# A self–injective cellular algebra is weakly symmetric

by

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## 1 Introduction

Textbooks on representation theory of finite groups or related topics often use the following hierarchy of finite dimensional algebras:

$$\begin{array}{c} \{semisimple\} \\ \cap \\ \{symmetric\} \\ \cap \\ \{weakly\ \ symmetric\} \\ \cap \\ \{quasi-Frobenius = self-injective\} \end{array}$$

Here self–injective means that each projective module is injective as well, whereas weakly symmetric says that the projective cover of any given simple module is the injective envelope of the same simple module. That is, the permutation $top(P) \mapsto soc(P)$ (for $P$ indecomposable projective–injective) is the identity. It is well–known that in general all these inclusions are proper.

The main result of this note states that inside the class of cellular algebras (which contains group algebras of symmetric groups, various kinds of Hecke algebras, Brauer algebras, Temperley–Lieb algebras, ..., see [2]) the third one of these inclusions is an identity.

**Theorem 1.1** *Let $A$ be a cellular algebra over a field. If $A$ is self–injective, then it is weakly symmetric (that is, the projective cover of a simple module is the injective envelope of the same simple module).*

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We show by examples that the other inclusions given above still are proper, if one restricts attention to cellular algebras.

As a consequence of the theorem we obtain:

**Corollary 1.2** *Let $A$ be a self–injective cellular algebra over a field. Let $P$ be an indecomposable projective $A$–module. Then $P$ either is simple or it has a cell filtration of length at least two. If a cell (=standard) module $\Delta$ is projective, then it must be simple.*

In section two we recall the definition of cellular algebras and then we prove theorem 1.1. Moreover we show by examples, that the other inclusions given above are proper. In section three we discuss some other properties of self–injective cellular algebras. This includes a proof of corollary 1.2.

## 2 Proof of theorem, and some examples

We first have to recall the two equivalent definitions of cellular algebras given in [2] and (later) in [3].

For simplicity we stick to the ground ring being an (arbitrary) field $k$. By algebra we always mean a finite dimensional associative algebra with unit.

**Definition 2.1** *(Graham and Lehrer, [2]) An associative $k$–algebra $A$ is called a* **cellular algebra** *with cell datum $(I, M, C, i)$ if the following conditions are satisfied:*

*(C1) The finite set $I$ is partially ordered. Associated with each $\lambda \in I$ there is a finite set $M(\lambda)$. The algebra $A$ has a $k$–basis $C^{\lambda}_{S,T}$ where $(S,T)$ runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in I$.*

*(C2) The map $i$ is a $k$–linear anti–automorphism of $A$ with $i^2 = id$ which sends $C^{\lambda}_{S,T}$ to $C^{\lambda}_{T,S}$.*

*(C3) For each $\lambda \in I$ and $S,T \in M(\lambda)$ and each $a \in A$ the product $aC^{\lambda}_{S,T}$ can be written as $(\sum_{U \in M(\lambda)} r_a(U,S) C^{\lambda}_{U,T}) + r'$ where $r'$ is a linear combination of basis elements with upper index $\mu$ strictly smaller than $\lambda$, and where the coefficients $r_a(U,S) \in k$ do not depend on $T$.*

In the following we shall call a $k$–linear anti–automorphism $i$ of $A$ with $i^2 = id$ an involution of $A$. In [3] it has been shown that the previous definition is equivalent to the following one.

**Definition 2.2** *([3]) Let $A$ be a $k$–algebra. Assume there is an antiautomorphism $i$ on $A$ with $i^2 = id$. A two–sided ideal $J$ in $A$ is called a* **cell ideal** *if and only if $i(J) = J$ and there exists a left ideal $\Delta \subset J$ such that $\Delta$ has finite $k$–dimension and that there is an isomorphism of $A$–bimodules $\alpha : J \simeq \Delta \otimes_k i(\Delta)$ (where $i(\Delta) \subset J$ is the $i$–image of $\Delta$) making the following diagram commutative:*

$$\begin{tikzcd} J \arrow[r, "\alpha"] \arrow[d, "i"'] & \Delta \otimes_k i(\Delta) \arrow[d, "x \otimes y \mapsto i(y) \otimes i(x)"] \\ J \arrow[r, "\alpha"] & \Delta \otimes_k i(\Delta) \end{tikzcd}$$

