PERFECT MATCHINGS IN ϵ-REGULAR GRAPHS AND THE BLOW-UP LEMMA

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Abstract. As proved in [2], every ϵ-regular graph G on vertex set $V_1 \cup V_2$, $|V_1| = |V_2| = n$, with density $d > 2\epsilon$ and all vertex degrees not too far from d, has about as many perfect matchings as a corresponding random bipartite graph, i.e. about $d^n n!$.

In this paper we utilize that result to prove that with probability quickly approaching one, a perfect matching drawn randomly from G is spread evenly, in the sense that for any large subsets of vertices $S \subset V_1$ and $T \subset V_2$, the number of edges of the matching spanned between S and T is close to $|S||T|/n$ (c.f. Lemma 1).

As an application we give an alternative proof of the Blow-up Lemma of Komlós, Sárközy and Szemerédi [10].

1. Introduction

For a graph G and two disjoint subsets U and W of its vertex set $V(G)$, denote by $e_G(U, W)$ the number of edges of G with one endpoint in U and the other in W, and set $d_G(U, W) = \frac{e_G(U,W)}{|U||W|}$ for *the density of the pair* (U, W) *in* G.

Given $\epsilon > 0$, a bipartite graph G with bipartition (V_1, V_2), $|V_1| = |V_2| = n$, is called *ϵ-regular* if for every pair of sets (U, W), $U \subset V_1$, $W \subset V_2$, $|U|, |W| > \epsilon n$, the inequality

$$|d_G(U, W) - d_G(V_1, V_2)| < \epsilon$$

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holds.

Given $\epsilon > 0$ and $0 < d < 1$, an ϵ-regular graph G is called *super* (d,ϵ)*-regular* if the minimum degree $\delta(G)$ and the maximum degree $\Delta(G)$ of G satisfy

$$(d-\epsilon)n \le \delta(G) \le \Delta(G) \le (d+\epsilon)n \ .$$

Remark. The last definition differs from that in [10], where this notion is less restrictive. Trivially, the super regularity as defined in [10] implies that defined here. On the other hand, every super regular graph as defined here contains a spanning subgraph which is super regular, maybe with other parameters, in the sense of [10] (cf. Fact 2 in the Appendix).

It is not hard to check that for $d > 2\epsilon$, every super (d,ϵ)-regular graph G satisfies the assumptions of the König-Hall theorem and therefore possesses at least one perfect matching. The following result was proved in [2].

Theorem 1. *Let G be a super (d,ϵ)-regular graph on $2n$ vertices, where $d > 2\epsilon$ and $n > n_0(\epsilon)$. Then the number $M(G)$ of perfect matchings of G satisfies*

$$(d-2\epsilon)^n n! \le M(G) \le (d+2\epsilon)^n n! \ .$$

It was also observed in [2] that for every ϵ-regular graph G with $d_G(V_1,V_2) = d$ and ϵ sufficiently small as a function of d, we have

$$M(G) < (d+3\epsilon)^n n! \ , \tag{1}$$

provided $n > n_0(\epsilon)$.

In this paper we study further properties of perfect matchings in ϵ-regular graphs. Section 2 contains two lemmas on random perfect matchings in a super (d,ϵ)-regular graph G. The first one says that with high probability a perfect matching drawn uniformly at random from G is, in some sense, spread evenly around the graph. The second lemma, which we call the Four Graphs Lemma, is utilized in an alternative proof of the Blow-up Lemma of Komlós, Sárközy and Szemerédi [10] presented in Section 3. Finally, three technical facts are proved in the Appendix.

2. Random perfect matchings in ϵ-regular graphs

With Theorem 1 in hand, we can study the properties of a random perfect matching $\sigma : V_1 \to V_2$ chosen uniformly from all perfect matchings of a super (d, ϵ)-regular graph G with vertex sets (V_1, V_2), $|V_1| = |V_2| = n$. We first show that σ is spread evenly, i.e. for any two large sets of vertices it is very likely that the right portion of σ goes between them.

Lemma 1. *For every choice of three real numbers $0 < d, d_1, d_2 < 1$ there exist $\epsilon > 0$, $c = c(\epsilon)$, $0 < c < 1$, $n_0(\epsilon)$ and a function $g(x) \to 0$ as $x \to 0$ such that the following holds. Let G be a super (d, ϵ)-regular graph with bipartition (V_1, V_2), $|V_1| = |V_2| = n > n_0(\epsilon)$, and let $S \subseteq V_1$, $|S| = d_1 n$, and $T \subseteq V_2$, $|T| = d_2 n$. If a perfect matching σ of G is drawn uniformly at random, then*

$$(d_1 d_2 - g(\epsilon))n < |\sigma(S) \cap T| < (d_1 d_2 + g(\epsilon))n \ ,$$

with probability at least $1 - c^n$.

Proof.

Consider the random variable $X = |\sigma(S) \cap T|$ and set $s = |S|$ and $t = |T|$. If G were a complete bipartite graph then X would have the hypergeometric distribution with expectation $st/n = d_1 d_2 n$ and the lemma would follow immediately from the Chernoff-Hoeffding bound (see below). Although we are not in such a comfortable position, using Theorem 1 and inequality (1), we will arrive at a point where the Chernoff-Hoeffding bound will be applicable.

