On the representation dimension of finite dimensional algebras

by

Changchang Xi

Department of Mathematics, Beijing normal University,
100875 Beijing, P.R.China; e-mail: xicc@bnu.edu.cn

1 Introduction

The importantce of the representation finite artin algebras for the whole representation theory of artin algebras is well understood and there is a large literature on the subject (see [1], [6] and [12]). These are artin algebras such that every indecomposable module is finitely generated and every module is a direct sum of indecomposable modules. Moreover there is a bijective correspondence between the class of representation finite artin algebras and that of artin algebras with global dimension smaller than or equal to 2. This result was established by Auslander in [1]. Motivated by this correspondence, M. Auslander introduced the concept of representation dimension for artin algebras to study the connection of arbitrary artin algebras with representation finite artin algebras. "It is hopped that this notion gives a reasonable way of measuring how far an artin algebra is from being representation finite type" ([1], p.134).

Representation dimension is a Morita-invariant of artin algebras and its definition involves homological dimensions and modules which are both generators and cogenerators. Unfortunately, little seems to be known about representation dimension. So there is a lot of essential questions on this invariant. Dealing with the computation and estimation of the representation dimension of an artin algebra, there is a few cases are known: M. Auslander proved that the representation dimension of an artin algebra is two if and only if the algebra is representation finite, and that the global dimension being at most 1 implies that the representation dimension is at most 3. Later, Fossum, Griffith and Reiten proved in [5] that for the 2×2 triangular matrix algebra over an algebra A the representation dimension is upper bounded by the representation dimension of A plus 2

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(see Th. 7.3 in [5]). The goal of this note is to enlarge the knowledge of representation dimensions. More precisely, in section three we are going to study the representation dimension of the tensor product of two algebras and give a upper bound of the representation dimension for tensor products. From this we obtain that the representation dimension of $n \times n$ triangular matrix algebra over an algebra A is still bounded by the representation dimension of A plus 2. In section four we investigate the representation dimensions of algebras and their factor algebras by powers of the radicals. In section five we consider the relationship of the representation dimension and the global dimension of an algebra. In the last section we discuss a special case of algebras which are one-point extensions. (Note that it is not yet known whether the representation dimension of an algebra is finite).

Throughout this paper we work with artin algebras which are finite dimensional k-algebras over a fixed field k with the identity 1. By a module we mean a finitely generated left module. The global dimension of an algebra A is denoted by $gl.dim(A)$. By D we denote the duality $Hom_k(-, k)$, and by A-mod the category of all A-modules.

2 Preliminaries and definitions

In this section we recall the definition of representation dimension from [1] and some relevant notion as well as some results which we need later.

Given a finite dimensional algebra A, we say that A has dominant dimension greater than or equal to n, denoted by dom.dim $(A) \geq n$, if there is an exact sequence

$$0 \longrightarrow_A A \longrightarrow X_1 \longrightarrow X_2 \longrightarrow ...$$

of A-modules such that X_i is projective and injective for $i = 1, ..., n$. We denote by $I_0(A)$ the module X_1.

M. Auslander proved the following theorem, which motivated him to introduce the notion of representation dimension, as a way of measuring how far a finite dimensional algebra is from being representation finite type.

Theorem 2.1 *Suppose A is a finite dimensional algebra with gl.dim $(A) \leq 2$. If P is a projective and injective A-module, then $End_A(P)$ has representation finite type. Further, up to Morita equivalence, all finite dimensional algebras of representation finite type are obtained in this way.*

The representation dimension is defined as follows.

Definition 2.2 *Let A be a finite dimensional algebra over a field k. Consider the finite dimensional algebra Λ of dominant dimension at least two such that $End_\Lambda(I_0(\Lambda))$ is Morita equivalent to A. Then the representation dimension of A is defined to be the minimum of the global dimension of all possible Λ, and denoted by rep.dim (A).*

In fact, M. Auslander also proved in [1] that the above definition is equivalent to the following definition:

$$rep.dim(A) = inf\{gl.dim(End_A(M)) \mid M \text{ is a generator-cogenerator}\}$$

Note that an A-module M is called a generator-cogenerator if every indecomposable projective module and also every indecomposable injective module is isomorphic to a summand of M.

The following lemma collects some known results on the representation dimension which we shall need in the sequel.

Lemma 2.3 *Let A be a non-semisimple k-algebra. Then*

(1) $rep.dim(A) = 2$ if and only if A is representation finite.

(2) If A is selfinjective algebra, then $rep.dim(A) \leq LL(A)$, where $LL(A)$ stands for the Loewy length of A.

(3) If $gl.dim\ (A) \leq 1$, then $rep.dim\ (A) \leq 3$.

