GENERIC IDEMPOTENT MODULES FOR A FINITE GROUP

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ABSTRACT. Let G be a finite group and k an algebraically closed field of characteristic p. Let $F_{\mathcal{U}}$ be the Rickard idempotent kG-module corresponding to the set $\mathcal{U}$ of subvarieties of the cohomology variety V_G which are not irreducible components. We show that $F_{\mathcal{U}}$ is a finite sum of *generic modules* corresponding to the irreducible components of V_G. In this context, a generic module is an indecomposable module of infinite length over kG but finite length as a module over its endomorphism ring.

1. Introduction

The purpose of this paper is to draw attention to a particular infinite dimensional module for a finite group algebra, which plays the role of a generic module in a certain precise sense which will be described. The module in question is a particular case of Rickard's construction [14] of idempotent modules in the stable category. These idempotent modules have played a pivotal role in some recent developments in modular representation theory, see for example [1, 3, 4, 5].

Theorem 1.1. *Let $\mathcal{U}$ denote the set of closed homogeneous irreducible subvarieties of the cohomology variety V_G which are not irreducible components, and let $F_{\mathcal{U}}$ be the corresponding idempotent module. Then $F_{\mathcal{U}}$ is endofinite.*

2. Endofinite modules in the stable category

For any ring Λ, a Λ-module M is said to be *endofinite* if it has finite length as a module over $\mathrm{End}_\Lambda(M)$. This concept is investigated by Crawley–Boevey in [9]. The following are some properties of endofinite modules over a finite dimensional algebra; their proofs can be found in Section 4 of [9].

(1) If M has finite length then M is endofinite.

(2) An arbitrary direct sum or direct product of (possibly infinitely many) copies of an endofinite module is endofinite.

(3) Every finite direct sum of endofinite modules is endofinite.

(4) Every direct summand of an endofinite module is endofinite.

(5) Every projective module and every injective module is endofinite. This follows from (2) and (3), since there are finitely many isomorphism classes of indecomposable projectives, and every projective is a direct sum of indecomposable projectives; and similarly for injectives.

(6) Every endofinite module is pure injective. It is even Σ-pure injective, since this is equivalent to d.c.c. on additive subgroups of finite definition (in the first order language of Λ-modules). Endofinite is equivalent to a.c.c. and d.c.c. on additive subgroups of finite definition.

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A *generic* module is defined to be an indecomposable endofinite module of infinite length. As an example, let $G = \langle g_1, g_2\rangle \cong \mathbb{Z}/2\times\mathbb{Z}/2$ and let k have characteristic two. Let t be an indeterminate, and let M be the kG-module with underlying vector space $k(t)\oplus k(t)$, with G acting via

$$g_1 \mapsto \left(\begin{smallmatrix} I & I \\ 0 & I\end{smallmatrix}\right) \qquad g_2 \mapsto \left(\begin{smallmatrix} I & T \\ 0 & I\end{smallmatrix}\right).$$

Here, I denotes the identity map on $k(t)$ and T denotes multiplication by t on $k(t)$. It is easy to see that M has infinite length over kG but length two over $\mathrm{End}_{kG}(M) \cong k(t)[u]/(u^2)$. Since M is also indecomposable, it is a generic module. In fact, it is the module $F_{\mathfrak{U}}$ for this group, and there are no other generic modules.

More generally, Crawley–Boevey has shown that for finite dimensional algebras over an algebraically closed field there is a strong connection between generic modules and the representation type; see [9]. Of course, for an algebra of finite representation type, there are no generic modules. For a finite dimensional algebra of tame representation type, there is a natural correspondence between generic modules and one parameter families of finite length modules. A finite dimensional algebra has wild representation type if and only if there exists a generic module such that the endomorphism ring modulo its radical is the full division ring of fractions of the free ring in two variables $k\langle x, y\rangle$. This division ring has infinite dimension over its center.

