SINCERE STABLE TUBES

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Introduction

The purpose of this paper is to prove the following result for an indecomposable finite dimensional algebra Λ (associative with an identity) over an algebraically closed field k: If the AR-quiver of Λ has a sincere family of stable tubes $\mathcal{C}$ such that there is no sequence $X \to Y \to Z$ of nonzero nonisomorphisms between indecomposable modules with X and Z in $\mathcal{C}$ and Y not in $\mathcal{C}$ then Λ must be concealed canonical. It is then a consequence that if the AR-quiver of Λ has a sincere stable tube $\mathcal{C}$ such that there is no sequence $X \to Y \to X$ of nonzero nonismorphisms between indecomposable modules, with $X \in \mathcal{C}$ and $Y \notin \mathcal{C}$ then Λ is concealed canonical. The concealed canonical algebras [LM] are by definition the endomorphism algebras of tilting bundles in the categories $\operatorname{coh}\mathbb{X}$ of coherent sheaves on weighted projective lines $\mathbb{X}$ in the sense of [GL], or equivalently tilts of canonical algebras introduced in [Ri] by tilting modules whose indecomposable direct summands have positive rank.

Stable tubes in AR-quivers occur frequently and have been extensively investigated. It has been proved in [Z] (see also [Li], [HPR]) that a regular component $\mathcal{C}$ of an AR-quiver contains an oriented cycle if and only if $\mathcal{C}$ is a stable tube. Moreover, if Λ is not a local Nakayama algebra, then every DTr-invariant indecomposable Λ-module lies in a stable tube of rank one [Ho]. Further, if Λ is of tame representation type, then by a result of [CB] all but finitely many indecomposable Λ-modules of any fixed dimension d lie in stable tubes of rank one. Finally, the AR-quivers of wild blocks of group algebras of finite groups have also many stable tubes. The above results show that assuming only the existence of a stable tube (or a family of stable tubes) $\mathcal{C}$ in the AR-quiver of an indecomposable algebra Λ one cannot expect strong implications on Λ. To have a closer relationship it is natural to assume that $\mathcal{C}$ is sincere, that is, every simple Λ-module occurs as a composition factor of a module in $\mathcal{C}$.

Another relationship between a family of stable tubes and the rest of the module category is concerned with existence or nonexistence of paths (or cycles). Recall that a sequence of nonzero nonisomorphisms $X = X_0 \to X_1 \to \cdots \to X_{n-1} \to X_n = Y$ between indecomposable modules is called a path of length n and it is called a cycle if $Y = X$. If $n = 2$ the path (or cycle) is said to be short [RSS]. Moreover, if $\mathcal{C}$ is a family of indecomposable Λ-modules and X, Y are in $\mathcal{C}$, while $X_1 \cdots, X_{n-1}$ do not lie in $\mathcal{C}$, we say that the path or cycle is external (with respect to $\mathcal{C}$). For symmetric algebras, including the group algebras, any indecomposable module X in a stable tube $\mathcal{C}$ lies on an external short cycle $X \to P \to X$ (with respect to $\mathcal{C}$) with P projective-injective. On the other hand, for a family of stable tubes of

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concealed canonical algebras there are no external paths [Ri]. What we prove in this paper is that excluding the possibility that the modules in a sincere family of stable tubes $\mathcal{C}$ of the AR-quiver of an indecomposable algebra Λ lie on external short paths forces the algebra Λ to be concealed canonical, and consequently that the tubes in $\mathcal{C}$ are generalized standard in the sense of [S1]. We also note that assuming that there is a sincere stable tube $\mathcal{C}$ which is generalized standard is not sufficient for the algebra to be concealed canonical. An example is here provided by the trivial extension of a tame hereditary algebra by an injective cogenerator (see [S4, 1.4]). Hence our result is in some sense best possible.

Our main result improves significantly the characterizations of concealed canonical algebras obtained in [LP], [S3], and generalizes a result proved for tame algebras in [S4].

The outline of the paper is as follows. In Section 1 we show the crucial result that a sincere family of stable tubes $\mathcal{C}$ without external short paths must be faithful. This fact implies homological properties as in [RSS], which are used to show in Section 2 that we can assume that there are no external paths for $\mathcal{C}$. Here we follow the proof of the corresponding result for ordinary cycles and short cycles from [HRS1]. In Section 3, applying the results of the previous sections and the main result of [HR], we show that if the AR-quiver of an indecomposable algebra Λ has a sincere family of stable tubes without external short paths, then Λ is quasitilted of canonical type. Finally, using the description of quasitilted algebras of canonical type from [LS], we get our main result.