(4) Let $T_2(A)$ denote the 2×2 triangular matrix algebra over A, then rep.dim $(T(A)) \leq rep.dim(A) + 2$.

The statements (1)–(3) are proved in [1], and the statement (4) was shown in [5], p.115.

3 Representation dimension of tensor products

Given two finite dimensional k-algebras A and B, we may form the tensor product $A \otimes_k B$ of A and B over k, which is still a finite dimensional k-algebra with the multiplication

$$(a \otimes b)(a_1 \otimes b_1) = aa_1 \otimes bb_1$$

for all $a, a_1 \in A$ and $b, b_1 \in B$. Here we write $\otimes$ for $\otimes_k$.

For any A-module M and B-module N, we have an $(A \otimes B)$-module $M \otimes N$ given by $(a \otimes b)(m \otimes n) = am \otimes bn$ for all $a \in A, b \in B, m \in M$ and $n \in N$. For $(A \otimes B)$-modules obtained in this way, we have the following properties.

Lemma 3.1 *Suppose that M is an A-module and N is a B-module. Then*

(1) If M is a projective A-module and N is a projective B-module, then $M \otimes N$ is a projective $(A \otimes B)$-module.

(2) If M is an injective A-module and N is an injective B-module, then $M \otimes N$ is an injective $(A \otimes B)$-module.

Proof. (1) follows immediately from the proposition 2.3 of Chapter IX in [3].

(2) Since $D(A \otimes B) = Hom_k(A \otimes B, k) \cong (DA) \otimes (DB)$, we see that the tensor product of the injective A-module DA and the injective B-module DB is an injective $(A \otimes B)$-module. This implies the statement (2).

One may consider the functor ${}_AM \otimes - : B\text{-mod} \longrightarrow (A \otimes B)\text{-mod}$. The following lemma shows that this functor has some nice properties.

Lemma 3.2 *For A-modules X, Y and B-modules M, N, we have*

$$Hom_A(X, Y) \otimes Hom_B(M, N) \cong Hom_{A \otimes B}(X \otimes M, Y \otimes N)$$

(as isomorphism of vector spaces).

For the proof of this lemma, see [3], Chapt. XI, Theorem 3.1, p. 209-210. From this lemma we get the following corollary.

Corollary 3.3 *(1) $End_{A \otimes B}(X \otimes M) = End_A(X) \otimes End_B(M)$.*

(2) If ${}_AX$ and ${}_BY$ are indecomposable, then $X \otimes Y$ is indecomposable.

(3) Let ${}_AX_i$, ${}_BM_i, i = 1, 2$, be indecomposable, then $X_1 \otimes M_1 \cong X_2 \otimes M_2$ as $(A \otimes B)$-module if and only if $X_1 \cong X_2$ and $M_1 \cong M_2$.

The following result on the global dimension is well-known.

Lemma 3.4 *Suppose k is a perfect field. Let A, B be two finite dimensional k-algebras. Then*

$$gl.dim(A \otimes B) = gl.dim(A) + gl.dim(B)$$

Combinating 3.1 and 3.3 we see that if ${}_AX$ and ${}_BM$ are indecomposable projective A-module and B-module respectively, then $X \otimes M$ is an indecomposable projective $(A \otimes B)$-module. The next result says that every indecomposable projective (respectively, injective) $(A \otimes B)$-module appears in this way.

Lemma 3.5 *Suppose k is a perfect field. If P is an indecomposable projective (respectively, injective) $(A \otimes B)$-module, then there is an indecomposable projective (respectively, injective) A-module ${}_AX$ and an indecomposable projective (respectively, injective) B-module M such that $P \cong X \otimes M$.*

Proof. Let $\{e_1, ..., e_n\}$ and $\{f_1, ..., f_m\}$ be complete sets of primitive orthogonal idempotents of A and B respectively. Then $\{e_i \otimes f_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$ is a complete set of primitive orthogonal idempotents for $A \otimes B$. This can be seen as follows: Since e_iAe_i and f_jBf_j are local algebras and k is a perfect field, the tensor product of local algebras is again a local algebra. Hence $e_i \otimes f_j$ is a primitive idempotent element in $A \otimes B$ and every indecomposable projective $(A \otimes B)$-module is isomorphic to $(A \otimes B)(e_i \otimes f_j) \cong Ae_i \otimes Bf_j$.

For the injective $(A \otimes B)$-module we just use the duality D. This finishes the proof.

Now let us prove the following result.