For an integer x, $0 \le x \le \min\{s, t\}$, let $M(G, x)$ be the number of perfect matchings σ of G for which $|\sigma(S) \cap T| = x$. Then $Prob(X = x) = \frac{M(G,x)}{M(G)}$. By Theorem 1, $M(G) \ge (d - 2\epsilon)^n n!$. On the other hand, denoting by $M(y)$ the maximum number of perfect matchings in any subgraph of G on $2y$ vertices,

$$M(G, x) \le \binom{s}{x}\binom{t}{x}\binom{n-t}{s-x}\binom{n-s}{t-x} \times M(x)M(s-x)M(t-x)M(n-s-t+x) \ .$$

If $y > \rho n$ then every subgraph of G with y vertices on each side is ϵ/ρ-regular with density at most $d+\epsilon$. Thus, if

$$\min\{x, s-x, t-x, n-x-t+x\} > \sqrt{\epsilon} n \ ,$$

then four applications of the upper bound given in (1) yield the following:

$$M(x) \le (d+\epsilon+3\sqrt{\epsilon})^x x! \ ,$$

$$M(s-x) \le (d+\epsilon+3\sqrt{\epsilon})^{s-x}(s-x)! \ ,$$

$$M(t-x) \le (d+\epsilon+3\sqrt{\epsilon})^{t-x}(t-x)! \ ,$$

and

$$M(n-s-t+x) \le (d+\epsilon+3\sqrt{\epsilon})^{n-s-t+x}(n-s-t+x)! \ .$$

Putting everything together we obtain, for some $\epsilon' = \epsilon'(\epsilon) \to 0$ as $\epsilon \to 0$,

$$\begin{aligned} Prob(X=x) \le & \left(\frac{d+\epsilon+3\sqrt{\epsilon}}{d-2\epsilon}\right)^n \times \\ & \binom{s}{x}\binom{t}{x}x!\binom{n-t}{s-x}(s-x)!\binom{n-s}{t-x}(t-x)!(n-s-t+x)!/n! \\ = & (1+\epsilon')^n \frac{\binom{t}{x}\binom{n-t}{s-x}}{\binom{n}{s}} \ , \end{aligned} \tag{2}$$

the last fraction being precisely the probability that a random variable with hypergeometric distribution and expectation st/n assumes the value of x.

If

$$\min\{x, s-x, t-x, n-x-t+x\} \le \sqrt{\epsilon} n \ ,$$

we obtain essentially the same bound, but with an extra factor of $(d+\epsilon+3\sqrt{\epsilon})^{-k\sqrt{\epsilon}n} < (1+\epsilon'')^n$, where $\epsilon'' = \epsilon''(\epsilon) \to 0$ as $\epsilon \to 0$, and $k \in \{1,2,3\}$ stands for the number of terms under the minimum which do not exceed $\sqrt{\epsilon}n$. This is because now we can use the inequality (1) only $4-k$ times, while for the k outstanding quantities we apply the obvious bound $M(y) \le y!$.

Finally, we quote a Chernoff-Hoeffding-type bound for the hypergeometric distribution [7, Thm. 2.6]. If Y is a random variable with the hypergeometric distribution then for every $m \geq 0$

$$Prob(Y \geq EY + m) \leq \exp\Big(-\frac{m^2}{2(EY + m/3)}\Big),$$
$$Prob(Y \leq EY - m) \leq \exp\Big(-\frac{m^2}{2EY}\Big).$$

The above inequalities with $EY = d_1 d_2 n$ and $m = g(\epsilon)n$, together with (2) yield that

$$\begin{aligned} Prob(|X - d_1 d_2 n| > g(\epsilon)n) &< e^{(\epsilon' + \epsilon'')n} Prob(|Y - EY| > t) \\ &\leq e^{(\epsilon' + \epsilon'')n} \left\{ e^{-\frac{g(\epsilon)^2 n}{2 d_1 d_2}} + e^{-\frac{g(\epsilon)^2 n}{2(d_1 d_2 + g(\epsilon)/3)}} \right\} . \end{aligned}$$

Any choice of $g(\epsilon)$ such that, say, $\epsilon' + \epsilon'' < g(\epsilon)^2/(3 d_1 d_2)$ but $g(\epsilon) \leq d_1 d_2$ guaranties that

$$Prob(|X - d_1 d_2 n| > g(\epsilon)n) < c^n$$

for some $0 < c = c(\epsilon) < 1$. $\square$

Our second result in this section will be the engine of an alternative proof of the Blow-up Lemma presented in the next section.

Let W_1, W_2, W_3 be disjoint sets of size n, and, for each pair i, j, $1 \leq i < j \leq 3$, let F_{ij} be a super (d_{ij}, ϵ)-regular graph with bipartition (W_i, W_j). In addition, let us assume that every edge $uv \in E(F_{12})$ satisfies the inequalities

$$(d_{13} d_{23} - \epsilon)n < |N_{F_{13}}(v) \cap N_{F_{23}}(u)| < (d_{13} d_{23} + \epsilon)n \ . \tag{3}$$

We call the triplet of graphs (F_{12}, F_{13}, F_{23}) *super* (d_{12}, d_{13}, d_{23})*-regular.* Moreover, for each perfect matching $\sigma : W_1 \to W_2$ in F_{12} we define a fourth graph A_σ with the bipartition (W_1, W_3), where $\{v, w\} \in E(A_\sigma)$ if and only if both $\{v, w\} \in E(F_{13})$ and $\{\sigma(v), w\} \in E(F_{23})$.

The Four Graphs Lemma. *For all choices of three real numbers $0 < d_{12}, d_{13}, d_{23} < 1$ there exist $\epsilon > 0$, $c_0 = c_0(\epsilon)$, $0 < c_0 < 1$, $n_0(\epsilon)$ and some function $h(x) \to 0$ as $x \to 0$ such that the following holds. Let (F_{12}, F_{13}, F_{23}) be a super (d_{12}, d_{13}, d_{23})-regular triplet of graphs with vertex sets W_1, W_2, W_3 of order $n > n_0(\epsilon)$ each. If a perfect matching σ is drawn uniformly at random, then A_σ is super $(d_{13}d_{23}, h(\epsilon))$-regular, with probability at least $1 - c_0^n$.*

Proof.

Let $deg_G(v) = |N_G(v)|$ denote the degree of a vertex v in a graph G. The definition of F_{12} guarantees that for every perfect matching σ of F_{12} and for each vertex $v \in W_1$,

$$(d_{13}d_{23} - \epsilon)n < deg_{A_\sigma}(v) < (d_{13}d_{23} + \epsilon)n \ , \tag{4}$$

which is just the first step toward the super $(d_{13}d_{23}, h(\epsilon))$-regularity of A_σ.