1. Sincere and faithful tubes

Let Λ be an indecomposable artin algebra, and let $\mathcal{C}$ be a sincere family of stable tubes in the AR-quiver of Λ having no external short paths. (A module in $\mathcal{C}$ is by definition indecomposable.) The aim of this section is to prove that $\mathcal{C}$ is faithful, that is, the annihilator $\operatorname{ann}\mathcal{C} = \cap_{C\in\mathcal{C}} \operatorname{ann} C$ of $\mathcal{C}$ is zero.

The following preliminary result will be useful. Here $\operatorname{Supp} M$ denotes the set of simple Λ-modules occurring as composition factors of a Λ-module M, and $\operatorname{Supp}\mathcal{C}$ the set of simple modules occurring as composition factors of modules in $\mathcal{C}$.

For a family of indecomposable modules $\mathcal{C}$ denote by $\operatorname{add}\mathcal{C}$ the family of modules which are direct sums of modules in $\mathcal{C}$.

Lemma 1.1. *Let $\mathcal{C}$ be a family of indecomposable Λ-modules. Then there is some M in add $\mathcal{C}$ such that* $\operatorname{ann} M = \operatorname{ann}\mathcal{C}$ *and* $\operatorname{Supp} M = \operatorname{Supp}\mathcal{C}$.

Proof. Choose K in $\operatorname{add}\mathcal{C}$ such that $\operatorname{ann} K$ is minimal. Then for any N in $\mathcal{C}$ we have $\operatorname{ann}(K \bigoplus N) = \operatorname{ann} K$ and also $\operatorname{ann}(K \bigoplus N) = \operatorname{ann} K \cap \operatorname{ann} N$. Hence we have $\operatorname{ann} K \subset \operatorname{ann} N$, and so $\operatorname{ann} K = \operatorname{ann}\mathcal{C}$.

Choose L in $\operatorname{add}\mathcal{C}$ such that $\operatorname{Supp} L$ is maximal. Then for any N in $\mathcal{C}$ we have $\operatorname{Supp}(L \bigoplus N) = \operatorname{Supp} L$ and $\operatorname{Supp}(L \bigoplus N) = \operatorname{Supp} L \cup \operatorname{Supp} N$, so that $\operatorname{Supp} N \subset \operatorname{Supp} L$. Then we get $\operatorname{Supp} L = \operatorname{Supp}\mathcal{C}$.

Putting $M = K \bigoplus L$ we get $\operatorname{ann} M = \operatorname{ann}\mathcal{C}$ and $\operatorname{Supp} M = \operatorname{Supp}\mathcal{C}$. □

We shall also need the following.

Lemma 1.2. *Let $\mathcal{C}$ be a stable tube in the AR-quiver of an artin algebra Λ. For any B and C in $\mathcal{C}$ there is some M in $\mathcal{C}$ with a monomorphism $B \to M$ and an epimorphism $M \to C$.*

Proof. This is well known (see [S2, Lemma 3.3]). $\square$

We shall need the following two results which are essentially proved in [RSS, Theorem 1.6, Proposition 3.1]. We include the proofs for the sake of completeness, since the corresponding statements in [RSS] are slightly different.

Lemma 1.3. *Assume that X, Y and Z are indecomposable Λ-modules such that* $\operatorname{rad}(Y, D\operatorname{Tr} X) \neq 0$ *and* $\operatorname{rad}(X, Z) \neq 0$. *Then there exists a short path $Y \to V \to Z$ in* $\operatorname{mod}\Lambda$, *where either $V = X$ or V is an indecomposable summand of the middle term E of an almost split sequence $0 \to D\operatorname{Tr} X \to E \to X \to 0$.*

Proof. Let $E = \bigoplus_{i=1}^{t} E_i$ where the E_i are indecomposable, and let $\alpha_i\colon D\operatorname{Tr} X \to E_i$ and $\beta_i\colon E_i \to X$ be the induced maps. Let $f\colon X \to Z$ and $g\colon Y \to D\operatorname{Tr} X$ be nonzero maps which are not isomorphisms. We choose i such that $\alpha_i g\colon Y \to E_i$ is nonzero. If $\beta_i\colon E_i \to X$ is a monomorphism, then also the composition $\beta_i\alpha_i g\colon Y \to X$ is nonzero and not an isomorphism, so that we get a short path $Y \to X \to Z$ with $V = X$.