Theorem 3.6 *Suppose k is a perfect field. Let A and B be two finite dimensional k-algebras. Then*

$$rep.dim(A \otimes B) \leq rep.dim(A) + rep.dim(B).$$

Proof. Let X be an A-module which contains each indecomposable projective module and each indecomposable injective module as direct summand such that $rep.dim(A) = gl.dim(End_A(X))$. Let N be a such B-module with the same property that $rep.dim(B) = gl.dim(End_B(N))$. Then, by 3.5, we know that $M \otimes N$ is an $(A \otimes B)$-module which contains each indecomposable projective and each indecomposable injective $(A \otimes B)$-module as a summand. Thus

$$\begin{aligned} rep.dim(A \otimes B) &\leq gl.dim(End_{A \otimes B}(M \otimes N)) \\ &= gl.dim(End_A(M) \otimes End_B(N)) \\ &= gl.dim(End_A(M)) + gl.dim(End_B(N)) \\ &= rep.dim(A) + rep.dim(B). \end{aligned}$$

As a consequence of the above result, we have the following corollary. Of course, the statement (1) of the corollary is a generalization of Theorem 7.3 in [5].

Corollary 3.7 *Suppose k is a perfect field. Let A be a finite dimensional k-algebra.*

(1) Let $T_n(A)$ be the $n \times n$ triangular matrix algebra with entries in A:

$$T_n(A) = \begin{pmatrix} A & A & \dots & A \\ 0 & A & \dots & A \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A \end{pmatrix}.$$

Then $rep.dimT_n(A) \leq rep.dim(A) + 2$.

(2) If B is a selfinjective k-algebra with $LL(B) = m$, then $rep.dim(A \otimes B) \leq rep.dim(A) + m$.

Proof. (2) follows from 2.3(2) and 3.6. As to (1), note just that $T_n(A) = T_n(k) \otimes A$ and that $rep.dimT_n(k) = 2$ for $n \geq 2$ and $rep.dimT_1(k) = 0$.

Clearly, we have that $rep.dim(A \oplus B) = max\{rep.dim(A), rep.dim(B)\}$. So if B is a direct sum of matrix algebras over an arbitrary field k, then, by definition, $rep.dim(B) =$

0. In this case , we have $rep.dim(A \otimes B) = rep.dim(A) + rep.dim(B)$. If B is not semisimple, the upper bound may not be attained. This can be seen from the following examples which show also that in general one cannot hope to have the equality in 3.6:

(1) If $A = B = T_2(k)$, then $rep.dim(A \otimes B) = 2 = rep.dim(A)$ by 2.3 .

(2) If $A = k[T]/(T^2)$, then since the Loewy length of $A \otimes A$ is 3, we have $rep.dim(A \otimes A) = 3$, but $rep.dim(A) = 2$ again by 2.3.

Proposition 3.8 *Let A and B be two selfinjective algebras over an arbitrary field k such that $A/rad(A)$ or $B/rad(B)$ is separable. Then*

$$rep.dim(A \otimes B) \leq LL(A) + LL(B) - 1,$$

where $LL(A)$ denotes the Loewy length of A.

Proof. Suppose $A/rad(A)$ is separable, that is, for any extension field L of k, the L-algebra $L \otimes (A/rad(A))$ is semisimple. Then the radical of $A \otimes B$ is $rad(A) \otimes B + A \otimes rad(B)$ (see [4], p.116). One can verify that the Loewy length of $A \otimes B$ is $LL(A) + LL(B) - 1$. Thus the proposition follows from 2.3 since the tensor product of two selfinjective algebras is again selfinjective by a result in [10].

One special case of tensor products is that one of the algebras A and B is a field extension of k. The following lemma is true by [8].

Lemma 3.9 *Let L be a finite dimensional separable extension of a field k. We denote by $rep.dim_k(A)$ the representation dimension of the k-algebra A. Then $rep.dim_k(A) = 2$ if and only if $rep.dim_L(L \otimes A) = 2$.*

Usually, the representation dimension of A changes, when we change the base field. This can be seen by the following example.

Let k be a non-perfect field of characteristic p. Then there is an extension field of k which is not separable. So there is a purely inseparable extension L of k of exponent one (i.e., $L = k(\alpha)$ with $\alpha \notin k$ and $\alpha^p \in k$). Then by [7], p.491, the tensor product $L \otimes L$ has a nilpotent element, so the L-algebra $L \otimes L$ is not semisimple. Note that L is a finite dimensional semisimple k-algebra. Thus $rep.dim_L(L \otimes L) \geq 2$ and $rep.dim_k(L) = 0$. For other examples one may see the paper [8].

Remark. It is unknown whether over a perfect field k the tensor product of two algebras A and B has representation dimension at least $max\{rep.dim(A), rep.dim(B)\}$.