Let Λ be a finite dimensional self-injective algebra. We write $\mathsf{Mod}(\Lambda)$ for the category of all Λ-modules and homomorphisms, and $\mathsf{mod}(\Lambda)$ for the full subcategory of finitely generated modules. The stable category $\mathsf{StMod}(\Lambda)$ has the same objects as $\mathsf{Mod}(\Lambda)$, but the arrows are $\mathrm{Hom}_\Lambda(M,N)$ modulo the homomorphisms which factor through a projective (= injective) module. The stable category of finitely generated modules is written $\mathsf{stmod}(\Lambda)$. The categories $\mathsf{StMod}(\Lambda)$ and $\mathsf{stmod}(\Lambda)$ are triangulated categories, in which the distinguished triangles correspond to short exact sequences in $\mathsf{Mod}(\Lambda)$. Recall from [11] that an object M in $\mathsf{StMod}(\Lambda)$ is said to be endofinite if $\underline{\mathrm{Hom}}_\Lambda(X,M)$ has finite length over $\underline{\mathrm{End}}_\Lambda(M)$ for all X in $\mathsf{stmod}(\Lambda)$. This does not conflict with the previous notion of endofiniteness, because of the following proposition.

Proposition 2.1. *The following are equivalent for a Λ-module M:*

(1) *M is an endofinite Λ-module;*
(2) *M is an endofinite object in $\mathsf{StMod}\Lambda$;*
(3) *$\underline{\mathrm{Hom}}_\Lambda(S,M)$ is of finite length over $\underline{\mathrm{End}}_\Lambda(M)$ for every simple Λ-module S.*

Proof. (1) $\Rightarrow$ (2) Suppose that M is endofinite and let $X \in \mathsf{mod}\Lambda$. Clearly, $\mathrm{Hom}_\Lambda(X,M)$ has finite length over $\mathrm{End}_\Lambda(M)$. Therefore the quotient $\underline{\mathrm{Hom}}_\Lambda(X,M)$ has finite length over $\underline{\mathrm{End}}_\Lambda(M)$ since it is of finite length over $\mathrm{End}_\Lambda(M)$.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) Direct limits of injective modules are injective. Therefore we can apply Zorn's lemma and find a maximal injective submodule I of M. Let $M = I\oplus M'$. Given any simple Λ-module S, we have

$$\underline{\mathrm{Hom}}_\Lambda(S,M) \cong \underline{\mathrm{Hom}}_\Lambda(S,M') = \mathrm{Hom}_\Lambda(S,M').$$

Assuming (3), this module is of finite length over $\mathrm{End}_\Lambda(M')$ since it is of finite length over $\underline{\mathrm{End}}_\Lambda(M') \cong \underline{\mathrm{End}}_\Lambda(M)$. Thus $\mathrm{Hom}_\Lambda(\Lambda, M')$ has finite length over $\mathrm{End}_\Lambda(M')$ since Λ is of finite length over Λ. We conclude that M' is endofinite. Every injective Λ-module is endofinite, so we conclude that $M = I \oplus M'$ is endofinite and therefore the proof is complete. $\square$

For a Λ-module M, we denote by $\mathrm{endol}(M)$ the length of M as a module over $\mathrm{End}_\Lambda(M)$.

Lemma 2.2. *Let H be a subgroup of a group G, and let k be a field.*
1. *If M is a kG-module then* $\operatorname{endol}(M\downarrow_H) \le \operatorname{endol}(M)$.
2. *If M is a kH-module then* $\operatorname{endol}(M\uparrow^G) \le \operatorname{endol}(M) \cdot |G:H|$.

Proof. (1) The action of $\operatorname{End}_{kH}(M\downarrow_H)$ on M extends the action of $\operatorname{End}_{kG}(M)$. Therefore $\operatorname{endol}(M)$ is a bound for $\operatorname{endol}(M\downarrow_H)$.

(2) The endomorphism ring $\operatorname{End}_{kH}(M)$ acts on $M\uparrow^G = kG \otimes_{kH} M$ via the ring homomorphism $\operatorname{End}_{kH}(M) \to \operatorname{End}_{kG}(M\uparrow^G)$ which sends an endomorphism α to the endomorphism $\mathrm{id} \otimes \alpha$. The kH-module kG is free of rank $|G:H|$, and therefore the length of $M\uparrow^G$ over $\operatorname{End}_{kH}(M)$ is equal to $\operatorname{endol}(M) \cdot |G:H|$. We conclude that the length of $M\uparrow^G$ over $\operatorname{End}_{kG}(M\uparrow^G)$ is bounded by $\operatorname{endol}(M) \cdot |G:H|$. □

Proposition 2.3. *Let G be a finite group and k be a field of characteristic p. Then a kG-module M is endofinite if and only if its restriction to a Sylow p-subgroup of G is endofinite.*