If $\beta_i\colon E_i \to X$ is an epimorphism, then the composition $f\beta_i\colon E_i \to Z$ is a nonzero nonisomorphism, so that we have a short path $Y \to E_i \to Z$ with $V = E_i$. $\square$

Lemma 1.4. *Let M be a Λ-module, $f\colon P \to Q$ a nonzero map between indecomposable projective Λ-modules such that* $\operatorname{Hom}_\Lambda(f, M)$ *is neither a monomorphism nor an epimorphism, and let* $X = \operatorname{Coker} f$. *Then* $\operatorname{Hom}_\Lambda(X, M) \neq 0$ *and* $\operatorname{Hom}_\Lambda(M, D\operatorname{Tr} X) \neq 0$.

Proof. Since $\operatorname{Hom}_\Lambda(f, M)$ is not a monomorphism, from the exact sequence

$$0 \to \operatorname{Hom}_\Lambda(X, M) \longrightarrow \operatorname{Hom}_\Lambda(Q, M) \xrightarrow{\operatorname{Hom}_\Lambda(f,M)} \operatorname{Hom}_\Lambda(f, M)$$

we get $\operatorname{Hom}_\Lambda(X, M) \neq 0$. For Y a finitely generated projective module, the natural map

$$D\operatorname{Hom}_\Lambda(Y, M) \to \operatorname{Hom}_\Lambda(Y^*, DM)$$

is an isomorphism, where $Y^* = \operatorname{Hom}_\Lambda(Y, \Lambda)$. We then have a commutative diagram

$$\begin{array}{ccc} D\operatorname{Hom}_\Lambda(P, M) & \xrightarrow{D\operatorname{Hom}(f,M)} & D\operatorname{Hom}_\Lambda(Q, M) \\ \simeq\big\downarrow & & \simeq\big\downarrow \\ \operatorname{Hom}_\Lambda(P^*, DM) & \xrightarrow{\operatorname{Hom}(f^*,DM)} & \operatorname{Hom}_\Lambda(Q^*, DM) \end{array}$$

where $f^* = \operatorname{Hom}_\Lambda(f, \Lambda)$. Since $\operatorname{Hom}_\Lambda(f, M)$ is not an epimorphism, $D\operatorname{Hom}_\Lambda(f, M)$ is not a monomorphism, and therefore $\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(f^*, DM)$ is not a monomorphism. Hence $\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\operatorname{Coker} f^*, DM) = \operatorname{Ker}\operatorname{Hom}(f^*, DM) \neq 0$. Note that $\operatorname{Coker} f^* \simeq \operatorname{Tr} X$, and so $\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(\operatorname{Tr} X, DM) \neq 0$. Therefore, $\operatorname{Hom}_\Lambda(M, D\operatorname{Tr} X) \neq 0$, and this finishes the proof. $\square$

The following will be useful for investigating the relationship between sincere and faithful families of stable tubes.

Proposition 1.5. *Let $\mathcal{C}$ be a family of stable tubes in Γ_Λ with no external short paths and let M be a module from* $\operatorname{add}\mathcal{C}$ *such that* $\operatorname{ann} M = \operatorname{ann}\mathcal{C}$. *Assume that $f\colon P \to Q$ is a nonzero morphism between indecomposable projective modules with* $\operatorname{Hom}_\Lambda(P, M) \neq 0$ *and* $\operatorname{Hom}_\Lambda(Q, M) \neq 0$. *Then* $\operatorname{Hom}_\Lambda(f, M) \neq 0$.

Proof. Let $I = \operatorname{ann}\mathcal{C}$ and $X = \operatorname{Coker} f$. Suppose $\operatorname{Hom}_\Lambda(f, M) = 0$. Then $\operatorname{Hom}_\Lambda(f, M)$ is neither a monomorphism nor an epimorphism, so applying Lemma 1.4 we get $\operatorname{Hom}_\Lambda(X, M) \neq 0$ and $\operatorname{Hom}_\Lambda(M, D\operatorname{Tr} X) \neq 0$. It then follows from Lemma 1.3 that there is a short path $M_1 \to V \to M_2$ with M_1 and M_2 indecomposable summands of M and where $V = X$ or V is an indecomposable direct summand in the middle term of an almost split sequence $0 \to D\operatorname{Tr} X \to E \to X \to 0$. By our assumption on $\mathcal{C}$, we conclude that V is in $\mathcal{C}$. Clearly then also X lies in $\mathcal{C}$. Consider the canonical exact sequence

