Uniqueness for Gibbs States of Quantum Lattices in Small Mass Regime

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Abstract

The model of interacting quantum particles performing anharmonic one–dimensional oscillations around their unstable equilibrium positions, which form the d– dimensional simple cubic lattice $\mathbb{Z}^d$, is considered. For this model, it is proved that for every fixed value of the temperature β^{-1}, there exists a positive $m_*(\beta)$ such that for the values of the physical mass of the particle $m \in (0, m_*(\beta))$, the set of tempered Gibbs states consists of exactly one element.

1 Introduction

In this paper, we consider the following quantum lattice model. To each point of the d–dimensional cubic lattice $\mathbb{Z}^d$ there is attached a quantum particle with the physical mass m, which has an unstable equilibrium position at this point. Such particles perform one–dimensional oscillations around their equilibrium positions and interact via an attractive potential. Similar objects have been studied for many years as quite realistic models of a crystalline substance undergoing structural phase transitions (see e.g. [11]). In our case, the phase transition is connected with the appearance of macroscopic displacements of the particles (a long–range order) if the dimension d, the mass m, the temperature β^{-1}, and the parameters of the potential energy satisfy certain conditions. These phenomena were studied mathematically in various papers, see e.g. [9], [14], [23], [27]. The essential problem in this context is to understand the role of quantum effects in the phase transitions in such models. As it was justified on the physical level [29] and observed experimentally (see [32] and Ch.2.5.4.3 of [11]), the quantum effects may suppress the long–range ordering. This was proved in [34]. Later on it was shown [3], [4] that not only the long–range order but also any critical anomaly of the displacements of particles from the equilibrium positions are suppressed if the model is "strongly quantum", which may occur in particular if the mass of the particle is small enough.

Another important problem of the mathematical theory of models which exhibit such phenomena is the description of their Gibbs states. The uniqueness of such a state implies the absence of all critical anomalies and all the more of the long– range ordering. Therefore, one may expect that the "strong quantumness" of the model implies the uniqueness for its Gibbs states. So far, the mentioned uniqueness for the model considered in this work was proved to occur under the conditions which do not contain the physical mass of the particle and thus are irrelevant to the "quantumness" of the model. It was done in [1] on the base of the logarithmic Sobolev in-

equalities. An alternative approach to the construction of the Gibbs states for such models lies in the use of cluster expansions. We note here the papers [8] and [25] where models of this type with "light" particles were studied by means of different versions of cluster expansions. In particular, in [8] the existence of temperature Gibbs states was shown by proving of the convergence of cluster expansions at certain fixed temperature and for small masses. In [25], for small values of the mass of the particle, the convergence of corresponding cluster expansions was shown to occur for all values of the temperature including zero. As a result, the existence of temperature and ground states was proved and a number of their properties have been described. It should be pointed out that for systems with unbounded oscillations, as in the case considered in this work, it is impossible to recover the uniqueness of the state from the convergence of the cluster expansion. However, this is possible in the case of quantum particles which move on compact manifolds. Here the uniqueness of the Euclidean Gibbs measure was proved to occur [7] as the consequence of the cluster expansion convergence.

In this paper, we present the proof of the uniqueness for the Gibbs states in the mentioned model for small values of the physical mass of the particle. We follow [1] and apply Dobrushin's technique up to the stage where the logarithmic Sobolev inequalities were used to estimate the variance of certain moments, which was crucial in satisfying Dobrushin's uniqueness criterion. Instead, we reduce this problem to the study of spectral properties of the one–particle Hamiltonian which describes the individual oscillations. This study is performed by means of a version of the lattice approximation technique known in the Euclidean quantum field theory [19], [20], [30]. A similar approach has already been used in our works [3], [4]. It enables us to involve the physical mass into the consideration, so that it appears explicitly in the uniqueness conditions. Unfortunately, it is impossible (at least so far) to remove the temperature dependence from these conditions, in contrast to the proof of the suppression of critical fluctuations given in [3], [4].

