsymbol{\lambda}]}$ (resp. $\mathbf{t}_{\boldsymbol{\lambda}}$) and $\tilde{w}_{(i)}$ with $\tilde{\mathbf{t}}^i \tilde{w}_{(i)} = \tilde{\mathbf{t}}_i$, then $\mathbf{t}^{\boldsymbol{\lambda}} w_{[\boldsymbol{\lambda}]} \tilde{w}_{(1)} \cdots \tilde{w}_{(m)} = \mathbf{t}_{\boldsymbol{\lambda}}$. We have, therefore

$$w_{\boldsymbol{\lambda}} = w_{(1)} \cdots w_{(m)} w_{[\boldsymbol{\lambda}]} = w_{[\boldsymbol{\lambda}]} \tilde{w}_{(m)} \cdots \tilde{w}_{(1)}, \quad w_{[\boldsymbol{\lambda}]}^{-1} w_{(i)} w_{[\boldsymbol{\lambda}]} = \tilde{w}_{(m-i+1)}. \tag{1.4}$$

Note that $w_{(i)} w_{(j)} = w_{(j)} w_{(i)}$ and $\tilde{w}_{(i)} \tilde{w}_{(j)} = \tilde{w}_{(j)} \tilde{w}_{(i)}$ for $i, j = 1, 2, \cdots, m$. Clearly, by (1.2-3), we have the following trivial intersection property

$$w_{\boldsymbol{\lambda}}^{-1} \mathfrak{S}_{\bar{\boldsymbol{\lambda}}} w_{\boldsymbol{\lambda}} \cap \mathfrak{S}_{\bar{\boldsymbol{\lambda}}'} = \{1\}. \tag{1.5}$$

The usual dominance order on $\Lambda^+(r)$ can also be generalized to $\Lambda_m^+(r)$. For m-partitions $\boldsymbol{\lambda} = (\lambda^{(1)}, \cdots, \lambda^{(m)})$ and $\boldsymbol{\mu} = (\mu^{(1)}, \cdots, \mu^{(m)})$ with $[\boldsymbol{\lambda}] = [a_0, \cdots, a_m]$ and $[\boldsymbol{\mu}] = [b_0, \cdots, b_m]$, we say that $\boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}$ if

$$\sum_{j=1}^{i} a_j + \sum_{k=1}^{l} \lambda_k^{(i+1)} \leqslant \sum_{j=1}^{i} b_j + \sum_{k=1}^{l} \mu_k^{(i+1)}$$

for all $i = 0, 1, \cdots, m-1$ and all k. The following lemma follows immediately from the definition.

(1.6) Lemma. *Suppose $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \Lambda_m^+(r)$ and $[\boldsymbol{\lambda}] = [\boldsymbol{\mu}]$. Then $\boldsymbol{\lambda} \trianglelefteq \boldsymbol{\mu}$ if and only if $\lambda^{(i)} \trianglelefteq \mu^{(i)}$ for all i.*

Let $W = (\mathbb{Z}/m\mathbb{Z}) \wr \mathfrak{S}_r$ be the wreath product of the cyclic group of order m and the symmetric group $\mathfrak{S}_r$. Then W is not a Coxeter group except for $m = 1, 2$. The Ariki-Koike algebra $\mathbf{H}_R = \mathbf{H}_{mR}^r$ over a *commutative ring R with 1* is an Iwahori-Hecke type algebra associated to W and parameters $q, q^{-1}, u_1, \cdots, u_m \in R$: it is associative with generators $T_0, T_1, \cdots, T_{r-1}$ subject to the relations:

$$(1.7)\qquad \begin{cases} (T_0 - u_1)\cdots(T_0 - u_m) = 0, & \\ T_0T_1T_0T_1 = T_1T_0T_1T_0, & \\ (T_i - q)(T_i + 1) = 0, & 1 \leqslant i \leqslant r-1 \\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, & 1 \leqslant i \leqslant r-2 \\ T_iT_j = T_jT_i, & 0 \leqslant i < j-1 \leqslant r-2. \end{cases}$$

Note that $\mathbf{H}_R$ is the group algebra of W when $q = 1$ and $u_1 = \xi$ is a primitive m-th root of unity and $u_i = \xi^i$. Let $\mathcal{H}_R = \mathcal{H}_R(r)$ denote the subalgebra of $\mathbf{H}_R$ generated by $T_1, \cdots, T_{r-1}$. Then $\mathcal{H}_R = \mathbf{H}_{1R}^r$ is the Iwahori-Hecke algebra associated to $\mathfrak{S}_r$. We will denote by $\mathcal{H}_R(\mathfrak{S}_\lambda)$ the subalgebra associated to the Young subgroup $\mathfrak{S}_\lambda$ of $\mathfrak{S}_r$. Let

$$(1.8)\qquad x_\lambda = \sum_{w \in \mathfrak{S}_\lambda} T_w \text{ and } y_\lambda = \sum_{w \in \mathfrak{S}_\lambda} (-q)^{-l(w)} T_w.$$

Then $x_\lambda \mathcal{H}_R$ and $y_\lambda \mathcal{H}_R$ define the associated q-permutation modules, and the Specht module [DJ] associated to a partition λ of r is the *submodule* $S^\lambda = x_\lambda T_{w_\lambda} y_{\lambda'} \mathcal{H}_R$ of $x_\lambda \mathcal{H}_R$.

We now come to define the notion of Specht modules for $\mathbf{H}_{mR}^r$. Let $L_1 = T_0$, $L_i = q^{-1} T_{i-1} L_{i-1} T_{i-1}$ for $i = 2, \cdots, r$, and put $\pi_0(x) = 1$, $\pi_a(x) = \prod_{j=1}^a (L_j - x)$ for any $x \in R$ and any positive integer a. We define, for $\boldsymbol{a} = [\boldsymbol{\lambda}] = [a_0, a_1, \cdots, a_m] \in \Lambda[m, r]$,

$$\pi_{\boldsymbol{a}} = \pi_{a_1}(u_2)\cdots\pi_{a_{m-1}}(u_m) \text{ and } \tilde{\pi}_{\boldsymbol{a}} = \pi_{a_1}(u_{m-1})\cdots\pi_{a_{m-1}}(u_1),$$

and, for $\boldsymbol{\lambda} \in \Lambda_m^+(r)$,

$$x_{\boldsymbol{\lambda}} = \pi_{[\boldsymbol{\lambda}]} x_{\bar{\boldsymbol{\lambda}}} = x_{\bar{\boldsymbol{\lambda}}} \pi_{[\boldsymbol{\lambda}]} \quad \text{and} \quad y_{\boldsymbol{\lambda}} = \tilde{\pi}_{[\boldsymbol{\lambda}]} y_{\bar{\boldsymbol{\lambda}}} = y_{\bar{\boldsymbol{\lambda}}} \tilde{\pi}_{[\boldsymbol{\lambda}]}.$$

(1.9) Lemma. *For any m-partition $\boldsymbol{\lambda} = (\lambda^{(1)}, \cdots, \lambda^{(m)})$, the R-submodule $x_{\boldsymbol{\lambda}} \mathbf{H} y_{\boldsymbol{\lambda}'} = R z_{\boldsymbol{\lambda}}$ is free of rank 1, where $z_{\boldsymbol{\lambda}} = x_{\boldsymbol{\lambda}} T_{w_{\boldsymbol{\lambda}}} y_{\boldsymbol{\lambda}'}$.*

Proof. From [DR, (2.8), (3.1)], we have $x_{\boldsymbol{\lambda}} \mathbf{H} y_{\boldsymbol{\lambda}'} = x_{\bar{\boldsymbol{\lambda}}} \mathcal{H}(\mathfrak{S}_{\boldsymbol{a}}) v_{\boldsymbol{a}} y_{\bar{\boldsymbol{\lambda}}'}$ where $\boldsymbol{a} = [\boldsymbol{\lambda}]$ and $v_{\boldsymbol{a}} = \pi_{\boldsymbol{a}} T_{w_{\boldsymbol{a}}} \tilde{\pi}_{\boldsymbol{a}'}$. Since $y_{\bar{\boldsymbol{\lambda}}'} = y_{\lambda^{(m)'} \vee \cdots \vee \lambda^{(1)'}}$ and $v_{\boldsymbol{a}} y_{\bar{\boldsymbol{\lambda}}'} = y_{\lambda^{(1)'} \vee \cdots \vee \lambda^{(m)'}} v_{\boldsymbol{a}}$, we have, by [DJ, (4.1)], the R-submodule $x_{\lambda^{(1)} \vee \cdots \vee \lambda^{(m)}} \mathcal{H}(\mathfrak{S}_{\boldsymbol{a}}) y_{\lambda^{(1)'} \vee \cdots \vee \lambda^{(m)'}}$ is generated by the element $Z_{\bar{\boldsymbol{\lambda}}} = x_{\lambda^{(1)} \vee \cdots \vee \lambda^{(m)}} T_{w_{(1)} \cdots w_{(m)}} y_{\lambda^{(1)'} \vee \cdots \vee \lambda^{(m)'}}$. Therefore, by [DR, (3.4)], the R-submodule $x_{\boldsymbol{\lambda}} \mathbf{H} y_{\boldsymbol{\lambda}'}$ is of rank 1, generated by $z_{\boldsymbol{\lambda}} = x_{\boldsymbol{\lambda}} T_{w_{\boldsymbol{\lambda}}} y_{\boldsymbol{\lambda}'} = Z_{\bar{\boldsymbol{\lambda}}} v_{\boldsymbol{a}}$. $\square$

(1.10) Definition. The right $\mathbf{H}_R$-module

$$S^{\boldsymbol{\lambda}} = x_{\boldsymbol{\lambda}}\mathbf{H}_R y_{\boldsymbol{\lambda}'}\mathbf{H}_R = x_{\boldsymbol{\lambda}} T_{w_{\boldsymbol{\lambda}}} y_{\boldsymbol{\lambda}'}\mathbf{H}_R$$

is called the *Specht module* associated to the m-partition $\boldsymbol{\lambda}$. Similarly, the right $\mathbf{H}_R$-module

$$\tilde{S}^{\boldsymbol{\lambda}} = y_{\boldsymbol{\lambda}}\mathbf{H}_R x_{\boldsymbol{\lambda}'}\mathbf{H}_R = y_{\boldsymbol{\lambda}} T_{w_{\boldsymbol{\lambda}}} x_{\boldsymbol{\lambda}'}\mathbf{H}_R$$

is called the *twisted Specht module* of $\mathbf{H}_R$ associated to the m-partition $\boldsymbol{\lambda}$.

The name will be justified in (2.5) since $\{S^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \Lambda_m^+(r)\}$ is a complete set of non-isomorphic simple modules for a semi-simple $\mathbf{H}_R$ [DR, (5.1)].

2. The Standard Basis Theorem. The standard basis theorem for Hecke algebra of type A refers to a basis for a Specht module S^λ indexed by the standard λ-tableaux. Recall from [DJ, §1], a λ-tableau of the from $\mathbf{t}^\lambda w$, $w \in \mathfrak{S}_r$, is called standard if it has increasing rows and columns. The following theorem is due to Dipper and James [DJ, (5.6)].

(2.1) Theorem. *Let λ be a partition of r. Then the Specht module $S^\lambda = z_\lambda \mathcal{H}_R$ with $z_\lambda = x_\lambda T_{w_\lambda} y_{\lambda'}$ is a free R-submodule with basis*

$$\{z_\lambda T_d \mid d \in \mathfrak{S}_r, \mathbf{t}_\lambda d \textit{ is standard}\} = \{z_\lambda T_d \mid d \in \mathfrak{S}_r, \mathbf{t}^{\lambda'} d \textit{ is standard}\}.$$

We are now ready to generalize this result to Ariki-Koike algebras. Let $\boldsymbol{\lambda}$ be an m-partition of r. A $\boldsymbol{\lambda}$-tableau $\mathbf{t}$ is said to be *standard* if each $\lambda^{(i)}$-subtableau has increasing rows and columns.

