Schemes of Tori and the Structure of Tame Restricted Lie Algebras

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0. Introduction

Much of the recent progress in the representation theory of infinitesimal group schemes rests on the application of algebro-geometric techniques related to the notion of cohomological support varieties (cf. [7, 10, 11, 12]). The noncohomological characterization of these varieties via the so-called *rank varieties* (see [22, 23]) involves schemes of additive subgroups that are the infinitesimal counterparts of the elementary abelian groups. In this note we introduce another geometric tool by considering schemes of tori of restricted Lie algebras. Our interest in these derives from the study of infinitesimal groups of tame representation type, whose determination [13] necessitates the results to be presented in sections 4 and 5 as well as techniques from abstract representation theory.

In contrast to the classical case of complex Lie algebras, the information on the structure of a restricted Lie algebra that can be extracted from its root systems is highly sensitive to the choice of the underlying maximal torus. Schemes of tori obviate this defect by allowing us to study algebraic families of root spaces. Accordingly, these schemes may also shed new light on various aspects of the structure theory of restricted Lie algebras. We intend to pursue these questions in a forthcoming paper [14], and focus here on first applications within representation theory.

Throughout, we will be working over an algebraically closed field k of characteristic $p > 0$. All k-Lie algebras are assumed to have finite dimension. To a pair (T, L) of restricted Lie algebras $(T, [p])$ and $(L, [p])$, we associate the scheme $\mathcal{T}_L : M_k \longrightarrow \mathrm{Ens}$ that assigns to every commutative k-algebra $R \in M_k$ the set

$$\mathcal{T}_L(R) := \mathrm{Lie}_{p-k}(T, L \otimes_k R)$$

of homomorphisms of restricted k-Lie algebras from T to $L \otimes_k R$. We recall that $L \otimes_k R$ obtains the structure of a restricted Lie algebra over R via

$$[x \otimes r, y \otimes s] := [x, y] \otimes rs \quad ; \quad (x \otimes r)^{[p]} := x^{[p]} \otimes r^p \quad \forall\, x, y \in L,\ r, s \in R.$$

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The reader is referred to [21] concerning basic facts of restricted Lie algebras.

Our paper is organized as follows. In §1 we show that the scheme $\mathcal{T}_L$ is smooth whenever T is a torus. In that case a certain open subscheme $\tilde{\mathcal{T}}_L$ of $\mathcal{T}_L$ is actually affine. After laying more groundwork by computing tangent spaces, we develop in §2 our main tool by collecting technical results on algebraic families of Lie algebras that are parametrized by the rational points of irreducible components of $\tilde{\mathcal{T}}_L$. The last three sections are concerned with applications pertaining to the structure of those restricted Lie algebras that admit a self-centralizing torus. In this case, our scheme $\tilde{\mathcal{T}}_L$ enables us to relate properties of root systems to those of the aforementioned rank varieties (cf. Theorem 3.5). In order to render the presentation of §3 consistent with the remainder of the paper, we have avoided the language of vector bundles. While the algebraic point of view has shortened the paper somewhat, it does occasionally veil the conceptual sources of some of our results, most notably Lemma 3.4. The last two results of §3 illustrate immediate applications of our methods to questions from representation theory. More specifically, we provide a noncohomological proof for the supersolvability of representation-finite infinitesimal groups (cf. [11, (2.1)]). Further consequences will be presented in another article [14].

Tame infinitesimal groups are known to have rank varieties of dimension ≤ 2 (see [19, 25]). In the investigation of their Lie algebras one essentially has to consider two classes: Lie algebras admitting a self-centralizing maximal torus, and those whose Cartan subalgebras generically have a one-dimensional rank variety. In §4 we address the former case and apply algebraic families of tori to classify all restricted Lie algebras with two-dimensional rank variety and self-centralizing torus. The relevant result, Theorem 4.2, is employed in §5 to study algebras of tame representation type. It turns out that $s\ell(2)$ is essentially the only restricted Lie algebra with self-centralizing torus, whose restricted enveloping algebra admits a tame principal block. By the same token, tameness occurs precisely when the heart of the projective cover of the trivial module is decomposable.

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1. Schemes of Tori

In this section we establish important properties of $\mathcal{T}_L$ and a certain open subscheme. Setting $n := \dim_k L$ and $r := \dim_k T$ we observe that $\mathcal{T}_L$ is, as a closed subscheme of the affine scheme $R \mapsto R^{r\times n}$, an affine algebraic k-scheme, whose function algebra we denote by A.

Let $u(T)$ be the restricted enveloping algebra of T. By definition, the $u(T)$-modules are precisely the restricted T-modules, i.e., those where $t^{[p]}m = t^p m$ for every $t \in T, m \in M$. Given a restricted T-module M, we denote by $H^n_*(T, M) := H^n(u(T), M)$ the n^{th} *restricted cohomology group* of T with coefficients in M.

Theorem 1.1 *Let $(T, [p])$ be a restricted Lie algebra. Then the following statements are equivalent:*

(1) *T is a torus.*

(2) *The affine algebraic scheme $\mathcal{S}_L$ is smooth for every p-subalgebra $S \subset T$ and every restricted Lie algebra $(L, [p])$.*

Proof. (1) ⇒ (2). Let $(L, [p])$ be a restricted Lie algebra. Since every p-subalgebra of T is a torus, we only have to verify that $\mathcal{T}_L$ is smooth. According to [2, (I.§4 (4.6))] and [2, (I.§4 (5.11))] it suffices to show that for every finite dimensional commutative k-algebra R and every ideal $I \subset R$ with $I^2 = (0)$ the canonical map

$$\mathcal{T}_L(R) \longrightarrow \mathcal{T}_L(R/I)$$

is surjective. To that end we let $\pi : R \longrightarrow R/I$ denote the canonical projection, and consider the exact sequence

$$(0) \longrightarrow L \otimes_k I \longrightarrow L \otimes_k R \longrightarrow L \otimes_k R/I \longrightarrow (0)$$

of restricted k-Lie algebras. Since $I^2 = (0)$ the algebra $L \otimes_k I$ is *strongly abelian*, that is, abelian with trivial p-map. Let $f : T \longrightarrow L \otimes_k R/I$ be a homomorphism of restricted k-Lie algebras. Then $T' := f(T)$ is a torus, and there results an exact sequence

$$(0) \longrightarrow L \otimes_k I \longrightarrow (id_L \otimes \pi)^{-1}(T') \longrightarrow T' \longrightarrow (0)$$

of restricted Lie algebras. Since $u(T')$ is semisimple, the restricted cohomology group $H^2_*(T', L \otimes_k I)$ vanishes. We may now apply [16, (3.3)] to see that the above sequence splits. Let $\gamma : T' \longrightarrow L \otimes_k R$ be a splitting, and consider $\hat{f} := \gamma \circ f$. Then $\hat{f} \in \mathcal{T}_L(R)$, and $(id_L \otimes \pi) \circ \hat{f} = (id_L \otimes \pi) \circ \gamma \circ f = f$. Consequently, the map $\mathcal{T}_L(\pi)$ is surjective.

(2) ⇒ (1). Suppose there is $z \in T \setminus \{0\}$ such that $z^{[p]} = 0$. Let $S := kz$ be the one-dimensional strongly abelian Lie algebra, and consider the abelian p-unipotent Lie algebra $L := ka \oplus ka^{[p]}$. Then the natural transformation $\psi : \mathrm{Spec}_k(k[X,Y]/(X^p)) \longrightarrow \mathcal{S}_L$, sending $h \in \mathrm{Spec}_k(k[X,Y]/(X^p))(R)$ to the homomorphism $\psi(R)(h) \in \mathcal{S}_L(R)$ that is given by $\psi(R)(h)(z) := a \otimes h(\bar{X}) + a^{[p]} \otimes h(\bar{Y})$, is an isomorphism. Since the algebra $k[X,Y]/(X^p)$ is not reduced, the scheme $\mathcal{S}_L$ is not smooth, a contradiction. We may now apply [21, (II.3.10)] to see that T is a torus. □

Remark. The interested reader may compare Theorem 1.1 to the following result from SGA3: Let $\mathcal{S}$ be a multiplicative group of finite type, $\mathcal{G}$ a smooth, affine group scheme over $\mathcal{S}$. Given any group scheme $\mathcal{H}$ over $\mathcal{S}$, the functor $\mathcal{HOM}_{\mathcal{S}}(\mathcal{H}, \mathcal{G})$ is representable and smooth (cf. [3, (Exp. XI, no.4 (4.1))]).

For the remainder of this section, we fix a pair (T, L) of restricted Lie algebras. Let B be a finitely generated integral domain over k. We consider a morphism $j : \mathrm{Spec}_k(B) \longrightarrow \mathcal{T}_L$ that associates to every algebra homomorphism $x : B \longrightarrow R$ the homomorphism

$$j(R)(x) : T \longrightarrow L \otimes_k R.$$

If T is a torus, we let $\mathrm{Tor}(T)$ be the set of subtori of T. Thanks to [21, (II.3.6)] this set is finite. Given a rational point $x \in \mathrm{Spec}_k(B)(k)$, the kernel of the homomorphism $j(k)(x) : T \longrightarrow L$ belongs to $\mathrm{Tor}(T)$.

Theorem 1.2 *Let T be a torus, B a finitely generated integral domain over k. Then the following statements hold:*

(1) *If $j : \mathrm{Spec}_k(B) \longrightarrow \mathcal{T}_L$ is a morphism, then the map $\mathrm{Spec}_k(B)(k) \longrightarrow \mathrm{Tor}(T) \;\; ; \;\; x \mapsto \ker j(k)(x)$ is constant.*

(2) *If $\mu : T \otimes_k B \longrightarrow L \otimes_k B$ is a homomorphism of restricted Lie algebras over B, then there exists a subtorus $T' \subset T$ such that $\ker \mu = T' \otimes_k B$.*

Proof. (1). From a faithful, finite dimensional representation of L, we obtain an embedding $\eta : L \hookrightarrow g\ell_n(k)$ of restricted Lie algebras. Thus, $\eta \otimes id_B$ defines an embedding $L \otimes_k B \hookrightarrow g\ell_n(B)$ of restricted B-Lie algebras. The representation $\varrho := (\eta \otimes id_B) \circ j(B)(id_B)$ endows $M := B^n$ with the structure of a restricted T-module, on which T acts via B-linear transformations. Consequently, there exists a weight space decomposition (cf. [2, (II.§2 (2.5))])

$$M = \bigoplus_{\lambda \in \Phi} M_\lambda,$$

where Φ is a finite set of k-algebra homomorphisms $u(T) \longrightarrow k$, and each weight space $M_\lambda := \{m \in M \; ; \; a \cdot m = \lambda(a)m \quad \forall \; a \in u(T), \, m \in M\}$ is a finitely generated projective B-module. The general characterization of projectives (cf. [4, (19.2)]) ensures the existence of elements $b_1, \ldots, b_q \in B$ such that

(a) there exist $x_1, \ldots, x_q$ with $1 = \sum_{i=1}^q x_i b_i$, and

(b) the localization $(M_\lambda)_{b_i}$ is a free B_{b_i}-module for every $\lambda \in \Phi$.