To have (4) satisfied (with some ϵ' instead of ϵ) by the vertices $w \in W_3$ as well, observe that

$$deg_{A_\sigma}(w) = |N_{F_{13}}(w) \cap \sigma^{-1}(N_{F_{23}}(w))| = |\sigma(N_{F_{13}}(w)) \cap N_{F_{23}}(w)| \ .$$

Thus, we need the property that for each $w \in W_3$ and some $\epsilon' > 0$,

$$(d_{13}d_{23} - \epsilon')n < |\sigma(N_{F_{13}}(w)) \cap N_{F_{23}}(w)| < (d_{13}d_{23} + \epsilon')n \ .$$

By Lemma 1, with $S = N_{F_{13}}(w)$ and $T = N_{F_{23}}(w)$, the probability that the last condition holds is at least $1 - nc^n$, where $\epsilon' = g(\epsilon)$ and c is the constant from Lemma 1.

It remains to show that, for some $h(\epsilon)$, the $h(\epsilon)$-regularity of A_σ holds with high probability. To achieve this task we employ the following handy criterion (see [1] and also [5, Proposition 2.5]).

Criterion. *If $G = (X, Y, E)$ is a bipartite graph, $|X| = |Y| = n$, with at least $(1-5\epsilon)n^2/2$ pairs of vertices $v, u \in X$ satisfying $deg_G(v), deg_G(u) > (d - \epsilon)n$ and $|N_G(v) \cap N_G(u)| < (d + \epsilon)^2 n$, then G is $(16\epsilon)^{1/5}$-regular.*

Call a pair $w_1, w_2 \in W_3$ *good* if

$$(d_{13}-\epsilon)^2 n < |N_{F_{13}}(w_1) \cap N_{F_{13}}(w_2)| < (d_{13}+\epsilon)^2 n$$

and

$$(d_{23}-\epsilon)^2 n < |N_{F_{23}}(w_1) \cap N_{F_{23}}(w_2)| < (d_{23}+\epsilon)^2 n \ .$$

By the ϵ-regularity of F_{13} and F_{23} there are at most $4\epsilon n^2$ pairs $w_1, w_2 \in W_3$ which are not good.

We apply Lemma 1 (upper bound) to pairs of sets $S = N_{F_{13}}(w_1) \cap N_{F_{13}}(w_2)$ and $T = N_{F_{23}}(w_1) \cap N_{F_{23}}(w_2)$, for all good pairs w_1, w_2, and conclude that with probability at least $1 - \binom{n}{2} c^n$,

$$|N_{A_\sigma}(w_1) \cap N_{A_\sigma}(w_2)| = |\sigma(S) \cap T| < (d_{13}d_{23} + g(\epsilon))^2 n \ .$$

Hence the graph A_σ is, by the Criterion, super $(d_{13}d_{23}, h(\epsilon))$-regular with probability at least $1 - nc^n - \binom{n}{2}c^n$, where $h(\epsilon) = (16\epsilon'')^{1/5}$ and $\epsilon'' = \max(g(\epsilon), 9\epsilon/5)$.

As for some $c < c_0 < 1$,

$$nc^n + \binom{n}{2} c^n < c_0^n \ ,$$

this completes the proof of the Four Graphs Lemma. $\square$

Comment. The Four Graphs Lemma remains valid even if $|W_3|$ differs slightly from $|W_1| = |W_2|$. (The definition of a super (d,ϵ)-regular graph with $|V_1| \neq |V_2|$ is an obvious modification of that presented in the Introduction.) Besides the balanced case, we will later apply this lemma also when $|W_3| = |W_1| - 1$.

3. A proof of the Blow-up Lemma

In 1983, Chvátal, Rödl, Szemerédi and Trotter [4], in the course of proving a linear bound for the Ramsey number of graphs with bounded maximum degree, showed how to embed a graph with bounded degree into a dense ϵ-regular graph whose order is only a constant multiple of the order of the smaller graph. Recently Komlós, Sárközy and Szemerédi [10] proved a powerful strengthening of that result which they called the Blow-up Lemma. This result allows one to embed any graph H with bounded maximum degree as a *spanning* subgraph of a dense, super (d,ϵ)-regular graph G. The Blow-up Lemma together with the Regularity Lemma of Szemerédi [14] enable one to tackle and solve difficult problems like the Pósa-Seymour conjecture on powers of hamiltonian cycles [8] or the Alon-Yuster conjecture on perfect F-matchings [9].

In [10] a slightly different, though equivalent (cf. Fact 2 in the Appendix) version of the following result was proved. For two graphs H and G on the same number of vertices, we call a bijection $f : V(H) \to V(G)$ *an embedding of* H *into* G if f maps every edge of H onto an edge of G. In other words, f is an isomorphism between H and a subgraph of G.

The Blow-up Lemma [10]. *For every choice of integers r and Δ and a real $d \in (0,1)$ there exist an $\epsilon > 0$ and an integer $n_0(\epsilon)$ such that the following is true. Consider an r-partite graph G with all partition sets $V_1, ..., V_r$ of order $n > n_0(\epsilon)$ and all $\binom{r}{2}$ bipartite subgraphs $G[V_i, V_j]$ super (d,ϵ)-regular. Then, for every r-partite graph H with maximum degree $\Delta(H) \le \Delta$ and all partition sets X_1,..., X_r of order n, there exists an embedding f of H into G with each set X_i mapped onto V_i, $i = 1, ..., r$.*

The idea of the original proof involves sequentially embedding the vertices of H into G until the current choices begin to threaten the feasibility of future options. When all but a small fraction of vertices are already embedded, the remaining vertices are all embedded at once using the König-Hall theorem.