(2.2) Standard Basis Theorem. *Let $\boldsymbol{\lambda}$ be an m-partition of r. Then the Specht module $S^{\boldsymbol{\lambda}} = z_{\boldsymbol{\lambda}}\mathbf{H}_R$ with $z_{\boldsymbol{\lambda}} = x_{\boldsymbol{\lambda}} T_{w_{\boldsymbol{\lambda}}} y_{\boldsymbol{\lambda}'}$ is a free R-submodule with basis $\{z_{\boldsymbol{\lambda}} T_d \mid d \in \mathfrak{S}_r, \mathbf{t}_{\boldsymbol{\lambda}} d \textit{ is standard }\} = \{z_{\boldsymbol{\lambda}} T_d \mid d \in \mathfrak{S}_r, \mathbf{t}^{\boldsymbol{\lambda}'} d \textit{ is standard }\}$.*

Proof. We first notice from the proof of (1.9) that

$$z_{\boldsymbol{\lambda}} = Z_{\bar{\boldsymbol{\lambda}}} v_{[\boldsymbol{\lambda}]} \text{ where } Z_{\bar{\boldsymbol{\lambda}}} = x_{\lambda^{(1)}\vee\cdots\vee\lambda^{(m)}} T_{w_{(1)}\cdots w_{(m)}} y_{\lambda^{(1)'}\vee\cdots\vee\lambda^{(m)'}} \tag{2.3}$$

Thus, [DR, (3.1)] implies $z_{\boldsymbol{\lambda}}\mathbf{H}_R = z_{\boldsymbol{\lambda}}\mathcal{H}_R$. Let $\lambda = (|\lambda^{(1)}|, \cdots, |\lambda^{(m)}|)$. By (2.1), the $\mathcal{H}_R(\mathfrak{S}_\lambda)$-module $Z_{\bar{\boldsymbol{\lambda}}}\mathcal{H}_R(\mathfrak{S}_\lambda)$ has basis $\{Z_{\bar{\boldsymbol{\lambda}}} T_d \mid d \in \mathfrak{S}_\lambda, \mathbf{t}_{\boldsymbol{\lambda}} d \text{ standard }\}$. Since, as $\mathcal{H}_R$-module, $z_{\boldsymbol{\lambda}}\mathcal{H}_R \cong Z_{\bar{\boldsymbol{\lambda}}} v_{[\boldsymbol{\lambda}]}\mathcal{H}_R(\mathfrak{S}_\lambda) \otimes_{\mathcal{H}_R(\mathfrak{S}_\lambda)} \mathcal{H}_R$ (see [DR, (3.4)]), $S^{\boldsymbol{\lambda}}$ has a basis $\{z_{\boldsymbol{\lambda}} T_d T_w \mid d \in \mathfrak{S}_\lambda, \mathbf{t}_{\boldsymbol{\lambda}} d \text{ standard }, w \in \mathcal{D}_\lambda\}$. Note that $w \in \mathcal{D}_\lambda$ if and only if $(a_i+1)w, \cdots, (a_{i+1})w$ is increasing for $i = 0, \cdots, m-1$, where $[a_0, \cdots, a_m] = [\boldsymbol{\lambda}]$. Thus, for $d \in \mathfrak{S}_\lambda$, $\mathbf{t}_{\boldsymbol{\lambda}} dw$ is standard for all $w \in \mathcal{D}_\lambda$ if $\mathbf{t}_{\boldsymbol{\lambda}} d$ is standard, and vice-versa. Therefore, this basis is the required basis. □

(2.4) Corollary. *The module $S^{\boldsymbol{\lambda}}$ (resp. $\tilde{S}^{\boldsymbol{\lambda}}$) is an R-pure submodule of $x_{\boldsymbol{\lambda}}\mathbf{H}_R$ (resp. $y_{\boldsymbol{\lambda}}\mathbf{H}_R$).*

Proof. This follows immediately from the purity of the type A Specht modules. □

(2.5) Remark. The standard basis given in (2.2) can also be obtained by using the Murphy type (or cellular) basis for $x_{\boldsymbol{\lambda}}\mathbf{H}_R$. See [DS, (5.2.1)] for the case $m=2$ and [DJM, (4.14)] in general. In these work, the bases are all defined for $x_{\boldsymbol{\lambda}'}\mathbf{H}_R$. However, the ring automorphism

$$\Phi : \mathbf{H}_R \to \mathbf{H}_R \tag{2.6}$$

defined by setting $\Phi(q)=q^{-1}$, $\Phi(u_i)=u_{m-i+1}$, for $1\leqslant i\leqslant m$, $\Phi(T_j)=-q^{-1}T_j$ for $1\leqslant j\leqslant r-1$ and $\Phi(T_0)=T_0$ (cf. (1.7)), will turn such a basis for $x_{\boldsymbol{\lambda}'}\mathbf{H}_R$ into a basis for $y_{\boldsymbol{\lambda}'}\mathbf{H}_R$. Denote this basis by

$$\{Y^{\boldsymbol{\mu}}_{\mathfrak{u}\mathbf{t}} \mid \boldsymbol{\mu}\trianglerighteq\boldsymbol{\lambda}', \mathbf{t}\in\mathbf{T}^s(\boldsymbol{\mu}), \mathfrak{u}\in\mathfrak{T}^{ss}(\boldsymbol{\mu},\boldsymbol{\lambda}')\},$$

where $\mathbf{T}^s(\boldsymbol{\mu})$ (resp. $\mathfrak{T}^{ss}(\boldsymbol{\mu},\boldsymbol{\lambda}')$) is the set of all standard $\boldsymbol{\mu}$-tableaux (semi-standard $\boldsymbol{\mu}$-tableaux of type $\boldsymbol{\lambda}'$). Note that, since $\Phi(\pi_{\boldsymbol{a}})=\tilde{\pi}_{\boldsymbol{a}}$ and $\Phi(x_{\bar{\boldsymbol{\lambda}}})=y_{\bar{\boldsymbol{\lambda}}}$, $Y^{\boldsymbol{\mu}}_{\mathfrak{u}\mathbf{t}}$ is of the form $hy_{\boldsymbol{\mu}}T_d$ for some $h\in\mathcal{H}_R$ and $d\in\mathfrak{S}_r$.

We now consider the homomorphism

$$\varphi : y_{\boldsymbol{\lambda}'}\mathbf{H}_R \to z_{\boldsymbol{\lambda}}\mathbf{H}_R;\quad y_{\boldsymbol{\lambda}'}h\mapsto z_{\boldsymbol{\lambda}}h = x_{\boldsymbol{\lambda}}T_{w_{[\boldsymbol{\lambda}]}}(y_{\boldsymbol{\lambda}'}h). \tag{2.7}$$

we claim that if $\varphi(Y^{\boldsymbol{\mu}}_{\mathfrak{u}\mathbf{t}})=x_{\boldsymbol{\lambda}}T_{w_{[\boldsymbol{\lambda}]}}Y^{\boldsymbol{\mu}}_{\mathfrak{u}\mathbf{t}}\neq 0$ then $\boldsymbol{\mu}=\boldsymbol{\lambda}'$. Using [DR, (2.8)], we see easily that $\varphi(Y^{\boldsymbol{\mu}}_{\mathfrak{u}\mathbf{t}})\neq 0$ implies $[\boldsymbol{\lambda}]\preccurlyeq[\boldsymbol{\mu}']$. Since $\boldsymbol{\mu}\trianglerighteq\boldsymbol{\lambda}'$, we have $[\boldsymbol{\mu}]\succcurlyeq[\boldsymbol{\lambda}']$ and hence $[\boldsymbol{\mu}']\preccurlyeq[\boldsymbol{\lambda}]$. So $[\boldsymbol{\mu}']=[\boldsymbol{\lambda}]$. Thus, if we write $Y^{\boldsymbol{\mu}}_{\mathfrak{u}\mathbf{t}}=hy_{\boldsymbol{\mu}}T_d$ as above, then, by [DR, (3.1a,c)] and noting $x_{\bar{\boldsymbol{\lambda}}}T_{w_{[\boldsymbol{\lambda}]}}=T_{w_{[\boldsymbol{\lambda}]}}x_{\lambda^{(m)}\vee\cdots\vee\lambda^{(1)}}$, we have $x_{\boldsymbol{\lambda}}T_{w_{[\boldsymbol{\lambda}]}}Y^{\boldsymbol{\mu}}_{\mathfrak{u}\mathbf{t}}=v_{[\boldsymbol{\lambda}]}x_{\lambda^{(m)}\vee\cdots\vee\lambda^{(1)}}h'y_{\bar{\boldsymbol{\mu}}}T_d$ for some $h'\in\mathcal{H}_R(\mathfrak{S}_{[\boldsymbol{\lambda}']})$. Since $[\boldsymbol{\mu}]=[\boldsymbol{\lambda}']$, this is non-zero implies, by [DJ, (4.1)], $\lambda^{(m-i+1)}\trianglelefteq\mu^{(i)\prime}$ for all $i=1,\cdots,m$. On the other hand, by (1.6), $\boldsymbol{\mu}\trianglerighteq\boldsymbol{\lambda}'$ and $[\boldsymbol{\mu}']=[\boldsymbol{\lambda}]$ implies $\lambda^{(m-i+1)\prime}\trianglelefteq\mu^{(i)}$, $1\leqslant i\leqslant m$. Hence $\lambda^{(m-i+1)\prime}=\mu^{(i)}$ for all i, and therefore, $\boldsymbol{\mu}=\boldsymbol{\lambda}'$. This proves that the image of the cellular basis for the Specht module defined as the quotient of $y_{\boldsymbol{\lambda}'}\mathbf{H}_R$ modulo the submodule spanned by all $Y^{\boldsymbol{\mu}}_{\mathfrak{u}\mathbf{t}}$ with $\boldsymbol{\mu}\triangleright\boldsymbol{\lambda}'$ is exactly the standard basis described in (2.2). Using a similar argument, we see that the twisted Specht modules $\tilde{S}^{\boldsymbol{\lambda}}$ defined in (1.10) are isomorphic to those defined in [DJM, (3.28)] as quotient modules of $x_{\boldsymbol{\lambda}'}\mathbf{H}_R$.

3. Branching rules, I. For a partition $\lambda=(\lambda_1,\cdots,\lambda_m)$ of r, we identify the boxes in the Young diagram $\mathcal{Y}(\lambda)$ with its position coordinates. Thus, we have

$$\mathcal{Y}(\lambda)=\{(i,j)\in\mathbb{Z}^+\times\mathbb{Z}^+\mid j\leqslant\lambda_i\}. \tag{3.1}$$

The elements of $\mathcal{Y}(\lambda)$ will be called *nodes.* A node of the form (i,λ_i) (resp. (i,λ_i+1)) is called *removable* (resp. *addable*) if $\lambda_i>\lambda_{i+1}$ (resp. $\lambda_{i-1}>\lambda_i$). Let $\boldsymbol{\lambda}=(\lambda^{(1)},\cdots,\lambda^{(m)})$ be an m-partition. Then its Young diagram $\mathcal{Y}(\boldsymbol{\lambda})$ is a union of the Young diagram $\mathcal{Y}(\lambda^{(k)})$, $1\leqslant k\leqslant m$. Thus, we have as a set of nodes

$$\mathcal{Y}(\boldsymbol{\lambda})=\{(i,j)_k\mid i,j\in\mathbb{Z}^+, j\leqslant\lambda^{(k)}_i, 1\leqslant k\leqslant m\}.$$

A node of $\mathcal{Y}(\boldsymbol{\lambda})$ is said to be removable (resp. addable) if it is a removable (resp. addable) node of $\mathcal{Y}(\lambda^{(k)})$ for some k. Let $\mathcal{R}_{\boldsymbol{\lambda}}$ denote the set of all removable nodes of $\mathcal{Y}(\boldsymbol{\lambda})$. Then $N = \#\mathcal{R}_{\boldsymbol{\lambda}} = \sum_{i=1}^m \#\mathcal{R}_{\lambda^{(i)}}$, and, if $\boldsymbol{\lambda}' = (\lambda^{(m)\prime}, \cdots, \lambda^{(1)\prime})$ is the m-partition dual to $\boldsymbol{\lambda}$, then we have a bijection $\tau : \mathcal{R}_{\boldsymbol{\lambda}} \to \mathcal{R}_{\boldsymbol{\lambda}'}; (i,j)_k \mapsto (j,i)_{m-k+1}$.