$$0 \to \operatorname{Im} f \to Q \xrightarrow{g} X \to 0.$$

Note that $\operatorname{Im} f \subset IQ$, because $I = \operatorname{ann} M$ and $\operatorname{Hom}_\Lambda(f, M) = 0$. On the other hand, since X belongs to $\mathcal{C}$ and $I = \operatorname{ann}\mathcal{C}$, we have $IX = 0$. Hence $IQ \subset \operatorname{Ker} g = \operatorname{Im} f$, and so $\operatorname{Im} f = IQ$. Consequently, $X = Q/IQ$. Finally, observe that $\mathcal{C}$ is a family of stable tubes of $\Gamma_{\Lambda/I}$ and Q/IQ is an indecomposable projective Λ/I-module, and so $X = Q/IQ$ cannot lie in $\mathcal{C}$, a contradiction. Therefore we have $\operatorname{Hom}_\Lambda(f, M) \neq 0$. □

We now have the following consequence.

Corollary 1.6. *Let $\mathcal{C}$ be a sincere family of stable tubes of Γ_Λ without external short paths. Then $\mathcal{C}$ is faithful.*

Proof. We know from Lemma 1.1 that there exists a sincere module M in $\operatorname{add}\mathcal{C}$ with $\operatorname{ann} M = \operatorname{ann}\mathcal{C}$. Suppose $I = \operatorname{ann} M \neq 0$. Write $1 = e_1 + \cdots + e_n$, where the e_i are primitive orthogonal idempotents. Then there are some i and j with $e_i I e_j \neq 0$. Choose $x \in I$ such that $e_i x e_j \neq 0$. Clearly, then x induces a nonzero morphism $f\colon P \to Q$, where $P = \Lambda e_i$ and $Q = \Lambda e_j$. On the other hand, it follows from our choice of f that $\operatorname{Hom}_\Lambda(f, M) = 0$, a contradiction to Proposition 1.5. Therefore $\mathcal{C}$ is faithful. □

Also when we have a family of stable tubes which is not sincere, we get useful information.

Proposition 1.7. *Let Λ be a basic artin algebra and $\mathcal{C}$ a family of stable tubes of Γ_Λ without external short paths. Write $\Lambda = P \oplus Q$, where the simple summands of $P/\operatorname{rad} P$ are exactly the simple composition factors of modules in $\mathcal{C}$. Denote by $\tau_Q(\Lambda)$ the ideal of Λ generated by the images of all morphisms from Q to Λ. Then $\operatorname{End}_\Lambda(P)^{\mathrm{op}} \simeq \Lambda/\tau_Q(\Lambda)$ and $\tau_Q(\Lambda) = \operatorname{ann}\mathcal{C}$.*

Proof. By Lemma 1.1 there is a module M in $\operatorname{add}\mathcal{C}$ with $\operatorname{ann} M = \operatorname{ann}\mathcal{C}$. Consider the natural functor $F\colon \operatorname{add} P \to \operatorname{add}\Lambda/\tau_Q(\Lambda)$ which assigns to each object P' of $\operatorname{add} P$ the object $F(P') = P'/\tau_Q(P')$. Let $f\colon P_1 \to P_2$ be a nonzero morphism with P_1 and P_2 indecomposable direct summands of P. We claim that $F(f) \neq 0$. Indeed, if $F(f) = 0$, then the image of f is contained in $\tau_Q(P_2)$, and we have that f is a composition of morphisms $P_1 \to Q' \to P_2$ with Q' from $\operatorname{add} Q$. Since $\operatorname{Hom}_\Lambda(Q, M) = 0$, the composition $\operatorname{Hom}_\Lambda(P_2, M) \to \operatorname{Hom}_\Lambda(Q, M) \to \operatorname{Hom}_\Lambda(P_1, M)$ is zero. Hence $\operatorname{Hom}_\Lambda(f, M) = 0$. On the other hand, $\operatorname{Hom}_\Lambda(P_1, M) \neq 0$ and $\operatorname{Hom}_\Lambda(P_2, M) \neq 0$ from our choice of P and since $\operatorname{ann} M = \operatorname{ann}\mathcal{C}$. For, since M is a faithful $\Lambda/\operatorname{ann}\mathcal{C}$-module, we have $P_i/(\operatorname{ann}\mathcal{C})P_i \hookrightarrow M^r$ for some r, and hence we have a nonzero map $P_i \to M$, for any indecomposable summand P_i of P. This leads to a contradiction to Proposition 1.5. Therefore, the functor F is faithful. Now obviously F is full and dense, hence this shows that F is an equivalence of categories, and consequently, $\operatorname{End}_\Lambda(P)^{\mathrm{op}} \simeq \Lambda/\tau_Q(\Lambda)$.