In our proof we embed H into G in just a constant number of rounds. A set of vertices of a graph is called *two-independent* if no two of them are adjacent or have a common neighbor. Observe that every pair of disjoint two-independent sets spans a matching. Applying the Hajnal-Szemerédi theorem (cf. [6]) to the square of H we partition the vertex set of H into $k = r\Delta^2$ two-independent sets of almost equal size. We randomly partition $V(G)$ in accordance with the partition of $V(H)$. We then embed H into G in just k steps, in each step finding a suitable perfect matching in some auxiliary graph built upon parts of G and H. These auxiliary graphs are updated in each step and indicate the candidates for future images of yet unembedded vertices of H. Their super regularity allows one to apply the Four Graphs Lemma, which in turn, guaranties that the super regularity of the auxiliary graphs (with varying parameters) is maintained through the rounds. As we will see later in the proof, it is crucial for this argument that the sets of vertices of H embedded into G in different rounds are connected to each other by matchings only.

Proof.

The proof consists of three, clearly separated parts. First we show how a suitable refinement of the original r-partitions results in a reformulation of the Blow-up Lemma (cf. Auxiliary Lemma below). Then we define the candidacy graphs A_i^j and show inductively that a sequence of embeddings can be found which preserves the super regularity of these graphs. For clarity, this is first shown in a somewhat special case when all the new partition sets are of equal size. Finally, in the third part of the proof we relax this simplification and show how to modify the argument to cover the general case as well. The proof of a technical fact (cf. Fact 1 below), needed on the way, is left out to the Appendix.

1. We begin with exhibiting the graph H in a simpler form. Recall that the Hajnal-Szemerédi Theorem assures that the vertex set of every graph F can be partitioned into $D+1$ independent sets of size $\lfloor \frac{|V(F)|}{D+1} \rfloor$ or $\lceil \frac{|V(F)|}{D+1} \rceil$, for each integer D satisfying $\Delta(F) \leq D < |V(F)|$.

For $i = 1, ..., r$ we apply the Hajnal-Szemerédi Theorem to the subgraph $H^2[X_i]$ of H^2, the square of H, where every pair of vertices at distance at most 2 in H is now joined by an edge. Since $\Delta(H^2[X_i]) \leq \Delta(\Delta - 1) < \Delta^2$, it follows that there is a finer partition of the vertex set of H into $r\Delta^2$ independent sets X_{ij}, $i = 1, ..., r$, $j = 1, ..., \Delta^2$, each of size $m = \lfloor \frac{n}{\Delta^2} \rfloor$ or $m + 1$, such that every pair of new sets induces a (possibly empty) matching in H.

We find a corresponding partition of $V(G)$ by randomly splitting each set V_i into sets V_{ij}, $i=1,...,r$, $j=1,...,\Delta^2$, of size $|V_{ij}|=|X_{ij}|$. Since the graphs $G[V_{i_1},V_{i_2}]$ are ϵ-regular, the graphs $G[V_{i_1,j_1},V_{i_2,j_2}]$ are, by definition, $\frac{n}{m}\epsilon$-regular, and thus, say, $2\Delta^2\epsilon$-regular.

Moreover, it is not hard to verify, using Chernoff-Hoeffding's inequality (cf. Section 2), that with high probability every pair V_{i_1,j_1},V_{i_2,j_2} with $i_1\neq i_2$ spans a subgraph $G[V_{i_1,j_1},V_{i_2,j_2}]$ satisfying

$$(d-2\epsilon)(m+1)\leq\delta(G[V_{i_1,j_1},V_{i_2,j_2}])\leq\Delta(G[V_{i_1,j_1},V_{i_2,j_2}])\leq(d+2\epsilon)m\ .$$

The asymmetry above is intended to make sure that in all four cases with respect to the sizes of V_{i_1,j_1} and V_{i_2,j_2} (each of them may have m or $m+1$ elements) the subgraph $G[V_{i_1,j_1},V_{i_2,j_2}]$ will be super $(d,2\Delta^2\epsilon)$-regular. Thus, there exists a required partition such that each graph $G[V_{i_1,j_1},V_{i_2,j_2}]$, $i_1\neq i_2$ is super $(d,2\Delta^2\epsilon)$-regular.

Note that subgraphs $G[V_{i,j_1},V_{i,j_2}]$ have density zero, while the density of the other pairs is close to d. Without loss of generality we may add edges to the zero density pairs and create a supergraph G' of G such that each pair V_{i_1,j_1},V_{i_2,j_2}, including those with $i_1=i_2$, is super $(d,2\Delta^2\epsilon)$-regular in G'.

On the other hand, by adding more edges to H, we can create a supergraph H' of H such that each pair X_{i_1,j_1},X_{i_2,j_2}, including those with $i_1=i_2$, induces in H' a matching which saturates at least one of the sets (it is a perfect matching when $|X_{i_1,j_1}|=|X_{i_2,j_2}|$). If one manages to embed H' into G' with every set X_{ij} going onto the corresponding set V_{ij}, the desired embedding of H into G follows.

2. Hence, we may start all over again and prove the following technical lemma which, by the above discussion, implies the Blow-up Lemma.

Auxiliary Lemma. *For every choice of an integer k and a real $d\in(0,1)$ there exist an $\epsilon>0$ and an integer $m_0(\epsilon)$ such that the following is true. Consider a k-partite graph G with all partition sets $V_1,...,V_k$ of order $m>m_0(\epsilon)$ or $m+1$ and all $\binom{k}{2}$ bipartite* *subgraphs $G[V_i, V_j]$ super (d, ϵ)-regular. Then, for every k-partite graph H with partition sets X_1,..., X_k satisfying $|X_i| = |V_i|$, $i = 1, 2, ..., k$, and such that all $\binom{k}{2}$ bipartite subgraphs $H[X_i, X_j]$ are saturating matchings, there exists an embedding f of H into G with each set X_i mapped onto V_i, $i = 1, ..., k$.*

Proof of the Auxiliary Lemma.

For a more transparent presentation of this proof we first assume that all the sets X_i and V_i, $i = 1, ..., k$, are of order m, and thus every graph $H[X_i, X_j]$ is a perfect matching. We will be inductively embedding X_i onto V_i, with the induction statement generalized a little. But first we introduce a few definitions.