The ordering on $\mathcal{R}_{\boldsymbol{\lambda}}$ will be fixed from top to bottom and from left to right: $\mathcal{R}_{\boldsymbol{\lambda}} = \{\mathfrak{n}_1, \cdots, \mathfrak{n}_N\}$, let $j_{\mathfrak{n}}$, $\mathfrak{n} \in \mathcal{R}_{\boldsymbol{\lambda}}$, be the number at the node $\mathfrak{n}$ in $\mathbf{t}_{\boldsymbol{\lambda}}$. Note that, if $[\boldsymbol{\lambda}] = [a_i]$ with $a_i = \sum_{j\leqslant i} |\lambda^{(j)}|$, then

$$j_{\mathfrak{n}_{N_i+1}} = r - a_i, \text{ where } N_0 = 0,\ N_i = \# \cup_{j\leqslant i} \mathcal{R}_{\lambda^{(j)}}, 1 \leqslant i \leqslant m-1. \tag{3.2}$$

For example, for $\boldsymbol{\lambda} = ((31),(22),(1))$, $\mathcal{R}_{\boldsymbol{\lambda}} = \{(1,3)_1, (2,1)_1, (2,2)_2, (1,1)_3\}$, and the corresponding $j_{\mathfrak{n}}$'s are 9,7,5,1. We have the following result. Recall for any $i \leqslant j$ the elements $s_{i,j} = \left(\begin{smallmatrix} i & i+1 & i+2 & \cdots & j \\ j & i & i+1 & \cdots & j-1 \end{smallmatrix}\right)$ if $i<j$ and $s_{i,i} = 1$.

(3.3) Lemma. *Let $\mathfrak{n} \in \mathcal{R}_{\boldsymbol{\lambda}}$, and let $\boldsymbol{\lambda}_{\mathfrak{n}}$ denote the multi-partition associated to the tableau obtained by removing $\mathfrak{n}$ from $\mathcal{Y}(\boldsymbol{\lambda})$. Let $\mathbf{T}^s(\boldsymbol{\lambda})$ be the set of all standard $\boldsymbol{\lambda}$-tableaux. Then:*

(a) The $\boldsymbol{\lambda}_{\mathfrak{n}}$-tableau $\mathbf{t}_{\boldsymbol{\lambda}_{\mathfrak{n}}}$ is the same as the tableau obtained by removing the entry r from $\mathbf{t}_{\boldsymbol{\lambda}} s_{j_{\mathfrak{n}},r}$. Likewise, the $\boldsymbol{\lambda}'_{\mathfrak{n}}$-tableau $\mathbf{t}^{\boldsymbol{\lambda}'_{\mathfrak{n}}}$ is the tableau obtained by removing the entry r from $\mathbf{t}^{\boldsymbol{\lambda}'} s_{j_{\mathfrak{n}},r}$.

(b) $w_{\boldsymbol{\lambda}} = s_{i_{\mathfrak{n}},r} w_{\boldsymbol{\lambda}_{\mathfrak{n}}} s_{r,j_{\mathfrak{n}}}$, where $i_{\mathfrak{n}}$ is defined by $(i_{\mathfrak{n}})w_{\boldsymbol{\lambda}} = j_{\mathfrak{n}}$.

(c) $s_{j_{\mathfrak{n}},r} \in \mathcal{D}_{\bar{\boldsymbol{\lambda}}',(r-1,1)}$ and $\mathfrak{S}_{\bar{\boldsymbol{\lambda}}'}^{s_{j_{\mathfrak{n}},r}} \cap \mathfrak{S}_{r-1} = \mathfrak{S}_{\bar{\boldsymbol{\lambda}}'_{\mathfrak{n}}}$.

Moreover, we have

$$\{d \in \mathfrak{S}_r \mid \mathbf{t}_{\boldsymbol{\lambda}} d \in \mathbf{T}^s(\boldsymbol{\lambda})\} = \bigcup_{\mathfrak{n}\in\mathcal{R}_{\boldsymbol{\lambda}}} \{s_{j_{\mathfrak{n}},r} x \mid x \in \mathfrak{S}_{r-1}, \mathbf{t}_{\boldsymbol{\lambda}_{\mathfrak{n}}} x \in \mathbf{T}^s(\boldsymbol{\lambda}_{\mathfrak{n}})\}.$$

Proof. The statement (a) follows immediately from the definition of $s_{i,r}$. Since $j_{\mathfrak{n}}$ is the number at $\mathfrak{n}$ in $\mathbf{t}_{\boldsymbol{\lambda}}$ and $i_{\mathfrak{n}}$ is the number at $\mathfrak{n}$ in $\mathbf{t}^{\boldsymbol{\lambda}}$ by (a), we see that $\mathbf{t}^{\boldsymbol{\lambda}_{\mathfrak{n}}}$ can be obtained by removing the entry r at $\mathfrak{n}$ from $\mathbf{t}^{\boldsymbol{\lambda}} s_{i_{\mathfrak{n}},r}$. On the other hand, since $\mathbf{t}^{\boldsymbol{\lambda}} w_{\boldsymbol{\lambda}} = \mathbf{t}_{\boldsymbol{\lambda}}$, it follows that $(\mathbf{t}^{\boldsymbol{\lambda}} s_{i_{\mathfrak{n}},r}) s_{r,i_{\mathfrak{n}}} w_{\boldsymbol{\lambda}} s_{j_{\mathfrak{n}},r} = \mathbf{t}^{\boldsymbol{\lambda}} w_{\boldsymbol{\lambda}} s_{j_{\mathfrak{n}},r} = \mathbf{t}_{\boldsymbol{\lambda}} s_{j_{\mathfrak{n}},r}$. Therefore, $s_{r,i_{\mathfrak{n}}} w_{\boldsymbol{\lambda}} s_{j_{\mathfrak{n}},r} = w_{\boldsymbol{\lambda}_{\mathfrak{n}}}$, proving (b). The first assertion in (c) follows from the relation

$$s_{r,j_{\mathfrak{n}}} s_i s_{j_{\mathfrak{n}},r} = \begin{cases} s_i, & \text{if } i < j_{\mathfrak{n}}, \\ s_{i-1}, & \text{if } i > j_{\mathfrak{n}}, \\ s_{r,j_{\mathfrak{n}}+1} s_{j_{\mathfrak{n}}} s_{j_{\mathfrak{n}}+1,r} & \text{if } i = j_{\mathfrak{n}}. \end{cases}$$

For d with $\mathbf{t}_{\boldsymbol{\lambda}} d \in \mathbf{T}^s(\boldsymbol{\lambda})$, write $d = d_1 x$ with $x \in \mathfrak{S}_{r-1}$ and d_1 distinguished relative to $\mathfrak{S}_{r-1}$. Then, $d_1 = s_{i,r}$ for some i. Since $\mathbf{t}_{\boldsymbol{\lambda}} d_1$ is again standard, it forces $i = j_{\mathfrak{n}}$ for some $\mathfrak{n} \in \mathcal{R}_{\boldsymbol{\lambda}}$. Thus, $\mathbf{t}_{\boldsymbol{\lambda}_{\mathfrak{n}}}$ is obtained by removing the entry r from $\mathbf{t}_{\boldsymbol{\lambda}} d_1$. Therefore, the tableau $\mathbf{t}_{\boldsymbol{\lambda}_{\mathfrak{n}}} x$ can be obtained from $\mathbf{t}_{\boldsymbol{\lambda}} s_{j_{\mathfrak{n}},r} x$ by removing the entry r. Since $\mathbf{t}_{\boldsymbol{\lambda}} s_{j_{\mathfrak{n}},r} x$ is standard, $\mathbf{t}_{\boldsymbol{\lambda}_{\mathfrak{n}}} x$ is standard, too, proving the inclusion "$\subseteq$". Since $\mathbf{t}_{\boldsymbol{\lambda}} s_{j_{\mathfrak{n}},r} x$ is the tableau obtained from $\mathbf{t}_{\boldsymbol{\lambda}_{\mathfrak{n}}} x$ by adding the entry r at the node $\mathfrak{n}$, $\mathbf{t}_{\boldsymbol{\lambda}} s_{j_{\mathfrak{n}},r} x$ is a standard tableau, proving the inclusion "$\supseteq$". Therefore, the required equality holds. $\square$

(3.4) Lemma. *(a) For positive integers j,k with $1 \leqslant j,k \leqslant r-1$, we have*

$$T_{j,r} L_k = \begin{cases} L_k T_{j,r}, & \text{if } k < j \\ L_{k+1} T_{k,r} - (q-1) L_{k+1} T_{k+1,r}, & \text{if } k = j \\ L_{k+1} T_{j,r} - (q-1) L_{k+1} T_{k+1,r} T_{j,k}, & \text{if } k > j, \end{cases}$$

where $T_{i,l} = T_{s_{i,l}}$.

For $\boldsymbol{a} = [0, a_1, \cdots, a_{m-1}, r] \in \Lambda[m, r]$, *let* $v_{\boldsymbol{a}} = \pi_{\boldsymbol{a}} T_{w_{\boldsymbol{a}}} \tilde{\pi}_{\boldsymbol{a}'}$. *Then*

(b) $v_{\boldsymbol{a}} T_i = T_{(i)w_{\boldsymbol{a}}^{-1}} v_{\boldsymbol{a}}$ *if* $i \neq r - a_j$ *for* $j = 1, \cdots, m$;

(c) $v_{\boldsymbol{a}} L_k = u_j q^{r-a_j+1-k} v_{\boldsymbol{a}} T_{k,r-a_j+1} T_{r-a_j+1,k}$, *where* j *satisfies* $r - a_j + 1 \leqslant k \leqslant r - a_{j-1}$.

Proof. For $k < j$, the result in (a) is clear. If $k = j$, then $T_{k,r} L_k = T_k L_k T_{k+1,r} = q L_{k+1} T_k^{-1} T_{k+1,r}$; if $k > j$, then

$$\begin{aligned} T_{j,r} L_k = T_{j,k+1} L_k T_{k+1,r} &= T_{j,k} T_k L_k T_{k+1,r} \\ &= q T_{j,k} L_{k+1} T_k^{-1} T_{k+1,r} = q L_{k+1} T_{j,k} T_k^{-1} T_{k+1,r}. \end{aligned}$$

Now the result (a) follows easily since $q T_k^{-1} = T_k - (q-1)$.

The statements (b) and (c) have been proved in [DR, (3.1)]. □

The Branching Theorem for symmetric groups can be found in [JK]. The following is the q-version (see [Jo, 3.4]).

(3.5) Lemma. *For any* $\lambda \in \Lambda^+(r)$, *let* $\mathfrak{n}_1, \cdots, \mathfrak{n}_k$ *be the removable nodes of* $\mathcal{Y}(\lambda)$ *counted from top to bottom, and define* $M_0 = 0$ *and* $M_t = z_\lambda T_{j_{\mathfrak{n}_t}, r} \mathcal{H}_R(\mathfrak{S}_{r-1}) + M_{t-1}$ *for* $t \geqslant 1$. *Then we have a filtration of* $\mathcal{H}_R(\mathfrak{S}_{r-1})$*-submodules for* $S_R^\lambda = z_\lambda \mathcal{H}_R$:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_k = S_R^\lambda$$

with sections of Specht modules: $M_t / M_{t-1} \cong S_R^{\lambda_{\mathfrak{n}_t}}$.