Setting $\mathcal{U}_i := \mathrm{Spec}_k(B_{b_i}) \subset \mathrm{Spec}_k(B)$ we propose to show that the map $x \mapsto \ker j(k)(x)$ is constant on $\mathcal{U}_i(k)$. We write $B_i := B_{b_i}$ and consider the canonical map $\zeta_i : B \longrightarrow B_i$. Owing to (b) there exists an inner automorphism $\sigma_i : g\ell_n(B_i) \longrightarrow g\ell_n(B_i)$ such that the image of $\varrho_i := \sigma_i \circ (id_{g\ell_n(k)} \otimes \zeta_i) \circ \varrho = \sigma_i \circ (\eta \otimes id_{B_i}) \circ j(B_i)(\zeta_i)$ is contained in the restricted Lie algebra $D_n(B_i)$ of diagonal $(n \times n)$-matrices over B_i.

Let $\mathrm{Lie}_p(T, D_n)$ be the scheme $\mathcal{T}_{D_n(k)}$, i.e.,

$$\mathrm{Lie}_p(T, D_n)(R) := \mathrm{Lie}_{p-k}(T, D_n(R)) \qquad \forall \; R \in M_k.$$

The homomorphism $\varrho_i : T \longrightarrow D_n(B_i)$ of restricted Lie algebras induces a morphism

$$f_i : \mathrm{Spec}_k(B_i) \longrightarrow \mathrm{Lie}_p(T, D_n) \;\; ; \;\; f_i(R)(x) := D_n(x) \circ \varrho_i$$

of affine algebraic schemes. By choosing toral bases of T and $D_n(k)$ one readily sees that the set $\mathrm{Lie}_p(T, D_n)(k)$ is finite. Since $\mathcal{U}_i(k)$ is, as an open set of the irreducible variety $\mathrm{Spec}_k(B)(k)$, irreducible, it follows that $f_i(k)$ is constant.

Given a rational point $x \in \mathcal{U}_i(k)$, the naturality of j implies

$$\ker f_i(k)(x) = \ker D_n(x) \circ \varrho_i = \ker[(id_L \otimes x) \circ j(B_i)(\zeta_i)] = \ker j(k)(x \circ \zeta_i).$$

As a result, the map $x \mapsto \ker j(k)(x)$ is constant on $\mathcal{U}_i(k)$.

By property (a) the $\mathcal{U}_i(k)$ form an open cover of $\mathrm{Spec}_k(B)(k)$. Since $\mathrm{Spec}_k(B)(k)$ is irreducible, the desired result follows.

(2). We consider the homomorphism $\nu : T \longrightarrow L \otimes_k B$ of restricted Lie algebras that is given by

$$\nu(t) := \mu(t \otimes 1) \quad \forall\ t \in T.$$

The homomorphism ν induces a morphism

$$j : \mathrm{Spec}_k(B) \longrightarrow \mathcal{T}_L \ \ ; \ \ j(R)(x) := (id_L \otimes x) \circ \nu.$$

Owing to (1) there exists a subtorus $T' \subset T$ such that $\ker[(id_L \otimes x) \circ \nu] = T'$ for every $x \in \mathrm{Spec}_k(B)(k)$. An application of Hilbert's Nullstellensatz now shows that $\mu(t \otimes 1) = 0$ for every $t \in T'$, whence $T' \otimes_k B \subset \ker \mu$. To verify the reverse inclusion, we pick a basis $\{t_1, \ldots, t_n\}$ of T such that $\{t_{m+1}, \ldots, t_n\}$ is a basis of T'. Given $v = \sum_{i=1}^n t_i \otimes b_i \in \ker \mu$, and $x \in \mathrm{Spec}_k(B)(k)$, we obtain

$$0 = [(id \otimes x) \circ \mu](v) = [(id \otimes x) \circ \nu](\sum_{i=1}^m x(b_i)t_i).$$

Hence $\sum_{i=1}^m x(b_i)t_i \in T'$ and $x(b_i) = 0$ for $1 \leq i \leq m$. Thanks to Hilbert's Nullstellensatz the elements $b_1, \ldots, b_m$ vanish, so that $v \in T' \otimes_k B$. $\square$

Let R be a commutative k-algebra. The natural embedding

$$\iota_R : L \hookrightarrow L \otimes_k R \ \ ; \ \ x \mapsto x \otimes 1$$

is a homomorphism of restricted Lie algebras. The corresponding restriction map

$$\mathrm{Hom}_{R-p-\mathrm{Lie}}(T \otimes_k R, L \otimes_k R) \longrightarrow \mathcal{T}_L(R) \ \ ; \ \ x \mapsto x \circ \iota_R$$

is readily seen to be an isomorphism between the set $\mathrm{Hom}_{R-p-\mathrm{Lie}}(T \otimes_k R, L \otimes_k R)$ of homomorphisms of restricted R-Lie algebras and $\mathcal{T}_L(R)$. We will henceforth identify the schemes $\mathcal{HOM}_{p-\mathrm{Lie}}(T, L)$ and $\mathcal{T}_L$ via the natural equivalence given above.

Let $\tilde{\mathcal{T}}_L : M_k \longrightarrow \mathrm{Ens}$ be the subscheme of $\mathcal{T}_L$ defined by

$$\tilde{\mathcal{T}}_L(R) := \{\varphi \in \mathrm{Hom}_{R-p-\mathrm{Lie}}(T \otimes_k R, L \otimes_k R) \ \ ; \ \ \varphi : T \otimes_k R \longrightarrow L \otimes_k R \text{ is split injective}\}.$$

Corollary 1.3 *Let T be a torus. Then $\tilde{\mathcal{T}}_L$ is a open, smooth, affine algebraic subscheme of $\mathcal{T}_L$.*

Proof. To see that $\tilde{\mathcal{T}}_L \subset \mathcal{T}_L$ is open, we consider an element $\varphi : T \otimes_k R \longrightarrow L \otimes_k R$ of $\mathcal{T}_L(R)$ along with its dual map $\varphi^* : (L \otimes_k R)^* \longrightarrow (T \otimes_k R)^*$. Let $I \subset R$ be the annihilator of $\mathrm{coker}\,\varphi^*$. Given a commutative R-algebra S, the element φ_S belongs to $\tilde{\mathcal{T}}_L(S)$ if and only if $I \cdot S = S$. In view of [2, (I.§1 (3.6))] this shows that $\tilde{\mathcal{T}}_L$ is open.

Since $\mathcal{T}_L$ is an affine algebraic scheme there exists an isomorphism $j : \mathrm{Spec}_k(A) \longrightarrow \mathcal{T}_L$. An irreducible component $\mathcal{Y}_L \subset \mathcal{T}_L$ corresponds to a subscheme $\mathrm{Spec}_k(B_{\mathcal{Y}_L})$, defined by a finitely generated integral domain $B_{\mathcal{Y}_L}$. Let $\mathcal{I}_L \subset \mathcal{T}_L$ be the union of those irreducible components $\mathcal{Y}_L$ of $\mathcal{T}_L$ for which $\ker j(k)(x) = (0)$ for some $x \in \mathrm{Spec}_k(B_{\mathcal{Y}_L})(k)$. Since $\mathcal{T}_L$ is smooth, the subscheme $\mathcal{Y}_L$ is also a connected component of $\mathcal{T}_L$ (cf. [2, (I.§4, no.4)], [2, (I.§1 (5.6))], and [2, (I.§1 (2.10))]). Consequently, $\mathcal{I}_L$ is an open, smooth, affine algebraic subscheme of $\mathcal{T}_L$.

Directly from the definitions we obtain $\tilde{\mathcal{T}}_L(k) \subset \mathcal{I}_L(k)$. The reverse inclusion is an immediate consequence of (1.2(1)). Since $\tilde{\mathcal{T}}_L$ and $\mathcal{I}_L$ are open subschemes of $\mathcal{T}_L$, we may apply [17, (1.5)] to see that $\tilde{\mathcal{T}}_L = \mathcal{I}_L$ has the desired properties. □

We continue by computing the tangent space $T_\varphi(\mathcal{T}_L)$ of the scheme $\mathcal{T}_L$ at the rational point $\varphi : T \longrightarrow L$. Given such a point φ, we denote by $L^{(\varphi)}$ the restricted T-module induced by φ, i.e.,

$$t \cdot x := [\varphi(t), x] \quad \forall\ t \in T,\ x \in L.$$

Note that the subspace of T-invariants of $L^{(\varphi)}$ coincides with $C_L(\varphi(T))$, the *centralizer* of $\varphi(T)$ in L. For a restricted T-module M we denote by $\mathrm{Der}_p(T, M)$ the space of all *p-derivations* of T with coefficients in M. By definition, these are the k-linear maps $\zeta : T \longrightarrow M$ such that

$$\zeta([s,t]) = s \cdot \zeta(t) - t \cdot \zeta(s) \ \ ; \ \ \zeta(t^{[p]}) = t^{p-1} \cdot \zeta(t) \qquad \forall\ s, t \in T.$$

Each element $m \in M$ induces a p-derivation $ad\, m : T \longrightarrow M\ ;\ t \mapsto t \cdot m$. The subspace consisting of all these *inner derivations* will be denoted $\mathrm{Innder}(T, M)$. Thanks to [16, (2.1)] the factor space $\mathrm{Der}_p(T, M)/\mathrm{Innder}(T, M)$ is isomorphic to $H^1_*(T, M)$.