4 Representation dimension and the radical of an algebra

M. Auslander proved in [1] that if $rep.dim(A/rad^{n-1}(A)) \leq 2$, then $rep.dim(A) \leq 3$. And then he asked a question whether $rep.dim(A) \leq rep.dim(A/rad^{n-1}(A)) + 1$ holds, where n is the nilpotent index of the radical $rad(A)$. Motivated by this question we shall provide a related bound in this section. The main result in this section is the following theorem which can be used to estimate the upper bound for representation dimensions of certain "complicated" algebras.

Theorem 4.1 *Let A be a finite dimensional k-algebra with Jacobson radical N of nilpotence index n, that is, $N^n = 0 \neq N^{n-1}$. Suppose that for each indecomposable injective A-module I with $N^{n-1}I \neq 0$ the indecomposable direct summands of $I/N^{n-1}I$ are either injective A/N^{n-1}-modules or projective A/N^{n-1}-modules, then $rep.dim(A) \leq rep.dim(A/N^{n-1}) + 3$.*

Before we give the proof of this theorem, let us first recall some definitions and facts needed.

Let $\mathcal{X}$ be a full subcategory of A-mod which is closed under direct sums and isomorphisms, and $X, M \in A$-mod. A homomorphism $f : X \longrightarrow M$ is called minimal if an endomorphism $g : X \longrightarrow X$ is an automorphism whenever $f = gf$. The morphism f is called a right $\mathcal{X}$-approximation of M if $X \in \mathcal{X}$ and for each homomorphism $g : Y \longrightarrow M$ with $Y \in \mathcal{X}$ there is a homomorphism $h : Y \longrightarrow X$ such that $g = hf$. If in addition f is minimal then we call f a minimal right $\mathcal{X}$-approximation of M.

The following lemma is true (see for example [2]).

Lemma 4.2 *Let $\mathcal{X}$ be a full subcategory of A-mod closed under direct sums and isomorphisms. If there are only finitely many indecomposable A-modules in $\mathcal{X}$ (up to isomorphisms), then for every A-module M in A-mod there is a minimal right $\mathcal{X}$-approximation of M. If ${}_AA \in \mathcal{X}$ then each minimal right $\mathcal{X}$-approximation is surjective.*

Let M be an A-module. We denote by $add(M)$ the full subcategory of A-mod whose objects are isomorphic to direct summands of direct sums of finite copies of M.

Let $\mathcal{C}$ be a skeletally small category. We denote by $\mathcal{C}^{op}$ the opposite category of $\mathcal{C}$ and by Funct$(\mathcal{C}^{op}, Ab)$ the abelian category of all functors from $\mathcal{C}^{op}$ to the category Ab of abelian groups. Let $\widehat{\mathcal{C}}$ be the full subcategory of Funct$(\mathcal{C}^{op}, Ab)$ consisting of all coherent functors G, that is, those functors G for which there is a morphism $C_1 \longrightarrow C_2$ in $\mathcal{C}$ such that the sequence

$$(\,, C_1) \longrightarrow (\,, C_2) \longrightarrow G \longrightarrow 0$$

is exact. Here and in the sequel we denote by $(\,, C)$ the Hom functor $\mathrm{Hom}_{\mathcal{C}}(\,, C) : \mathcal{C}^{op} \longrightarrow Ab$ for $C \in \mathcal{C}$.

The following lemma is proved in [1].

Lemma 4.3 *Let M be in A-mod. Then the category $\widehat{add(M)}$ and $End(M)$-mod are equivalent. In particular, $gl.dim(End_A(M)) = gl.dim(\widehat{add(M)})$.*

Note that for each module $X \in A$-mod the functor $(, X)$ belongs to $\widehat{add(M)}$ by 4.2.

Now we prove the theorem 4.1.

Let $B = A/N^{n-1}$ and V_0 be a B-module such that $gl.dim(End_B(V_0)) = rep.dim(B)$. We may assume that $rep.dim(B) = m < \infty$. (Otherwise there is nothing to prove). Let $P_1, ..., P_t$ be a complete set of non-isomorphic indecomposable projective A-modules and $I_1, ..., I_s$ a complete set of non-isomorphic indecomposable injective A-modules. Put $V := V_0 \oplus \bigoplus_{i=1}^{t} P_i \oplus \bigoplus_{j=1}^{s} I_j$ and define $\mathcal{X} = add(V_0)$. We want to show that $gl.dim(End_A(V)) \leq rep.dim(B) + 3$.