It is also clear that $\tau_Q(\Lambda)$ is contained in $\operatorname{ann} M$. Moreover, $\mathcal{C}$ is a sincere family of stable tubes of $\Gamma_{\Lambda/\tau_Q(\Lambda)}$ without external short paths in $\operatorname{mod} \Lambda/\tau_Q(\Lambda)$. Then by Corollary 1.6, $\mathcal{C}$ is a faithful family of tubes of $\Gamma_{\Lambda/\tau_Q(\Lambda)}$, and consequently $\tau_Q(A) = \operatorname{ann} M$. This finishes the proof. $\square$

2. Shortening of paths

In this section we prove that if a faithful family of stable tubes $\mathcal{C}$ has an external path, then it has an external short path.

If we are given a path between indecomposable modules $X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots \to X_{n-1} \xrightarrow{f_n} X_n = Y$ with $n > 2$, there are sufficient conditions for being able to find a shorter path from X to Y, so that we eventually get a short path [HRS1]. An important sufficient condition is here that if C is indecomposable and $\operatorname{Hom}_\Lambda(X, C) \neq 0$, then $\operatorname{id}_\Lambda C \leq 1$. The relevant proofs are given in [HRS1] (see also [ARS]). We shall need a slight variation of these results. We start with X being in our family $\mathcal{C}$, and assume that $X_1, \ldots, X_{n-1}$ are not in $\mathcal{C}$. Then we also want that the shortened path has the corresponding property.

We start by giving a result on modules of projective or injective dimension at most one, putting us in the position of being able to shorten paths.

Lemma 2.1. *Let $\mathcal{C}$ be a faithful family of stable tubes without external short paths and let X be an indecomposable Λ-module.*

(a) *If $\operatorname{Hom}_\Lambda(X, N) \neq 0$ for some module N from $\mathcal{C}$ then $\operatorname{pd}_\Lambda X \leq 1$.*
(b) *If $\operatorname{Hom}_\Lambda(N, X) \neq 0$ for some module N from $\mathcal{C}$ then $\operatorname{id}_\Lambda X \leq 1$.*

In particular, $\operatorname{pd}_\Lambda Z = 1 = \operatorname{id}_\Lambda Z$ for any module Z from $\mathcal{C}$.

Proof. We know by Lemma 1.1 that $\operatorname{add} \mathcal{C}$ contains a faithful module M, which is then also sincere. We shall only prove (a) because the proof of (b) is dual. Suppose $\operatorname{Hom}_\Lambda(X, N) \neq 0$ for some module N from $\mathcal{C}$ and assume $\operatorname{pd}_\Lambda X \geq 2$. Then we have $\operatorname{Hom}_\Lambda(D\Lambda, D\operatorname{Tr} X) \neq 0$. We have two cases to consider.

Assume first that X belongs to $\mathcal{C}$. There exists an indecomposable injective module I such that $\operatorname{Hom}_\Lambda(I, D\operatorname{Tr} X) \neq 0$. Moreover, since M is sincere, we have also $\operatorname{Hom}_\Lambda(M, I) \neq 0$. We then have a path $M' \to I \to D\operatorname{Tr} X$ with M' an indecomposable summand of M and $D\operatorname{Tr} X$ in $\mathcal{C}$, and I not in $\mathcal{C}$, which contradicts the fact that $\mathcal{C}$ has no external short paths.

Assume X does not belong to $\mathcal{C}$. Invoking the fact that M is faithful we get an epimorphism $M^r \to D(\Lambda)$ for some positive integer r. Since $\operatorname{Hom}_\Lambda(D\Lambda, D\operatorname{Tr} X) \neq 0$ we obtain $\operatorname{Hom}_\Lambda(M, D\operatorname{Tr} X) \neq 0$. Hence we get that $\operatorname{rad}(M, D\operatorname{Tr} X) \neq 0$ and $\operatorname{rad}(X, N) \neq 0$. Applying Lemma 1.3 we conclude that there is a path $M' \to V \to N$ in $\operatorname{mod} \Lambda$ where M' is an indecomposable summand of M and $V = X$ or V is an indecomposable direct summand of the middle term of an almost split sequence $0 \to D\operatorname{Tr} X \to E \to X \to 0$. Clearly then V does not belong to $\mathcal{C}$, and we have a contradiction. Therefore, $\operatorname{pd}_\Lambda X \leq 1$. $\square$

We now give the results on shortening paths.