For every $1 \le i \le k-1$ and each $x \in X_{i+1} \cup ... \cup X_r$, let $N_i(x)$ denote the set of precisely i neighbors of x which belong to $X_1 \cup ... \cup X_i$. For any bijection f_i between $X_1 \cup ... \cup X_i$ and $V_1 \cup ... \cup V_i$, set $M_i(x) = f_i(N_i(x))$. Given f_i, for each j, $j = i+1, ..., k$, we define a bipartite auxiliary graph A_i^j, with bipartition (X_j, V_j) and the edge set

$$E(A_i^j) = \{xv : x \in X_j, v \in V_j \text{ and } uv \in E(G) \text{ for each } u \in M_i(x)\} \ .$$

So, the edges of A_i^j join a vertex $x \in X_j$ to all vertices of V_j which, after embedding the sets $X_1, ..., X_i$ onto $V_1, ..., V_i$, are still good candidates to be the image of x. We will call the graphs A_i^j *candidacy graphs.*

Let $H_i = H[X_1 \cup ... \cup X_i]$ and, similarly, $G_i = G[V_1 \cup ... \cup V_i]$. We will prove by induction that the following statement $\mathbf{S}(i)$ holds for $i = 1, ..., k$.

$\mathbf{S}(i)$: *There exists an embedding f_i of H_i into G_i such that $f_i(X_s) = V_s$, $s = 1, ..., i$, and for each j, $i+1 \le j \le k$, the candidacy graph A_i^j is super (d^i, δ_i)-regular, for some $\delta_i = \delta_i(\epsilon) \to 0$ as $\epsilon \to 0$.*

The instance $\mathbf{S}(k)$ of the above statement (where the latter condition is vacuously satisfied) yields the Auxiliary Lemma. Statement $\mathbf{S}(1)$ is trivial with $\delta_1 = \epsilon$, since A_1^j is isomorphic to $G[V_1, V_j]$ and any bijection $f_1 : X_1 \to V_1$ will do. One consequence of the assumption that the subgraphs $H[X_i, X_j]$ are perfect matchings is that the approximate densities of the candidacy graphs decrease in each step by the factor of d. The super regularity of these graphs will follow from the Four Graphs Lemma.

Let us assume the truth of $\mathbf{S}(i)$ for some fixed i, $1 \le i < r$, and prove $\mathbf{S}(i+1)$. Furthermore, let f_i be as described in the statement $\mathbf{S}(i)$.

Consider graphs: $F'_{12} = A_i^{i+1}$, and, for each $i+2 \le j \le r$, $F_{23}^j = G[V_{i+1}, V_j]$ and $F_{13}^j = (X_{i+1}, V_j, E)$, where $\{x,w\} \in E$ if and only if $w \in V_j$ and $\{y,w\} \in E(A_i^j)$, where $N_H(x) \cap X_j = \{y\}$.

Observe that, since $H[X_{i+1}, X_j]$ is a perfect matching, the graph F_{13}^j is isomorphic to A_i^j under the isomorphism which keeps the elements of V_j fixed, while mapping each $x \in X_{i+1}$ onto its unique neighbor $y \in X_j$.

By the validity of $\mathbf{S}(i)$, graphs F'_{12} and F_{13}^j are super (d^i, δ_i)-regular, while F_{23}^j is super (d,ϵ)-regular. Every perfect matching $\sigma : X_{i+1} \to V_{i+1}$ of F'_{12} defines an embedding f_{i+1} by the formula

$$f_{i+1}(x) = \begin{cases} \sigma(x) \text{ if } x \in X_{i+1} \\ f_i(x) \text{ if } x \in X_1 \cup \ldots \cup X_i \end{cases} \tag{5}$$

The graph A_σ as defined in the Four Graphs Lemma is clearly isomorphic to A_{i+1}^j. But we cannot apply that lemma directly to the triple of graphs F'_{12}, F_{13}^j and F_{23}^j, because we do not know if condition (3) holds.

Therefore we define a subgraph $F_{12} \subseteq F'_{12}$. Since in our case $d_{13}d_{23} = d^{i+1}$, an edge $xv \in E(F'_{12})$ belongs to $E(F_{12})$ if and only if, for all $j = i+2, \ldots, k$,

$$(d^{i+1} - 2\delta_i)m < |N_{F_{13}^j}(x) \cap N_{F_{23}^j}(v)| < (d^{i+1} + 2\delta_i)m \ .$$

By this definition, F_{12} satisfies (3) with $\epsilon = 2\delta_i$. Our next fact, whose proof is presented in the Appendix, guarantees that F_{12} satisfies the other assumption of the Four Graphs Lemma.

Fact 1. *F_{12} is super $(d^i, 2k\sqrt{\delta_i})$-regular.*

We now, for each $j = i+2, \ldots, k$, apply the Four Graphs Lemma with $W_1 = X_{i+1}$, $W_2 = V_{i+1}$ and $W_3 = V_j$, to the graphs F_{12}, F_{13}^j and F_{23}^j. We conclude that, for each

$j = i+2, ..., k$, most of the perfect matchings of F_{12} define, via (5), such embeddings f_{i+1} which make the graph A^j_{i+1} super (d^{i+1}, δ_{i+1})-regular, where $\delta_{i+1} = h(2k\sqrt{\delta_i})$ and h is as in the Four Graphs Lemma. Thus, most perfect matchings of F_{12} make, indeed, all graphs A^j_{i+1}, $i+2 \le j \le k$, super (d^{i+1}, δ_{i+1})-regular, and the statement $\mathbf{S}(i+1)$ follows.