We are now ready to prove the (ordinary) Branching Theorem for Ariki-Koike algebra. A version of this result has been given by Ariki-Mathas [AM, 1.5] with a proof using cellular basis. If we turn their result into a result for $y_{\boldsymbol{\lambda}'} \mathbf{H}$ using (2.6) and apply the homomorphism φ given in (2.7), the theorem below follows immediately from theirs. However, the proof supplied here can be viewed as a proof in our context without using cellular bases, following the idea of lifting from level 1 to level m. Recall from (2.2) that $z_{\boldsymbol{\lambda}} = x_{\boldsymbol{\lambda}} T_{w_{\boldsymbol{\lambda}}} y_{\boldsymbol{\lambda}'}$.

(3.6) Branching Theorem. *Let* $\boldsymbol{\lambda}$ *be an* m*-partition of* r, *and let* $\mathfrak{n}_1, \cdots, \mathfrak{n}_N$ *be the removable nodes of* $\mathcal{Y}(\boldsymbol{\lambda})$ *counted from top to bottom and from left to right. Define* $\boldsymbol{M}_0 = 0$ *and* $\boldsymbol{M}_t = z_{\boldsymbol{\lambda}} T_{j_{\mathfrak{n}_t}, r} \mathbf{H}_{mR}^{r-1} + \boldsymbol{M}_{t-1}$ *for* $t \geqslant 1$. *Then these modules form a filtration of* $\mathbf{H}_{mR}^{r-1}$*-submodules for* $S_R^{\boldsymbol{\lambda}} = z_{\boldsymbol{\lambda}} \mathbf{H}_R$:

$$0 = \boldsymbol{M}_0 \subset \boldsymbol{M}_1 \subset \cdots \subset \boldsymbol{M}_N = S_R^{\boldsymbol{\lambda}}$$

with sections of Specht modules: $\boldsymbol{M}_t / \boldsymbol{M}_{t-1} \cong S_R^{\boldsymbol{\lambda}_{\mathfrak{n}_t}}$.

Proof. Let $M_0 = 0$ and $M_t = z_{\boldsymbol{\lambda}} T_{j_{\mathfrak{n}_t}, r} \mathcal{H}_R(\mathfrak{S}_{r-1}) + M_{t-1}$ for $t \geqslant 1$. We claim that $\boldsymbol{M}_t = M_t$ for all t. Indeed, since M_t is a right $\mathcal{H}_R(\mathfrak{S}_{r-1})$-module, we need only prove

$$z_{\boldsymbol{\lambda}} T_{j_{\mathfrak{n}_t}, r} L_k \in M_t, \quad \forall k \leqslant r - 1. \tag{3.7}$$

Let $[\boldsymbol{\lambda}] = [a_0, a_1, \cdots, a_m]$, and choose i such that $r - a_{i+1} < j_{\mathfrak{n}_t} \leqslant r - a_i$. Thus, by (3.2), $\mathfrak{n}_t$ is a removable node of the $(i+1)$-th Young subdiagram $\mathcal{Y}(\lambda^{(i+1)})$. By (2.3) and (3.4b), we have

$$z_{\boldsymbol{\lambda}} T_{j_{\mathfrak{n}_t},r} L_k = Z_{\bar{\lambda}} v_{[\boldsymbol{\lambda}]} T_{j_{\mathfrak{n}_t},r} L_k = z_{\bar{\lambda}} T_{(j_{\mathfrak{n}_t}) w_{[\boldsymbol{\lambda}]}^{-1}} \cdots T_{(r-a_i-1) w_{[\boldsymbol{\lambda}]}^{-1}} v_{[\boldsymbol{\lambda}]} T_{r-a_i,r} L_k.$$

Put $b_j = r - a_j$ for all j. By (3.4a,c), we have the following equalities:

$$(3.8)\quad z_{\boldsymbol{\lambda}} T_{j_{\mathfrak{n}_t},r} L_k = \begin{cases} u_j q^{b_j+1-k} z_{\boldsymbol{\lambda}} T_{j_{\mathfrak{n}_t},r} T_{k,b_j+1} T_{b_j+1,k}, & \text{if } k < b_i, b_j < k \leqslant b_{j-1} \\ u_i z_{\boldsymbol{\lambda}} T_{j_{\mathfrak{n}_t},r} - h_1, & \text{if } k = b_i \\ u_j q^{b_j-k} z_{\boldsymbol{\lambda}} T_{j_{\mathfrak{n}_t},r} T_{k,b_j} T_{b_j,k} - h_2, & \text{if } k > b_i, b_j \leqslant k < b_{j-1} \end{cases}$$

where

$$h_1 = u_i(q-1) z_{\boldsymbol{\lambda}} T_{j_{\mathfrak{n}_t},b_i} T_{b_i+1,r} = u_i(q-1) z_{\boldsymbol{\lambda}} T_{b_i+1,r} T_{j_{\mathfrak{n}_t},b_i},$$
$$h_2 = u_j q^{b_j-k}(q-1) z_{\boldsymbol{\lambda}} T_{k+1,b_j+1} T_{b_j+1,r} T_{j_{\mathfrak{n}},k}.$$

Since $T_{k+1,b_j+1} T_{b_j+1,r} \in \sum_{l \geqslant k > j_{\mathfrak{n}}} T_{l,r} \mathcal{H}_R(\mathfrak{S}_{r-1})$, and $j_{\mathfrak{n}_1}, \cdots, j_{\mathfrak{n}_N}$ is decreasing, we have $h_1, h_2 \in M_{t-1}$ by induction, proving (3.7), and hence the claim.

Put $z_{\boldsymbol{\lambda}} = v_{[\boldsymbol{\lambda}]} \tilde{Z}_{\bar{\boldsymbol{\lambda}}^\circ}$ (cf. (2.3)), where $\boldsymbol{\lambda}^\circ = (\lambda^{(m)}, \cdots, \lambda^{(1)})$ and

$$\tilde{Z}_{\bar{\boldsymbol{\lambda}}^\circ} = \tilde{Z}_{\lambda^{(m)} \vee \cdots \vee \lambda^{(1)}} = x_{\lambda^{(m)} \vee \cdots \vee \lambda^{(1)}} T_{\tilde{w}_{(m)} \cdots \tilde{w}_{(1)}} y_{\lambda^{(m)\prime} \vee \cdots \vee \lambda^{(1)\prime}}.$$

Note that $y_{\lambda^{(m)\prime} \vee \cdots \vee \lambda^{(1)\prime}} = y_{\bar{\boldsymbol{\lambda}}'}$. Since $\tilde{w}_{(i)} \in \mathfrak{S}_{\{r-a_i+1, \cdots, r-a_{i-1}\}}$ (see the definition above (1.4)) and, for $a < b$, $\mathfrak{S}^{s_{a,b}}_{\{a+1,\cdots,b\}} \cap \mathfrak{S}_{\{a,\cdots,b-1\}} = \mathfrak{S}_{\{a,\cdots,b-1\}}$, (3.3c) for $m = 1$ implies that $y_{\bar{\boldsymbol{\lambda}}'} T_{j_{\mathfrak{n}_t},r} = h_0 T_{j_{\mathfrak{n}_t},r} y_{\bar{\boldsymbol{\lambda}}'_{\mathfrak{n}_t}}$ for some $h_0 \in \mathcal{H}_R(\mathfrak{S}_{\{r-a_{i+1}+1,\cdots,r-a_i\}})$. On the other hand, by (3.3b),

$$\begin{aligned} &T_{\tilde{w}_{(m)} \cdots \tilde{w}_{(1)}} h_0 T_{j_{\mathfrak{n}_t},r} \\ &= T_{\tilde{w}_{(m)} \cdots \tilde{w}_{(i+2)}} (T_{\tilde{w}_{(i+1)}} h_0 T_{j_{\mathfrak{n}_t},r-a_i})(T_{\tilde{w}_{(i)}} T_{r-a_i,r-a_{i-1}}) \cdots (T_{\tilde{w}_{(1)}} T_{r-a_1,r}) \\ &= T_{\tilde{y}_{(m)} \cdots \tilde{y}_{(i+2)}} [(T_{\tilde{w}_{(i+1)}} h_0 T_{j_{\mathfrak{n}_t},r-a_i}) T_{r-a_i,r}] T_{\tilde{y}_{(i)} \cdots \tilde{y}_{(1)}}, \end{aligned}$$

where $\tilde{y}_{(i)}$ is the $\tilde{w}_{(i)}$ defined relative to $\boldsymbol{\lambda}_{\mathfrak{n}_t}$ as in (1.4), and $x_{\lambda^{(m)} \vee \cdots \vee \lambda^{(1)}} T_{r-a_i,r} = x_{\lambda^{(m)} \vee \cdots \vee \lambda^{(i+1)}} T_{r-a_i,r} \tilde{x}_{\lambda^{(i)}_{\mathfrak{n}_t} \vee \cdots \vee \lambda^{(1)}_{\mathfrak{n}_t}}$, where $\tilde{x}_{\lambda^{(i)}_{\mathfrak{n}_t} \vee \cdots \vee \lambda^{(1)}_{\mathfrak{n}_t}}$ is the sum of T_w's with $w \in \mathfrak{S}_{\{r-a_i,\cdots,r-a_{i-1}-1\}} \times \cdots \times \mathfrak{S}_{\{r-a_1,\cdots,r-1\}}$. So, noting $\lambda^{(j)} = \lambda^{(j)}_{\mathfrak{n}_t}$ for all $j \neq i+1$, we eventually see that

$$Z_{\bar{\boldsymbol{\lambda}}^\circ} T_{j_{\mathfrak{n}_t},r} = \tilde{Z}_{\lambda^{(m)}_{\mathfrak{n}_t} \vee \cdots \vee \lambda^{(i+2)}_{\mathfrak{n}_t}} [(\tilde{x}_{\lambda^{(i+1)}} T_{\tilde{w}_{(i+1)}} \tilde{y}_{\lambda^{(i+1)\prime}} T_{j_{\mathfrak{n}_t},r-a_i}) T_{r-a_i,r}] \tilde{Z}_{\lambda^{(i)}_{\mathfrak{n}_t} \vee \cdots \vee \lambda^{(1)}_{\mathfrak{n}_t}},$$