Lemma 2.2. *Let $\mathcal{C}$ be a family of stable tubes in Γ_Λ with no external short paths. Let M and N be indecomposable Λ-modules with $\operatorname{Hom}_\Lambda(M, N) = 0$ and M in $\mathcal{C}$.*

Let $M \xrightarrow{f} X \xrightarrow{g} N$ *be a path in* $\operatorname{mod}\Lambda$ *with* $X \notin \mathcal{C}$ *and* X *having smallest possible length. Then* $\operatorname{Ker} g$ *is indecomposable.*

Proof. We follow the proof of the corresponding result of [HRS1] (see [ARS, p. 323]), and only explain which changes one must do. Let $K = \operatorname{Ker} g$ and write $K = K_1 \oplus K_2$ where K_1 is indecomposable. We have a pushout diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & K_1 \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0 \end{tikzcd}$$

where $Z = \operatorname{Im} g$. It is shown that there is an indecomposable summand Y_1 of Y such that the composition map $M \to K \to K_1 \to Y \to Y_1$ is not zero (and we also have a nonzero map $Y_1 \to Y \to Z \to N$). If $K_2 \neq 0$, a path $M \to Y_1 \to N$ is obtained with $l(Y_1) < l(X)$. We note that the composition map $M \to Y_1$ factors as $M \to X \to Y_1$. Since $M \in \mathcal{C}$ and $X \notin \mathcal{C}$, and and there are no external short paths for $\mathcal{C}$, we also have $Y_1 \notin \mathcal{C}$. Hence we get a contradiction. □

Proposition 2.3. *Let* $\mathcal{C}$ *be a faithful family of stable tubes in* Γ_Λ *with no external short paths. Assume that there is a path* $M \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} N$ *in* $\operatorname{mod}\Lambda$ *with* $M \in \mathcal{C}$ *and* X *and* Y *not in* $\mathcal{C}$*. Then there is a path* $M \to Z \to N$ *in* $\operatorname{mod}\Lambda$ *with* Z *not in* $\mathcal{C}$*.*

Proof. We follow closely the proof of the corresponding result in [HRS1] (see [ARS, p. 324]), and only explain the differences. We start with a path with $l(X) + l(Y)$ smallest possible.

A crucial property is that if $\operatorname{Hom}_\Lambda(M, K) \neq 0$ for some indecomposable module K, then $\operatorname{id}_\Lambda K \leq 1$, which in our case follows from Lemma 2.1. The second point is that for the indecomposable modules U_i replacing X or Y in considered paths from M to N there is a path $M \to X \to U_i$. Since $X \notin \mathcal{C}$ and $\mathcal{C}$ is without external short paths, it also follows that $U_i \notin \mathcal{C}$. The rest of the proof is the same as in [HRS1]. □

We can now give the main result in this section.

Theorem 2.4. *Let* $\mathcal{C}$ *be a sincere family of stable tubes without external short paths. Then* $\mathcal{C}$ *has no external paths.*

Proof. Assume that $X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots \to X_{n-1} \xrightarrow{f_n} X_n = Y$ with $n \geq 3$ and X and Y in $\mathcal{C}$ is an external path of $\mathcal{C}$, and with n smallest possible. Applying Proposition 2.3 we can shorten the path to finally get the contradiction that $\mathcal{C}$ has no external short paths. □

3. Reduction to concealed canonical algebras

Assume now that Λ is an indecomposable finite dimensional algebra over an algebraically closed field k. The aim of this section is to complete the proof of the following main result. We shall use the results of the previous sections, together with [HR], [LS].

Theorem 3.1. *Let* Λ *be an indecomposable finite dimensional algebra over an algebraically closed field* k*. Assume that the AR-quiver of* Λ *has a sincere family* $\mathcal{C}$ *of stable tubes without external short paths. Then* Λ *is concealed canonical.*

Proof. Denote by $\mathcal{T}$ the additive category generated by the indecomposable modules $X \notin \mathcal{C}$ for which there is some path $C \to \cdots \to X$ with $C \in \mathcal{C}$. Denote by $\mathcal{F}$ the additive category generated by the remaining indecomposable modules. Then clearly $(\mathcal{T}, \mathcal{F})$ is a split torsion pair, since we know by Theorem 2.4 that the modules in $\mathcal{C}$ do not lie on any external path. Since $\mathcal{C}$ is sincere, the projective modules are in $\mathcal{F}$ and the injective modules are in $\mathcal{T}$. For C indecomposable in $\mathcal{F}$ we clearly have $\operatorname{Hom}_\Lambda(D\Lambda, D\operatorname{Tr} C) = 0$ and hence $\operatorname{pd}_\Lambda C \leq 1$. Similarly, $\operatorname{id}_\Lambda C \leq 1$ for C in $\mathcal{T}$ and for $C \in \mathcal{C}$. It follows that all predecessors of indecomposable projective modules have projective dimension at most one, so that Λ is quasitilted by [HRS2, II Theorem 1.14].