Proof. Let P be a Sylow p-subgroup of G. If M is endofinite then Lemma 2.2 (1) shows that $M\downarrow_P$ is endofinite. Conversely, if $M\downarrow_P$ is endofinite then by Lemma 2.2 (2), $M\downarrow_P\uparrow^G$ is endofinite. But since $|G:P|$ is coprime to p, M is a direct summand of $M\downarrow_P\uparrow^G$. □

3. Idempotent modules

Definition 3.1. Let G be a finite group and k be a field.
1. A kG-module E is called *counital idempotent* if there exists a map $\varepsilon\colon E \to k$ such that $\varepsilon \otimes \mathrm{id}_E$ induces an isomorphism $E \otimes_k E \to E$ in $\mathsf{StMod}kG$.
2. A kG-module F is called *unital idempotent* if there exists a map $\phi\colon k \to F$ such that $\phi \otimes \mathrm{id}_F$ induces an isomorphism $F \to F \otimes_k F$ in $\mathsf{StMod}kG$.

Suppose that E is a counital idempotent kG-module. Then the kG-modules M such that $\varepsilon \otimes \mathrm{id}_M$ induces an isomorphism $E \otimes_k M \to M$ in $\mathsf{StMod}kG$ form a localizing subcategory $\mathcal{L}_E$ of $\mathsf{StMod}kG$. Recall that a full subcategory of $\mathsf{StMod}kG$ is *localizing* if it is a triangulated subcategory which is closed under arbitrary direct sums. Moreover, the inclusion functor $\mathcal{L}_E \to \mathsf{StMod}kG$ has a right adjoint which preserves arbitrary direct sums; it sends a kG-module M to $E \otimes_k M$. Note that $\mathcal{L}_E$ does not depend on the map $\varepsilon\colon E \to k$. In fact, if $\varepsilon'\colon E \to k$ is a second map inducing an isomorphism $E \otimes_k E \to E$ in $\mathsf{StMod}kG$ then there exists an isomorphism $\alpha\colon E \to E$ such that $\varepsilon' = \varepsilon\alpha$ in $\mathsf{StMod}kG$.

Lemma 3.2. *Let $\phi\colon k \to F$ be a map in $\mathsf{StMod}kG$ such that $\phi \otimes \mathrm{id}_F$ is an isomorphism. Then ϕ induces an isomorphism $\underline{\operatorname{Hom}}_{kG}(F,F) \to \underline{\operatorname{Hom}}_{kG}(k,F)$ of $\underline{\operatorname{End}}_{kG}(F)$-modules.*

Proof. Complete the map ϕ to a triangle $E \to k \to F \to \Omega^{-1}(E)$. Tensoring this triangle with F gives a new triangle which shows that $F \otimes_k E = 0$ (in the stable category). It follows that $\underline{\operatorname{Hom}}_{kG}(E,F) = 0$ since any map $E \to F$ induces a commutative square

$$\begin{tikzcd}
k \otimes_k E \arrow[r] \arrow[d] & k \otimes_k F \arrow[d, "\cong"] \\
F \otimes_k E \arrow[r] & F \otimes_k F
\end{tikzcd}$$

The same argument shows that $\underline{\operatorname{Hom}}_{kG}(\Omega^{-1}(E),F) = 0$. Applying $\underline{\operatorname{Hom}}_{kG}(-,F)$ to the above triangle then shows that $\underline{\operatorname{Hom}}_{kG}(F,F) \cong \underline{\operatorname{Hom}}_{kG}(k,F)$. □

Theorem 3.3. *Let G be a p-group. Then a unital idempotent kG-module F is endofinite if and only if $\underline{\mathrm{End}}_{kG}(F)$ is an Artinian ring.*

Proof. We may assume that the field k is of characteristic p, and therefore the trivial kG-module k is, up to isomorphism, the unique simple kG-module since G is a p-group. Lemma 3.2 shows that $\underline{\mathrm{Hom}}_{kG}(k, F)$ has finite length over $\underline{\mathrm{End}}_{kG}(F)$ if and only if $\underline{\mathrm{End}}_{kG}(F)$ is Artinian. Therefore the assertion follows from the above criterion for endofiniteness. □

Remark 3.4. A unital idempotent kG-module F with Artinian endomorphism ring $\underline{\mathrm{End}}_{kG}(F)$ has a unique decomposition $F = F_1 \oplus \cdots \oplus F_n$ into finitely many indecomposable objects in $\mathsf{StMod}kG$. Moreover, each F_i is a unital idempotent module and $F_i \otimes_k F_j = 0$ for all $i \neq j$; see [11].