2 Temperature loop formalism for Gibbs states

As in [1] and [3], we consider a translation invariant system of interacting quantum particles performing one–dimensional oscillations, described by the following formal Hamiltonian

$$H = \sum_k H_k + \frac{J}{4} \sum_{<j,k>} (x_j - x_k)^2, \tag{2.1}$$

where the indices j, k run over the d–dimensional lattice $\mathbb{Z}^d \subset \mathbb{R}^d$ with the Euclidean distance $|\ .\ |$. The second sum in (2.1) is taken over the nearest neighbors, that is restricted to the pairs with $|j-k|=1$. It describes ferroelectric ($J>0$) interactions of the particles whereas the first term is a sum of one–particle Hamiltonians

$$H_k = -\frac{1}{2m}\frac{d^2}{dx^2} + \frac{1}{2}x_k^2 + V(x_k), \tag{2.2}$$

where $m>0$ is the physical mass of the particle and

$$\begin{aligned} V(x) &= P_{2p}(x) - \frac{1}{2}ax^2 \\ P_{2p}(x) &= \sum_{l=2}^{p} b_l x^{2l},\ p \geq 2. \end{aligned} \tag{2.3}$$

Here we suppose: $a \in \mathbb{R}$, $b_p > 0$, and $b_l \geq 0$ for all $l \leq p$. For $d \geq 2$ and $a > 0$, the model has a critical point and the long–range order behaviour (see e.g. [9], [23]).

Each Hamiltonian H_k is an essentially self–adjoint lower semi–bounded operator on the complex Hilbert space $L^2(\mathbb{R}, dx)$ (see e.g. [10]). We will study temperature Euclidean Gibbs states of the system, i.e. the Gibbs states constructed using a probability measure on a loop space, see [2], [22], [18] for this approach. Thus the Gibbs state at the inverse temperature β is described by the Euclidean Gibbs measure on the space of periodic trajectories (loops) taking values in $\mathbb{R}^{\mathbb{Z}^d}$.

Consider the space of continuous periodic functions

$$\mathcal{C}_\beta \stackrel{\text{def}}{=} \{\omega \in C([0,\beta] \to \mathbb{R}) \mid \omega(0) = \omega(\beta)\}$$

and let $\mathcal{H}_\beta$ be the Hilbert space of real–valued functions on $[0,\beta]$ which are square integrable with respect to the Lebesgue measure. Let also I_β be the interval $[0,\beta]$ with the ends identified (i.e. a circle of length β). The temperature loop space (TLS) is

$$\begin{aligned} \Omega_\beta &\stackrel{\text{def}}{=} {\mathcal{C}_\beta}^{\mathbb{Z}^d} \\ &= \{\omega = (\omega_k)_{k\in\mathbb{Z}^d} \mid \omega_k \in \mathcal{C}_\beta\}. \end{aligned} \tag{2.4}$$

For $\omega \in \Omega_\beta$ and $\Lambda \subset \mathbb{Z}^d$, we set $\omega_\Lambda = (\omega_k)_{k\in\Lambda}$. We shall also consider

$$\Omega_{\beta,\Lambda} \stackrel{\text{def}}{=} \{\omega_\Lambda = (\omega_k)_{k\in\Lambda} \mid \omega_k \in \mathcal{C}_\beta\}, \tag{2.5}$$

defined for all subsets $\Lambda \subset \mathbb{Z}^d$. The spaces $\Omega_{\beta,\Lambda}$, Ω_β are equipped with the product topology and with the σ–algebra $\mathcal{B}(\Omega_{\beta,\Lambda})$ generated by the cylinder sets

$$\{\omega \mid (\omega_k)_{k\in\Delta} \in B_\Delta,\ \Delta \in \mathcal{L}\},$$

where $\mathcal{L}$ is the set of finite subsets of $\mathbb{Z}^d$, B_Δ is a direct product of Borel subsets $\{B_k \subset \mathcal{C}_\beta \mid k \in \Delta\}$. Consider again $\mathcal{H}_\beta$ equipped with the usual inner product $(.,.)_\beta$ and the norm $\|.\|_\beta$. Define on $\mathcal{H}_\beta$ a strictly positive trace class operator $S_\beta = (-m\Delta_\beta + 1)^{-1}$, where Δ_β stands for the Laplace–Beltrami operator on the circle I_β, and m is the physical mass of the particle. In this setting, one can define the Gaussian measure γ_β on $\mathcal{H}_\beta$ which has zero mean and S_β as a covariance operator. This measure is uniquely determined by its Fourier transform

$$\int_{\mathcal{H}_\beta} \exp(i(\varphi,\omega)_\beta)\gamma_\beta(d\omega) = \exp(-\frac{1}{2}(\varphi, S_\beta\varphi)_\beta), \quad \varphi \in \mathcal{H}_\beta. \tag{2.6}$$