*The algebra $A$ (with the involution $i$) is called* **cellular** *if and only if there is a vector space decomposition $A = J'_1 \oplus J'_2 \oplus \ldots \oplus J'_n$ (for some $n$) with $i(J'_j) = J'_j$ for each $j$ and such that setting $J_j = \oplus_{l=1}^{j} J'_l$ gives a chain of two sided ideals of $A$: $0 = J_0 \subset J_1 \subset J_2 \subset \ldots \subset J_n = A$ (each of them fixed by $i$) and for each $j$ ($j = 1, \ldots, n$) the quotient $J'_j = J_j/J_{j-1}$ is a cell ideal (with respect to the involution induced by $i$ on the quotient) of $A/J_{j-1}$.*

The modules $\Delta(j), 1 \leq j \leq n$, are called **standard** modules of the cellular algebra $A$, and the above chain in $A$ is called a **cell chain**. (Standard modules are called cell modules in [2]).

Before we can prove the theorem 1.1 we have to demonstrate a lemma.

**Lemma 2.1** *Let $A$ be an algebra (over a field) with cell ideal $J$ (with respect to some involution $i$). Then all indecomposable projective–injective left $A$–ideals intersecting non–trivially with $J$ are isomorphic.*

**Proof.** Let $P_a$ be a projective–injective left $A$–ideal which intersects $J$ non–trivially. We can write $P$ as $Ae$ for some idempotent $e = e^2 \in A$. By definition, $J$ has the form $J \simeq \Delta \otimes_k i(\Delta)$. Hence the intersection $Ae \cap J$ is isomorphic to $\Delta \otimes_k i(\Delta)e$ which is a direct sum of copies of $\Delta$. But the socle $L$ of $P$ is simple. Hence $J \cap Ae$ has precisely one direct summand $\Delta$, and this module must have simple socle which is isomorphic to$L$ as well. Thus if $\Delta$ is a submodule of an injective module, say $I$, then $I$ must be isomorphic to $P$. ■

**Proof of 1.1.**

We fix a self–injective cellular algebra $A$ over a field $k$. We fix a cell chain $0 = J_0 \subset J_1 \subset \ldots \subset J_n = A$ and we fix the involution $i$. We write the unit element as a complete sum $1 = \sum_j e_j$ of pairwise orthogonal non–equivalent idempotents, and we decompose the algebra into $A = \oplus_{j,l} e_j A e_l$. (Note that the idempotents $e_j$ are not required to be primitive. We only have to separate the equivalence classes of idempotents from each other.)

We denote the indecomposable projective left $A$–modules by $P_1, \ldots, P_n$, where the ordering is compatible with the partial order in the cell structure in the following way: Let $P_a$ and $P_b$ be two projective left ideals, and let $J$ be the smallest ideal in the cell chain intersecting $P_a$ non–trivially. Then if $J$ intersects $P_b$ non–trivially, we must have $b \leq a$. In fact, let $J = J_s$ be the smallest ideal in the cell chain with the property that $J \cap P_a \neq 0$ and consider the factor algebra $\bar{A} := A/J_{s-1}$. We denote by $\bar{J}$ (respectively, $\bar{a}$) the image of $J$ (respectively, $a \in A$) under the canonical surjective mapping from $A$ to $A/J_{s-1}$. Thus $\bar{J}$ is a cell ideal in $\bar{A}$, and it gives us a standard module $\Delta$. Since $P_a \cap J_{s-1} = 0$, the projective $A$-module $P_a = Ae_a$ is also a projective-injective $\bar{A}$-module, where $e_a$ is a primitive idempotent element in $A$. Thus we are in the situation of lemma 2.1 which implies $\Delta$ cannot be a submodule of any other injective $\bar{A}$-module. This implies that if $P_b, b \neq a$, is another indecomposable projective–injective $A$-module with $P_b \cap J \neq 0$, then $P_b \cap J_{s-1} \neq 0$, and hence the required compatibility is possible.

By assumption, each $P_a$ is injective, i.e. $P_a \simeq I_{\sigma(a)}$ where the index $\sigma(a)$ is given by some permution $\sigma$. We want to show: $a = \sigma(a)$ for every $a$. We proceed inductively.