As in [HRS2, II Theorem 2.3] (see also remark in [Ha2, p.62]), we can "tilt" with respect to the split torsion pair $(\mathcal{T}, \mathcal{F})$ to get a hereditary abelian k-category $\mathcal{H}$ derived equivalent to $\operatorname{mod}\Lambda$, with torsion pair $(\mathcal{F}[1], \mathcal{T})$. The objects in $\mathcal{H}$ are by definition complexes $X^\bullet : \cdots \to X_{-1} \to X_0 \to X_1 \to \cdots$ in $\operatorname{mod}\Lambda$ with $H^{-1}(X^\bullet) \in \mathcal{F}$ and $H^0(X^\bullet) \in \mathcal{T}$ [HRS2, I Corollary 2.2]. We know that each such complex $X^\bullet$ is quasiisomorphic to a complex $I^\bullet : 0 \to I_{-1} \xrightarrow{f} I_0 \xrightarrow{g} I_1 \to I_2 \to \cdots \to I_n \to 0$ of injective Λ-modules (see [Ha1]).

Let now S be a quasisimple object of smallest length in $\mathcal{C}$, and consider the corresponding object $S[1] \in \mathcal{C}[1] \subset \mathcal{F}[1]$ of $\mathcal{H}$. So $S[1]$ is a stalk complex concentrated in degree -1. Let $I^\bullet$ be an indecomposable object in $\mathcal{H}$ with $\operatorname{Hom}_\mathcal{H}(S[1], I^\bullet) \neq 0$. We can assume that $I^\bullet$ has no split exact direct summand. We then have a nonzero map $h\colon S \to H^{-1}(I^\bullet) = \operatorname{Ker} f \in \mathcal{F}$ in $\operatorname{mod}\Lambda$. We can write $\operatorname{Ker} f = K \oplus L$ with K indecomposable and $\operatorname{Hom}_\Lambda(S, K) \neq 0$. Then we have $K \in \mathcal{C}$ and $\operatorname{id}_\Lambda K \leq 1$. The minimal injective copresentation $0 \to K \oplus L \to I_{-1} \xrightarrow{f} I_0$ is a direct sum of minimal injective copresentations $0 \to K \to I'_{-1} \xrightarrow{f'} I'_0 \to 0$ and $0 \to L \to I''_{-1} \to I''_0$. This shows that the complex $I'_{-1} \xrightarrow{f'} I'_0$ is a nonzero summand of $I^\bullet$, and hence $I^\bullet$ is quasiisomorphic to $K[1]$.

If $S[1]$ was not a simple object in $\mathcal{H}$, we would have a proper epimorphism $t\colon S[1] \to K[1]$, with $K \in \mathcal{C}$, which is clearly impossible. Since $\mathcal{H}$ is a connected hereditary abelian k-category, with k-algebraically closed, and finite dimensional homomorphism and extension spaces, it follows from [HR] that $\mathcal{H}$ is derived equivalent to the module category of a canonical algebra, since $\mathcal{H}$ has a simple object. Hence Λ is a quasitilted algebra derived equivalent to a canonical algebra.

We now want to show that Λ is concealed canonical. It follows from the structure theorem of quasitilted algebras of canonical type [LS, Theorem 3.4] that Λ is a semiregular branch enlargement of a concealed canonical algebra C and Γ_Λ has a trisection $\Gamma_\Lambda = \mathcal{P} \vee \mathcal{T} \vee \mathcal{Q}$ where $\mathcal{T}$ is a semiregular family of tubes separating $\mathcal{P}$ from $\mathcal{Q}$ (in the sense of [Ri]), obtained from a separating $\mathbb{P}_1(K)$-family of stable tubes in Γ_C by ray and coray insertions. Here, $\mathcal{T}$ semiregular means that no tube in $\mathcal{T}$ has both a projective and an injective module. If $\mathcal{T}$ is a family of stable tubes then $\Lambda = C$, and consequently Λ is a concealed canonical algebra. Assume $\mathcal{T}$ contains an injective module. Since $\mathcal{T}$ is a family of pairwise orthogonal tubes and the family $\mathcal{C}$ is sincere, we conclude that $\mathcal{P}$ contains a stable tube $\mathcal{C}_1$ from $\mathcal{C}$. Then it follows from [LS], [M], [Ri] that C is a tame concealed algebra and there is a tubular algebra C_1, being a tubular coextension of C, such that