Actually, the set of continuous loops is a full measure set (i.e. $\gamma_\beta(\mathcal{C}_\beta) = 1$), and the measure γ_β is canonically generated by the oscillator bridge process of length β [31]. By means of this Gaussian measure one can define

$$\sigma_\beta(d\omega) = \frac{1}{X_\beta}\exp\{-\int_0^\beta V(\omega(\tau))d\tau\}\gamma_\beta(d\omega) \tag{2.7}$$

as a probability measure on $\mathcal{H}_\beta$, supported on $\mathcal{C}_\beta$. Here X_β is the normalizing constant and V is given by eq.(2.3). Semi–boundedness of the latter implies that the measure σ_β, along with γ_β, has all moments, that is

$$\int_{\mathcal{C}_\beta} \|\omega\|^l_{\mathcal{C}_\beta}\sigma_\beta(d\omega) < \infty, \quad \forall l \in I\!N. \tag{2.8}$$

For every $\zeta \in \Omega_\beta$ and each finite subset $\Lambda \subset \mathbb{Z}^d$, we define the local Gibbs measure in Λ, subject to a chosen ζ, as the probability measure

$$\nu_{\beta,\Lambda}(d\omega_\Lambda|\zeta) \stackrel{\text{def}}{=} \frac{1}{Z_{\beta,\Lambda}(\zeta)}\exp\{-\Phi_{\beta,\Lambda}(\omega_\Lambda|\zeta)\}\prod_{k\in\Lambda}\sigma_\beta(d\omega_k), \tag{2.9}$$

on the loop space $\Omega_{\beta,\Lambda}$. Here

$$\begin{aligned} &\Phi_{\beta,\Lambda}(\omega_\Lambda|\zeta) \\ \stackrel{\text{def}}{=}\ &\frac{J}{4}\int_0^\beta \left\{\sum_{j,k\in\Lambda}(\omega_j(\tau)-\omega_k(\tau))^2 + 2\sum_{j\in\Lambda,k\in\Lambda^c}(\omega_j(\tau)-\zeta_k(\tau))^2\right\}d\tau, \end{aligned} \tag{2.10}$$

is the pair interaction in Λ, prescribed by the Hamiltonian (2.1), under the external boundary condition

$$\zeta_{\Lambda^c} \in {\mathcal{C}_\beta}^{\Lambda^c}, \quad \Lambda^c \stackrel{\text{def}}{=} \mathbb{Z}^d \setminus \Lambda.$$

All sums in (2.10) are taken under the condition $|j-k|=1$, and

$$Z_{\beta,\Lambda}(\zeta) \stackrel{\text{def}}{=} \int_{\Omega_{\beta,\Lambda}} \exp\{-\Phi_{\beta,\Lambda}(\omega_\Lambda|\zeta)\} \prod_{k\in\Lambda} \sigma_\beta(d\omega_k), \tag{2.11}$$

is the local partition function subject to the external boundary condition ζ_{Λ^c}. For $B \in \mathcal{B}(\Omega_\beta)$ and $\omega \in \Omega_\beta$, let $\mathbb{1}_B(\omega)$ take values 1 resp. 0 if ω belongs resp. does not belong to B. Then one can introduce a family of probability kernels $\{\pi_{\beta,\Lambda} \mid \Lambda \in \mathcal{L}\}$ between $(\Omega_\beta, \mathcal{B}(\Omega_\beta))$ and $(\Omega_\beta, \mathcal{B}(\Omega_\beta))$ by

$$\pi_{\beta,\Lambda}(B|\zeta) \stackrel{\text{def}}{=} \int_{\Omega_{\beta,\Lambda}} \mathbb{1}_B(\omega_\Lambda \times \zeta_{\Lambda^c}) \nu_{\beta,\Lambda}(d\omega_\Lambda|\zeta). \tag{2.12}$$

These kernels satisfy the consistency conditions which are set as follows (see e.g. [16]). For arbitrary pair $\Lambda' \subset \Lambda$, each $B \in \mathcal{B}(\Omega_\beta)$, and $\zeta \in \Omega_\beta$,

$$\begin{aligned} \pi_{\beta,\Lambda}\pi_{\beta,\Lambda'}(B|\zeta) &\stackrel{\text{def}}{=} \int_{\Omega_\beta} \pi_{\beta,\Lambda}(d\omega|\zeta)\pi_{\beta,\Lambda'}(B|\omega) \\ &= \pi_{\beta,\Lambda}(B|\zeta). \end{aligned} \tag{2.13}$$