Proposition 1.4 *Let $\varphi \in \mathcal{T}_L(k)$ be a rational point.*

(1) *There is an isomorphism $T_\varphi(\mathcal{T}_L) \cong \mathrm{Der}_p(T, L^{(\varphi)})$ of k-vector spaces.*

(2) $\dim_k T_\varphi(\mathcal{T}_L) = \dim_k H^1_*(T, L^{(\varphi)}) + \dim_k L - \dim_k C_L(\varphi(T))$.

Proof. (1). Let $k[\tau]$ be the algebra of dual numbers,

$$\pi : k[\tau] \longrightarrow k \ \ ; \ \ a + b\tau \mapsto a$$

the canonical projection. It follows from [2, (II.§4(3.3))] that $T_\varphi(\mathcal{T}_L)$ corresponds bijectively to the set

$$D_\varphi := \{\lambda \in \mathcal{T}_L(k[\tau]) \ ; \ \mathcal{T}_L(\pi)(\lambda) = \varphi\}.$$

Moreover, the vector space structure of D_φ can be read off from any closed embedding $\mathcal{T}_L \hookrightarrow \mathrm{Spec}_k(B)$. Here we consider the embedding $\mathcal{T}_L \hookrightarrow \mathcal{HOM}(T, L)$, where $\mathcal{HOM}(T, L)(R) := \mathrm{Hom}_k(T, L \otimes_k R)$ is the space of k-linear maps from T to $L \otimes_k R$.

By definition, each element $\lambda \in D_\varphi$ is of the form $\lambda := \varphi + \lambda_1\tau$, where $\lambda_1 : T \longrightarrow L$ is k-linear. In view of the foregoing observations

$$\Omega : D_\varphi \longrightarrow \mathrm{Hom}_k(T, L) \ \ ; \ \ \lambda \mapsto \lambda_1$$

is an injective homomorphism of k-vector spaces. Direct computation shows that $\Omega(\lambda)$ is a derivation from T to $L^{(\varphi)}$. Given $t \in T$, we have, observing $\tau^2 = 0$,

$$\varphi(t^{[p]}) + \lambda_1(t^{[p]})\tau = \lambda(t^{[p]}) = (\varphi(t) + \lambda_1(t)\tau)^{[p]} = \varphi(t)^{[p]} + (ad(\varphi(t))^{p-1}(\lambda_1(t))\tau,$$

so that

$$\lambda_1(t^{[p]}) = t^{p-1} \cdot \lambda_1(t).$$

Consequently, $\Omega(\lambda) \in \mathrm{Der}_p(T, L^{(\varphi)})$.

(2). This follows directly from (1) and our remarks preceding (1.4). □

Corollary 1.5 *Let T be a torus, $\mathcal{X}_L \subset \tilde{\mathcal{T}}_L$ an irreducible component. Then we have*

$$\dim \mathcal{X}_L = \dim_k L - \dim_k C_L(\varphi(T))$$

for every $\varphi \in \mathcal{X}_L(k)$.

Proof. According to (1.3) the scheme $\tilde{\mathcal{T}}_L$ is a smooth, open subscheme of $\mathcal{T}_L$. Thanks to [2, (I.§4, no.4)], [2, (I.§1 (5.6))], and [2, (I.§1 (2.10))] $\mathcal{X}_L$ is also a connected component of $\tilde{\mathcal{T}}_L$. In particular, $\mathcal{X}_L$ is open and smooth, so that $\dim \mathcal{X}_L = \dim_k T_\varphi(\mathcal{X}_L) = \dim_k T_\varphi(\mathcal{T}_L)$ for all $\varphi \in \mathcal{X}_L(k)$. Since T is a torus, we have $H^1_*(T, L^{(\varphi)}) = (0)$, and the assertion now follows from (2) of (1.4). $\square$

2. Algebraic Families of Modules

Throughout this section we let A be a finitely generated, reduced k-algebra with associated scheme $\mathcal{X} := \operatorname{Spec}_k(A)$. Given a k-vector space V, we consider the free A-module $V \otimes_k A$. For an A-submodule $M \subset V \otimes_k A$, and $x \in \mathcal{X}(k)$ we denote by $M(x) := (id_V \otimes x)(M) \subset V$ the subspace of V obtained by specialization along x. If M is a direct summand of the A-module $V \otimes_k A$, then $M(x) \cong M \otimes_A k_x$, where k_x denotes the one-dimensional A-module afforded by x. We will be studying intersections $I_U(M) := \bigcap_{x\in U} M(x)$ for certain A-direct summands $M \subset V \otimes_k A$ and subsets $U \subset \mathcal{X}(k)$.

Definition. Let V be a finite dimensional vector space, $\mathcal{F}$ a finite set of irreducible subvarieties of V. A family $(W_i)_{i\in I}$ of subspaces of V is said to be *$\mathcal{F}$-regular* if every subspace W_i contains an element of $\mathcal{F}$.

Lemma 2.1 *Suppose that A is an integral domain, and let $\mathcal{F}$ be a finite set of irreducible subvarieties of the finite dimensional vector space V. If $P \subset V \otimes_k A$ is an A-submodule such that*

(a) *P is an A-direct summand of $V \otimes_k A$, and*
(b) *there exists a dense subset $U \subset \mathcal{X}(k)$ such that $(P(x))_{x\in U}$ is $\mathcal{F}$-regular,*

then the subspace $I_{\mathcal{X}(k)}(P) \subset V$ is $\mathcal{F}$-regular.

Proof. We write $\mathcal{F} = \{Y_1, \dots, Y_n\}$. In view of (a) there exists an A-submodule $Q \subset V \otimes_k A$ such that $V \otimes_k A = P \oplus Q$. Note that this implies $V = P(x) \oplus Q(x)$ for every $x \in \mathcal{X}(k)$. We consider an element $y \in Y_j \subset V \otimes_k k \cdot 1$ and write $y = p + q$ with $p \in P$ and $q \in Q$. Then $y \in P(x)$ if and only if $(id_V \otimes x)(q) = 0$. By choosing a basis $\{v_1, \dots, v_m\}$ of V and writing $q = \sum_{i=1}^m v_i \otimes a_i$, we see that the latter condition is equivalent to $x(a_i) = 0$ for $1 \le i \le m$. Consequently, the set $Z_j[y] := \{x \in \mathcal{X}(k) \ ; \ y \in P(x)\}$ is closed, and $Z_j := \{x \in \mathcal{X}(k) \ ; \ Y_j \subset P(x)\} = \bigcap_{y\in Y_j} Z_j[y]$ has the same property. Condition (b) implies $\mathcal{X}(k) = \bar{U} \subset \bigcup_{j=1}^n Z_j$. Since A is an integral domain, the variety $\mathcal{X}(k)$ is irreducible, and there exists $j_0 \in \{1, \dots, n\}$ such that $Z_{j_0} = \mathcal{X}(k)$. It follows that $Y_{j_0} \subset I_{\mathcal{X}(k)}(P)$. $\square$

Let $\mathcal{G}$ be an affine algebraic k-group, V a $\mathcal{G}$-module. We assume that $\mathcal{G}$ acts on A via algebra homomorphisms. Consider the $\mathcal{G}$-module $V \otimes_k A$ with the diagonal operation

$$g \cdot [(v \otimes_k r) \otimes_R (a \otimes_k s)] := g(v \otimes_k r) \otimes_R g(a \otimes_k s)$$

for $g \in \mathcal{G}(R)$, $r, s \in R$, $v \in V$, $a \in A$.

Lemma 2.2 *Let $P \subset V \otimes_k A$ be a $\mathcal{G}$-stable A-submodule that is also an A-direct summand of $V \otimes_k A$. If $U \subset \mathcal{X}(k)$ is dense, then $I_U(P) \subset V$ is a $\mathcal{G}$-submodule.*

Proof. Let $\mathcal{O}(\mathcal{G})$ be the function algebra of $\mathcal{G}$. Since $V \cong V \otimes_k k \cdot 1$ is $\mathcal{G}$-invariant and k is a field, V is an $\mathcal{O}(\mathcal{G})$-subcomodule of $V \otimes_k A$. By the same token, $V \cap P \subset V \otimes_k A$ is, as a subcomodule of $V \otimes_k A$, also $\mathcal{G}$-invariant. We will conclude the proof by showing that

$$V \cap P = I_U(P).$$

Observing the obvious inclusion $V \cap P \subset I_U(P)$, we proceed by showing that $I_U(P) \subset V \cap P$. By assumption there exists an A-module Q such that $V \otimes_k A = P \oplus Q$. Recall that this implies $V = P(x) \oplus Q(x)$ for every $x \in \mathcal{X}(k)$.

Now let v be an element of $I_U(P)$, and write $v = p + q$ with $p \in P$ and $q \in Q$. It follows that $v = (id_V \otimes x)(p) + (id_V \otimes x)(q) \quad \forall\ x \in \mathcal{X}(k)$. As v belongs to $I_U(P)$, the element q vanishes after specialization along $x \in U$. Letting $(v_j)_{j \in J}$ be a basis of V, we write $q = \sum_{j \in J} v_j \otimes a_j$. By the above, the ideal $S := \sum_{j \in J} A a_j \subset A$ is annihilated by every $x \in U$. Thus, U is contained in the zero locus $Z(S)$ of S. Since U is a dense subset of $\mathcal{X}(k)$, it follows from Hilbert's Nullstellensatz that $S = (0)$. Consequently, $q = 0$ and v belongs to $V \cap P$. $\square$

We conclude this section by recording subsidiary results that will be useful in our future work [13]. Recall the following basic fact:

Lemma 2.3 *Let $\varphi : X \longrightarrow Y$ be a dominant morphism of affine varieties. If Y is irreducible, then there exists a nonempty open subset $U \subset Y$ such that the function $y \mapsto \dim \varphi^{-1}(y)$ is constant on U.* $\square$

Proposition 2.4 *Let A be an integral domain, $P \subset V \otimes_k A$ an A-direct summand. If $Y \subset V$ is an affine subvariety such that $P(x) \cap Y \neq \emptyset$ for every element $x \in \mathcal{X}(k)$, then there exists an open subset $\emptyset \neq U \subset \mathcal{X}(k)$ such that the map $x \mapsto \dim P(x) \cap Y$ is constant on U.*

Proof. Consider the subset Z of $\mathcal{X}(k) \times Y$ given by

$$Z := \{(x, y) \ ; \ \exists\ p \in P \text{ such that } (id_V \otimes x)(p) = y\}.$$