$$\Gamma_{C_1} = \mathcal{P}_0 \vee \mathcal{T}_0 \vee (\vee_{q \in \mathbb{Q}^+} \mathcal{T}_q) \vee \mathcal{T}_\infty \vee \mathcal{Q}_\infty,$$

where $\mathcal{P}_0$ is a preprojective component, $\mathcal{T}_0$ is a $\mathbb{P}_1(K)$-family of ray tubes, $\mathcal{T}_\infty$ is a $\mathbb{P}_1(K)$-family of coray tubes, $\mathcal{Q}_\infty$ is a preinjective component, and, for each $q \in \mathbb{Q}^+$, $\mathcal{T}_q$ is a $\mathbb{P}_1(K)$-family of stable tubes,

$$\mathcal{P} = \mathcal{P}_0 \vee \mathcal{T}_0 \vee (\vee_{q\in\mathbb{Q}^+} \mathcal{T}_q)$$

and all coray tubes from $\mathcal{T}$ are in the family $\mathcal{T}_\infty$. Dually, if $\mathcal{T}$ contains a projective module then $\mathcal{Q}$ contains a stable tube $\mathcal{C}_2$ from $\mathcal{C}$, C is a tame concealed algebra and there is a tubular algebra C_2, being a tubular extension of C, such that

$$\Gamma_{C_2} = \mathcal{P}'_0 \vee \mathcal{T}'_0 \vee (\vee_{q\in\mathbb{Q}^+} \mathcal{T}'_q) \vee \mathcal{T}'_\infty \vee \mathcal{Q}'_\infty$$

where $\mathcal{P}'_0$ is a preprojective component, $\mathcal{T}'_0$ is a $\mathbb{P}_1(K)$-family of ray tubes, $\mathcal{T}'_\infty$ is a $\mathbb{P}_1(K)$-family of coray tubes, $\mathcal{Q}'_\infty$ is a preinjective component, and, for each $q \in \mathbb{Q}^+$, $\mathcal{T}'_q$ is a $\mathbb{P}_1(K)$-family of stable tubes,

$$\mathcal{Q} = (\vee_{q\in\mathbb{Q}^+} \mathcal{T}'_q) \vee \mathcal{T}'_\infty \vee \mathcal{Q}'_\infty,$$

and all ray tubes from $\mathcal{T}$ are in the family $\mathcal{T}'_0$. Since any tubular algebra is concealed canonical, in order to prove the theorem, it is enough to show that $\mathcal{T}$ has not both a projective module and an injective module. Suppose it is not the case. Let Γ be a tube from $\mathcal{T}$ containing a projective module, and so not from the family $\mathcal{C}$. Then there is a stable tube Ω in Γ_C such that Γ is obtained from Ω by ray insertions. Note that Ω is a stable tube in the family $\mathcal{T}_\infty$. Invoking now the structure of the module category of a tubular algebra [Ri, 5.2] we conclude that there exist indecomposable modules $X \epsilon\, \mathcal{C}_1$, Y_1, $Y_2 \,\epsilon\, \Omega$ and Z in $\mathcal{C}_2$ such that $\mathrm{Hom}_\Lambda(X, Y_1) \neq 0$ and $\mathrm{Hom}_\Lambda(Y_2, Z) \neq 0$. Since Ω is a stable tube of Γ_C there exists some Y in Ω and a monomorphism $Y_1 \to Y$ and an epimorphism $Y \to Y_2$ (see Lemma 1.2). Hence, we get a short path $X \to Y \to Z$ with X and Z from the family $\mathcal{C}$. Since Y lies in a nonstable tube Γ of Γ_Λ, Y does not belong to $\mathcal{C}$, and so the above path is external, a contradiction. □

We have the following consequence.

Corollary 3.2. *Let Λ be an indecomposable finite dimensional algebra over an algebraically closed field k. Assume that Λ has a sincere stable tube $\mathcal{C}$ in its AR-quiver, such that no module in $\mathcal{C}$ lies on an external short cycle. Then Λ is concealed canonical.*

Proof. By Theorem 3.1 it is enough to show that there is no external short path for $\mathcal{C}$. So assume to the contrary that there is an external short path $C \to X \to B$ with B and C in $\mathcal{C}$ and X not in $\mathcal{C}$. By Lemma 1.2 there is some M in $\mathcal{C}$ such that we have a monomorphism $B \to M$ and an epimorphism $M \to C$. Hence we obtain an external short cycle $M \to X \to M$, which gives a contradiction. □

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