A NOTE ON THE ARRANGEMENT OF SUBGROUPS IN THE AUTOMORPHISM GROUPS OF SUBMODULE LATTICES OF FREE MODULES

Alexandre A. Panin

Department of Mathematics and Mechanics
St.Petersburg State University
2 Bibliotechnaya square,
St.Petersburg 198904, Russia
e-mail: alex@ap2707.spb.edu

Abstract. A complete description of subgroups in the general linear group over a semilocal ring containing the group of diagonal matrices was obtained by Z.I.Borewicz and N.A.Vavilov. It is shown in the present paper that a similar description holds for the intermediate subgroups of the group of all automorphisms of the lattice of right submodules of a free finite rank R–module over a simple artinian ring containing the group consisting of those automorphisms which leave invariant an appropriate sublattice. Bibliography: 11 titles.

§ 1. Introduction

The following result is due to Z.I.Borewicz and N.A.Vavilov (see [BV] for a preliminary version).

Theorem [V1]. *Let R be a semilocal ring such that the decomposition of the factor-ring $R/J(R)$ in the direct sum of simple artinian rings doesn't include either fields containing less than seven elements, or the full matrix ring* $\mathrm{M}(2,\mathbb{F}_2)$. *Then for every intermediate subgroup F,* $\mathrm{D}(n,R) \leqslant F \leqslant \mathrm{GL}(n,R)$, *there exists a unique*

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D–net σ of two-sided ideals in R of order n such that $G(\sigma) \leqslant F \leqslant \mathcal{N}(\sigma)$, where $\mathcal{N}(\sigma)$ is the normalizer of the net subgroup $G(\sigma)$ in $\mathrm{GL}(n, R)$.

One may find a wealth of background information and many further related references in the surveys [V2], [V3].

This theorem was generalized by A.Z.Simonian [S], A.A.Panin and A.V.Yakovlev [PY], A.A.Panin [P1], [P2]. It was shown that these results can be stated in terms of Galois correspondence between sublattices of the lattice of (right) submodules of a free module R^n and subgroups of $\mathrm{GL}(n, R)$ considered as automorphism groups of this lattice.

One can try to describe subgroups of the group of *all* automorphisms of this lattice containing the group of "diagonal" automorphisms (i.e., the group consisting of those automorphisms which leave invariant its appropriate sublattice).

We solve this problem for a simple artinian ring in the present paper.

§ 2. Preliminaries

Let L be a lattice and let G be a subgroup of the group $\mathrm{Aut}(L)$ of all automorhisms of the lattice L. Consider a subgroup F of the group G and a sublattice M of the lattice L. By definition, put

$$L(F) = \{l \in L \text{ such that } f(l) = l \text{ for every } f \in F\},$$

$$G(M) = \{g \in G \text{ such that } g(m) = m \text{ for every } m \in M\}.$$

Let L be a complete modular lattice and L_0 be a sublattice of L which is a finite Boolean algebra such that $1_L = 1_{L_0}$ and $0_L = 0_{L_0}$. Consider a subgroup G of the group $\mathrm{Aut}(L)$. Let $H = G(L_0)$, $L_0^{'} = L(H)$. By $e_1, e_2, \dots, e_n$ we denote the atoms of L_0.

For every $x \in L$ the collection $([x]_1, \dots, [x]_n)$, where $[x]_i = (x + \sum\limits_{j \neq i} e_j) \cdot e_i$, is called the *support* of x and is denoted by $[x]$. The support can be defined equivalently as the minimal (with respect to the natural ordering) collection $(x_1, \dots, x_n)$, where $x_i \leqslant e_i$ and $x \leqslant \sum\limits_{i=1}^{n} x_i$ (see [PY], [P2]).

Certain subgroups of type $G(M)$, where M is a sublattice of $L_0^{'}$, are of special interest. The following definition was introduced in [PY].

Definition. *A net collection in $L_0^{'}$ is a collection of elements $\tau = (\tau_{ij})_{i,j=1}^{n}$ such that for every indices i, j, k and $g \in G$:*

(*i*) $\tau_{ij} \leqslant e_j$

(*ii*) $\tau_{ii} = e_i$

(*iii*) $\tau_{ij} \in L_0^{'}$

(*iv*) $[g(e_i)]_j \leqslant \tau_{ij}$ *iff* $[g(\tau_{ki})]_j \leqslant \tau_{kj}$.

For every net collection $\tau = (\tau_{ij})$ in $L_0^{'}$ the lattice generated by elements $\sum\limits_{j=1}^{n} \tau_{ij},\ i = 1, \ldots, n$ is called the lattice *associated* with τ and is denoted by K_τ.

§ 3. Main theorem

Let L be the lattice of right submodules of a free R–module M of rank n, where R is a semilocal ring such that the decomposition of the factor-ring $R/J(R)$ in the direct sum of simple artinian rings doesn't include either fields containing less than seven elements, or the full matrix ring $\mathrm{M}(2, \mathbb{F}_2)$. Consider a basis $\bar{e}_1, \ldots, \bar{e}_n$ of M and let e_i be spanned by $\bar{e}_i$. Let L_0 be the sublattice of L generated by $e_1, \ldots, e_n$. One can consider $\mathrm{PGL}(n, R)$ as a subgroup of $\mathrm{Aut}(L)$. Since we are only interested in description of the *intermediate* subgroups, it does not make a difference to speak about $\mathrm{PGL}(n, R)$ or $G = \mathrm{GL}(n, R)$ here, with $H = G(L_0)$ playing the role of $\mathrm{D}(n, R)$ in our case.

It was proved in [P2] that for every net collection σ in $L_0^{'}$ the collection of ideals σ^{T} is a D–net in R, and the net subgroup $G(\sigma^{\mathrm{T}})$ equals $G(K_\sigma)$.