Definition 2.1 *A probability measure μ_β on $(\Omega_\beta, \mathcal{B}(\Omega_\beta))$ is said to be an Euclidean Gibbs state of the lattice model (2.1), (2.2), (2.3) at the inverse temperature β if it satisfies the Dobrushin–Lanford–Ruelle (DLR) equilibrium equations:*

$$\mu_\beta \pi_{\beta,\Lambda} = \mu_\beta,$$

that is

$$\int_{\Omega_\beta} \mu_\beta(d\omega)\pi_{\beta,\Lambda}(B|\omega) = \mu_\beta(B), \tag{2.14}$$

for all $\Lambda \in \mathcal{L}$ and $B \in \mathcal{B}(\Omega_\beta)$.

In order to exclude the states with no physical relevance we impose some *a priori* conditions restricting the growth of the sequences of moments (see [1], [20]). To this end we introduce

$$\mathcal{S}'(\mathbb{Z}^d) \stackrel{\text{def}}{=} \operatorname{ind}\lim_{p\in\mathbb{N}} l^2_{-p}(\mathbb{Z}^d),$$

where

$$l^2_{-p} \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R}^{\mathbb{Z}^d} \mid \sum_{k\in\mathbb{Z}^d} (1+|k|)^{-p} x_k^2 < \infty \right\}. \tag{2.15}$$

Definition 2.2 *The class of tempered Gibbs measures $\mathcal{G}_\beta$, for the model considered, consists of the Euclidean Gibbs states defined above which are such that their moment sequences obey the condition*

$$(< \|\omega_k\|_\beta >_{\mu_\beta})_{k\in\mathbb{Z}^d} \in \mathcal{S}'(\mathbb{Z}^d), \tag{2.16}$$

where $\| \, . \, \|_\beta$ is the norm in the Hilbert space $\mathcal{H}_\beta$.

Here and further on we write

$$< f >_\mu = \int f d\mu \tag{2.17}$$

if the integral makes sense.

For the study of the existence of the Euclidean Gibbs measures, we refer to [1] and the references therein. Here we only remark that the class $\mathcal{G}_\beta$ is actually nonempty. Our main aim in this paper is to refine the uniqueness condition obtained in [1] by involving explicitly the parameters which describe the quantum character of the model.

3 Uniqueness theorem

Theorem 3.1 *For the lattice model (2.1), (2.2), (2.3), for every β, there exists a positive $m_*(\beta)$ such that for all values of the physical mass $m \in (0, m_*(\beta))$, the class $\mathcal{G}_\beta$ consists of exactly one element, that is $|\mathcal{G}_\beta| = 1$.*

Remark 3.1 *The value of m_* depends also on the concrete choice of the function V (2.3) as well as on the value of the interaction intensity J.*

The uniqueness stated above will be shown in the context of the metric fixed point theory methods. The DLR equation (2.14) may used to define an operator in a metric space of measures. The contractivity property of this operator would imply the uniqueness for its fixed points. Thus we metrize the space of measures as follows. Let $(\mathcal{X}, \rho)$ be a complete separable metric space, $\mathcal{M}$ be the set of all probability measures on $\mathcal{B}(\mathcal{X})$, and

$$\mathcal{M}_1 \stackrel{\text{def}}{=} \left\{ \mu \in \mathcal{M} \mid \int_{\mathcal{X}} \rho(x, x_0)\mu(dx) < \infty \right\}, \tag{3.1}$$

for some $x_0 \in \mathcal{X}$. Further, let $\mathrm{Lip}(\mathcal{X})$ denote the set of all Lipschitz functions $f : \mathcal{X} \to \mathbb{R}$, and

$$[f]_{\mathrm{Lip}} \stackrel{\text{def}}{=} \sup \left\{ \frac{|f(x) - f(y)|}{\rho(x, y)} \mid x, y \in \mathcal{X}, \; x \neq y \right\}, \tag{3.2}$$

$$\mathrm{Lip}_1(\mathcal{X}) \stackrel{\mathrm{def}}{=} \{f \in \mathrm{Lip}(\mathcal{X}) \mid [f]_{\mathrm{Lip}} \leq 1\}. \tag{3.3}$$