By assumption there exists an A-submodule $Q \subset V \otimes_k A$ such that $V \otimes_k A = P \oplus Q$. As a result $V = P(x) \oplus Q(x)$ for every $x \in \mathcal{X}(k)$. Letting pr_Q denote the projection onto Q, we conclude that $Z = \{(x, y) \ ; \ [(id_V \otimes x) \circ pr_Q](y) = 0\}$. In order to show that Z is a Zariski-closed subset of $\mathcal{X}(k) \times Y$, we pick a basis $\{v_1, \ldots, v_m\} \subset V$ of V and consider the projections

$pr_\ell : V \longrightarrow k \ ; \ \sum_{j=1}^m \alpha_j v_j \mapsto \alpha_\ell$. Let $(a_{ij})_{1\le i,j\le m} \in Mat_m(A)$ be the $(m \times m)$-matrix given by

$$pr_Q(v_j \otimes 1) = \sum_{i=1}^m v_i \otimes a_{ij} \qquad 1 \le i, j \le m.$$

Then we have

$$(x, y) \in Z \Leftrightarrow 0 = \sum_{j=1}^m pr_j(y)x(a_{ij}) \qquad 1 \le i \le m.$$

Since $pr_1|_Y, \ldots, pr_m|_Y \in k[Y]$, and for $a \in A$ the map

$$\tilde{a} : \mathcal{X}(k) \longrightarrow k \ \ ; \ \ x \mapsto x(a)$$

belongs to $k[\mathcal{X}(k)]$, we see that

$$Z = \{(x, y) \ ; \ (\sum_{j=1}^m pr_j \otimes \tilde{a}_{ij})(x, y) = 0 \quad 1 \le i \le m\}$$

is a closed subset of $\mathcal{X}(k) \times Y$.

By assumption the restriction $\pi : Z \longrightarrow \mathcal{X}(k)$ of the projection onto the first factor is surjective. As $\mathcal{X}(k)$ is irreducible, Lemma 2.3 ensures the existence of a nonempty open subset $U \subset \mathcal{X}(k)$ such that $x \mapsto \dim \pi^{-1}(x)$ is constant on U. Since $\pi^{-1}(x) \cong P(x) \cap Y$, our assertion follows. □

Corollary 2.5 *Let A be an integral domain, $P \subset V \otimes_k A$ an A-direct summand. If $Y \subset V$ is a conical variety, then there exists an open subset $\emptyset \neq U \subset \mathcal{X}(k)$ such that the map $x \mapsto \dim P(x) \cap Y$ is constant on U.*

Proof. As Y is conical, we have $\{0\} \subset P(x) \cap Y$ for every $x \in \mathcal{X}(k)$. □

Remark. In the situation of (2.5) the map $x \mapsto \dim P(x) \cap Y$ can be shown to be upper semicontinuous with the constant value being the minimum.

3. Lie Algebras with Self-Centralizing Tori

Throughout this section we fix a restricted Lie algebra $(L, [p])$. Given a maximal torus $T \subset L$ we recall that the centralizer $C_L(T)$ of T in L is a Cartan subalgebra of L (cf. [21, (II.4.1)]). There exists a finite set $\Phi \subset T^* \setminus \{0\}$ such that

$$L = C_L(T) \oplus \bigoplus_{\alpha \in \Phi} L_\alpha.$$

Here $L_\alpha := \{x \in L \ ; \ [t, x] = \alpha(t)x \ \ \forall \, t \in T\}$ is the *root space* of L relative to T corresponding to α. Let $\mathcal{V}_L := \{x \in L \ ; \ x^{[p]} = 0\}$ be the *rank variety* of L. Observe that $\mathcal{V}_L$ is an affine variety, whose dimension will be referred to as the *α_p-rank* of L. In this section we will discuss the relationship between $\mathcal{V}_L$ and the structure of L in case the latter admits a self-centralizing (maximal) torus. By the above remarks this is equivalent to L possessing a toral Cartan subalgebra. Restricted Lie algebras of classical type [20, §3] and those associated to reductive groups enjoy this property.

We denote by $T(L)$ the *toral radical* of L, that is, the largest toral ideal of L. Note that $T(L)$ is contained in the center $C(L)$ of L.

Lemma 3.1 *The canonical projection $L \longrightarrow L/T(L)$ induces a bijective morphism $\mathcal{V}_L \longrightarrow \mathcal{V}_{L/T(L)}$ of affine varieties.*

Proof. The canonical projection obviously induces a morphism $\pi : \mathcal{V}_L \longrightarrow \mathcal{V}_{L/T(L)}$. If $x, y \in \mathcal{V}_L$ satisfy $\pi(x) = \pi(y)$, then there exists $z \in T(L)$ with $x = y + z$. Consequently, $0 = x^{[p]} = (y + z)^{[p]} = y^{[p]} + z^{[p]} = z^{[p]}$, so that $z = 0$. If $x \in L$ is an element such that $(x + T(L))^{[p]} = 0$, then there exists $z \in T(L)$ such that $x^{[p]} = z^{[p]}$. Consequently, $x - z \in \mathcal{V}_L$ is a pre-image of $x + T(L)$. $\square$

We let $\mathrm{GF}(p)$ denote the Galois field with p elements.

Lemma 3.2 *Let $T \subset L$ be a self-centralizing torus. Then the following statements hold:*

(1) $\dim_k L_\alpha \le \dim \mathcal{V}_L \quad \forall\ \alpha \in \Phi$.

(2) *If there exists $\alpha_0 \in \Phi$ such that $\dim_k L_{\alpha_0} = \dim \mathcal{V}_L$, then $\dim(T \oplus L_{\alpha_0}) \cap \mathcal{V}_L = \dim \mathcal{V}_L$.*

Proof. (1). We put $T^{(\alpha)} := \ker \alpha$ and observe that $T^{(\alpha)}$ is the toral radical of the p-subalgebra $H^{(\alpha)} := \bigoplus_{i \in \mathrm{GF}(p)} L_{i\alpha}$. Consider the restricted Lie algebra $K^{(\alpha)} := H^{(\alpha)}/T^{(\alpha)}$. By (3.1) the canonical projection $H^{(\alpha)} \longrightarrow K^{(\alpha)}$ induces a bijective morphism $\pi_\alpha : \mathcal{V}_{H^{(\alpha)}} \longrightarrow \mathcal{V}_{K^{(\alpha)}}$. Moreover, direct computation shows that the subspace $K^{(\alpha)}_\alpha \cong L_\alpha$ is contained in $\mathcal{V}_{K^{(\alpha)}}$. Consequently, $\dim_k L_\alpha \le \dim \mathcal{V}_{K^{(\alpha)}} = \dim \mathcal{V}_{H^{(\alpha)}} \le \dim \mathcal{V}_L$.

(2). As π_{α_0} is a bijective morphism, $\pi_{\alpha_0}^{-1}(K^{(\alpha_0)}_{\alpha_0}) \subset \mathcal{V}_L \cap (T \oplus L_{\alpha_0})$ is an irreducible subset of dimension $\dim_k L_{\alpha_0}$. Consequently,

$$\dim \mathcal{V}_L = \dim_k L_{\alpha_0} = \dim \pi_{\alpha_0}^{-1}(K^{(\alpha_0)}_{\alpha_0}) \le \dim \mathcal{V}_L \cap (T \oplus L_{\alpha_0}) \le \dim \mathcal{V}_L,$$

so that equality holds throughout. $\square$

Given a torus $T \subset L$, we consider the scheme $\mathcal{T}_L$. According to (1.3) the open subscheme $\tilde{\mathcal{T}}_L$ is a smooth affine algebraic scheme. Let $\mathcal{X} \subset \tilde{\mathcal{T}}_L$ be the irreducible component such that the given embedding $T \hookrightarrow L$ corresponds to a rational point $x_0 \in \mathcal{X}(k)$. Then there exists a finitely generated integral domain A over k, and an isomorphism $\Psi : \mathrm{Spec}_k(A) \longrightarrow \mathcal{X}$.

Let $j : T \hookrightarrow L \otimes_k A$ be the *universal embedding* corresponding to id_A, i.e., $j = \Psi(A)(id_A)$. By definition, any element $\varphi \in \mathcal{X}(R)$ is obtained by specialization, i.e., there exists a unique algebra homomorphism $x : A \longrightarrow R$ such that

$$\varphi = (id_L \otimes x) \circ j.$$

The torus $j(T)$ operates on $\tilde{L} := L \otimes_k A$ via A-linear transformations. Hence there exists a finite subset $\tilde{\Phi} \subset T^* \setminus \{0\}$ such that

$$\tilde{L} = \tilde{L}_0 \oplus \bigoplus_{\alpha \in \tilde{\Phi}} \tilde{L}_\alpha,$$

where $\tilde{L}_\alpha = \{y \in \tilde{L} \ ; \ [j(t), y] = \alpha(t)y \ \ \forall \ t \in T\}$ is a finitely generated, projective A-module for every $\alpha \in \tilde{\Phi} \cup \{0\}$. Specialization along x_0 yields

$$L = \tilde{L}_0(x_0) \oplus \bigoplus_{\alpha \in \tilde{\Phi}} \tilde{L}_\alpha(x_0) = C_L(T) \oplus \bigoplus_{\alpha \in \Phi} L_\alpha,$$

so that $\Phi = \tilde{\Phi}$ and $L_\alpha = \tilde{L}_\alpha(x_0)$ for every $\alpha \in \Phi \cup \{0\}$. Since each $\tilde{L}_\alpha$ is a finitely generated projective module over the integral domain A, we have

$$(*) \qquad \dim_k \tilde{L}_\alpha(x) = \dim_k L_\alpha \quad \forall \ \alpha \in \Phi \cup \{0\}, \ x \in \mathcal{X}(k).$$

For future reference we record the following immediate consequence

Lemma 3.3 *Suppose that $T \subset L$ is self-centralizing. Then we have*

$$\tilde{L}_0 = Aj(T).$$

In particular, the torus $j(T)(x)$ is self-centralizing for every $x \in \mathcal{X}(k)$.