where $\tilde{x}$ and $\tilde{y}$ are defined over $\mathfrak{S}_{\{r-a_{i+1}+1,\cdots,r-a_i\}}$ and the last $\tilde{Z}$ is defined over $\mathfrak{S}_{\{r-a_i,\cdots,r-a_{i-1}-1\}} \times \cdots \times \mathfrak{S}_{\{r-a_1,\cdots,r-1\}}$. By Lemma (3.5), we have an epimorphism from $[(\tilde{x}_{\lambda^{(i+1)}} T_{\tilde{w}_{(i+1)}} \tilde{y}_{\lambda^{(i+1)\prime}} T_{j_{\mathfrak{n}_t},r-a_i}) T_{r-a_i,r}] \mathcal{H}_R(\mathfrak{S}_{\{r-a_{i+1}+1,\cdots,r-a_i-1\}})$ onto $\tilde{z}_{\lambda^{(i+1)}_{\mathfrak{n}_t}} \mathcal{H}_R(\mathfrak{S}_{\{r-a_{i+1}+1,\cdots,r-a_i-1\}})$, where $\tilde{z}_{\lambda^{(i+1)}_{\mathfrak{n}_t}} = \tilde{x}_{\lambda^{(i+1)}_{\mathfrak{n}_t}} T_{\tilde{y}_{(i+1)}} \tilde{y}_{\lambda^{(i+1)}_{\mathfrak{n}_t}}$ is defined over $\mathfrak{S}_{\{r-a_{i+1}+1,\cdots,r-a_i-1\}}$. This results in an epimorphism of $\mathcal{H}_R(\mathfrak{S}_{[\boldsymbol{\lambda}^\circ_{\mathfrak{n}_t}]})$-modules $\tilde{Z}_{\bar{\boldsymbol{\lambda}}^\circ} T_{j_{\mathfrak{n}_t},r} \mathcal{H}_R(\mathfrak{S}_{[\boldsymbol{\lambda}^\circ_{\mathfrak{n}_t}]}) \twoheadrightarrow \tilde{Z}_{\bar{\boldsymbol{\lambda}}^\circ_{\mathfrak{n}_t}} \mathcal{H}_R(\mathfrak{S}_{[\boldsymbol{\lambda}^\circ_{\mathfrak{n}_t}]})$, and hence, an epimorphism of $\mathcal{H}_R(\mathfrak{S}_{r-1})$-modules $\tilde{Z}_{\bar{\boldsymbol{\lambda}}^\circ} T_{j_{\mathfrak{n}_t},r} \mathcal{H}_R(\mathfrak{S}_{r-1}) \twoheadrightarrow \tilde{Z}_{\bar{\boldsymbol{\lambda}}^\circ_{\mathfrak{n}_t}} \mathcal{H}_R(\mathfrak{S}_{r-1})$. Observing from (3.5) the kernel of the first epimorphism above, we see the kernel of this epimorphism is the intersection $\tilde{Z}_{\bar{\boldsymbol{\lambda}}\circ}T_{j_{\mathfrak{n}_t},r}\mathcal{H}_R(\mathfrak{S}_{r-1})\cap N_{t-1}$, where N_i is defined as $N_i = \tilde{Z}_{\bar{\boldsymbol{\lambda}}\circ}T_{j_{\mathfrak{n}_i},r}\mathcal{H}_R(\mathfrak{S}_{r-1}) + N_{i-1}$ for $i \geqslant 1$, and $N_0 = 0$. (Thus, $M_i = v_{[\boldsymbol{\lambda}]}N_i$ for all i.) Since the left multiplications by $v_{[\boldsymbol{\lambda}]}$ and $v_{[\boldsymbol{\lambda}_{\mathfrak{n}_t}]}$ induce isomorphisms by [DR, (3.4)], it yields an epimorphism of $\mathcal{H}_R(\mathfrak{S}_{r-1})$-modules $z_{\boldsymbol{\lambda}}T_{j_{\mathfrak{n}_t},r}\mathcal{H}_R(\mathfrak{S}_{r-1}) \twoheadrightarrow S^{\boldsymbol{\lambda}_{\mathfrak{n}_t}}$ with kernel $z_{\boldsymbol{\lambda}}T_{j_{\mathfrak{n}_t},r}\mathcal{H}_R(\mathfrak{S}_{r-1}) \cap M_{t-1}$. Therefore, we finally obtain an $\mathcal{H}_R(\mathfrak{S}_{r-1})$-module isomorphism $f_t : S^{\boldsymbol{\lambda}_{\mathfrak{n}_t}} \to M_t/M_{t-1}$ which maps $z_{\boldsymbol{\lambda}_{\mathfrak{n}_t}}h$ to $z_{\boldsymbol{\lambda}}T_{j_{\mathfrak{n}_t},r}h + M_{t-1}$ for all $h \in \mathcal{H}_R(\mathfrak{S}_{r-1})$.

By the claim, it remains to prove that f_t is an $\mathbf{H}^{r-1}_{mR}$-module isomorphism. We first note from (3.4b,c) that (3.8) holds if $z_{\boldsymbol{\lambda}}T_{j_{\mathfrak{n}_t},r}$ is replaced by $z_{\boldsymbol{\lambda}_{\mathfrak{n}_t}}$ and h_i by zeros. Also, note that, for any $x \in \mathfrak{S}_{r-1}$, $T_xT_0 = L_kh$ for $h \in \mathcal{H}_R(\mathfrak{S}_{r-1})$. So we have for some $h' \in \mathcal{H}_R(\mathfrak{S}_{r-1})$

$$f_t(z_{\boldsymbol{\lambda}_{\mathfrak{n}_t}}T_xT_0) = f_t(z_{\boldsymbol{\lambda}_{\mathfrak{n}_t}}L_kh) = f_t(z_{\boldsymbol{\lambda}_{\mathfrak{n}_t}}h'h) = f_t(z_{\boldsymbol{\lambda}_{\mathfrak{n}_t}})h'h = f_t(z_{\boldsymbol{\lambda}_{\mathfrak{n}_t}}T_x)T_0.$$

Therefore, f_t is an $\mathbf{H}^{r-1}_{mR}$-module isomorphism, and the theorem is proven. □

(3.9) Corollary. *Let R be a field and assume that $\mathbf{H}^r_{mR}$ is semi-simple. Then we have an isomorphism of $\mathbf{H}^{r-1}_{mR}$-modules: $S^{\boldsymbol{\lambda}}_R|_{\mathbf{H}^{r-1}_{mR}} \cong \oplus_{\mathfrak{n}\in\mathcal{R}_{\boldsymbol{\lambda}}} S^{\boldsymbol{\lambda}_{\mathfrak{n}}}_R$.*

4. Branching rules, II. In this section, we shall describe the modular branching theorem for an Ariki-Koike algebras $\mathbf{H} = \mathbf{H}^r_{mR}$ over a *field* R which has a semi-simple "bottom", that is, satisfying the assumption

$$f_{m,r} = \prod_{i=1}^{m-1}\prod_{j=i+1}^{m}\prod_{k=1-r}^{r-1}(u_iq^k - u_j) \neq 0. \tag{4.1}$$

Let l be the smallest integer a such that $1 + q + \cdots + q^{a-1} = 0$. (Note that, if $q = 1$, then l is the characteristic of R.) If such an integer does not exist, then we set $l = \infty$. A partition of r is said to be l-regular if λ has *no* non-zero part occurring l or more times. An m-partition $\boldsymbol{\lambda} = (\lambda^{(1)}, \cdots, \lambda^{(m)})$ is said to be l-regular if each $\lambda^{(i)}$ is l-regular. Unless otherwise specified, we shall assume in this section that R is a *field* containing the element $u_1, \cdots, u_m$ and $q \neq 0$, and the subscript R in $\mathbf{H}_R$ etc. will be dropped for notational simplicity.

(4.2) Theorem. *Let $\mathbf{H}$ be the Ariki-Koike algebra over a field R such that $f_{m,r} \neq 0$. Then the set $\{y_{\boldsymbol{\lambda}'}T_{w_{\boldsymbol{\lambda}'}}z_{\boldsymbol{\lambda}}\mathbf{H} \mid \boldsymbol{\lambda} \in \Lambda^+_m(r)$ l-regular$\}$ is a complete set of non-isomorphic simple $\mathbf{H}$-modules.*

Proof. With our hypothesis, recall from [DR, (3.9),(4.14)] that there exist orthogonal idempotents $e_{\boldsymbol{a}} = v_{\boldsymbol{a}}T_{w_{\boldsymbol{a}'}}c_{\boldsymbol{a}}$ of $\mathbf{H}$, (Note that $c_{\boldsymbol{a}}$ is denoted $z_{\boldsymbol{a}}$ in [DR].) $\boldsymbol{a} \in \Lambda[m,r]$, such that the categories of $\mathbf{H}$-modules and $\epsilon\mathbf{H}\epsilon$-modules are Morita equivalent, where $\epsilon = \sum_{\boldsymbol{a}\in\Lambda[m,r]} e_{\boldsymbol{a}}$. Let $D^{\boldsymbol{\lambda}} = y_{\boldsymbol{\lambda}'}T_{w_{\boldsymbol{\lambda}'}}x_{\boldsymbol{\lambda}}T_{w_{\boldsymbol{\lambda}}}y_{\boldsymbol{\lambda}'}\mathbf{H}$. By the Morita equivalence, it suffices to prove that the set

$$\{D^{\boldsymbol{\lambda}}\epsilon \mid \boldsymbol{\lambda} \in \Lambda^+_m(r) \text{ } l\text{-regular}\}$$

is a complete set of non-isomorphic simple $\epsilon\mathbf{H}\epsilon$-modules.

Suppose $\boldsymbol{\lambda} = (\lambda^{(1)}, \cdots, \lambda^{(m)})$ is an l-regular m-partition and write $\boldsymbol{a} = [a_i] = [\boldsymbol{\lambda}]$. Then, by (2.3), $y_{\boldsymbol{\lambda}'} T_{w_{\boldsymbol{\lambda}'}} z_{\boldsymbol{\lambda}} = y_{\boldsymbol{\lambda}'} T_{w_{\boldsymbol{\lambda}'}} Z_{\bar{\boldsymbol{\lambda}}} v_{\boldsymbol{a}}$. Since $v_{\boldsymbol{a}}\mathbf{H} = e_{\boldsymbol{a}}\mathbf{H}$, we have by [DR, (3.10),(3.8)]

$$
\begin{aligned}
D^{\boldsymbol{\lambda}}\epsilon &= y_{\boldsymbol{\lambda}'} T_{w_{\boldsymbol{\lambda}'}} Z_{\bar{\boldsymbol{\lambda}}}(e_{\boldsymbol{a}}\mathbf{H}e_{\boldsymbol{a}}) \\
&= \tilde{\pi}_{\boldsymbol{a}'} T_{w_{\boldsymbol{a}}^{-1}} (y_{\lambda^{(1)\prime}\vee\cdots\vee\lambda^{(m)\prime}} T_{w(1)^{-1}\cdots w(m)^{-1}} Z_{\bar{\boldsymbol{\lambda}}})(e_{\boldsymbol{a}}\mathbf{H}e_{\boldsymbol{a}}) \\
&= \tilde{\pi}_{\boldsymbol{a}'} T_{w_{\boldsymbol{a}}^{-1}} D^{\bar{\boldsymbol{\lambda}}} e_{\boldsymbol{a}},
\end{aligned}
\tag{4.3}
$$

where $D^{\bar{\boldsymbol{\lambda}}} = (y_{\lambda^{(1)\prime}\vee\cdots\vee\lambda^{(m)\prime}} T_{w(1)^{-1}\cdots w(m)^{-1}} Z_{\bar{\boldsymbol{\lambda}}})\mathcal{H}(\mathfrak{S}_{\boldsymbol{a}})$. Now, it is known ([J, 8.1i]), the $\mathcal{H}(\mathfrak{S}_{\{a_{i-1}+1,\cdots,a_i\}})$-module $D^{\lambda^{(i)}} = y_{\lambda^{(i)\prime}} T_{w_{(i)}^{-1}} x_{\lambda^{(i)}} T_{w_{(i)}} y_{\lambda^{(i)\prime}} \mathcal{H}(\mathfrak{S}_{\{a_{i-1}+1,\cdots,a_i\}})$ is simple for every $i = 1, \cdots, m$, where $x_{\lambda^{(i)}}$ and $y_{\lambda^{(i)}}$ are defined as in (1.8) using the subgroup $\mathfrak{S}_{\{a_{i-1}+1,\cdots,a_i\}}$. So $D^{\bar{\boldsymbol{\lambda}}} = D^{\lambda^{(1)}} \cdots D^{\lambda^{(m)}}$ is a simple $\mathcal{H}(\mathfrak{S}_{\boldsymbol{a}})$-module, and the set

$$
\{D^{\bar{\boldsymbol{\lambda}}} \mid \boldsymbol{\lambda} \in \Lambda_m^+(r)\ l\text{-regular}, [\boldsymbol{\lambda}] = \boldsymbol{a}\}
\tag{4.4}
$$

forms a complete set of non-isomorphic simple $\mathcal{H}(\mathfrak{S}_{\boldsymbol{a}})$-modules.

