On the automorphism group of the rank two free group

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Let Φ be the free group of rank two on generators ξ and η, and call F the group of those automorphisms of Φ which fix the commutator of ξ and η. In this note we present a simple criterion which characterizes the elements of F. It in some sense counterparts the paper of Osborne-Zieschang [3].

We fix some notation:

We understand the commutator of ξ and η to be $\lambda = \eta^{-1}\xi^{-1}\eta\xi$.

For $\omega \in \Phi$ let $\tilde{\omega}$ denote the word in Φ which arises from ω by replacing ξ by ξ^{-1} and η by η^{-1}, The mapping $\omega \longrightarrow \tilde{\omega}$ is an automorphism of Φ, it conjugates λ to $\tilde{\lambda} = (\xi\eta)\lambda(\xi\eta)^{-1}$, and its square is the identity. We call ω a palindrome if it is such in common language understanding:

$$\tilde{\omega} \cdot \omega = 1.$$

Note that a palindrome is cyclically reduced if and only if it is a power of ξ or η, the case $\xi^0 = \eta^0 = 1$ being included.

We define an extension of Φ by an infinite cyclic group with generator τ by requiring

$$\tau\xi\tau^{-1} = (\xi\eta)^{-1}\xi^{-1}(\xi\eta),\ \tau\eta\tau^{-1} = (\xi\eta)^{-1}\eta^{-1}(\xi\eta).$$

We have

$$\tau^2\xi\tau^{-2} = \lambda\xi\lambda^{-1},\ \tau^2\eta\tau^{-2} = \lambda\eta\lambda^{-1},$$

which means that τ^2 acting on Φ via conjugation by λ may be identified to an element of F. The fact that τ may in some sense be viewed as a square root of λ plays an important rôle in geometry, see [2].

Lemma. Let $\alpha, \beta \in \Phi$ such that

$$\xi \longrightarrow \alpha,\ \eta \longrightarrow \beta$$

defines an element of F, the group of automorphisms of Φ which fix λ. Then

$$\tilde{\alpha} = \eta\xi\alpha^{-1}\eta^{-1}\xi^{-1},\ \tilde{\beta} = \xi\eta\beta^{-1}\xi^{-1}\eta^{-1},$$

or, in other words: $\eta\alpha\xi^{-1}$ and $\xi\beta\eta^{-1}$ are palindromes.

Proof. We recall that the group F is generated by

$$f_1 :\ \xi \longrightarrow \xi,\ \eta \longrightarrow \xi\eta,$$

$$f_2 :\ \xi \longrightarrow \eta\xi,\ \eta \longrightarrow \eta.$$

The statement of the lemma is easily checked for powers of f_1 and f_2. So we have to show that assuming it is true for α and β it will remain true if this pair is replaced by α and $\alpha\beta$, or α and $\alpha^{-1}\beta$, or $\beta\alpha$ and β, or $\beta^{-1}\alpha$ and β.

We compute, in the first case:

$$\widetilde{(\alpha\beta)} = \tilde{\alpha}\tilde{\beta} = \eta\xi\alpha^{-1}\eta^{-1}\xi^{-1} \cdot \xi\eta\beta^{-1}\xi^{-1}\eta^{-1} = \eta\xi\alpha^{-1}\beta^{-1}\xi^{-1}\eta^{-1}.$$

But we have assumed

$$\beta^{-1}\alpha^{-1}\beta\alpha = \eta^{-1}\xi^{-1}\eta\xi,$$

so

$$\widetilde{(\alpha\beta)} = \eta\xi\xi^{-1}\eta^{-1}\xi\eta\beta^{-1}\alpha^{-1}\xi^{-1}\eta^{-1} = \xi\eta(\alpha\beta)^{-1}\xi^{-1}\eta^{-1}.$$

Similarly in the third case:

$$\widetilde{(\beta\alpha)} = \tilde{\beta}\tilde{\alpha} = \xi\eta\beta^{-1}\xi^{-1}\eta^{-1} \cdot \eta\xi\alpha^{-1}\eta^{-1}\xi^{-1}$$

$$= \xi\eta\eta^{-1}\xi^{-1}\eta\xi\alpha^{-1}\beta^{-1}\eta^{-1}\xi^{-1} = \eta\xi(\beta\alpha)^{-1}\eta^{-1}\xi^{-1}.$$

The second and fourth cases are dealt with the same way.

So the lemma is proven.

Now we can state the

Theorem. Let $\alpha, \beta \in \Phi$. Then the assignment

$$\xi \longrightarrow \alpha,\ \eta \longrightarrow \beta$$

defines an automorphism of Φ which fixes λ if and only if $\eta\alpha\xi^{-1}, \xi\beta\eta^{-1}$, and $\eta\beta\alpha\xi^{-1}$ are all palindromes. (Here $\eta\beta\alpha\xi^{-1}$ may be replaced by $\xi\alpha\beta\eta^{-1}$.)

Proof. The necessity of the conditions on α and β has been the subject of the lemma. So we show sufficiency. By a well known theorem of Nielsen's (see [1]) we have to show only that the commutator $\beta^{-1}\alpha^{-1}\beta\alpha$ of α and β is λ. This is a matter of straightforward computation:

$$1 = \widetilde{(\eta\beta\alpha\xi^{-1})} \cdot (\eta\beta\alpha\xi^{-1}) = \eta^{-1}\tilde{\beta}\tilde{\alpha}\xi \cdot \eta\beta\alpha\xi^{-1}$$
$$= \eta^{-1}\xi\eta\beta^{-1}\xi^{-1}\eta^{-1}\eta\xi\alpha^{-1}\eta^{-1}\xi^{-1}\xi\eta\beta\alpha\xi^{-1} = \eta^{-1}\xi\eta\beta^{-1}\alpha^{-1}\beta\alpha\xi^{-1}.$$

So the commutators coincide; the computation is completely analoguous in the alternative case mentioned in the statement. So the theorem is proven.

The theorem may be given the following form: Let

$$P = \{(\omega, \pi) \mid \omega, \pi, \text{ and } \omega^{-1}\eta\xi^{-1}\pi \text{ are palindromes}\}.$$

Then the assignment

$$P \ni (\omega, \pi) \to f \in F$$

where by definition

$$f(\xi) = \eta^{-1}\omega\xi, \ f(\eta) = \xi^{-1}\pi\eta$$

establishes a bijection between P and F.

There are palindromes which are useless for constructing elements of F; $\omega = \eta\xi\eta^{-1}\xi\eta$ is an example: There is no palindrome ω^* such that $\alpha = \eta^{-1}\omega\xi$ and $\beta = \xi^{-1}\omega^*\eta$ would constitute an automorphism of Φ which leaves invariant λ. On the other hand: The palindromes which fit in this context if $\omega = \xi^m$, m arbitrary integral, is given are $\omega^* = \xi(\eta^{-1}\xi^{m+1})^n\eta^{-1}\xi$, n arbitrary integral.

References

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[2] T. Jørgensen: *On pairs of once punctured tori*, unpublished manuscript

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