3. The above proof was executed under the simplifying assumption that all the sets V_i and X_i had common cardinality m. In the more realistic case, when some sets X_i (and thus V_i) are of size $m+1$ and the others are of size m, we order them in the descending manner, so that for some $1 \le l \le k-1$, we have $|X_i| = |V_i| = m+1$ for $j = 1, ..., l$ and $|X_i| = |V_i| = m$ for $j > l$. This way it is still guaranteed that each vertex $x \in X_j$, $j > i$, has precisely i neighbors in $X_1 \cup ... \cup X_i$. The only difference is that now for each pair of indices (i, j), $i < l < j$, there is precisely one vertex $x_j \in X_{i+1}$ with no neighbor in X_j, and so one cannot define the neighbors of x_j in the graph F^j_{13}.

But it is irrelevant for the super regularity of A^j_{i+1} how one matches x_j in σ. Thus, to complete the construction of F^j_{13}, for each $j > l$ (and $i < l$), we arbitrarily choose $\lfloor d^i m \rfloor$ vertices of V_j to become the neighbors of x_j in F^j_{13}. Since $F^j_{13} \setminus \{x_j\}$ is isomorphic to A^j_i, it is super (d^i, δ_i)-regular, and after adding to it vertex x_j and its neighbors, it remains, say, super $(d^i, 2\delta_i)$-regular.

Having defined the graph F^j_{13} for each $j = i+2, ..., k$, we can now define F'_{12} and F^j_{23}, and then F_{12} as before, and, for each j, apply the Four Graphs Lemma to the triad $(F_{12}, F^j_{13}, F^j_{23})$ (with $|W_1| = |W_2| = |W_3| + 1$ whenever $i < l < j$). As before, we find a perfect matching σ of F_{12} such that all the graphs A^j_σ are super $(d^{i+1}, h(2k\sqrt{2\delta_i}))$-regular. Each graph A^j_{i+1} is isomorphic to $A^j_\sigma \setminus \{x_j\}$. This deletion costs us (again, with room to spare) another factor of 2 and we conclude that the embedding f_{i+1}, defined by (5), makes all graphs A^j_{i+1}, $j = i+2, ..., k$, super (d^{i+1}, δ_{i+1})-regular, where $\delta_{i+1} = 2h(2k\sqrt{2\delta_i})$. Thus the statement $\mathbf{S}(i+1)$ is proved in the general (unbalanced) case as well. $\square$

Comments.

- It follows from our proof that for some $f(x) \to 0$ as $x \to 0$, there are at least
$$(d - f(\epsilon))^{\binom{k}{2}(m+1)} m!^k > (d - f(\epsilon))^{(r\Delta)^2 n} (n/\Delta^2)!^{r\Delta^2}$$
different embeddings of H into G.
- In [11] an algorithmic polynomial time version of the proof from [10] was given. The above proof can also be derandomized yielding a polynomial time embedding algorithm (c.f. [12]). Let us mention that the Blow-up Lemma proved algorithmically in [11] assumes that G is ϵ-regular with minimum degree $\delta(G) > cn$ for some $c > 0$. This is a more restrictive assumption than that in [10], but still less so than the one in this paper. Nevertheless, based on Fact 3 in the Appendix, in such a graph G one can find a super regular spanning subgraph in polynomial time.
- In a forthcoming paper [13] we apply a similar approach to embed a bounded degree 3-uniform hypergraph into a 3-uniform hypergraph with dense, regular structure.

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Appendix

In order to complete the proof of the Auxiliary Lemma, and, consequently, of the Blow-up Lemma, we owe to the Reader the proof of Fact 1.

Proof of Fact 1.

Let us fix $x \in X_{i+1}$ and $j > i+1$, and define
$$N^{\leq}_{x,j} = \{v \in V_{i+1} : |N_{F^j_{13}}(x) \cap N_{F^j_{23}}(v)| \leq (d^{i+1} - 2\delta_i)m\} \ .$$

By the super (d^i, δ_i)-regularity of F^j_{13}, $|N_{F^j_{13}}(x)| > d^i - \delta_i$. Thus,

$$d_{F^j_{23}}(N^{\leq}_{x,j}, N_{F^j_{13}}(x)) < \frac{d^{i+1} - 2\delta_i}{d^i - \delta_i} \leq d - \delta_i \ .$$

Since $\epsilon \leq \delta_i$, the graph F^j_{23} is δ_i-regular and, by choosing ϵ so small that $d^i - \delta_i > \epsilon$, it forces that $|N^{\leq}_{x,j}| \leq \epsilon$. Similarly $|N^{\geq}_{x,j}| \leq \epsilon$, where

$$N^{\geq}_{x,j} = \{v \in V_{i+1} : |N_{F^j_{13}}(x) \cap N_{F^j_{23}}(v)| \geq (d^{i+1} + 2\delta_i)m\} \ .$$

When creating F_{12} we have removed an edge xv from F'_{12} if and only if $v \in N^{\leq}_{x,j} \cup N^{\geq}_{x,j}$ for at least one j, $j = i+2, ..., k$. Hence, for each $x \in X_{i+1}$,

$$deg_{F'_{12}}(x) - deg_{F_{12}}(x) \leq 2(k - i + 1)\delta_i m < 2k\delta_i m \ , \tag{6}$$

Similarly, by the δ_i-regularity of F^j_{13}, (6) remains true for each $v \in V_{i+1}$. Fact 1 now follows by a simple observation.

Observation. *Let $0 < \alpha < (\frac{k}{k+1})^2$ and $k \geq 2$. If F is a spanning subgraph of a super (d, α)-regular graph F' with $2m$ vertices, where for each $v \in V(F)$, $deg_{F'}(v) - deg_F(v) < k\alpha m$, then F is super $(d, k\sqrt{\alpha})$-regular.*

Indeed, for every pair of sets (U, W), $U \subset V_1$, $W \subset V_2$, $|U|, |W| > k\sqrt{\alpha}m$, we have

$$d_F(U, W) \geq d_{F'}(U, W) - \frac{k\alpha m}{|W|} \geq d - \alpha - \sqrt{\alpha} \geq d - k\sqrt{\alpha}$$

and, trivially, $d_F(U, W) \leq d_{F'}(U, W) \leq d + \alpha \leq d + k\sqrt{\alpha}$, which proves that F is $k\sqrt{\alpha}$-regular. For each vertex its degree in F does not exceed its degree in F', thus it does not exceed $(d + \alpha)m$. On the other hand, $\delta(F) \geq (d - \alpha)m - k\alpha m \geq (d - k\sqrt{\alpha})m$. This proves the observation, and therefore completes the proof of Fact 1. $\square$

Next, we present an elementary, though tedious argument showing that the Blow-up Lemma proved in this paper is equivalent to the original one formulated in [10]. The two statements differ in the definition of a super regular graph. For the sake of distinction between these two definitions let us use the phrase "half-super regular" for the notion defined in [10].