Since $e_{\boldsymbol{a}} = v_{\boldsymbol{a}} T_{w_{\boldsymbol{a}'}} c_{\boldsymbol{a}}^{-1}$, where $T_{w_{\boldsymbol{a}'}}$ and $c_{\boldsymbol{a}}$ are invertible and $c_{\boldsymbol{a}}$ is in the center of $\mathcal{H}(\mathfrak{S}_{\boldsymbol{a}})$, [DR, (3.4)] implies that the map $D^{\bar{\boldsymbol{\lambda}}} \to D^{\bar{\boldsymbol{\lambda}}} e_{\boldsymbol{a}}$ sending x to $xe_{\boldsymbol{a}}$ for $x \in D^{\bar{\boldsymbol{\lambda}}}$ is an R-module isomorphism. Obviously, this is an $\mathcal{H}(\mathfrak{S}_{\boldsymbol{a}})$-module isomorphism. So $D^{\bar{\boldsymbol{\lambda}}} e_{\boldsymbol{a}}$ is a simple $\mathcal{H}(\mathfrak{S}_{\boldsymbol{a}})$-module. By (4.3), there is an epimorphism from $D^{\bar{\boldsymbol{\lambda}}} e_{\boldsymbol{a}}$ to $D^{\boldsymbol{\lambda}}\epsilon = D^{\boldsymbol{\lambda}} e_{\boldsymbol{a}}$. Therefore, $D^{\boldsymbol{\lambda}}\epsilon$ is a simple $\epsilon\mathbf{H}\epsilon$-module, and consequently, by (4.4), all $D^{\boldsymbol{\lambda}}\epsilon$ form a complete set of non-isomorphic $\epsilon\mathbf{H}\epsilon$-modules. $\square$

To state the modular branching rule for Ariki-Koike algebra, we need the notion of normal and good nodes. Since Ariki-Koike algebra is semi-simple if $f_{m,r} \neq 0$ and $l > r$ ([A1] or [DR, (5.2)]) and the branching rule in this semi-simple case has been done in (3.9), we assume in the rest of this section $l \leqslant r$ and $f_{m,r} \neq 0$.

Recall from (3.1) that the boxes in the Young diagram $\mathcal{Y}(\lambda)$ can be identified with its position coordinates, called nodes, $\mathfrak{n} = (i, j)$. The node $\mathfrak{n} = (i, \lambda_i)$ is called a removable node of λ if $\lambda_i > \lambda_{i+1}$ and the node $(i, \lambda_i + 1)$ is called an addable node if $\lambda_{i-1} > \lambda_i$. The node $\mathfrak{n}$ is called a removable (resp. addable) node for m-partition $\boldsymbol{\lambda} = (\lambda^{(1)}, \cdots, \lambda^{(m)})$ if it is a removable (resp. addable) node of $\lambda^{(i)}$ for some i.

For each node (i, j) of λ, define the l-residue of $\mathfrak{n}$

$$
\operatorname{res}(\mathfrak{n}) = \begin{cases} \operatorname{rem}(j - i) & \text{if } q = 1, \\ q^{j-i} & \text{otherwise,} \end{cases}
$$

where $\operatorname{rem}(j - i)$ is the remainder when $j - i$ is divided by l. A removable node $\mathfrak{n}$ of λ is called *normal* if for every addable node $\mathfrak{m}$ with $\operatorname{res}(\mathfrak{m}) = \operatorname{res}(\mathfrak{n})$, there exists a removable node $\mathfrak{n}'(\mathfrak{m})$ strictly between $\mathfrak{n}$ and $\mathfrak{m}$ with $\operatorname{res}(\mathfrak{n}'(\mathfrak{m})) = \operatorname{res}(\mathfrak{n})$, and $\mathfrak{m} \neq \mathfrak{m}'$ implies $\mathfrak{n}'(\mathfrak{m}) = \mathfrak{n}'(\mathfrak{m}')$. A removable node is called *good* if it is the lowest among the normal nodes of a fixed residue. (A node (i, j) is lower than the node (i', j') if $i > i'$.) Let $\mathcal{R}_{normal}(\lambda)$ (resp. $R_{good}(\lambda)$) be the set of all normal (resp. good) nodes of $\lambda \in \Lambda^+(r)$. The following result, generalizing Kleshchev's result [K] for symmetric groups, is due to Brundan [B, 2.5-6].

(4.5) Theorem. *Let $\mathcal{H}(r)$ be the Hecke algebra of type A over the field R. If $\lambda \in \Lambda^+(r)$ and $\mu \in \Lambda^+(r-1)$ are l-regular partitions, then*

(a) $\mathrm{Hom}_{\mathcal{H}(r-1)}(S^\mu, D^\lambda|_{\mathcal{H}(r-1)}) = \begin{cases} R, & \text{if } \mu = \lambda_{\mathfrak{n}} \text{ for some } \mathfrak{n} \in \mathcal{R}_{normal}(\lambda), \\ 0, & \text{otherwise.} \end{cases}$

(b) $\mathrm{Hom}_{\mathcal{H}(r-1)}(D^\mu, D^\lambda|_{\mathcal{H}(r-1)}) = \begin{cases} R, & \text{if } \mu = \lambda_{\mathfrak{n}} \text{ for some } \mathfrak{n} \in \mathcal{R}_{good}(\lambda), \\ 0, & \text{otherwise.} \end{cases}$.

Therefore, the socle of the restriction of D^λ to $\mathcal{H}(r-1)$ is $\oplus_{\mathfrak{n}\in\mathcal{R}_{good}(\lambda)} D^{\lambda_{\mathfrak{n}}}$.

We are going to generalize this result to the Ariki-Koike algebra $\mathbf{H}$ satisfying (4.1).

Let $\boldsymbol{\lambda}$ be an m-partition of r and $\mathfrak{n} = (i,j)_k$ a node of $\boldsymbol{\lambda}$. The residue of $\mathfrak{n}$ is defined as

$$\mathrm{res}(\mathfrak{n}) = \begin{cases} \mathrm{rem}(j-i)\xi^k & \text{if } q = 1, u_k = \xi^k \\ q^{j-i}u_k & \text{otherwise,} \end{cases}$$

where $\mathrm{rem}(j-i)$ is defined as above and ξ is the m-th primitive root of unity. We say that node $(i,j)_k$ is *lower* than the node $(i_1,j_1)_{k_1}$ if either $k > k_1$ or $k = k_1$ and $i > i_1$. Thus, any two removable nodes are comparable. So, we defined normal and good (removable) nodes similarly as in the $m = 1$ case, and let $\mathcal{R}_{normal}(\boldsymbol{\lambda})$ (resp. $\mathcal{R}_{good}(\boldsymbol{\lambda})$) be the set of all normal (resp. good) nodes of $\boldsymbol{\lambda} \in \Lambda_m^+(r)$.

(4.6) Lemma. *Let $f_{m,r} \neq 0$ and let $\boldsymbol{\lambda} \in \Lambda_m^+(r)$. If $\mathfrak{n}$ and $\mathfrak{n}'$ are two nodes of $\mathcal{Y}(\boldsymbol{\lambda})$ with same residues, then $\mathfrak{n}, \mathfrak{n}'$ are in $\mathcal{Y}(\lambda^{(k)})$ for some k. Therefore, $\mathfrak{n}$ is a normal (resp. good) removable node of $\mathcal{Y}(\boldsymbol{\lambda})$ with residue q^iu_k if and only if $\mathfrak{n}$ is a normal (resp. good) removable node of $\mathcal{Y}(\lambda^{(k)})$.*

Proof. Suppose $\mathrm{res}(\mathfrak{n}) = q^a u_k$ and $\mathrm{res}(\mathfrak{n}') = q^{a'}u_{k'}$ with $0 \leqslant a, a' \leqslant l-1$ and $1 \leqslant k, k' \leqslant m$. Since $f_{m,r} \neq 0$, $u_i/u_j \neq q^b$ for any $i \neq j$, $1 \leqslant i,j \leqslant m$ and $0 \leqslant b \leqslant l-1 \leqslant r-1$. Thus, if $\mathfrak{n}$ and $\mathfrak{n}'$ have the same residue, that is, $q^a u_k = q^{a'}u_{k'}$, then $k = k'$ (and hence, $a = a'$). So they are in $\mathcal{Y}(\lambda^{(k)})$. The last assertion follows immediately from the definition. □

In the proof of modular branching rules below, we shall follow the notations used in [DR]. Thus, if $\boldsymbol{a} = [a_0, a_1, \cdots, a_m] \in \Lambda[m,r]$, then we define [DR, (1.2)] $\boldsymbol{a}' = [0, r-a_{m-1}, \cdots, r-a_1, r]$, and $\boldsymbol{a}_\dashv = [a_0, a_1, \cdots, a_{j-1}, r-1, \cdots, r-1]$, where j is the minimal index such that $a_j = r$. We also define [DR, (1.9)]

$$\boldsymbol{a}_i = \boldsymbol{a}_\dashv + \mathbf{1}_i, \text{ where } \mathbf{1}_i = [0, \underbrace{0, \cdots, 0}_{i-1}, 1, \cdots, 1] \in \Lambda[m,1].$$

(4.7) Lemma. *Let $\mathbf{H}$ be the Ariki-Koike algebra over a commutative ring R. For $\boldsymbol{a} \in \Lambda[m,r]$, write $\boldsymbol{a}_\dashv = [b_0, b_1, \cdots, b_m] \in \Lambda[m, r-1]$, and let $U_i = \mathbf{H}\pi_{\boldsymbol{a}_i}T_{b_i+1,r}T_{w_{\boldsymbol{a}_\dashv}}\tilde{\pi}_{(\boldsymbol{a}_\dashv)'}$ for $1 \leqslant i \leqslant m$. Then*

(a) $U_1 = \mathbf{H}v_{\boldsymbol{a}_1}$ and $U_m = \mathbf{H}v_{\boldsymbol{a}_\dashv}$, and

(b) U_i is a free R-submodule with basis $\{T_w L_{b_i+1}^c \pi_{\boldsymbol{a}_i} T_{b_i+1,r} T_{w_{\boldsymbol{a}_\dashv}} \tilde{\pi}_{(\boldsymbol{a}_\dashv)'} \mid 0 \leqslant c \leqslant i-1, w \in \mathfrak{S}_r\}$. So the rank of U_i is $i \cdot r!$.

If, in addition, R is an integral domain in which $f_{m,r}$ is a unit, then *(c)* $\mathbf{H}v_{\boldsymbol{a}}$ *is a projective* $\mathbf{H}$*-module, and, for each* $i = 2, \cdots, m$*, we have short exact sequence* $0 \to U_{i-1} \to U_i \xrightarrow{\varphi_i} \mathbf{H}v_{\boldsymbol{a}_i} \to 0$*. Therefore,* $\mathbf{H}v_{\boldsymbol{a}_\dashv} \cong \oplus_{i=1}^m \mathbf{H}v_{\boldsymbol{a}_i}$*.*

Proof. By the definition, we have $(\boldsymbol{a}_\dashv)' = [0, r-1-b_{m-1}, \cdots, r-1-b_1, r-1]$. So, if $(\boldsymbol{a}_\dashv)' = \boldsymbol{c}_\dashv$ for some $\boldsymbol{c} \in \Lambda[m,r]$, then

$$\boldsymbol{c}_i = [0, \underbrace{r-1-b_{m-1}, \cdots, r-1-b_{m-i+1}}_{i-1}, r-b_{m-i}, \cdots, r-b_1, r],$$

and therefore, $(\boldsymbol{c}_i)' = [0, b_1, \cdots, b_{m-i}, b_{m-i+1}+1, \cdots, b_{m-1}+1, r] = \boldsymbol{a}_{m-i+1}$. Now, using the V_i defined by $\boldsymbol{c}$ in [DR, (4.7)], we see that

$$U_i = \Phi(\iota(V_{m-i+1})), \tag{4.8}$$

where ι is the R-linear anti-involution on $\mathbf{H}$ sending T_i to T_i (see [GL, (5.5)]) and Φ is defined in (2.6). Note that $\Phi(\iota(v_{\boldsymbol{b}})) = q^a v_{\boldsymbol{b}'}$ for some $a \in \mathbb{Z}$. Thus, $U_1 = \Phi(\iota(V_m)) = \Phi(\iota(v_{\boldsymbol{c}_m}\mathbf{H})) = \mathbf{H}v_{\boldsymbol{c}'_m} = \mathbf{H}v_{\boldsymbol{a}_1}$, and $U_m = \Phi(\iota(V_1)) = \Phi(\iota(v_{\boldsymbol{c}_\dashv}\mathbf{H})) = \mathbf{H}v_{\boldsymbol{c}'_\dashv} = \mathbf{H}v_{\boldsymbol{a}_\dashv}$, proving (a). (Note that (a) can be also seen directly from the definition of U_i.) The statements (b) and (c) follow from [DR, (4.7b),(4.12),(4.14)] and the relation (4.8). $\square$

Recall from the proof of (4.2) that we have the isomorphism $D^{\boldsymbol{\lambda}}\epsilon \cong D^{\bar{\boldsymbol{\lambda}}}e_{[\boldsymbol{\lambda}]}$ of irreducible $\epsilon\mathbf{H}\epsilon$-modules. Since $D^{\bar{\boldsymbol{\lambda}}}e_{[\boldsymbol{\lambda}]} = (y_{\lambda^{(1)\prime}\vee\cdots\vee\lambda^{(m)\prime}}T_{w(1)^{-1}\cdots w(m)^{-1}}Z_{\bar{\boldsymbol{\lambda}}}v_{[\boldsymbol{\lambda}]}\mathbf{H})e_{[\boldsymbol{\lambda}]}$, where $Z_{\bar{\boldsymbol{\lambda}}}$ is given in (2.3), the module in parentheses is irreducible, and hence isomorphic to $D^{\boldsymbol{\lambda}}$. In the rest of the section, we put

$$D^{\boldsymbol{\lambda}} = y_{\lambda^{(1)\prime}\vee\cdots\vee\lambda^{(m)\prime}}T_{w(1)^{-1}\cdots w(m)^{-1}}Z_{\bar{\boldsymbol{\lambda}}}v_{[\boldsymbol{\lambda}]}\mathbf{H}. \tag{4.9}$$

Recall also that $\Lambda_m^+(r)$ is the set of all m-partitions of r. The following result is the modular branching rule for the Ariki-Koike algebra under the assumption (4.1), i.e., $f_{m,r} \neq 0$ in R.