Induction start: $a = 1$. By assumption $J = J_1$ intersects $P_1$ non–trivially. Thus $\Delta_1$ has simple socle $L_{\sigma(1)}$ and this socle is generated by an element $x_1$ which is a path from 1 to $\sigma(1)$. This means, $x_1$ can be written as a product $y_{\sigma(1),l_1} \cdots y_{l_j,l_{j+1}} \cdots y_{l_m,1}$ where each element $y_{l_j,l_{j+1}}$ is in

$$e_{l_j} \cdot (rad(A) - rad^2(A)) \cdot e_{l_{j+1}}$$

(Note that the 'arrows' $y_{l_j,l_{j+1}}$ in such a path start and end at equivalence classes of primitive idempotents, not at primitive idempotents. Note, moreover, that the order in the product is chosen in such a way that the starting point 1 is at the very right and the ending point $\sigma(1)$ is at the very left.) It is known ([3], proposition 5.1) that the involution $i$ sends an idempotent to an idempotent in the same equivalence class. Thus the $i$–image $i(x_1)$ is a path from $\sigma(1)$ to 1 in the same sense. But $J$ intersects all other projective (left or right) ideals trivially, hence $fJ = Jf = 0$ for all primitive idempotents $f$ with $Af \not\simeq P_1$, since for example $fJ \neq 0$ would imply $J \cap fA \neq 0$, a contradiction. Since $J$ is fixed by $i$, the element $i(x_1)$ must be in $J$ again. This shows that the socle of $J$ has a direct summand isomorphic to $L(1)$. But we know already that the socle of $J$ is a direct sum of copies of $L(\sigma(1))$ since $P_1 \simeq I_{\sigma(1)}$. This implies $1 = \sigma(1)$.

Induction step: We have to consider the case of $P_j$. We know already that the projectives $P_1, \ldots, P_{j-1}$ are self–dual. Hence $\sigma(j)$ must be greater or equal $j$. Let $J_s$ be the smallest ideal in the cell chain with $P_j \cap J_s \neq 0$. Let $\bar{A} = A/J_{s-1}$. As above we get that the standard module $\Delta$ corresponding to the cell ideal $\bar{J}=J_s/J_{s-1}$ can be considered as a submodule of $P_j$ with a simple socle $L_{\sigma(j)}$ and that $\bar{J}$ intersects the (projective–injective) modules $P_{j+1}, \ldots, P_n$ trivially. (Note that all the $P_t$, with $t \geq j+1$, have trivial intersection with $J_s$, so they are $\bar{A}$-modules as well.) The socle $L$ of $P_j$ is generated by an element $x$ which is a path (as defined above) from $j$ to $\sigma(j)$.

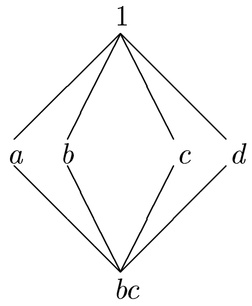
We claim that multiplying $x$ on the right by any radical element gives zero. In fact, such a right multiplication defines a homomorphism $\varphi$ from $P_j$ to $\bar{A}$ which has image in $rad(\bar{A})$. If $\varphi$ is an injective map, then its image (being isomorphic to the injective module $P_j$) splits off and $\varphi$ is an isomorphism. This contradicts the choice of $\varphi$. Hence $\varphi$ cannot be injective and its kernel must contain the socle of $P_j$, which proves the claim. (Of course, multiplication on the left also gives zero, since $x$ is in the socle.)

Hence multiplying $i(x)$ on the left by any radical element of $\bar{A}$ also gives zero, which means that it must be in the socle of $\bar{A}$. Since $\bar{J}$ is fixed by $i$, the element $i(x)$ stays inside $J$, the socle of which is a direct sum of copies of $L$. But this implies that $i(x)$ is a path ending at $\sigma(j)$ whereas by construction the path should end at $j$. Therefore $j$ equals $\sigma(j)$. ∎

The following examples show that the theorem is optimal. More precisely, the other two inclusions discussed in the introduction are proper. Furthermore, the class of self–injective cellular algebras, which are not semisimple, is properly contained in the class of cellular algebras of infinite global dimension.

Group algebras of symmetric groups provide example of cellular algebras, which are self–injective, but not semisimple.