Suppose M is an indecomposable A-module. We show that there is an exact sequence $0 \longrightarrow X_{m+3} \longrightarrow ... \longrightarrow X_3 \longrightarrow X_2 \longrightarrow M \longrightarrow 0$ with all $X_i \in add(V)$ such that the induced sequence

$$0 \longrightarrow (X, X_{m+3}) \longrightarrow ... \longrightarrow (X, X_3) \longrightarrow (X, X_2) \longrightarrow (X, M) \longrightarrow 0$$

is exact for all $X \in \mathrm{add}(V)$

1) If $M \in add(V)$ then we simply define $X_2 = M$ and $X_3 = ... = X_{m+3} = 0$ and the morphism $X_2 \longrightarrow M$ the identity.

2) Suppose that M is not in $add(V)$. There are two cases to be considered.

The first case: M is a B-module. Then we consider the functor $(, M) : \mathcal{X}^{op} \longrightarrow Ab$. Since $gl.dim(End_B(V_0)) = gl.dim(\widehat{add(V_0)}) = m$ by 4.3, we have a minimal projective resolution for $(, M)$:

$$0 \longrightarrow (, X_{m+2}) \longrightarrow ... \longrightarrow (, X_3) \longrightarrow (, X_2) \longrightarrow (, M) \longrightarrow 0$$

with all $X_i \in \mathcal{X}$. Since $B \in \mathcal{X}$, we have an exact sequence of B-modules

$$0 \longrightarrow X_{m+2} \longrightarrow ... \longrightarrow X_3 \longrightarrow X_2 \longrightarrow M \longrightarrow 0$$

We will show that this is a desired sequence . If X is a projective A-module, then the sequence

$$(*) \quad 0 \longrightarrow (X, X_{m+2}) \longrightarrow ... \longrightarrow (X, X_3) \longrightarrow (X, X_2) \longrightarrow (X, M) \longrightarrow 0$$

is exact. Now assume that X is an indecomposable injective A-module. If X is also a B-module, then X is a direct summand of V_0, thus in $\mathcal{X}$. So there is nothing to prove. We assume that X is not a B-module. Then $0 \neq N^{n-1}X$ is the socle of X and $X/Soc(X)$ is either an injective or a projective B-module by assumption. Since V_0 contains all indecomposable injective B-module and $(X, M_1) = (X/Soc(X), M_1)$ for any indecomposable B-module M_1, we see that $(*)$ is exact if X is an indecomposable injective A-module with $N^{n-1}X \neq 0$. Hence for all $X \in add(V)$ the sequence $(*)$ is exact.

The second case: M is not a B-module. We take $M' = \{m \in M \mid N^{n-1}m = 0\}$. Then M' is a B-module. Suppose that $l : X_0 \longrightarrow M'$ is a right minimal $\mathcal{X}$-approximation of M' and $g : P \longrightarrow M/M'$ a projective cover of A-modules and $h : P \longrightarrow M$ a lifting such that $g = h\pi$, where π is the canonical homomorphism $M \longrightarrow M/M'$. Define $f : X_0 \oplus P \longrightarrow M$ by $(x,p) \mapsto l(x) + h(p)$. If $(x,p) \in Ker(f)$ with $x \in X_0$ and $p \in P$ then $p \in Ker(g)$. Since P is a projective cover, we have $Ker(g) \subset NP$. Hence $Ker(f)$ is a B-module because $N^{n-1}(x,p) \subset (N^{n-1}x, N^{n-1}p) = 0$. Now we show that for any $X \in add(V)$ the induced map $(X, X_0 \oplus P) \longrightarrow (X, M)$ is surjective. If $X \in \mathcal{X}$ then the image of any homomorphism from X to M is a B-module, thus lies in M'. This implies that the induced map is surjective. If X is projective then there is nothing to show. So we assume that X is an indecomposable injective A-module not in $\mathcal{X}$. Since $\mathcal{X}$ contains all indecomposable injective B-modules, we know that X is not a B-module and hence $N^{n-1}X \neq 0$. Let g be a homomorphism from X to M. Then g is not injective. Otherwise we would have $M \cong X \in add(V)$. This yields that $N^{n-1}X = Soc(X) \subset Ker(g)$. Hence $(X, M) = (X/Soc(X), M) = (X/Soc(X), M')$. Since every homomorphism from $X/Soc(X)$ to M' factors through $l : X_0 \longrightarrow M'$, it follows that the map $(X, X_0 \oplus P) \longrightarrow (X, M)$ is surjective. Hence we have proved that for all $X \in \mathrm{add}(V)$ the induced map is surjective.