Proof. Directly from the definition we obtain $Aj(T) \subset \tilde{L}_0$. As j belongs to $\tilde{\mathcal{T}}_L(A)$, the A-module $Aj(T)$ is a direct summand of $\tilde{L}_0$. By $(*)$ we therefore have

$$\dim_k(Aj(T))(x) = \dim_k T = \dim_k \tilde{L}_0(x)$$

for every rational point $x \in \mathcal{X}(k)$. Thus, $(Aj(T))(x) = \tilde{L}_0(x) \ \ \forall \ x \in \mathcal{X}(k)$. Since $Aj(T)$ is an A-direct summand of $\tilde{L}_0$, an application of Hilbert's Nullstellensatz yields the desired identity. □

Remark. Let $P \subset \tilde{L}$ be a p-subalgebra which is also an A-direct summand. According to (2.5) there exists a dense open subset $U \subset \mathcal{X}(k)$ such that the function $x \mapsto \dim \mathcal{V}_{P(x)}$ is constant on U. In the special case where $P = \tilde{L}_0$, we will denote this constant by $c_0(\mathcal{X})$. A restricted Lie algebra $(L, [p])$ admits a self-centralizing torus if and only if $c_0(\mathcal{X}) = 0$ for some irreducible component $\mathcal{X} \subset \tilde{\mathcal{T}}_L$.

Let $\mathcal{G} := \mathcal{AUT}(L)^\circ$ be the connected component of the automorphism scheme of the restricted Lie algebra $(L, [p])$. The natural operation of $\mathcal{G}$ on L induces an action of $\mathcal{G}$ on $\tilde{\mathcal{T}}_L$. For $g \in \mathcal{G}(R)$ and $\varphi \in \tilde{\mathcal{T}}_L(R)$ we have $g \cdot \varphi := g \circ \varphi$.

Lemma 3.4 *The following statements hold:*

1. *The irreducible component $\mathcal{X} \subset \tilde{\mathcal{T}}_L$ is $\mathcal{G}$-invariant.*
2. *The induced diagonal action of $\mathcal{G}$ on $\tilde{L}$ fixes $j(T)$ pointwise.*

Proof. (1). Since $\tilde{\mathcal{T}}_L$ is smooth, the irreducible component $\mathcal{X}$ is a connected component (cf. [2, (I.§4,no.4),(I.§1(5.6)),(I.§1(2.10))]). Thanks to [2, (I.§4 (6.11))] the scheme $\mathcal{G} \times \mathcal{X}$ is connected, so that the same holds for its image under the morphism $\mathcal{G} \times \mathcal{X} \longrightarrow \tilde{\mathcal{T}}_L$. Since this image contains points of $\mathcal{X}$, the desired result follows.

(2). We note that the group $\mathcal{G}(k)$ of k-rational points of $\mathcal{G}$ operates on the set of morphisms $M_k(\mathrm{Spec}_k(A), \tilde{\mathcal{T}}_L)$ via

$$g \cdot \Phi := g_{\tilde{\mathcal{T}}_L} \circ \Phi \circ g^{-1}_{\mathrm{Spec_k}(A)} = g_{\tilde{\mathcal{T}}_L} \circ \Phi \circ \mathrm{Spec_k}(g_A).$$

Here $g_{\tilde{\mathcal{T}}_L}$ denotes the natural transformation $\tilde{\mathcal{T}}_L \longrightarrow \tilde{\mathcal{T}}_L$ given by $g_{\tilde{\mathcal{T}}_L}(R)(\varphi) = g_R \cdot \varphi$ for every $\varphi : T \longrightarrow L \otimes_k R$, and $g_R \in \mathcal{G}(R)$ is the image of g under the canonical map $\mathcal{G}(k) \longrightarrow \mathcal{G}(R)$.

The group $\mathcal{G}(k)$ also acts on the set $\tilde{\mathcal{T}}_L(A)$ by means of

$$g \cdot \varphi := g_{\tilde{\mathcal{T}}_L}(A)(\tilde{\mathcal{T}}_L(g_A)(\varphi)).$$

Since the Yoneda isomorphism $M_k(\mathrm{Spec}_k(A), \mathcal{X}) \cong \mathcal{X}(A)$ is functorial in A and $\mathcal{X}$, it follows that the bijection

$$\Theta : M_k(\mathrm{Spec}_k(A), \tilde{\mathcal{T}}_L) \longrightarrow \tilde{\mathcal{T}}_L(A)$$

is $\mathcal{G}(k)$-equivariant.

Next, we observe that the isomorphism $\Psi : \mathrm{Spec}_k(A) \longrightarrow \tilde{\mathcal{T}}_L$ is $\mathcal{G}(k)$-covariant, and thereby an invariant element of $M_k(\mathrm{Spec}_k(A), \tilde{\mathcal{T}}_L)$. Consequently, $j = \Theta(\Psi)$ enjoys the same property. Since $j_{A\otimes_k R}$ is the image of $\Psi_R : \mathrm{Spec}_R(A \otimes_k R) \longrightarrow (\tilde{\mathcal{T}}_L)_R$ under the isomorphism $\Theta_R : M_R(\mathrm{Spec}_R(A \otimes_k R), (\tilde{\mathcal{T}}_L)_R) \longrightarrow (\tilde{\mathcal{T}}_L)_R(A \otimes_k R)$, it follows that $j_{A\otimes_k R}$ is invariant under $\mathcal{G}(R)$. Consequently, $j(T) \subset \tilde{L}$ is fixed pointwise. □

We are now in a position to establish our main structural result on restricted Lie algebras with self-centralizing tori.

Theorem 3.5 *Suppose that $T \subset L$ is a self-centralizing torus. If there exists $\alpha_0 \in \Phi$ such that $\dim_k L_{\alpha_0} = \dim \mathcal{V}_L$, then $\Phi = \{\alpha_0\}$.*

Proof. Thanks to (3.1) we may pass to $L/T(L)$ and assume that the Lie algebra L has a trivial center. As before, we consider an irreducible component $\mathcal{X} \subset \tilde{\mathcal{T}}_L$ such that the given embedding $T \hookrightarrow L$ corresponds to a rational point $x_0 \in \mathcal{X}(k)$. Thus, $\mathcal{X} \cong \mathrm{Spec}_k(A)$ for some finitely generated integral domain A.

We let $\mathcal{G} = \mathcal{AUT}(L)^\circ$ be the connected component of the automorphism scheme of the restricted Lie algebra $(L, [p])$. Owing to (3.4) the natural operation of $\mathcal{G}$ on $\tilde{\mathcal{T}}_L$ leaves $\mathcal{X}$ invariant, and $\mathcal{G}$ operates on A via algebra automorphisms. By the same token, the induced action of $\mathcal{G}$ on $\tilde{L}$ fixes $j(T)$. Since $\mathcal{G}$ operates on $\tilde{L}$ via homomorphisms of k-Lie algebras, all root spaces $\tilde{L}_\alpha$ of $\tilde{L}$ are $\mathcal{G}$-invariant. In particular, $P := Aj(T) \oplus \tilde{L}_\alpha$ satisfies the conditions of (2.2), so that $I := I_{\mathcal{X}(k)}(P)$ is an ideal of L that is contained in $T \oplus L_{\alpha_0}$.

Let x be an element of $\mathcal{X}(k)$. Owing to (3.3) the torus $[Aj(T)](x) \subset L$ is self-centralizing. Since $\dim_k \tilde{L}_{\alpha_0}(x) = \dim_k L_{\alpha_0}(x_0) = \dim_k L_{\alpha_0} = \dim \mathcal{V}_L$, part (2) of (3.2) shows that $\dim P(x) \cap \mathcal{V}_L = \dim \mathcal{V}_L$. Hence, if $\mathcal{F}$ is the set of irreducible components of $\mathcal{V}_L$ of dimension $\dim \mathcal{V}_L$, then the algebraic family $(P(x))_{x\in\mathcal{X}(k)}$ is $\mathcal{F}$-regular. Thanks to (2.1) the ideal I also enjoys this property.

We decompose the ideal I into its homogeneous parts relative to T. Observing $I \subset T \oplus L_{\alpha_0}$, we obtain $I = (I \cap T) \oplus (I \cap L_{\alpha_0}) = I_0 \oplus I_{\alpha_0}$. Since the gradation given by the root space decomposition is compatible with the p-map, we have $L_{\alpha_0}^{[p]} \subset T$. The identity $[x^{[p]}, x] = 0$ then shows that $L_{\alpha_0}^{[p]} \subset \ker \alpha_0$. Accordingly, $Z := I_0 + \langle I_{\alpha_0}^{[p]} \rangle$ is a subspace of T. Let $Z^{(p)} \subset T$ be the smallest $[p]$-stable subspace of T containing Z. Then $I^{(p)} := Z^{(p)} \oplus I_{\alpha_0}$ is a p-ideal containing I. As its toral radical is contained in $T(L) = (0)$, it readily follows that $I_{\alpha_0}^{[p]} = \{0\}$, whence $\mathcal{V}_{I^{(p)}} = I_{\alpha_0}$. By the preceding paragraph we have

$$\dim_k L_{\alpha_0} = \dim \mathcal{V}_L = \dim \mathcal{V}_{I^{(p)}} = \dim_k I_{\alpha_0},$$

proving that $L_{\alpha_0} = I_{\alpha_0}$. Let $\beta \neq \alpha_0, -\alpha_0$ be a root. Then $[L_\beta, L_{\alpha_0}] \subset I_{\alpha_0+\beta} = (0)$, so that $L_\beta^{[p]}$ is contained in the torus $\ker \alpha_0 \cap \ker \beta$. If $x \in L_\beta$, then there exists $t \in \ker \alpha_0 \cap \ker \beta$ such that $x^{[p]} = t^{[p]}$. Consequently, $k(x-t) \oplus L_{\alpha_0} \subset \mathcal{V}_L$. We conclude that $x - t = 0$, whence $x = 0$. As a result, $L_\beta = (0)$, a contradiction. For $p = 2$, our assertion follows. Alternatively, we have $[L_{\alpha_0}, L_{\alpha_0}] \subset I_{2\alpha_0} = (0)$, and $T(I^{(p)}) = (0)$ implies $L_{\alpha_0} = I^{(p)}$ or $L_{\alpha_0} = [I^{(p)}, I^{(p)}]$. In either case the root space L_{α_0} is an ideal of L, and the foregoing arguments also show that $\beta := -\alpha_0$ is not a root. Consequently, $\Phi = \{\alpha_0\}$, as desired. □

By way of illustration, we retrieve the following well-known result (cf. [7, (3.2),(4.3)]), which we will apply repeatedly in the sequel.