4. The module $F_{\mathcal{U}}$

Let G be a finite group and k an algebraically closed field of characteristic p. Let V_G be the maximal ideal spectrum of the graded cohomology ring $H^*(G, k) = \mathrm{Ext}^*_{kG}(k, k)$ (Note that this ring is only graded commutative; if this bothers you, notice that for $p = 2$ this means it is commutative; for p odd, elements of odd degree square to zero, so modulo the nil radical this ring is again commutative). Let $V_1, \ldots, V_t$ be the irreducible components of V_G, and let $\mathcal{U}$ be the collection of closed homogeneous irreducible subvarieties of V_G which are not irreducible components. This collection $\mathcal{U}$ is clearly closed under specialization, in the sense that if $V \in \mathcal{U}$ and $W \subseteq V$ then $W \in \mathcal{U}$. Denote by $\mathcal{C}(\mathcal{U})$ the thick subcategory of $\mathsf{stmod}(kG)$ consisting of finitely generated kG-modules M such that $V_G(M) \in \mathcal{U}$. Let

$$E_{\mathcal{U}} \to k \to F_{\mathcal{U}} \to \Omega^{-1} E_{\mathcal{U}}$$

be the corresponding distinguished triangle of idempotent modules, as constructed by Rickard ([14], Section 5). Let $\mathcal{C}(\mathcal{U})^{\oplus}$ denote the triangulated subcategory of $\mathsf{StMod}(kG)$ generated by (possibly infinite) direct sums of objects in $\mathcal{C}(\mathcal{U})$. Then passage to the quotient category $\mathsf{StMod}(kG) \to \mathsf{StMod}(kG)/\mathcal{C}(\mathcal{U})^{\oplus}$ has a left adjoint, which takes a module M to $F(\mathcal{U}) \otimes_k M$. Furthermore, the natural functor $\mathsf{stmod}(kG)/\mathcal{C}(\mathcal{U}) \to \mathsf{StMod}(kG)/\mathcal{C}(\mathcal{U})^{\oplus}$ is fully faithful.

Proposition 4.1. *The stable endomorphism ring $\underline{\mathrm{End}}_{kG}(F_{\mathcal{U}})$ is Artinian. It is a direct product of local rings, $E_1 \times \cdots \times E_t$, corresponding to the irreducible components $V_1, \ldots, V_t$. The quotient $E_i/J(E_i)$ of E_i by its radical is isomorphic to the function field $k(V_i)$.*

Proof. By the above remarks, the ring $\underline{\mathrm{End}}_{kG}(F_{\mathcal{U}})$ is isomorphic to the endomorphism ring of k in the quotient category $\mathsf{StMod}(kG)/\mathcal{C}(\mathcal{U})^{\oplus}$, or equivalently in $\mathsf{stmod}(kG)/\mathcal{C}(\mathcal{U})$. So the method of Carlson, Donovan and Wheeler [7] can be used to give the following explicit description of $\underline{\mathrm{End}}_{kG}(F_{\mathcal{U}})$. Let $S_{\mathcal{U}}$ be the multiplicative subset of $H^*(G, k)$ consisting of elements which are not identically zero on any irreducible component of V_G, or equivalently, elements which do not belong to any minimal prime ideal in $H^*(G, k)$. Then $\underline{\mathrm{End}}_{kG}(F_{\mathcal{U}})$ is isomorphic to the ring of degree zero elements in the homogeneous localization $S_{\mathcal{U}}^{-1} H^*(G, k)$. The effect of this localization is that the homogeneous prime ideals in $S_{\mathcal{U}}^{-1} H^*(G, k)$ correspond precisely to the minimal prime ideals in $H^*(G, k)$, namely to the irreducible components of V_G. The prime ideals in $\underline{\mathrm{End}}_{kG}(F_{\mathcal{U}})$ are then obtained by intersecting these homogeneous primes with the degree zero part. It follows that in $\underline{\mathrm{End}}_{kG}(F_{\mathcal{U}})$, every prime ideal is maximal. So this is a zero dimensional Noetherian graded commutative ring, and is hence Artinian.