Now by the previous result we have an exact sequence

$$0 \longrightarrow X_{m+3} \longrightarrow X_{m+2} \longrightarrow \dots \longrightarrow X_3 \longrightarrow Ker(f) \longrightarrow 0$$

with all $X_i \in \mathcal{X} \subset \mathrm{add}(V)$ such that for all $X \in \mathrm{add}(V)$ the sequence

$$0 \longrightarrow (X, X_{m+3}) \longrightarrow (X, X_{m+2}) \longrightarrow \dots \longrightarrow (X, X_3) \longrightarrow (X, Ker(f)) \longrightarrow 0$$

is exact for all $X \in add(V)$. Define $X_2 := X_0 \oplus P$, then we obtain an exact sequence

$$0 \longrightarrow (X, X_{m+3}) \longrightarrow (X, X_{m+2}) \longrightarrow \dots \longrightarrow (X, X_3) \longrightarrow (X, X_2) \longrightarrow (X, M) \longrightarrow 0$$

for all $X \in \mathrm{add}(V)$, where all X_i are in $add(V)$.

Now we establish that $gl.dim(\widehat{add(V)}) \leq m + 3$. Take a functor G in $\widehat{add(V)}$. Then there is a morphism $f : X_1 \longrightarrow X_0$ in $\mathrm{add}(V)$ such that $(, X_1) \longrightarrow (, X_0) \longrightarrow G \longrightarrow 0$ is exact. Letting $M = Ker(f)$, we know that there is an exact sequence

$$0 \longrightarrow X_{m+3} \longrightarrow X_{m+2} \longrightarrow \dots \longrightarrow X_2 \longrightarrow M \longrightarrow 0$$

is exact with all $X_i \in add(V)$ such that the induced sequence

$$0 \longrightarrow (X, X_{m+3}) \longrightarrow (X, X_{m+2}) \longrightarrow \dots \longrightarrow (X, X_2) \longrightarrow (X, M) \longrightarrow 0$$

is exact for all $X \in add(V)$. Thus the sequence

$$0 \longrightarrow (,X_{m+3}) \longrightarrow ... \longrightarrow (,X_2) \longrightarrow (,X_1) \longrightarrow (,X_0) \longrightarrow G \longrightarrow 0$$

is exact in $add(V)$. This shows that $proj.dim(G) \leq m+3$. By 4.3, we get $gl.dim(End_A(V)) \leq m+3 = rep.dim(B)+3$. Thus the proof is completed.

Remark. From the proof we may formulate the following fact: Let M be an A/N^{n-1}-module such that $rep.dim(A/N^{n-1}) = gl.dim(End_{A/N^{n-1}}(M))$. If each injective indecomposable A-module I with $N^{n-1}I \neq 0$ has the factor module $I/N^{n-1}I$ in $add(M)$, then $rep.dim(A) \leq rep.dim(A/N^{n-1}) + 3$. In particular, if $rep.dim(A/N^{n-1}) \leq 2$, then $rep.dim(A) \leq 5$. (Compare this with Auslander's result).

As a consequence of the theorem, we obtain the following result of M.Auslander [1].

Corollary 4.4 *Let A be a finite dimensional k-algebra. If $rad^2(A) = 0$, then $rep.dim(A) \leq 3$.*

Finally, let us give an example of a finite dimensional algebra satisfying the condition in the theorem 4.1.

Example. Consider the k-algebra A with the following decomposition

$$_AA = \begin{matrix} & 1 & \\ 1 & 1 & 2 \\ & 1 & \\ & 2 & \end{matrix} \oplus \begin{matrix} 2 \\ 1 \\ 2 \end{matrix}$$

Then the injective module $D(A_A)$ looks as follows

$$D(A_A) = \begin{matrix} & 1 & \\ 1 & 1 & 2 \\ & 1 & \end{matrix} \oplus \begin{matrix} & 1 & \\ 1 & 1 & 2 \\ & 1 & \\ & 2 & \end{matrix}$$

We can see that the condition in 4.1 is satisfied. Hence $rep.dim(A) \leq rep.dim(A/N^3)+3$. It follows further from 2.3 that $rep.dim(A) \leq 6$.

5 Representation dimension and global dimension

In this section we will discuss the relationship between the global dimension and the representation dimension of a finite dimensional algebra under certain conditions. Our result generalises a result of Auslander which says that if $gl.dim(A) \leq 1$ then $rep.dim(A) \leq 3$.

Let M be an A-module. We denote by $\mathrm{Fac}(M)$ the full subcategory of A-mod consisting of all modules which are generated by M. Recall that a full subcategory of A-mod closed under direct summands is said to be of finite type if it contains only finitely many indecomposable A-modules (up to isomorphisms).