We say that a bipartite graph G with bipartition (V_1, V_2), $|V_1| = |V_2| = n$, is called *half-super* (d, ϵ)*-regular* if $\delta(G) \geq dn$ and, for every pair of sets (U, W), $U \subset V_1$, $W \subset V_2$, $|U|, |W| > \epsilon n$, the inequality $d_G(U, W) \geq d$ holds. Compared with the super regularity defined earlier in this paper, here both upper bounds, for $d_G(U, W)$ and for $\Delta(G)$, are dropped. Thus, clearly, every super (d, ϵ)-regular graph is half-super (d, ϵ)-regular, indicating that the Blow-up Lemma from [10] might be stronger than our version of it. However, the following result implies that the two statements are equivalent.

Fact 2. *For all $0 < d < 1$ and $\epsilon > 0$ there exists an $\tilde{\epsilon} > 0$ such that every half-super $(d, \tilde{\epsilon})$-regular graph $\tilde{G}$ on $2n$ vertices, $n > n_0(\epsilon)$, contains a spanning subgraph G which is super $(d^2/2, \epsilon)$-regular.*

In the proof we will rely on the Szemerédi Regularity Lemma [14] in the following, special form.

Szemerédi's Regularity Lemma. *For all $\epsilon > 0$ there exist integers $N = N(\epsilon)$ and $T = T(\epsilon)$ such that for every bipartite graph with bipartition (V_1, V_2), $|V_1| = |V_2| = n > N$, there exists $t < T$ and a refined partition $V_1 = V_1^{(1)} \cup \ldots \cup V_1^{(t)}$, $V_2 = V_2^{(1)} \cup \ldots \cup V_2^{(t)}$ such that for $s = 1, 2$ and $i = 1, \ldots, t$,*

$$\lfloor n/t \rfloor \leq |V_s^{(i)}| \leq \lceil n/t \rceil$$

and all but at most ϵt^2 pairs $(V_1^{(i)}, V_2^{(j)})$ are ϵ-regular.

Proof of Fact 2.

Let $\hat{\epsilon}$ be as defined in Fact 3 below (with $c = d^2/2$), $\hat{\epsilon} < 1/5$, and let $T = T(\hat{\epsilon}^4)$ be the upper bound for the number of partition classes defined via Szemerédi's Regularity Lemma. Choose $\tilde{\epsilon}$ smaller than $\min\{1/T, \hat{\epsilon}^4\}$ and apply Szemerédi's Regularity Lemma with $\hat{\epsilon}^4$ to a given half-super $(d, \tilde{\epsilon})$-regular graph $\tilde{G}$ on $2n$ vertices. Let, for some $t < T$,

$V_1 = V_1^{(1)} \cup \ldots \cup V_1^{(t)}$, $V_2 = V_2^{(1)} \cup \ldots \cup V_2^{(t)}$ be the obtained partition with the set sizes at most one apart, where all but at most $\hat{\epsilon}^4 t^2$ pairs $(V_1^{(i)}, V_2^{(j)})$ are $\hat{\epsilon}^4$-regular in $\tilde{G}$.

For every $\hat{\epsilon}^4$-regular pair $(V_1^{(i)}, V_2^{(j)})$, we remove a subset of $e_G(V_1^{(i)}, V_2^{(j)}) - d|V_1^{(i)}||V_2^{(j)}|$ edges in such a way that, in the remaining graph $\hat{G}$, the pair is still $\hat{\epsilon}^4$-regular, $\delta(\hat{G}) > d^2 n/2$, and for every pair of sets (U, W), $U \subset V_1$, $W \subset V_2$, $|U|, |W| > \epsilon n$, the inequality $d_{\hat{G}}(U, W) \geq d - \hat{\epsilon}$ holds. The existence of such subsets can be proved by considering random selections.

If we can prove that $\hat{G}$ is $\hat{\epsilon}$-regular then Fact 3 below (with $c = d^2/2$) finishes off the proof of Fact 2.

Let $X \subseteq V_1$, $Y \subseteq V_2$, $|X|, |Y| > \hat{\epsilon} n$. Since $\epsilon' < \hat{\epsilon}$, the lower bound

$$e_{\hat{G}}(X, Y) \geq (d - \hat{\epsilon})|X||Y| \ .$$

follows by the definition of $\hat{G}$.

For the upper bound, denote $X^{(i)} = X \cap V_1^{(i)}$ and $Y^{(i)} = Y \cap V_2^{(i)}$ and consider the expression

$$e_{\hat{G}}(X, Y) = \sum_i \sum_j e_{\hat{G}}(X^{(i)}, Y^{(i)}) \ .$$

Let us split the above summation into three parts:

(A) Over all pairs (i, j) such that $(V_1^{(i)}, V_2^{(j)})$ is $\hat{\epsilon}^4$-irregular. They contribute at most $\hat{\epsilon}^4 t^2 (n/t + 1)^2 < 2\hat{\epsilon}^4 n^2$.

(B) Over all pairs (i, j) such that $\min\{|X^{(i)}|, |Y^{(i)}|\} < \tilde{\epsilon} n$. They contribute at most $t^2 \tilde{\epsilon} (n/t + 1)^2 < 2\tilde{\epsilon} n^2$.