A commutative Artinian ring is necessarily a direct product of local rings with nilpotent Jacobson radicals. For $\underline{\mathrm{End}}_{kG}(F_{\mathcal{U}})$, these local rings correspond to the homogeneous maximal ideals in

$S_{\mathcal{U}}^{-1}H^*(G,k)$. Furthermore, by definition, the function field of V_i is equal to the field of fractions of the integral domain of degree zero elements in the quotient of $H^*(G,k)$ by the corresponding minimal prime ideal. This is the same as the degree zero elements in the quotient of $S_{\mathcal{U}}^{-1}H^*(G,k)$ by the corresponding maximal ideal. This in turn is the same as the quotient of E_i by its Jacobson radical. □

Remark 4.2. Greenlees and Lyubeznik [10] (Corollary 7.4) have shown that the ring $S_{\mathcal{U}}^{-1}H^*(G,k)$ is a finite product of Gorenstein local rings. It is tempting to suppose that it follows from this that the degree zero part $\underline{\mathrm{End}}_{kG}(F_{\mathcal{U}})$ is a symmetric algebra. That this is not the case may be seen by examining the case where p is odd and G is an elementary abelian p-group of rank three. In this case, $\underline{\mathrm{End}}_{kG}(F_{\mathcal{U}})$ is an Artinian local ring of length four and square zero radical, so it cannot be a symmetric algebra. The theorem of Greenlees and Lyubeznik should therefore be interpreted as applying to the *graded* endomorphism ring

$$\underline{\mathrm{End}}^*_{kG}(F_{\mathcal{U}}) = \bigoplus_{n\in\mathbb{Z}} \underline{\mathrm{Hom}}_{kG}(\Omega^n F_{\mathcal{U}}, F_{\mathcal{U}}).$$

Remark 4.3. For other collections $\mathcal{V}$ of closed homogeneous irreducible subvarieties of V_G, closed under specialization, the structure of the stable endomorphism ring of $F_{\mathcal{V}}$ can be deduced from the work of Carlson and Wheeler [8].

Theorem 4.4. *The module $F_{\mathcal{U}}$ is endofinite.*

Proof. By Proposition 2.3, it suffices to show that the restriction of $F_{\mathcal{U}}$ to a Sylow p-subgroup P of G is endofinite. But the restriction of $F_{\mathcal{U}}$ to P is stably isomorphic to the corresponding module $F_{\mathcal{U}}$ there. By Proposition 4.1 this module has Artinian stable endomorphism ring, so by Theorem 3.3 it is endofinite. □

5. Local control

In this section, we use the techniques of [2] to reduce the description of the summands of $F_{\mathcal{U}}$ down to local subgroups, and there we give an explicit construction for this module.

By Proposition 4.1, $F_{\mathcal{U}}$ decomposes into indecomposable pieces $F_1 \oplus \cdots \oplus F_t$ corresponding to the irreducible components $V_1, \dots, V_t$. The correspondence has the property that

$$\underline{\mathrm{End}}_{kG}(F_i)/J\underline{\mathrm{End}}_{kG}(F_i) \cong k(V_i).$$

By a theorem of Quillen [12, 13], the irreducible components of V_G are in one-one correspondence with the conjugacy classes of elementary abelian p-subgroups of G. Let E_i be a representative of the conjugacy class corresponding to V_i, and set $N_i = N_G(E_i)$.

Lemma 5.1. *The module F_i is isomorphic to the module induced from the corresponding module $F_{\mathcal{U}_i}$ for N_i.*

Proof. We apply the methods of [2], but with $\mathcal{M}_{r-1}$ replaced by $\mathcal{C}(\mathcal{U})$, and $\mathcal{M}_{r,r-1}$ replaced by $\mathsf{stmod}(kG)/\mathcal{C}(\mathcal{U})$. The arguments work equally well in this context, and show the following. The category $\mathsf{stmod}(kG)/\mathcal{C}(\mathcal{U})$ decomposes as a direct sum of triangulated categories $\mathcal{M}_{G,E_i}$, with the E_i as above. Induction and restriction between N_i and G establishes an equivalence of categories $\mathsf{StMod}(kN_i)/\mathcal{C}(\mathcal{U}_i) \rightleftharpoons \mathcal{M}_{G,E_i}$, under which the trivial module corresponds to the module called $\pi_{G,E_i}(k)$ in [2].