**Example 1.** Let $k$ be a field and $\lambda \in k$. Let $A = k < a, b, c, d > /I$, where $I$ is generated by $a^2, b^2, c^2, d^2, ab, ac, ba, bd, ca, cd, db, dc,\ cb - \lambda bc, ad - bc, da - bc$. As a left module over itself this algebra looks as follows:




Then this algebra is symmetric if and only if $\lambda = 1$. Indeed, suppose $f : A \longrightarrow k$ is a non-zero symmetric linear form whose kernel does not contain any non-zero left ideal in $A$. Since the socle of $A$ is one-dimensional and spanned by $bc$, we have $0 \neq f(bc) = f(cb) = \lambda f(bc)$ and $\lambda = 1$ . Conversely, if $\lambda = 1$, then we define a non-degenerate linear form $f$ by $f(1) = f(a) = f(b){=}f(c) = f(d) = 0$ and $f(bc) = 1$, and this makes $A$ a symmetric algebra for $\lambda = 1$.

Moreover, there is an involution $i$ on $A$, namely, $i$ fixes $a$ and $d$, but interchanges $b$ and $c$. With respect to this involution the algebra $A$ is a cellular algebra. Let us exhibit a cell chain $J_1 \subseteq J_2 \subseteq J_3 = A$ of $A$ by giving a basis in square form:

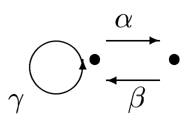
$$J_1: \quad bc; \qquad J_2/J_1: \quad \begin{matrix} a, & b, \\ c, & d; \end{matrix} \qquad J_3/J_2: \quad 1$$

Note that we can use a result in [4] to verify that this chain is really a cell chain of $A$.

If we choose $\lambda$ different from 1, then we get a cellular algebra which is weakly symmetric but not symmetric.

**Example 2.** Now we give an example of a cellular algebra of infinite global dimension, but not self–injective.

Consider the algebra $A$ which is given by the following quiver




with relations $\gamma^2 = \alpha\beta, \beta\alpha = 0, \beta\gamma = 0$.

Thus $A$ has the following decomposition as a left module over itself:

$$ {}_AA = \begin{array}{ccc} & 1 & \\ 1 & & 2 \\ & 1 & \end{array} \oplus \begin{array}{c} \\ 2 \\ 1 \end{array} $$

Clearly, this algebra has infinite global dimension and is cellular, but it is not a self–injective algebra.

## 3 Properties of self–injective cellular algebras

Let us first prove corollary 1.2.

**Proposition 3.1** *Let $A$ be a cellular algebra over a field $k$ (with respect to an involution $i$), $P$ an indecomposable projective–injective $A$–ideal with simple top $L$ and $J$ a cell ideal with associated standard module $\Delta$. Suppose the intersection $J \cap P$ is not zero. Then the composition multiplicity $[\Delta : L]$ equals one.*

**Proof.** We can write the bimodule $J$ as $J \simeq \Delta \otimes_k i(\Delta)$ and $P = Ae$ for some primitive idempotent $e \in A$. Then $J \cap P$ equals $Je$ which is isomorphic (as left module) to $\Delta \otimes_k i(\Delta)e$. Denote by $l$ the $k$–dimension of $e\Delta$ which equals that of $i(\Delta)e$. Then $J \cap P$ is a direct sum of $l$ copies of $\Delta$. But being contained in the injective module $P$, it must have simple socle. Hence $l$ equals one. By definition $l$ also equals $[\Delta : L]$. ∎

**Proof of 1.2.** Let $A$ and $P$ be as in the statement of 1.2. Denote by $L$ the simple top of $P$. By theorem 1.1, $L$ is isomorphic to the socle of $P$. Since $A$ is cellular, its projective module $P$ has a cell filtration. Assume this filtration has length one, that is, $P$ equals a standard module, say $\Delta$. Then by proposition 3.1, the composition multiplicity $[\Delta : L]$ equals one. But $L$ occurs both as top and as socle, thus $P$ must be simple. ∎

For a cellular self–injective algebra, we also have the following property.