Theorem 5.1 *Let A be a finite dimensional algebra. If $Fac(D(A))$ is of finite type and if $Hom_A(X, M) = 0$ for all $X \in Fac(D(A))$ and all indecomposable module $M \notin Fac(D(A))$, then $rep.dim(A) \leq gl.dim(A) + 2$.*

Proof. Let $P_1, ..., P_t$ be a complete set of all non-isomorphic indecomposable projective A-modules. Since $\mathrm{Fac}(D(A))$ is of finite type, we may assume that $\{N_1, ..., N_s\}$ is a complete set of non-isomorphic indecomposable modules in $\mathrm{Fac}(D(A))$. Define $V = P_1 \oplus ... \oplus P_t \oplus N_1 \oplus ... \oplus N_s$. Let M be an A-module. We decompose M into $M_1 \oplus M_2$ where $M_2 \in add(V)$, and M_1 has no indecomposable direct summands in $\mathrm{add}(V)$. We take a projective cover $P_0(M_1) \longrightarrow M_1$ and define $X_2 = P_0(M_1) \oplus M_2$ as well as the canonical surjective homomorphism $f_2 : X_2 \longrightarrow M$. We demonstrate that for any indecomposable $X \in add(V)$ the induced map $(X, X_2) \longrightarrow (X, M)$ is surjective. If X is projective there is nothing to prove. Now let X be some module N_i. Since M_1 does not belong to $add(V)$, we know that $\mathrm{Hom}(N_i, M_1) = 0$ by the hypothesis. So each homomorphism from N_i to M is, in fact, a homomorphism from N_i to M_2, this implies that for $X = N_i$ the induced map is surjective. Hence for all $X \in add(V)$ the induced map is surjective. Note that the module $P_0(M_1)$ is a direct summand of the projective cover P_0 of M.

We can assume that $gl.dim(A) = m < \infty$. To prove the theorem, we show that for each module M we can find an exact sequence

$$0 \longrightarrow X_{m+2} \longrightarrow X_{m+1} \longrightarrow ... \longrightarrow X_3 \longrightarrow X_2 \longrightarrow M \longrightarrow 0$$

with all $X_i \in add(V)$ such that for each $X \in add(V)$ the induced sequence

$$0 \longrightarrow (X, X_{m+2}) \longrightarrow (X, X_{m+1}) \longrightarrow ... \longrightarrow (X, X_3) \longrightarrow (X, X_2) \longrightarrow (X, M) \longrightarrow 0$$

is exact.

Let M be an A-module, and let $0 \longrightarrow P_m \longrightarrow ... \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ be a minimal projective resolution for M. We have defined the module X_2 and the homomorphism $f_2 : X_2 \longrightarrow M$. Consider the kernel of f_2. This kernel is in fact the first syzygy of M_1. Replacing M by the kernel of f_2 and repeating the above procedure, we define a module X_3 and the corresponding homomorphism $f_3 : X_3 \longrightarrow Ker(f_2)$. All these datums have the desired properties. Note again that $P_0(Ker(f_2)_1)$ is a direct summand of P_1. Since the global dimension of A is m, the above procedure must stop at most after m steps, so we have an exact sequence

$$0 \longrightarrow X_{m+2} \longrightarrow X_{m+1} \longrightarrow ... \longrightarrow X_3 \longrightarrow X_2 \longrightarrow M \longrightarrow 0$$

with $X_i \in \text{add}(V)$ and the desired property.

As in the proof of 4.1, we get that $rep.dim(A) \leq m + 2 = gl.dim(A) + 2$. The proof is finished.

Let us mention that hereditary algebras and tame concealed algebras satisfy the conditions in the theorem 5.1.

Corollary 5.2 *(1) If A is a hereditary algebra, then $rep.dim(A) \leq 3$.*

(2) If A is a tame concealed algebra, then $3 \leq rep.dim(A) \leq 4$.

For the definition of tame concealed algebras and also the definition of tubular algebras below, we refer to [11].

Finally, we give another bound of representation dimension related to the global dimension.

Proposition 5.3 *Let A be a finite dimensional k-algebra. If $proj.dim(TrD(A)) \leq 1$, then $rep.dim(A) \leq 1 + 2gl.dim(A)$, where TrD stands for the inverse of Auslander-Reiten translation of the algebra A. In particular, the representation dimension of a tubular algebra is at most 5.*

Proof. One just needs to use a result in [9], p.246 and notes that the condition in the proposition implies that $Hom_A(D(A), A) = 0$.

6 Representation dimension of one-point extensions

In this section we shall consider the representation dimension of algebras which are one-point extensions. These are algebras of the form

$$A = \Lambda[M] = \begin{pmatrix} \Lambda & M \\ 0 & k \end{pmatrix},$$

where Λ is a finite dimensional k-algebra, and M is a Λ-module. The multiplication and the addition are defined as the usual ones of matrices. This algebra A is called the one-point extension of Λ by the Λ-module M (see [11], p. 90).