Now we apply the adjunction mentioned before the proof of Proposition 4.1. This takes the trivial module in $\mathsf{stmod}(kG)/\mathcal{C}(\mathcal{U})$ to $F_{\mathcal{U}}$, and the module $\pi_{G,E_i}(k)$ to F_i. The corresponding adjunction at the level of N_i takes the trivial module in $\mathsf{stmod}(kN_i)/\mathcal{C}(\mathcal{U}_i)$ to $F_{\mathcal{U}_i}$. Since passage to the quotient category commutes with induction, it follows that $F_{\mathcal{U}_i}\uparrow^G \cong F_i$. □

Now N_i has a unique maximal elementary abelian p-subgroup, namely E_i. The map from the cohomology variety of N_i to the cohomology variety of G induces a birational equivalence $V_{N_i} \to V_i$. In this situation, there is an explicit construction for $F_{\mathcal{U}_i}$ which we now describe. This generalizes the construction given in Theorems 5.1.2 and 5.4.6 of [6].

So from now on, let G be a group with a unique maximal elementary abelian p-subgroup E, and assume that E is normal in G. Let r be the p-rank of E, and let $X_1, \dots, X_{r-1}$ be indeterminates. Choose elements $\zeta_1, \dots, \zeta_r$ forming a homogeneous system of parameters for $H^*(G,k)$ and having the same degree n. Set $K = k(X_1, \dots, X_{r-1})$, and for $1 \le j \le r-1$, set

$$\xi_j = X_j\zeta_r - \zeta_j \in H^n(G,K).$$

Represent ξ_j by a map $\hat{\xi}_j : K \to \Omega^{-n}K$, and write L_j for the cokernel. Write $\phi_j : k \to L_j$ for the composite $k \hookrightarrow K \xrightarrow{\hat{\zeta}_r} \Omega^{-n}K \to L_j$.

Theorem 5.2. *The kG-module $F_{\mathcal{U}}$ is isomorphic to the tensor product $L_1 \otimes_K \cdots \otimes_K L_{r-1}$, with the coefficients restricted to k, and with the map $\phi_1 \otimes \cdots \otimes \phi_{r-1}$ as $\phi : k \to F_{\mathcal{U}}$.*

Proof. Since E is the unique maximal elementary abelian subgroup of G, it suffices to restrict to $E = \langle g_1, \dots, g_r\rangle$ and use the theory developed in [4]. Namely, let V be a closed homogeneous irreducible subvariety of the rank variety $V_E^r \cong \mathbb{A}^r(k)$, and write L for the tensor product $L_1 \otimes_K \cdots \otimes_K L_{r-1}$. We need to complete the map $\phi : k \to L$ to a triangle $X \to k \to L \to \Omega^{-1}X$, and then tensor with the function field $k(V)$. We then need to check that the triangle restricts to $k(V) \to k(V) \to 0 \to \Omega^{-1}k(V)$ if V is a proper subvariety of V_E^r, and to $0 \to k(V) \to k(V) \to 0$ otherwise. In other words, we need to show that the rank variety of L is $\{V_E^r\}$, while that of X is the complement of this.

Let $H^*(E,k)/(\text{nilpotents}) = k[x_1, \dots, x_r]$ with the x_j in degree one if $p = 2$ and two if p is odd. Let m denote n if $p = 2$ and $n/2$ if p is odd. Then $\mathrm{res}_{G,E}(\zeta_j)$ (mod nilpotents) is equal to some homogeneous polynomial $f_j(x_1, \dots, x_r)$ of degree m. The polynomials $f_1, \dots, f_r$ form a regular sequence in $k[x_1, \dots, x_r]$.

Denote by $t_1, \dots, t_r$ the coordinate functions on V_E^r, regarded as elements of $k(V)$. Thus if $V = V_E^r$ then the t_j are algebraically independent, while if V is a proper subvariety of V_E^r then there exists a homogeneous irreducible polynomial in the elements $t_1, \dots, t_r$ which equals zero in $k(V)$.