The main result of this article is the following

Theorem 1. *Let $R = \mathrm{M}(m, T)$ be a full matrix ring over a skew field T not equal to $\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_4, \mathbb{F}_5$, and $G = \mathrm{Aut}(L)$ be the group of all automorphisms of L, $H = G(L_0)$, $L_0^{'} = L(H)$. Then for every subgroup F, $H \leqslant F \leqslant G$, there exists a unique D–net σ of two-sided ideals in R of order n such that $G(K_\sigma) \leqslant F \leqslant \mathcal{N}_G G(K_\sigma)$, where K_σ is the lattice associated with the net collection σ^{T} in $L_0^{'}$.*

Proof. First, let $R = T$ be a skew field.

If $n \geqslant 3$, then it follows from the Fundamental Theorem of Projective Geometry [A] that the group $\mathrm{Aut}(L)$ is equal to the semidirect product of the normal subgroup $\mathrm{PGL}(n, T)$ and $\mathrm{Aut}(T)$. If, again, we do not distinguish GL and PGL, then $G = \mathrm{GL}(n, T) \leftthreetimes \mathrm{Aut}(T)$ and $H = \mathrm{D}(n, T) \leftthreetimes \mathrm{Aut}(T)$. Thus each intermediate subgroup of G containing H is of the form $F \leftthreetimes \mathrm{Aut}(T)$, where $\mathrm{D}(n, T) \leqslant F \leqslant \mathrm{GL}(n, T)$. If the skew field T satisfies the conditions of Theorem 1, then for every such F there exists a unique D–net σ of ideals in T of order n such that $G(\sigma) \leqslant F \leqslant \mathcal{N}(\sigma)$. It is clear that $G(K_\sigma) = G(\sigma) \leftthreetimes \mathrm{Aut}(T)$. We have to prove that $G(K_\sigma)$ is normal in $F \leftthreetimes \mathrm{Aut}(T)$ (it is not evident since $\mathrm{Aut}(T)$ is not a normal subgroup of G). It was proved in [P2] that $f(\sum\limits_{i=1}^{n} \sigma_{ij}) \in L_0$ for every $f \in F$. Hence $f^{-1}\varphi f \in G(K_\sigma)$ for every $\varphi \in \mathrm{Aut}(T)$. The rest is trivial.

The Fundamental Theorem of Projective Geometry fails for $n = 2$, so one has to consider this case separately. Every proper nonzero subspace of M is an atom of the lattice L, so the automorphism group of L is organized very simply: it's just the symmetric group acting on the set of atoms of L. Let T be a skew field not equal to $\mathbb{F}_2$ and $\mathbb{F}_3$. Then L contains at least 5 atoms. It's easy to verify that there are only 5 intermediate subgroups: $H, \mathcal{N}_G H, G(M_1)$,

$G(M_2), G$, where $M_1 = \{e_1\}, M_2 = \{e_2\}$, and all subgroups except H are self-normalizable. Hence Theorem 1 for skew fields is proved.

The case of simple artinian rings is easily reduced to the case of skew fields since the lattice of right $\mathrm{M}(m,T)$–submodules of $\mathrm{M}(m,T)^n$ is isomorphic to the lattice of right T–subspaces of T^{nm}. This completes the proof of Theorem 1.

§ 4. Final remarks

Remark 1. The Fundamental Theorem of Projective Geometry holds in more general settings [OjS], [SV]. Therefore it is natural to try to generalize Theorem 1 to semilocal rings, at least commutative ones.

Remark 2. The assertion of Theorem 1 for $n = 2$ and skew fields can be deduced from a more general theorem on the automorphisms of certain complete modular lattices [P2].

Definition. *For every $i \neq j$ and $x \leqslant e_j$ the set of $f \in G$ such that*

(i) $f(e_s) = e_s$ *for* $s \neq i$,

(ii) $[f(e_i)]_k = \begin{cases} 0, & k \neq i, j; \\ e_i, & k = i; \\ x, & k = j \end{cases}$

is denoted by $H_{ij}(x)$ and its elements are called "transvections".

The following result is a corollary of Theorem 3.1 [P2] (see also [PY] for a slightly weaker result).

Theorem 2. *Let L be a modular lattice of finite length, L_0 a sublattice of the same length which is a Boolean algebra with atoms $e_1, \dots, e_n$, let G be a subgroup of the group of all automorphisms of the lattice L, $H = G(L_0)$.*

Assuming that the conditions $1^0 - 4^0$ stated below are satisfied, for every subgroup $F \geqslant H$ of the group G the group $G(K_\sigma)$ is normal in F.

1^0. *For every i there exists at least one automorphism of H which changes all atoms $x \in L \setminus \{e_i\}$ such that $[x]_i = e_i$ and leaves invariant all atoms $x \in L$ with $[x]_i = 0$.*

2^0. *If $x, y \in L$ are atoms with $[x] = [y]$, then there exists $h \in H$ such that $h(x) = y$.*

3^0. *For every $i \neq j$ the set $H_{ij}(e_j)$ is not empty.*

4^0. *For every $a \in G$ and $i \neq j$ the subgroup generated by $h \in H$ such that $H_{ij}([aha^{-1}(e_i)]_j) \cap \langle a, H\rangle \neq \varnothing$, and $H_{ij}([ah^{-1}a^{-1}(e_i)]_j) \cap \langle a, H\rangle \neq \varnothing$ is equal to H.*

It is worth mentioning that the description of the intermediate subgroups of the full linear group, containing the group of diagonal matrices, can also be deduced from this theorem. Moreover, it is likely that methods and results of [PY], [P1], [P2] can bring further insight in our understanding of linear groups.

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