(4.10) Theorem. *Let* $\mathbf{H}_m^r$ *be the Ariki-Koike algebra over a field* R *in which* $f_{m,r} \neq 0$*. If* $\boldsymbol{\lambda} \in \Lambda_m^+(r)$ *and* $\boldsymbol{\rho} \in \Lambda_m^+(r-1)$ *are* l*-regular* m*-partitions, then*

(a) $\mathrm{Hom}_{\mathbf{H}_m^{r-1}}(S^{\boldsymbol{\rho}}, D^{\boldsymbol{\lambda}}|_{\mathbf{H}_m^{r-1}}) \cong \begin{cases} R, & \text{if } \boldsymbol{\rho} = \boldsymbol{\lambda}_{\mathfrak{n}} \text{ for some } \mathfrak{n} \in \mathcal{R}_{normal}(\boldsymbol{\lambda}), \\ 0, & \text{otherwise.} \end{cases}$

(b) $\mathrm{Hom}_{\mathbf{H}_m^{r-1}}(D^{\boldsymbol{\rho}}, D^{\boldsymbol{\lambda}}|_{\mathbf{H}_m^{r-1}}) \cong \begin{cases} R, & \text{if } \boldsymbol{\rho} = \boldsymbol{\lambda}_{\mathfrak{n}} \text{ for some } \mathfrak{n} \in \mathcal{R}_{good}(\boldsymbol{\lambda}), \\ 0, & \text{otherwise.} \end{cases}$.

Therefore, the socle of the restriction of $D^{\boldsymbol{\lambda}}$ *to* $\mathbf{H}_m^{r-1}$ *is* $\oplus_{\mathfrak{n}\in\mathcal{R}_{good}(\boldsymbol{\lambda})}D^{\boldsymbol{\lambda}_{\mathfrak{n}}}$*.*

Proof. Because $f_{m,r-1}$ is a factor of $f_{m,r}$ and $f_{m,r} \neq 0$, we have $f_{m-1,r} \neq 0$. So, by [DR, (4.14c)], we have a Morita equivalence between the categories of $\mathbf{H}_m^{r-1}$-modules and $\epsilon\mathbf{H}_m^{r-1}\epsilon$-modules, where $\epsilon = \sum_{\boldsymbol{b}\in\Lambda[m,r-1]} e_{\boldsymbol{b}}$. Note that $\{e_{\boldsymbol{b}}\}$ is a set of orthogonal idempotents. Using a standard result on Morita equivalence (see, e.g., [AF]) and noting [DR, (3.10)], we have, for $M = S^{\boldsymbol{\rho}}$ or $D^{\boldsymbol{\rho}}$, the linear isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathbf{H}_m^{r-1}}(M, D^{\boldsymbol{\lambda}}|_{\mathbf{H}_m^{r-1}}) &\cong \mathrm{Hom}_{\epsilon\mathbf{H}_m^{r-1}\epsilon}(M\epsilon, D^{\boldsymbol{\lambda}}\epsilon) \\ &\cong \mathrm{Hom}_{e_{[\boldsymbol{\rho}]}\mathbf{H}_m^{r-1}e_{[\boldsymbol{\rho}]}}(Me_{[\boldsymbol{\rho}]}, D^{\boldsymbol{\lambda}}e_{[\boldsymbol{\rho}]}).\end{aligned} \tag{4.11}$$

Since $e_{[\boldsymbol\rho]}\mathbf{H}_m^{r-1}e_{[\boldsymbol\rho]} = e_{[\boldsymbol\rho]}\mathcal{H}(\mathfrak{S}_{[\boldsymbol\rho]}) = \mathcal{H}(\mathfrak{S}_{[\boldsymbol\rho]})e_{[\boldsymbol\rho]} \cong \mathcal{H}(\mathfrak{S}_{[\boldsymbol\rho]})$ we may twist the action on $Me_{[\boldsymbol\rho]}$ via this isomorphism and obtain isomorphisms of $\mathcal{H}(\mathfrak{S}_{[\boldsymbol\rho]})$-modules: $S^{\boldsymbol\rho}e_{[\boldsymbol\rho]} = S^{\bar{\boldsymbol\rho}}e_{[\boldsymbol\rho]} \cong S^{\bar{\boldsymbol\rho}}$ and $D^{\boldsymbol\rho}e_{[\boldsymbol\rho]} = D^{\bar{\boldsymbol\rho}}e_{[\boldsymbol\rho]} \cong D^{\bar{\boldsymbol\rho}}$. So the calculation of the Hom sets above is reduced to calculate the module $D^{\boldsymbol\lambda}e_{[\boldsymbol\rho]}$.

¿From (4.9), we first have to manipulate the set $v_{[\boldsymbol\lambda]}\mathbf{H}e_{[\boldsymbol\rho]} = v_{[\boldsymbol\lambda]}\mathbf{H}v_{[\boldsymbol\rho]}T_{w_{\boldsymbol a'}}c_{[\boldsymbol\rho]}^{-1}$ (see line 2 after (4.4)). Choose $\boldsymbol a \in \Lambda[m,r]$ such that $\boldsymbol a_\dashv = [\boldsymbol\rho]$. Then $v_{[\boldsymbol\lambda]}\mathbf{H}v_{[\boldsymbol\rho]} = v_{[\boldsymbol\lambda]}\mathbf{H}v_{\boldsymbol a_\dashv}$. Suppose $v_{[\boldsymbol\lambda]}\mathbf{H}v_{\boldsymbol a_\dashv} \neq 0$. Then, the isomorphism $\mathbf{H}v_{\boldsymbol a_\dashv} \cong \oplus_{i=1}^m \mathbf{H}v_{\boldsymbol a_i}$ in (4.7c) together with the orthogonal property [DR, (3.10)] implies that there is a unique i such that $[\boldsymbol\lambda] = \boldsymbol a_i$. Thus, $v_{[\boldsymbol\lambda]}\mathbf{H}v_{\boldsymbol a_\dashv} = v_{\boldsymbol a_i}\mathbf{H}v_{\boldsymbol a_\dashv} = v_{\boldsymbol a_i}U_i$, since $U_i \cong \oplus_{j=i}^m \mathbf{H}v_{\boldsymbol a_j}$. Now $\tilde\pi_{\boldsymbol a_i'}\mathcal{H}\pi_{\boldsymbol a_i} = \mathcal{H}(\mathfrak{S}_{\boldsymbol a_i'})\iota(v_{\boldsymbol a_i})$ by [DR, (3.1)], it follows that

$$\begin{aligned}
v_{\boldsymbol a_i}U_i &= \pi_{\boldsymbol a_i}T_{w_{\boldsymbol a_i}}(\tilde\pi_{\boldsymbol a_i'}\mathbf{H}\pi_{\boldsymbol a_i})T_{b_i+1,r}T_{w_{\boldsymbol a_\dashv}}\tilde\pi_{(\boldsymbol a_\dashv)'}\\
&= \mathcal{H}(\mathfrak{S}_{\boldsymbol a_i})\pi_{\boldsymbol a_i}T_{w_{\boldsymbol a_i}}\iota(v_{\boldsymbol a_i})T_{b_i+1,r}T_{w_{\boldsymbol a_\dashv}}\tilde\pi_{(\boldsymbol a_\dashv)'}\\
&= \mathcal{H}(\mathfrak{S}_{\boldsymbol a_i})v_{\boldsymbol a_i}T_{w_{\boldsymbol a_i'}}\pi_{\boldsymbol a_i}T_{b_i+1,r}T_{w_{\boldsymbol a_\dashv}}\tilde\pi_{(\boldsymbol a_\dashv)'}.
\end{aligned}$$

Write $\pi_{\boldsymbol a_i}T_{b_i+1,r} = h_i\pi_{\boldsymbol a_\dashv}$, where

$$\begin{aligned}
h_i &= (L_{b_i+1} - u_{i+1})T_{b_i+1,b_{i+1}+1}\cdots(L_{b_{m-1}+1} - u_m)T_{b_{m-1}+1,r}\\
&= \prod_{j=i}^{m-1}(L_{b_j+1} - u_{j+1})T_{b_j+1,b_{j+1}+1}
\end{aligned}$$

(see [DR, (2.7b)]). Then, $v_{\boldsymbol a_i}\mathbf{H}v_{\boldsymbol a_\dashv} = \mathcal{H}(\mathfrak{S}_{\boldsymbol a_i})v_{\boldsymbol a_i}T_{w_{\boldsymbol a_i'}}h_iv_{\boldsymbol a_\dashv}$, and therefore, $D^{\boldsymbol\lambda}e_{\boldsymbol a_\dashv} = D^{\bar\lambda}v_{\boldsymbol a_i}T_{w_{\boldsymbol a_i'}}h_ie_{\boldsymbol a_\dashv}$. Thus the isomorphism $\mathcal{H}(\mathfrak{S}_{\boldsymbol a_\dashv}) \cong e_{\boldsymbol a_\dashv}\mathbf{H}_m^{r-1}e_{\boldsymbol a_\dashv} = e_{\boldsymbol a_\dashv}\mathcal{H}(\mathfrak{S}_{\boldsymbol a_\dashv})$ will turn as above the $e_{\boldsymbol a_\dashv}\mathcal{H}(\mathfrak{S}_{\boldsymbol a_\dashv})$-module $D^{\bar\lambda}v_{\boldsymbol a_i}T_{w_{\boldsymbol a_i'}}h_ie_{\boldsymbol a_\dashv}$ into an $\mathcal{H}(\mathfrak{S}_{\boldsymbol a_\dashv})$-module with the action

$$(xv_{\boldsymbol a_i}T_{w_{\boldsymbol a_i'}}h_ie_{\boldsymbol a_\dashv}) * T_w = xv_{\boldsymbol a_i}T_{w'_{\boldsymbol a_i}}h_iT_we_{\boldsymbol a_\dashv} \tag{4.12}$$

for all $x \in D^{\bar\lambda}$ and $w \in \mathfrak{S}_{\boldsymbol a_\dashv}$.