The generic cyclic shifted subgroup of E corresponding to V is generated by

$$u_V = 1 + t_1(g_1 - 1) + \cdots + t_r(g_r - 1) \in k(V)E.$$

We must restrict the map $k \to L$ to u_V, being careful to observe that tensor product does not commute with restriction. To get around this difficulty, we form the tensor product of the complexes

$$0 \to K \xrightarrow{\hat{\xi}_j} \Omega^{-n}K \to 0$$

to form a Koszul-like complex whose homology is L. The point of this is that Yoneda composition of cohomology elements does commute with restriction. So we are restricting a tensor product of complexes of the form

$$0 \to k(V) \otimes_k K \xrightarrow{1\otimes\hat{\xi}_j} k(V) \otimes_k \Omega^{-n}K \to 0$$

and then taking the homology. We can act as though tensor product commutes with restriction.

Let ζ be a nonzero element of $H^2(\langle u_V\rangle, k(V))$ if p is odd, and of $H^1(\langle u_V\rangle, k(V))$ if $p=2$, chosen so that $\mathrm{res}_{E,\langle u_V\rangle}(x_j) = t_j\zeta$. Then

$$\mathrm{res}_{G,\langle u_V\rangle}(\zeta_j) = \mathrm{res}_{E,\langle u_V\rangle}(f_j(x_1,\dots,x_r)) = f_j(t_1,\dots,t_r)\zeta^m,$$

and

$$\mathrm{res}_{G,\langle u_V\rangle}(\xi_j) = (X_j f_r(t_1,\dots,t_r) - f_j(t_1,\dots,t_r))\zeta^m.$$

These are the elements involved in the restriction to $\langle u_V\rangle$ of the above Koszul-like complex.

If $f_r(t_1,\dots,t_r)=0$, so that V is contained in the corresponding hypersurface, then

$$\mathrm{res}_{G,\langle u_V\rangle}(\xi_j) = -f_j(t_1,\dots,t_r)\zeta^m$$

does not involve any of the X_j. So we may perform the construction of the triangle over $k(V)$ and restrict, and then tensor with K at the end. Since some $f_j(t_1,\dots,t_r)$ is nonzero, the corresponding $\hat{\xi}_j$ restricts to a map with projective cokernel, and so $k(V)\otimes_k L{\downarrow_{\langle u_V\rangle}}$ is projective. So from now on, assume that $f_r(t_1,\dots,t_r)\neq 0$.

Using $\hat{\zeta}^m$ to identify $k(V)$ with $\Omega^{-m}k(V)$, it follows that up to irrelevant projective summands, the restriction is equal to the Koszul complex obtained using the elements $X_jf_r - f_j$ on the ring $k(V)\otimes_k K$. Since $f_r(t_1,\dots,t_r)\neq 0$, these elements form a regular sequence, in this ring, because each uses an X_j not used in any of the other terms. So the homology is equal to the quotient

$$k(V)\otimes_k K/(X_1f_r - f_1,\dots,X_{r-1}f_r - f_{r-1}).$$

If $V=V_E$, so that the t_j are algebraically independent, the map (given by restricting $\phi_1\otimes\cdots\otimes\phi_{r-1}$) from $k(V)$ to this quotient is an isomorphism, since each occurrence of X_j can be replaced by f_j/f_r. On the other hand, if V is a proper subvariety, so that there is a nontrivial homogeneous relation between the elements $t_1,\dots,t_r$, then there is also a nontrivial homogeneous relation between the elements $f_1,\dots,f_r$, say $g(f_1,\dots,f_r)=0$. But then

$$\begin{aligned}1\otimes 1 &= 1\otimes g(X_1,\dots,X_{r-1},1)/g(X_1,\dots,X_{r-1},1)\\ &= g(f_1/f_r,\dots,f_{r-1}/f_r,1)\otimes 1/g(X_1,\dots,X_{r-1},1) = 0,\end{aligned}$$

and so the quotient is zero, and the restriction of L to $\langle u_V\rangle$ is projective.

The conclusion of this calculation is that the rank variety of L is equal to $\{V_E^r\}$, while the rank variety of X is the complement of this. Rickard's characterization of $F_{\mathfrak{U}}$ (see the remark after Proposition 5.7 of [14]) now completes the proof of the theorem. ∎

Remark 5.3. This theorem can be used as the basis for an alternative proof that $F_{\mathfrak{U}}$ is endofinite.

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