Corollary 3.6 *Let $(L, [p])$ be a restricted Lie algebra that possesses a self-centralizing torus. Then L has α_p-rank 1 if and only if $L/T(L)$ is the two-dimensional non-abelian Lie algebra.*

Proof. Let T be a self-centralizing torus of L. Since $\dim \mathcal{V}_L = 1$, the set Φ of roots is not empty. From (3.5) we obtain $\Phi = \{\alpha\}$ with $\dim_k L_\alpha = 1$. Let $t \in T$ be an element such that $t^{[p]} = t$ and $\alpha(t) = 1$. Then $L/T(L) \cong (kt \oplus L_\alpha)$ has the asserted structure. □

We conclude this section with an application of our techniques to representation-finite infinitesimal groups. Recall that an infinitesimal k-group $\mathcal{G}$ is *supersolvable* if it admits a chain of normal subgroups such that the associated factor groups are isomorphic to μ_p or α_p. An associative algebra which possesses only finitely many isoclasses of indecomposable modules is said to have *finite representation type*. Given an infinitesimal k-group $\mathcal{G}$, we let $H(\mathcal{G})$ be the *distribution algebra* of $\mathcal{G}$. By definition, $H(\mathcal{G})$ is the dual of the function algebra $\mathcal{O}(\mathcal{G})$. If $H(\mathcal{G})$ has finite representation type, then we say that $\mathcal{G}$ is representation-finite.

Proposition 3.7 (cf. (2.1) of [11]) *Let $\mathcal{G}$ be an infinitesimal k-group. If $\mathcal{G}$ is representation-finite, then $\mathcal{G}$ is supersolvable.*

Proof. Let $L = \operatorname{Lie}(\mathcal{G})$ be the Lie algebra of $\mathcal{G}$. We propose to show that $\dim \mathcal{V}_L \leq 1$. To that end, we let $\mathcal{G}'$ and $\mathcal{G}''$ be two subgroups of type α_p such that $\mathcal{G}' \cap \mathcal{G}'' = e_k$. The right translation of $\mathcal{G}'$ on $\mathcal{G}$ induces an operation $\mathcal{O}(\mathcal{G}) \times \mathcal{G}' \longrightarrow \mathcal{O}(\mathcal{G})$ of $\mathcal{G}'$ on the function algebra $\mathcal{O}(\mathcal{G})$ via algebra homomorphisms. We denote by $B := \mathcal{O}(\mathcal{G})^{\mathcal{G}'}$ the subalgebra of invariants. The left translation of $\mathcal{G}''$ on $\mathcal{G}$ induces a left operation of $\mathcal{G}''$ on $\mathcal{O}(\mathcal{G})$ that sends B to B. We let $C := {}^{\mathcal{G}''}B = {}^{\mathcal{G}''}\mathcal{O}(\mathcal{G}) \cap \mathcal{O}(\mathcal{G})^{\mathcal{G}'}$ be the algebra of $\mathcal{G}''$-invariant functions of

B. According to [2, (III.§2 (6.1))] $\mathcal{O}(\mathcal{G})$ is a free B-module, and [2, (III.§1 (2.2))] shows that $\mathcal{Y} := \mathcal{G}/\mathcal{G}' \cong \operatorname{Spec}_k(B)$ is representable. Owing to [2, (II.§2 (2.2))] the group $\mathcal{G}''$ operates freely on $\mathcal{Y}$. In particular, we have $H(\mathcal{G}''_R \cap g\mathcal{G}'_R g^{-1}) = R$ for every commutative k-algebra R and every $g \in \mathcal{G}(R)$. A consecutive application of [2, (III.§2 (6.1))] and [2, (III.§1 (2.2)] then yields the representability of the functor $\mathcal{G}''\backslash\mathcal{Y} \cong \operatorname{Spec}_k(C)$. We may now apply [24, (II.8.4 (2))] to the trivial $H(\mathcal{G}')$-module k and obtain an isomorphism

$$\mathcal{O}(\mathcal{G}) \otimes_C H(\mathcal{G}) \otimes_{H(\mathcal{G}')} k \cong H(\mathcal{G}''_{\mathcal{O}(\mathcal{G})}) \otimes_{\mathcal{O}(\mathcal{G})} \mathcal{O}(\mathcal{G}) \cong H(\mathcal{G}'') \otimes_k \mathcal{O}(\mathcal{G})$$

of $(H(\mathcal{G}'') \otimes_k C)$-modules. Consequently, the induced module $H(\mathcal{G}) \otimes_{H(\mathcal{G}')} k$ is projective over $H(\mathcal{G}'')$. Since k is a direct summand of $(H(\mathcal{G}) \otimes_{H(\mathcal{G}'')} k)|_{H(\mathcal{G}'')}$, we see that the two induced modules $H(\mathcal{G}) \otimes_{H(\mathcal{G}')} k$ and $H(\mathcal{G}) \otimes_{H(\mathcal{G}'')} k$ are not isomorphic. As $H(\mathcal{G})$ has finite representation type, the set $\operatorname{Proj}(\mathcal{V}_L)$ is therefore finite, and we thus obtain $\dim \mathcal{V}_L \leq 1$.

We proceed by showing that $\mathcal{G}$ contains a normal subgroup of type μ_p or α_p. According to [2, (IV.§1 (4.4)),(IV.§3 (1.1))] the multiplicative center $\mathcal{M}({}_{\mathcal{F}}\mathcal{G})$ of the first Frobenius kernel ${}_{\mathcal{F}}\mathcal{G}$ of $\mathcal{G}$ is contained in the center of $\mathcal{G}$. We may therefore assume without loss of generality that $\mathcal{M}({}_{\mathcal{F}}\mathcal{G}) = e_k$. This implies $T(L) = (0)$.

To see that $\mathcal{G}$ contains a normal subgroup that is isomorphic to α_p, we will show that L contains a one-dimensional, unipotent p-ideal that is invariant under the adjoint representation of $\mathcal{G}$. In view of (3.6) the ideal $[L, L]$ has this property in case there exists a self-centralizing torus $T \subset L$. We therefore assume that there exists a maximal torus $T \subset L$ that is not self-centralizing. Let $\mathcal{X} \subset \tilde{\mathcal{T}}_L$ be the corresponding irreducible component, and write $\mathcal{X} \cong \operatorname{Spec}_k(A)$. As before, we let $j : T \hookrightarrow \tilde{L}$ be the universal embedding. Thanks to (3.3) we have $\tilde{L}_0(x) \neq j(T)(x)$ for every $x \in \mathcal{X}(k)$. An application of (2.1) then ensures that $I := I_{\mathcal{X}(k)}(\tilde{L}_0)$ is a $\mathcal{G}$-invariant p-ideal of L of α_p-rank 1 that is contained in the Cartan subalgebra $C_L(T)$. In particular, I is nilpotent, so that its center $C(I)$ is a unipotent, commutative, $\mathcal{G}$-invariant p-ideal of α_p-rank 1. According to [2, (IV.§2 (2.14))] there exists an element $x \in L \setminus \{0\}$ such that $x^{[p]^{n+1}} = 0 \neq x^{[p]^n}$ and $C(I) = \sum_{i=0}^{n} kx^{[p]^i}$. Thus, the ideal $J := kx^{[p]^n}$ has the desired properties.

By the foregoing discussion, the group $\mathcal{G}$ contains a normal subgroup $\mathcal{N}$ that is isomorphic to μ_p or α_p. Since $\mathcal{G}/\mathcal{N}$ has finite representation type, we may assume inductively that $\mathcal{G}/\mathcal{N}$ is supersolvable. Consequently, $\mathcal{G}$ is also supersolvable. □

4. Lie Algebras of α_p-Rank 2

In this section we apply Theorem 3.5 to Lie algebras of α_p-rank 2. Our results provide first steps towards the classification of infinitesimal groups of tame representation type. We recall that $\mathrm{GF}(p)$ denotes the Galois field with p elements, and consider the following $p+1$ classes of restricted Lie algebras:

1. $L_1(r) := kt \oplus kx \oplus ky$; $[t, x] = x,\ [t, y] = ry,\ [x, y] = 0$; $t^{[p]} = t,\ x^{[p]} = 0 = y^{[p]}$, where $r \in \mathrm{GF}(p) \setminus \{0\}$

2. $L_2 := s\ell(2)$.

3. $L_3 := B \oplus B$, where $B := kt \oplus kx$; $[t,x] = x$; $t^{[p]} = t$, $x^{[p]} = 0$ is the two-dimensional non-abelian Lie algebra.

Note that all Lie algebras involved have a self-centralizing torus, are centerless, and of α_p-rank 2. In the following we will show that the above list comprises all restricted Lie algebras with these properties.

Recall that a Lie algebra L with trivial center and all derivations being inner is referred to as *complete*. Throughout, we let $(L, [p])$ be a restricted Lie algebra.

Lemma 4.1 *Suppose that L is centerless of α_p-rank 2, and let $T \subset L$ be a self-centralizing torus T such that*

(a) $\dim_k L_\alpha = 1 \quad \forall\, \alpha \in \Phi$, *and*

(b) $\Phi \subset \mathrm{GF}(p)\alpha \quad \forall\, \alpha \in \Phi$.

Then $p \neq 2$, and either L is isomorphic to $L_1(r)$ for some $r \in \mathrm{GF}(p) \setminus \{0,1\}$, or $L \cong L_2$.

Proof. As before, we let $\mathcal{X} \cong \mathrm{Spec}_k(A)$ be the irreducible component of $\tilde{\mathcal{T}}_L$ containing the given embedding $T \hookrightarrow L$ as a rational point $x_0 \in \mathcal{X}(k)$. If $p = 2$, then $L = T \oplus L_\alpha$ has α_p-rank 1, a contradiction. We distinguish between the following cases:

1. *There is $\alpha \in \Phi$ such that $[L_\alpha, L_{-\alpha}] \neq (0)$.* Consider the A-direct summand $P := Aj(T) \oplus \tilde{L}_\alpha \oplus \tilde{L}_{-\alpha}$ of $\tilde{L}$. By assumption, we have $\tilde{L}_\alpha(x) \neq (0) \neq \tilde{L}_{-\alpha}(x)$ for every $x \in \mathcal{X}(k)$. In view of (3.6), the algebraic family $(P(x))_{x \in \mathcal{X}(k)}$ of p-subalgebras is regular with respect to the set of two-dimensional irreducible components of $\mathcal{V}_L$. Thanks to (3.4) and (2.1) the p-ideal $I_{\mathcal{X}(k)}(P) \subset T \oplus L_\alpha \oplus L_{-\alpha} \cong s\ell(2)$ enjoys the same property. As $p \geq 3$ the latter algebra is simple and complete. Consequently, $I_{\mathcal{X}(k)}(P) \cong s\ell(2)$, and $L = I_{\mathcal{X}(k)}(P) \oplus C_L(I_{\mathcal{X}(k)}(P))$. Since the first summand has α_p-rank 2, the second is, as a Lie algebra of α_p-rank 0, a torus (cf. [21, (II.3.10)]). As L is centerless, we see that $L = I_{\mathcal{X}(k)}(P) \cong L_2$.