We now claim that this $\mathcal{H}(\mathfrak{S}_{\boldsymbol a_\dashv})$-module is isomorphic to the $\mathcal{H}(\mathfrak{S}_{\boldsymbol a_\dashv})$-module $D^{\bar\lambda}T_{b_i+1,r}$. Indeed, write $w = w_1\cdots w_m$, where $w_j \in \mathfrak{S}_{\{b_j+1,\cdots,b_{j+1}\}}$. Since

$$s_{b_j+1,b_{j+1}+1}\mathfrak{S}_{\{b_j+1,\cdots,b_{j+1}\}} = \mathfrak{S}_{\{b_j+2,\cdots,b_{j+1}+1\}}s_{b_j+1,b_{j+1}+1}$$

and the product has length additivity, we have

$$T_{b_j+1,b_{j+1}+1}T_{w_j} = T_{w_j^*}T_{b_j+1,b_{j+1}+1}$$

where $w_j^* = s_{b_j+1,b_{j+1}+1}w_js_{b_{j+1}+1,b_j+1}$. Thus, from (4.12), we have by the commuting relations between L_i's and T_j's ([DR, (2.5)])

$$\begin{aligned}
xv_{\boldsymbol a_i}T_{w_{\boldsymbol a_i'}}h_iT_we_{\boldsymbol a_\dashv} &= xv_{\boldsymbol a_i}T_{w_{\boldsymbol a_i'}}T_{w_1,\cdots w_{i-1}}\prod_{j=i}^{m-1}(L_{b_j+1} - u_{j+1})(T_{b_j+1,b_{j+1}+1}T_{w_j})\\
&= xv_{\boldsymbol a_i}T_{w_{\boldsymbol a_i'}}T_{w_1,\cdots w_{i-1}w_i^*\cdots w_m^*}h_i\\
&= xT_{w_1,\cdots w_{i-1}w_i^*\cdots w_m^*}v_{\boldsymbol a_i}T_{w'_{\boldsymbol a_i}}h_ie_{\boldsymbol a_\dashv}.
\end{aligned}$$

Here the last equality follows from a repeated use of [DR, (1.8)] and [DR, (2.7a)] noting $w_1, \cdots w_{i-1}w_i^* \cdots w_m^* \in \mathfrak{S}_{\boldsymbol{a}_i}$. On the other hand, we have

$$xT_{b_i+1,r}T_w = xT_{w_1,\cdots w_{i-1}w_i^*\cdots w_m^*}T_{b_i+1,r}. \tag{4.13}$$

Therefore, the linear map

$$f : D^{\bar{\lambda}}T_{b_i+1,r} \to D^{\bar{\lambda}}v_{\boldsymbol{a}_i}T_{w_{\boldsymbol{a}_i'}}h_i e_{\boldsymbol{a}_\dashv}$$

defined by sending $xT_{b_i+1,r}$ to $xv_{\boldsymbol{a}_i}T_{w'_{\boldsymbol{a}_i}}h_i e_{\boldsymbol{a}_\dashv}$ is a surjective $\mathcal{H}(\mathfrak{S}_{\boldsymbol{a}_\dashv})$-module homomorphism, and hence an isomorphism by a comparison of dimensions, proving the claim.

Let $[\lambda] = \boldsymbol{a}_i = [c_0, \cdots, c_m]$. Then, $c_j = b_j$ if $0 \leqslant j \leqslant i-1$, or b_j+1 if $i \leqslant j \leqslant m$. Since $D^{\bar{\lambda}} = D^{\lambda^{(1)}} \cdots D^{\lambda^{(m)}}$ where $D^{\lambda^{(j)}}$ is the corresponding irreducible $\mathcal{H}(\mathfrak{S}_{\{c_{j-1}+1,\cdots,c_j\}})$-module, (4.13) implies the $\mathcal{H}(\boldsymbol{a}_\dashv)$-module isomorphism

$$D^{\bar{\lambda}}T_{b_i+1,r} \cong D^{\lambda^{(1)}} \otimes \cdots \otimes D^{\lambda^{(i-1)}} \otimes (D^{\lambda^{(i)}}|_{\mathcal{H}(\mathfrak{S}_{b_{i-1}+1,\cdots,b_i})}) \otimes \tilde{D}^{\lambda^{(i+1)}} \otimes \cdots \otimes \tilde{D}^{\lambda^{(r)}},$$

where $D^{\lambda^{(i)}}|_{\mathcal{H}(\mathfrak{S}_{\{b_{i-1}+1,\cdots,b_i\}})\}}$ denotes the module by restricting the action from $\mathcal{H}(\mathfrak{S}_{\{c_{i-1}+1,\cdots,c_i\}})$ to $\mathcal{H}(\mathfrak{S}_{c_{i-1}+1,\cdots,b_i})$, and $\tilde{D}^{\lambda^{(j)}}$, for $j = i+1, \cdots, m$, denotes the irreducible $\mathcal{H}(\mathfrak{S}_{\{b_{j-1}+1,\cdots,b_j\}})$-module obtained by twisting the $\mathcal{H}(\mathfrak{S}_{\{c_{j-1}+1,\cdots,c_j\}})$-module structure on $D^{\lambda^{(j)}}$ through the canonical isomorphism $\mathcal{H}(\mathfrak{S}_{\{c_{j-1}+1,\cdots,c_j\}}) \cong \mathcal{H}(\mathfrak{S}_{\{b_{j-1}+1,\cdots,b_j\}})$.

With what we obtained above, (4.11) becomes (recalling $[\boldsymbol{\lambda}] = \boldsymbol{a}_i$)

$$\begin{aligned}
&\mathrm{Hom}_{e_{[\boldsymbol{\rho}]}\mathbf{H}_m^{r-1}e_{[\boldsymbol{\rho}]}}(S^{\bar{\boldsymbol{\rho}}}e_{[\boldsymbol{\rho}]}, D^{\boldsymbol{\lambda}}e_{[\boldsymbol{\rho}]}) \cong \mathrm{Hom}_{\mathcal{H}(\mathfrak{S}_{[\boldsymbol{\rho})]})}(S^{\bar{\boldsymbol{\rho}}}, D^{\bar{\boldsymbol{\lambda}}}T_{b_i+1,r})\\
&\cong \otimes_{j=1}^{i-1}\mathrm{Hom}_{\mathcal{H}(\mathfrak{S}_{\{b_{j-1}+1,\cdots,b_j\}})}(S^{\mu^{(j)}}, D^{\lambda^{(j)}})\\
&\quad \otimes \mathrm{Hom}_{\mathcal{H}(\mathfrak{S}_{\{b_{i-1}+1,\cdots,b_i\}})}(S^{\mu^{(i)}}, D^{\lambda^{(i)}}|_{\mathcal{H}(\mathfrak{S}_{\{b_{i-1}+1,\cdots,b_i\}})})\otimes\\
&\quad \otimes_{j=i+1}^{m} \mathrm{Hom}_{\mathcal{H}(\mathfrak{S}_{\{b_{j-1}+1,\cdots,b_j\}})}(S^{\mu^{(j)}}, \tilde{D}^{\lambda^{(j)}}),
\end{aligned} \tag{4.14}$$

and similarly,

$$\begin{aligned}
&\mathrm{Hom}_{e_{[\boldsymbol{\rho}]}\mathbf{H}_m^{r-1}e_{[\boldsymbol{\rho}]}}(D^{\bar{\boldsymbol{\rho}}}e_{[\boldsymbol{\rho}]}, D^{\boldsymbol{\lambda}}e_{[\boldsymbol{\rho}]}) \cong \mathrm{Hom}_{\mathcal{H}(\mathfrak{S}_{[\boldsymbol{\rho}]})}(D^{\bar{\boldsymbol{\rho}}}, D^{\bar{\boldsymbol{\lambda}}}T_{b_i+1,r})\\
&\cong \otimes_{j=1}^{i-1}\mathrm{Hom}_{\mathcal{H}(\mathfrak{S}_{\{b_{j-1}+1,\cdots,b_j\}})}(D^{\mu^{(j)}}, D^{\lambda^{(j)}})\\
&\quad \otimes \mathrm{Hom}_{\mathcal{H}(\mathfrak{S}_{\{b_{i-1}+1,\cdots,b_i\}})}(D^{\mu^{(i)}}, D^{\lambda^{(i)}}|_{\mathcal{H}(\mathfrak{S}_{\{b_{i-1}+1,\cdots,b_i\}})})\otimes\\
&\quad \otimes_{j=i+1}^{m} \mathrm{Hom}_{\mathcal{H}(\mathfrak{S}_{\{b_{j-1}+1,\cdots,b_j\}})}(D^{\mu^{(j)}}, \tilde{D}^{\lambda^{(j)}}).
\end{aligned} \tag{4.15}$$

Clearly, the Hom set in (4.14) (resp. (4.15)) is isomorphic to R if and only if $\mu^{(j)} = \lambda^{(j)}$ for all $j \neq i$, and $\mathrm{Hom}_{\mathcal{H}(\mathfrak{S}_{\{b_{i-1}+1,\cdots,b_i\}})}(S^{\mu^{(i)}}, D^{\lambda^{(i)}}|_{\mathcal{H}(\mathfrak{S}_{\{b_{i-1}+1,\cdots,b_i\}})}) \cong R$ (resp. $\mathrm{Hom}_{\mathcal{H}(\mathfrak{S}_{\{b_{i-1}+1,\cdots,b_i\}})}(D^{\mu^{(i)}}, D^{\lambda^{(i)}}|_{\mathcal{H}(\mathfrak{S}_{\{b_{i-1}+1,\cdots,b_i\}})}) \cong R$). However, by (4.5), the latter is equivalent to $\mu^{(i)} = \lambda_{\mathfrak{n}}^{(i)}$ for $\mathfrak{n} \in \mathcal{R}_{normal}(\lambda^{(i)})$ (resp. $\mathfrak{n} \in \mathcal{R}_{good}(\lambda^{(i)})$). Now, applying (4.6), the theorem is proved. $\square$

Using the very recent work [A2] on the classification of the irreducible modules, we propose the following conjecture.

(4.16) Conjecture. *Replacing the l-regular multipartitions by Kleshchev multipartitions, (4.10) holds in general.*

References

[AF] F. Anderson, and K. Fuller, *Rings and Categories of Modules*, Springer, New York, 1973.

[A1] S. Ariki, *On the semi-simplicity of the Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$*, J. Algebra **169** (1994), 216-225.

[A2] S. Ariki, *On the classification of simple modules for cyclotomic Hecke algebras of type $G(m,1,n)$ and Kleshchev multipartitions*, preprint.

[AK] S. Ariki and K. Koike, *A Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$ and the construction of its irreducible representations*, Adv. Math. **106** (1994), 216-243.

[AM] S. Ariki and A. Mathas, *The number of simple modules of the Hecke algebras of type $G(r,1,n)$*, preprint.

[B] J. Brundan, *Modular branching rules and the Mullineux map for Hecke algebras of type A*, Proc. London. Math. Soc. (3) **77** (1998), 551-581.

[C] H. Can, *Representations of the generalized symmetric groups*, Beiträge Alg. Geo. **37** (1996), 289-307.

[CPS] E. Cline, B. Parshall and L. Scott, *Stratifying Endomorphism algebras*, Mem. Amer. Math. Soc. **591**, 1996.

[DJ] R. Dipper and G. D. James, *Representations of Hecke algebras of general linear groups*, Proc. London Math. Soc. **52**, 20-52.

[DJM] R. Dipper and G. D. James and A. Mathas, *Cyclotomic q-Schur algebras*, Math. Zeit **229** (1998), 385-416.

[DR] J. Du and H. Rui, *Ariki-Koike algebras with semi-simple bottoms*, Math. Zeit. (provisionally accepted).

[GL] J. Graham and G. Lehrer, *Cellular algebras*, Invent. Math. **126** (1996), 1-34.

[J] G. D. James, *The irreducible representations of the finite general linear groups*, Proc. London Math. Soc. **52** (1986), 236-268.

[Jo] T. Jost, *Morita equivalence for blocks of Hecke algebras of symmetric groups*, J. Algebra **194** (1997), 201–203.

[K] A. S. Kleshchev, *Branching rules for symmetric groups and applications*, in Newton Institiute Volume.

[JK] G. James and A. Kerber, *The representation theory of the symmetric group*, Addison-Wesley, London, 1981.