While we can not say much about the bounds of the representation dimension for arbitrary one-point extensions, we get some results when we impose certain conditions on M. Our result in this direction is

Proposition 6.1 *Let Λ be a finite dimensional algebra over an arbitrary field k, and M a Λ-module. Suppose A is the one-point extension of Λ by M. If M is a simple injective Λ-module, then*

$$rep.dim(\Lambda) \leq rep.dim(A) \leq rep.dim(\Lambda) + 2.$$

Proof. Let X be a Λ-module such that $rep.dim(\Lambda) = gl.dim(End_\Lambda(X))$. Let $P(\omega)$ be the projective A-module with $rad(P(\omega)) = M$. Then $P(\omega)/M$ is an injective A-module, we denote this simple injective module by $I(\omega)$. Now set $Y = X \oplus P(\omega) \oplus I(\omega)$ and consider the endomorphism algebras of ${}_AY$. An easy computation shows that

$$End_A(Y) = \begin{pmatrix} End_\Lambda(X) & Hom_\Lambda(X, P(\omega)) & 0 \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}.$$

Let B be the algebra $T_2(k)$, and V be the $\mathrm{End}_\Lambda(X)$-B bimodule $Hom_\Lambda(X, P(\omega)), 0)$. Then we can rewrite $\mathrm{End}_A(Y)$ as

$$End_A(Y) \cong \begin{pmatrix} End_\Lambda(X) & V \\ 0 & B \end{pmatrix}.$$

By Proposition 5.1 of [9], p.246, we have

$$\begin{aligned} gl.dimEnd_A(Y) & \leq max\{gl.dimEnd_\Lambda(X) + proj.dimV_B + 1, gl.dim(B)\} \\ & \leq max\{rep.dim(\Lambda) + 1 + 1, 1\} = rep.dim(\Lambda) + 2. \end{aligned}$$

Note that the last inequality follows from the fact that $gl.dim(B) = 1$. Note that the condition on M implies that each indecomposable injective A-module is either an injective Λ-module or isomorphic to $P(\omega)$. Since Y contains all non-isomorphic indecomposable projective A-modules and all indecomposable injective A-modules as direct summands, we have $rep.dim(A) \leq gl.dim(\Lambda) + 2$ by definition.

To prove the first inequality of the theorem, we note that if X_0 is an indecomposable A-module with composition factors $I(\omega)$ then the top $X_0/rad(X_0)$ lies in add $I(\omega)$ and the radical of X_0 has no composition factor isomorphic to $I(\omega)$. Hence any generator-cogenerator for the A-mod is of the form $Y = X \oplus P(\omega) \oplus I(\omega) \oplus X_0$, where X is a generator-cogenerator for Λ-mod and where X_0 is a direct sum of indecomposable A-modules such that X_0 has no direct summand isomorphic to $P(\omega)$ or $I(\omega)$ and such that each direct summand of X_0 has top in add $I(\omega)$. Let us consider the endomorphism algebra of Y. Since $Hom_A(X_0, X) = 0 = Hom_A(I(\omega), X_0)$ and $Hom_A(X_0, P(\omega)) = 0$, the endomorphism algebra of Y is isomorphic to the triangular matrix algebra:

$$\begin{pmatrix} End_\Lambda(X) & Hom_\Lambda(X, P(\omega)) & 0 & Hom_\Lambda(X, X_0) \\ 0 & k & k & Hom_A(P(\omega), X_0) \\ 0 & 0 & k & 0 \\ 0 & 0 & Hom_A(X_0, I(\omega)) & End_A(X_0) \end{pmatrix}.$$

Again by [9], p.246, it holds that $gl.dimEnd_A(Y) \geq gl.dimEnd_\Lambda(X)$. This implies that $gl.dimEnd_A(Y) \geq rep.dim(\Lambda)$ and that $rep.dim(A) \geq rep.dim(\Lambda)$ because Y is an arbitrary generator-cogenerator. The proof is finished.

Dually, we have the notion of one-point coextension $[M]\Lambda$ of Λ by a Λ-module M defined by

$$A := [M]\,\Lambda = \begin{pmatrix} k & DM \\ 0 & \Lambda \end{pmatrix} \cong (\Lambda^{op}[DM])^{op},$$

Where Λ^{op} denotes the opposite algebra of Λ. In this case, we have the same statement as in the above proposition. This follows from the fact that $rep.dim(A) = rep.dim(A^{op})$ for all finite dimensional algebra A.

Proposition 6.2 *If M is a simple projective Λ-module, then*

$$rep.dim(\Lambda) \leq rep.dim[M]\Lambda \leq rep.dim(\Lambda) + 2.$$

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