2. *We have $[L_\alpha, L_{-\alpha}] = (0) \quad \forall\, \alpha \in \Phi$.* In this case $J := \bigoplus_{\alpha \in \Phi} L_\alpha$ is a p-ideal of L of dimension $\leq p-1$. Since $L_\alpha^{[p]} = (0)$ for every $\alpha \in \Phi$, we may apply the Engel-Jacobson Theorem (cf. [21, (I.3.1)]) to the Lie-set $\bigcup_{\alpha \in \Phi} L_\alpha$ to see that J is nilpotent. Thus, $((ad\, x_1) \circ (ad\, x_2) \circ \cdots \circ (ad\, x_{p-1}))|_J = 0$ for every $x_1, \ldots, x_{p-1} \in J$, so that the p-mapping is additive on J. Consequently, $J^{[p]} = (0)$, and $\dim_k J \leq \dim \mathcal{V}_L = 2$. In view of (3.6) the ideal J is not one-dimensional. Hence there exists an element $r \in \mathrm{GF}(p) \setminus \{0,1\}$ with $L \cong L_1(r)$. □

Theorem 4.2 *Let $p \geq 3$. Suppose that L is centerless and of α_p-rank 2. If L admits a self-centralizing torus, then L is isomorphic to $L_1(r)$ for some $r \in \mathrm{GF}(p) \setminus \{0\}$, L_2, or L_3.*

Proof. Let $T \subset L$ be a self-centralizing torus. If L admits a root space L_α of dimension 2, then Theorem 3.5 yields $L = T \oplus L_\alpha$. As L is centerless, we see that $\dim_k T = 1$. Consequently, $L \cong L_1(1)$.

In the sequel we assume that $\dim_k L_\alpha = 1$ for every $\alpha \in \Phi$. We let $\mathcal{X} \cong \mathrm{Spec}_k(A)$ be the irreducible component of $\tilde{\mathcal{T}}_L$ such that $x_0 \in \mathcal{X}(k)$ is the given embedding $T \hookrightarrow L$. Observing (3.3) we recall the root space decomposition

$$\tilde{L} = Aj(T) \oplus \bigoplus_{\alpha \in \Phi} \tilde{L}_\alpha$$

of $\tilde{L} := L \otimes_k A$. As before, we let $\mathcal{G} := \mathcal{AUT}(L)^\circ$ be the connected component of the automorphism scheme of the restricted Lie algebra $(L, [p])$.

Given a root $\alpha \in \Phi$, we consider the p-subalgebra $\tilde{H}^{(\alpha)} := Aj(T) \oplus \bigoplus_{i \in GF(p)\setminus\{0\}} \tilde{L}_{i\alpha}$ of $\tilde{L}$. According to (3.4) each $\tilde{H}^{(\alpha)}$ is a $\mathcal{G}$-invariant A-direct summand of $\tilde{L}$. By (2.2) $I^{(\alpha)} := I_{\mathcal{X}(k)}(\tilde{H}^{(\alpha)})$ is a $\mathcal{G}$-stable subspace of L, and thus in particular a p-ideal.

For $x \in \mathcal{X}(k)$ we consider the restricted Lie algebra $Q^{(\alpha)}(x) := \tilde{H}^{(\alpha)}(x)/C(\tilde{H}^{(\alpha)}(x))$. The kernel of the canonical map $I^{(\alpha)} \longrightarrow Q^{(\alpha)}(x)$ is, as a toral ideal of $I^{(\alpha)}$, contained in $T(L)$. Consequently, we have an embedding $\iota_\alpha(x) : I^{(\alpha)} \hookrightarrow Q^{(\alpha)}(x)$ of restricted Lie algebras.

We proceed by considering the following cases:

(a) *There exists a root* $\alpha \in \Phi$ *such that* $\dim \mathcal{V}_L \cap \tilde{H}^{(\alpha)}(x) = 2$ *for every* $x \in \mathcal{X}(k)$.

Thanks to (2.1) the ideal $I^{(\alpha)}$ contains an irreducible component of $\mathcal{V}_L$. Let $x \in \mathcal{X}(k)$. By Lemma 3.1 and Lemma 4.1 the restricted Lie algebra $Q^{(\alpha)}(x)$ is either isomorphic to $s\ell(2)$ or to $L_1(r(x))$ for some $r(x) \in GF(p) \setminus \{0, 1\}$.

If there exists an element $y_0 \in \mathcal{X}(k)$ such that $Q^{(\alpha)}(y_0) \cong s\ell(2)$, then $\iota_\alpha(y_0)(I^\alpha) \subset s\ell(2)$ is a p-subalgebra of α_p-rank 2. Hence $\iota_\alpha(y_0)$ is an isomorphism. As a result, $I^{(\alpha)} \subset L$ is a complete ideal of L, and there results a decomposition $L = I^{(\alpha)} \oplus C_L(I^{(\alpha)})$. Since $\mathcal{V}_{C_L(I^{(\alpha)})}(k) = \{0\}$, the second summand is, as a toral ideal of L, trivial. Thus, $L = I^{(\alpha)} \cong L_2$ in that case.

Alternatively, each $Q^{(\alpha)}(x)$ is of type $L_1(r(x))$ for some $r(x) \in \mathrm{GF}(p) \setminus \{0, 1\}$. Setting $r := r(x_0)$, we see that $\tilde{H}^{(\alpha)}(x_0) = T \oplus L_\alpha \oplus L_{r\alpha}$. Since $I^{(\alpha)} \subset \tilde{H}^{(\alpha)}(x_0)$ has α_p-rank 2, Corollary 3.6 implies that $L_\alpha \oplus L_{r\alpha} \subset I^{(\alpha)}$. Let $\beta \in \Phi \setminus \mathrm{GF}(p)\alpha$ be a root. Then we have $[L_\beta, I^{(\alpha)}] = (0)$, so that $L_\beta^{[p]}$ is contained in the torus $\ker\alpha \cap \ker\beta$. Given $x \in L_\beta$, there thus exists $t \in \ker\alpha \cap \ker\beta$ such that $k(x - t) \subset \mathcal{V}_L$. Since the element $x - t$ centralizes $I^{(\alpha)}$, it follows that $k(x - t) \times \mathcal{V}_{I^{(\alpha)}} \subset \mathcal{V}_L$. From the condition $\dim \mathcal{V}_L = 2 = \dim \mathcal{V}_{I^{(\alpha)}}$ we therefore obtain $x - t = 0$. Consequently, $L_\beta = (0)$, a contradiction. As a result, $\Phi \subset \mathrm{GF}(p)\alpha$, and Lemma 4.1 yields $L \cong L_1(r)$.

(b) *For every* $\alpha \in \Phi$ *there exists* $x_\alpha \in \mathcal{X}(k)$ *such that* $\dim \mathcal{V}_L \cap \tilde{H}^{(\alpha)}(x_\alpha) = 1$.

Thanks to (3.6) there are isomorphisms $\tilde{H}^{(\alpha)}(x_\alpha) \cong j(T)(x_\alpha) \oplus \tilde{L}_\alpha(x_\alpha)$ for $\alpha \in \Phi$. Since the function $x \mapsto \dim_k \tilde{L}_\alpha(x)$ is constant, it follows that $\tilde{H}^{(\alpha)} = Aj(T) \oplus \tilde{L}_\alpha$ for every root $\alpha \in \Phi$. In particular, any two roots $\alpha \neq \beta \in \Phi$ are linearly independent over $\mathrm{GF}(p)$. If $[\tilde{L}_\alpha, \tilde{L}_\beta] \neq (0)$ for all $\alpha \neq \beta$, then $\alpha + \beta$ and $2(\alpha + \beta)$ are roots, a contradiction. Consequently, there exists a pair $(\alpha, \beta) \in \Phi \times \Phi$ with $\alpha \neq \beta$ and such that $[\tilde{L}_\alpha, \tilde{L}_\beta] = (0)$.

Consider the p-subalgebra $\tilde{K} := Aj(T) \oplus \tilde{L}_\alpha \oplus \tilde{L}_\beta$. By (3.6) the p-subalgebra $\tilde{K}(x) \subset L$ has α_p-rank 2 for every $x \in \mathcal{X}(k)$, so that an application of (2.1) shows that the p-ideal $I_{\mathcal{X}(k)}(\tilde{K})$ inherits this property. Since $I := I_{\mathcal{X}(k)}(\tilde{K}) \subset T \oplus L_\alpha \oplus L_\beta$ is the direct sum of its root spaces, the condition $\dim \mathcal{V}_{I_{\mathcal{X}(k)}(\tilde{K})} = 2$ implies $L_\alpha \oplus L_\beta \subset I_{\mathcal{X}(k)}(\tilde{K})$. Neither $-\alpha$ nor $-\beta$ are roots, so $L_\alpha \oplus L_\beta$ is an ideal of L. For $\zeta \in \{\alpha, \beta\}$ we have $L_\zeta^{[p]} \subset I \cap \ker\alpha \cap \ker\beta = T(I) = (0)$. If $\gamma \in \Phi \setminus \{\alpha, \beta\}$ is a root such that $[L_\gamma, L_\alpha \oplus L_\beta] = (0)$, then $L_\gamma^{[p]} \subset \ker\alpha \cap \ker\beta \cap \ker\gamma$. Given $x \in L_\gamma$ there thus exists $t \in \ker\alpha \cap \ker\beta \cap \ker\gamma$ such that $k(x - t) \oplus L_\alpha \oplus L_\beta \subset \mathcal{V}_L$. Consequently, $L_\gamma = (0)$, so that $\Phi \subset \{\alpha, \beta, \alpha - \beta, \beta - \alpha\}$. Since $\alpha - \beta$ and $\beta - \alpha$ cannot both be roots, we may assume that $\Phi \subset \{\alpha, \beta, \alpha - \beta\}$. Setting $J := L_\alpha \oplus L_\beta \oplus L_{\alpha-\beta}$, we have $L_{\alpha-\beta}^{[p]} \subset \ker\alpha \cap \ker(\alpha - \beta) = (0) = [J, [J, J]]$, so that $J \subset \mathcal{V}_L$. Hence $L_{\alpha-\beta} = (0)$ and

$L \cong L_3$. $\square$

Remark. By combining (1.5), (3.6) and (4.2) we obtain the estimate $\dim \mathcal{X} \leq \dim \mathcal{V}_L$ for the Lie algebras considered.