(C) Over all other pairs (big chunks of $\hat{\epsilon}^4$-regular pairs). They contribute at most $(d + \hat{\epsilon}^4)|X||Y|$.

Altogether, (A)+(B)+(C) and the inequality $\hat{\epsilon} < 1/5$ imply that

$$e_{\hat{G}}(X, Y) \leq \left[2\hat{\epsilon}^2 + 2\tilde{\epsilon}/\hat{\epsilon}^2 + (d + \hat{\epsilon}^4)\right] |X||Y| < (d + \hat{\epsilon})|X||Y| \ .$$

This shows that $\hat{G}$ is $\hat{\epsilon}$-regular. $\square$

We wrap up this technical appendix with a result which on one hand supplements the proof of Fact 2, and on the other, constitutes an independent argument showing that the algorithmic version of the Blow-up Lemma given in [11] is equivalent to ours. A quick glance into the proof of Fact 3 reveals that it can be derandomized in polynomial time.

Fact 3. *For all $0 < c < 1$ and all $\epsilon > 0$ there exists an $\hat{\epsilon} > 0$ such that every $\hat{\epsilon}$-regular graph $\hat{G}$ on $2n$ vertices, $n > n_0(\epsilon)$, and with minimum degree $\delta(\hat{G}) > cn$ contains a spanning subgraph G which is super (c, ϵ)-regular.*

Proof of Fact 3.

Let us choose $\hat{\epsilon} = 0.1\epsilon^2$ and denote by $d = d(n) = d_{\hat{G}}(V_1, V_2)$ the density of an $\hat{\epsilon}$-regular graph $\hat{G}$ with $\delta(\hat{G}) > cn$. Note that $d > c$.

For $i = 1, 2$ denote by V_i' the subset of V_i consisting of the vertices of degree between $(d-\hat{\epsilon})n$ and $(d+\hat{\epsilon})n$. Let G' be the subgraph of $\hat{G}$ spanned by $V_1' \cup V_2'$. By the $\hat{\epsilon}$- regularity of $\hat{G}$, $|V_i'| \geq n - 2\hat{\epsilon}n$, $i = 1, 2$. Also, the graph G' is ϵ'-regular for

$$\epsilon' = \max\left(\frac{\hat{\epsilon}}{1 - 2\hat{\epsilon}}, 2\hat{\epsilon}\right) = 2\hat{\epsilon} \ ,$$

and

$$(d - 3\hat{\epsilon})n \leq \delta(G') \leq \Delta(G') \leq (d + \hat{\epsilon})n \ .$$

Take a random subgraph G_p' of G' with $p = c/d$. Using the standard Chernoff-Hoeffding inequalities for the tails of binomial distributions (which are identical to those presented in Section 2), one can show that, with positive probability (in fact, with probability tending to 1 as $n \to \infty$), G_p' is $2\hat{\epsilon}$-regular and satisfies inequalities

$$(c - 3\hat{\epsilon})n \leq \delta(G_p') \leq \Delta(G_p') \leq (c + \hat{\epsilon})n \ .$$

So, there exists a spanning subgraph G'' of G' with the above properties.

We create G from G'' by adding back the vertices of $(V_1 \cup V_2) \setminus (V_1' \cup V_2')$ and joining each of them arbitrarily to exactly $\lceil (c - 2\hat{\epsilon})n \rceil$ of their original neighbors in $\hat{G}$ not belonging to

$V(G'') = V_1' \cup V_2'$. This operation may increase the degrees of other vertices by at most $2\hat{\epsilon}n$. Thus,

$$(c - 3\hat{\epsilon})n \le \delta(G) \le \Delta(G) \le (c + 3\hat{\epsilon})n \ .$$

It remains to prove that G is ϵ-regular.

Let $U_i \subset V_i'$, $W_i \subset V_i \setminus V_i'$, $|U_i| = u_i$, $|W_i| = w_i$, $u_i + w_i \ge \epsilon n$, $w_i \le 2\hat{\epsilon}$, $i = 1, 2$. By the $2\hat{\epsilon}$-regularity of G'', we have

$$\begin{aligned} d_G(U_1 \cup W_1, U_2 \cup W_2) &\ge \frac{e_{G''}(U_1, U_2)}{(u_1 + w_1)(u_2 + w_2)} = \frac{d_{G''}(U_1, U_2)}{(1 + w_1/u_1)(1 + w_2/u_2)} \\ &\ge \frac{d_{G''}(V_1, V_2) - 2\hat{\epsilon}}{(1 + 2\hat{\epsilon}/(\epsilon - 2\hat{\epsilon}))^2} \ge (c - 5\hat{\epsilon}) \left(1 - \frac{2\hat{\epsilon}}{\epsilon}\right)^2 > c - 5\hat{\epsilon} - \frac{4\hat{\epsilon}}{\epsilon} > c - \epsilon \ . \end{aligned} \tag{7}$$

Now, set $x = e_G(U_1, W_2) + e_G(U_2, W_1)$ and $y = \frac{x}{(u_1+w_1)(u_2+w_2)}$. Again by the $2\hat{\epsilon}$-regularity of G'', we have

$$d_G(U_1 \cup W_1, U_2 \cup W_2) \le d_{G''}(U_1, U_2) + y < c + 2\hat{\epsilon} + y \ .$$

Suppose that $d_G(U_1 \cup W_1, U_2 \cup W_2) > c + \epsilon$, so that $y > \epsilon - 2\hat{\epsilon}$. But then, by the $\hat{\epsilon}$-regularity of $\hat{G}$, similarly to (7), one can show that

$$d_{\hat{G}}(U_1 \cup W_1, U_2 \cup W_2) \ge \frac{d - \hat{\epsilon}}{(1 + 2\hat{\epsilon}/(\epsilon - 2\hat{\epsilon}))^2} + y > d - \frac{\epsilon}{2} + y > d + \hat{\epsilon} \ ,$$

which contradicts the $\hat{\epsilon}$-regularity of the graph $\hat{G}$. □

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