**Proposition 3.2** *Let $\{C^{\lambda}_{s,t} \mid s,t \in M(\lambda)\}$ be a cell basis of a cellular algebra $A$. If $A$ is self–injective, then $AC^{\lambda}_{s,t}$ is indecomposable for $\lambda$ a minimal element and $s,t \in M(\lambda)$.*

**Proof.** Since $A$ is self–injective, we have that $Hom_A(Ax, I) = xA \cap I$ for any left ideal $I$ in $A$ and $x \in A$. In particular, $End_A(AC^{\lambda}_{s,s}) \simeq C^{\lambda}_{s,s}A \cap AC^{\lambda}_{s,s} \simeq kC^{\lambda}_{s,s}$ since $\lambda$ is minimal. This shows that $AC^{\lambda}_{s,s}$ is indecomposable. Now our proposition follows immediately from the following fact which holds true for all cell ideals.

**Lemma 3.3** *Let $A$ be a $k$-algebra with an involution $i$. Suppose $J$ is a cell ideal in $A$ with a cell basis $\{C_{s,t} \mid 1 \leq s, t \leq n\}$, Then $AC_{j,s} \simeq AC_{j,t}$ for all $j, s$ and $t$.*

**Proof.** We define $r_j = (C_{j,1}, C_{j,2}, ..., C_{j,n})^{tr}$ , the column vector of the elements $C_{j,1}, C_{j,2}, ..., C_{j,n}$. Then for any element $a \in A$ we have

$$ar_j = \begin{pmatrix} aC_{j,1} \\ aC_{j,2} \\ \vdots \\ aC_{j,n} \end{pmatrix} = r_a(1,j)r_1 + r_a(2,j)r_2 + ... + r_a(n,j)r_n,$$

where $r_a(j,l)$ is an element in $k$ for all $j, l$. Now we define a mapping

$$\varphi : AC_{j,s} \longrightarrow AC_{j,t}, \quad \sum_l r_a(l,j)C_{l,s} \mapsto \sum_l r_a(l,j)C_{j,t}$$

Since $J$ is a cell ideal, this mapping is well-defined. Clearly, it is injective and surjective. Moreover $\varphi$ is also an $A$-homomorphism by the definition of a cell ideal. ■

Under the assumption of $A$ being self–injective, we can say more about this situation.

**Proposition 3.4** *If $A$ is a self–injective algebra with an involution $i$ and $J$ is a cell ideal with a basis $\{C_{s,t}\}$ such that $i(C_{s,t}) = C_{t,s}$, then each non–zero homomorphism from $AC_{s,s}$ to $AC_{t,t}$ is an isomorphism.*

**Proof.** Since $A$ is self–injective, we have that $Hom_A(AC_{s,s}, AC_{t,t}) \cong C_{s,s}A \cap AC_{t,t} \subset kC_{s,t}$. Let $f_0$ be a non-zero homomorphism from $AC_{s,s}$ to $AC_{t,t}$. Then $C_{s,s}A \cap AC_{t,t} = kC_{s,t}$ and there are two elements $a, b \in A$ such that $C_{s,s}a = C_{s,t} = bC_{t,t}$. We define a $A$-homomorphism $f : AC_{s,s} \longrightarrow AC_{t,t}$ by $xC_{s,s} \longmapsto xC_{s,s}a$. It is clear that $f$ is an $A$–homomorphism and that $f_0$ is a scalar multiple of $f$. Thus it is sufficient to show that $f$ is an isomorphism.

Now we define $g : AC_{t,t} \longrightarrow AC_{s,s}$ by $yC_{t,t} \longmapsto yC_{t,t}i(b)$. One can check that $g$ is also an $A$–homomorphism. Since $C_{s,t} = bC_{t,t} = \sum_l r_b(l,t)C_{l,t}$, we have that $r_b(s,t) = 1, r_b(l,t) = 0$ for $l \neq s$ and $bC_{t,t}i(b) = C_{s,t}i(b) = \sum_j r_b(j,t)C_{s,j} = r_b(s,t)C_{s,s}$. This implies that the composition of $f$ and $g$ is an non–zero endomorphism of $AC_{s,s}$. Thus the statement follows from $End_A(AC_{s,s}) \cong k$. ■

Many known examples of quasi–hereditary algebras arise as endomorphism rings of sums of modules satisfying certain ordering conditions. This leads us to the following problem.

**Question**. Let $A$ be a weakly symmetric cellular algebra. When is the endomorphism algebra $End(\oplus_{\lambda\in\Lambda,S\in M(\lambda)}AC^{\lambda}_{S,S})$ a quasi-hereditary algebra?

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