Corollary 4.3 *Let $p \geq 3$ and suppose that L has a self-centralizing torus. If $\dim \mathcal{V}_L \leq 2$, then either L is supersolvable or $L/T(L) \cong s\ell(2)$.*

Proof. Owing to (3.1) the Lie algebra $L/T(L)$ has α_p-rank ≤ 2. In view of (3.6) and (4.2) this implies that $L/T(L)$ is either supersolvable or isomorphic to $s\ell(2)$. In the former case, the Lie algebra L is also supersolvable. $\square$

5. Tame Principal Blocks of Enveloping Algebras

It is well-known (cf. [7, §4]) that the restricted enveloping algebra $u(L)$ has finite representation type if and only if its principal block $\mathcal{B}_0(L)$ enjoys this property. In this section we shall employ our previous results in order to determine those restricted Lie algebras with self-centralizing torus, whose principal block is tame. Recall that a representation-infinite algebra is *tame* whenever its isoclasses of indecomposable modules of any given dimension can be parametrized by a one-dimensional variety. The reader is referred to [5] concerning the precise definition of tameness. We shall see below that this property is closely related to the decomposability of certain modules as well as to structural features of the stable Auslander-Reiten quiver of $u(L)$.

Let $(L, [p])$ be a restricted Lie algebra. We denote by $\Gamma_s(L)$ the stable Auslander-Reiten quiver of the self-injective algebra $u(L)$. The reader may consult [1] for the definition and basic properties of $\Gamma_s(L)$. Roughly speaking, $\Gamma_s(L)$ is a directed graph with the isoclasses of the non-projective indecomposable $u(L)$-modules as vertices. The connected component containing the isoclass $[k]$ of the trivial $u(L)$-module will be denoted Θ_0. Closely related to the position of $[k]$ within Θ_0 is the structure of the projective cover P_0 of k. We denote by $H(P_0) := \mathrm{Rad}(P_0)/\mathrm{Soc}(P_0)$ the *heart* of P_0, that is, the quotient of the radical by the socle.

Theorem 5.1 *Let $(L, [p])$ be a restricted Lie algebra of characteristic $p \geq 3$ with self-centralizing torus. Then the following statements are equivalent:*

(1) *$H(P_0)$ is decomposable.*
(2) *$L/T(L) \cong s\ell(2)$.*
(3) *$\mathcal{B}_0(L) \cong \mathcal{B}_0(s\ell(2))$.*
(4) *$\mathcal{B}_0(L)$ is tame.*
(5) *$\Theta_0 \cong \mathbb{Z}[\tilde{A}_{12}]$.*

Proof. (1) $\Rightarrow$ (2). Suppose that $\dim \mathcal{V}_L \leq 1$. According to [7, (3.2)] the principal block $\mathcal{B}_0(L)$ is a Nakayama algebra. Hence $H(P_0)$ is indecomposable, a contradiction. In case $\dim \mathcal{V}_L \geq 3$ we reach the same contradiction by applying [9, (2.1)] and [9, (4.1)] consecutively. Thus, $\dim \mathcal{V}_L = 2$, and by (4.3) the Lie algebra $L/T(L)$ is either supersolvable or isomorphic to $s\ell(2)$. Thanks to [9, (4.2)] and [9, (4.1)] the former alternative does not apply.

(2) ⇒ (3). Let $I := u(L)T(L)$ be the two-sided ideal of $u(L)$ that is generated by $T(L)$. The canonical projection $L \longrightarrow L/T(L)$ induces an isomorphism $u(L)/I \cong u(L/T(L))$. Since $u(T(L))$ is, as a commutative semisimple algebra, a product of copies of the ground field k, the augmentation ideal $u(T(L))^\dagger$ is a sum of blocks. In other words, there exist orthogonal idempotents $e_1, \ldots, e_r \in u(T(L))$ such that $u(T(L))^\dagger = \bigoplus_{i=1}^r u(T(L))e_i$. Since $T(L)$ is contained in the center of L, each e_i is a central idempotent of $u(L)$. It follows that $I = u(L)u(T(L))^\dagger = \bigoplus_{i=1}^r u(L)e_i$ is a sum of blocks of $u(L)$. Consequently, there exist blocks $\mathcal{B}_1, \ldots, \mathcal{B}_s$ of $u(L)$ such that the projection $u(L) \longrightarrow u(L/T(L))$ induces an isomorphism $\bigoplus_{i=1}^s \mathcal{B}_i \cong u(L/T(L))$. Since the projection is a map of augmented algebras, it follows that $\mathcal{B}_0(L) \cong \mathcal{B}_0(L/T(L)) \cong \mathcal{B}_0(s\ell(2))$.

(3) ⇒ (4). By work of Fischer [15] the principal block of $u(s\ell(2))$ is tame.

(4) ⇒ (5). Suppose that $\mathcal{B}_0(L)$ is tame. It follows from Rickard's Theorem [19, Thm.2] (see also [25, Thm.1]) and [7, (3.2)] that $\dim \mathcal{V}_L = 2$. Owing to (4.3) the Lie algebra L is either supersolvable, or $L/T(L)$ is isomorphic to $s\ell(2)$. In the former case, [25, Thm.3] and [8, (4.1)] imply that $\mathcal{B}_0(L)$ is not tame. Hence $L/T(L) \cong s\ell(2)$, and we may argue as above to see that $\mathcal{B}_0(L) \cong \mathcal{B}_0(s\ell(2))$. By combining [18, Thm.1] with general results on the structure of $\Gamma_s(L)$ (cf. [6, Thm.1] or [7, (5.7)]) we conclude that Θ_0 is isomorphic to $\mathbb{Z}[\tilde{A}_{12}]$.

(5) ⇒ (1). Let Ω be the Heller operator of the self-injective algebra $u(L)$. By assumption every vertex of the component $\Omega(\Theta_0) \cong \Theta_0 \cong \mathbb{Z}[\tilde{A}_{12}]$ has a successor of multiplicity 2. In particular, $[\mathrm{Rad}(P_0)] = \Omega([k])$ has this property. According to general theory, the successors of $[\mathrm{Rad}(P_0)]$ can be read off from the non-projective part of the middle term of the standard almost split sequence

$$(0) \longrightarrow \mathrm{Rad}(P_0) \longrightarrow H(P_0) \oplus P_0 \longrightarrow P_0/\mathrm{Soc}(P_0) \longrightarrow (0).$$

Consequently, $H(P_0)$ has two isomorphic indecomposable constituents. □

Remark. If $u(L)$ is tame, then its principal block is tame or representation-finite. The results of [7, §4] rule out the latter alternative. We also note that the structure of $\mathcal{B}_0(s\ell(2))$ is well understood (cf. [15]). In particular, $\mathcal{B}_0(s\ell(2))$ is a special biserial algebra (see [5, (II.1.1)] for the definition).

Our final result concerns the structure of the Hopf algebra $H(\mathcal{G})$ of an infinitesimal group scheme $\mathcal{G}$. Given such a scheme $\mathcal{G}$, we let $R(\mathcal{G})$ be its *solvable radical.* An infinitesimal group with trivial solvable radical will be referred to as *semisimple.*

Corollary 5.2 *Let $\mathcal{G}$ be a semisimple infinitesimal group of characteristic $p \geq 3$. Suppose that* $\mathrm{Lie}(\mathcal{G})$ *possesses a self-centralizing torus. If* $\dim \mathcal{V}_{\mathrm{Lie}(\mathcal{G})} \leq 2$*, then $\mathcal{G}$ is isomorphic to a subgroup of* $\mathrm{SL}(2)$ *that contains the first Frobenius kernel* $\mathrm{SL}(2)_1$.

Proof. According to (4.3) the Lie algebra $\mathrm{Lie}(\mathcal{G})$ is either supersolvable or the factor algebra $\mathrm{Lie}(\mathcal{G})/T(\mathrm{Lie}(\mathcal{G}))$ is isomorphic to $s\ell(2)$. In the former case induction on the height of $\mathcal{G}$ in conjunction with [2, (IV.§4(2.6)] yields the solvability of $\mathcal{G}$, a contradiction.

Next, we observe that $C(\mathrm{Lie}(\mathcal{G}))$ and $T(\mathrm{Lie}(\mathcal{G}))$ are invariant under the adjoint representation $Ad : \mathcal{G} \longrightarrow \mathcal{AUT}(\mathrm{Lie}(\mathcal{G}))$. Since $\mathcal{G}$ is connected, [26, (7.7)] implies that $\mathcal{G}$ acts trivially on $T(\mathrm{Lie}(\mathcal{G}))$. Thus, $T(\mathrm{Lie}(\mathcal{G})) \subset \mathrm{Lie}(\mathcal{G})^{\mathcal{G}} = \mathrm{Lie}(Z(\mathcal{G}))$ is contained in the Lie algebra of the center of $\mathcal{G}$. Since $\mathcal{G}$ is semisimple, the toral radical $T(\mathrm{Lie}(\mathcal{G}))$ is therefore trivial, and $\mathrm{Lie}(\mathcal{G}) \cong s\ell(2)$.

Consequently, the adjoint representation of $\mathcal{G}$ gives rise to a homomorphism

$$\varrho : \mathcal{G} \longrightarrow \mathcal{AUT}(s\ell(2)) \cong \mathrm{PSL}(2)$$

of group schemes whose differential $\mathrm{Lie}(\varrho) : s\ell(2) \longrightarrow s\ell(2)$ is bijective. Hence ϱ is a closed embedding, and in view of [2, (II.§7 (4.1))] the image $\varrho(\mathcal{G})$ contains the first Frobenius kernel of PSL(2). As $\mathcal{G}$ is infinitesimal, the homomorphism ϱ factors through a suitable Frobenius kernel of PSL(2). But PSL(2) is a factor group of SL(2) by an étale subgroup, so $\mathrm{PSL}(2)_r \cong \mathrm{SL}(2)_r$ for every $r \geq 1$. $\square$

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