For $\mu_1,\ \mu_2 \in \mathcal{M}_1$, we set

$$R(\mu_1,\mu_2) \tag{3.4}$$
$$\stackrel{\mathrm{def}}{=} \quad \sup\left\{\left|\int_{\mathcal{X}} f(x)\mu_1(dx) - \int_{\mathcal{X}} f(x)\mu_2(dx)\right| \ \mid \ f \in \mathrm{Lip}_1(\mathcal{X})\right\}.$$

The operator mentioned above appears here in the form of Dobrushin's matrix. It is defined in terms of the measures $\nu_{\beta,\Lambda}$ given by eqs. (2.9) – (2.11) with the one–point subset $\Lambda = \{k\}$. To simplify notations we set

$$\xi_{\{j\}^c} = \xi_j^c, \quad \nu_{\beta,\{k\}}(B|\xi) = \nu_k(B|\xi). \tag{3.5}$$

The elements of Dobrushin's matrix $C_{k,j};\ k,j \in \mathbb{Z}^d$ are:

$$C_{kj} \stackrel{\mathrm{def}}{=} \sup\left\{\frac{R(\nu_k(.|\xi),\nu_k(.|\eta))}{\|\xi_j - \eta_j\|_\beta} \mid \xi,\eta \in \Omega_\beta,\ \xi_j^c = \eta_j^c\right\}. \tag{3.6}$$

They are used to check Dobrushin's uniqueness criterion [12], [13], [16], [24].

Proposition 3.1 (Dobrushin's uniqueness criterion) *For the model considered, let the following condition be satisfied*

$$\sup\left\{\sum_{j\in\mathbb{Z}^d\setminus\{k\}} C_{kj} \mid k \in \mathbb{Z}^d\right\} < 1. \tag{3.7}$$

Then there exists exactly one tempered Gibbs measure, that is $|\mathcal{G}_\beta| = 1$.

Directly from the definitions (2.9) – (2.11), (3.5) one obtains

$$\nu_k(d\omega_k|\xi) = \frac{1}{Z_k(\xi)} \exp\left\{-Jd\|\omega_k\|_\beta^2 + J\sum_{|i-k|=1}(\omega_k,\xi_i)_\beta\right\}\sigma_\beta(d\omega_k). \tag{3.8}$$

This implies that $\nu_k(d\omega_k|\xi) = \nu_k(d\omega_k|\eta)$ if $\xi_j^c = \eta_j^c$ and $|k-j| \neq 1$. Hence

$$R(\nu_k(.|\xi),\nu_k(.|\eta) = 0, \quad C_{kj} = 0,$$

in this case. For $|k-j| = 1$, all C_{kj} are the same, thus we set

$$C \stackrel{\mathrm{def}}{=} C_{kj}, \quad \text{for} \ \ |k-j| = 1. \tag{3.9}$$

Therefore Dobrushin's uniqueness condition (3.7) is satisfied if:

$$C < \frac{1}{2d}. \tag{3.10}$$

The translation invariance of the model implies that each measure $\nu_k(.|\xi)$ is a copy of the following measure defined on the probability space $(\mathcal{H}_\beta, \mathcal{B}(\mathcal{H}_\beta))$:

$$\begin{aligned} \nu^x(d\omega) &= \frac{1}{Z_x} \exp\{-Jd\|\omega\|_\beta^2 + J(\omega, x)_\beta\}\sigma_\beta(d\omega), \\ Z_x &= \int_{\mathcal{H}_\beta} \exp\{-Jd\|\omega\|_\beta^2 + J(\omega, x)_\beta\}\sigma_\beta(d\omega), \end{aligned} \tag{3.11}$$

with

$$\mathcal{H}_\beta \ni x = \sum_{i:\ |i-k|=1} \xi_i.$$

Hence the Dobrushin coefficients (3.9) may be rewritten as

$$C = \sup\left\{ \frac{R(\nu^x, \nu^y)}{\|x-y\|_\beta} \mid x, y \in \mathcal{H}_\beta,\ x \neq y \right\}. \tag{3.12}$$

For certain fixed $f \in \mathrm{Lip}_1(\mathcal{H}_\beta)$, let us consider the real–valued function

$$\mathcal{H}_\beta \ni x \mapsto < f >_{\nu^x} = \int_{\mathcal{H}_\beta} f(\omega)\nu^x(d\omega) \in I\!R. \tag{3.13}$$

We use it to estimate the metric $R(\nu^x, \nu^y)$ defined by (3.4). To this end we apply the mean value theorem to the function (3.13), which is Fréchet differentiable due to (2.8). Its derivative in the direction $\varphi \in \mathcal{H}_\beta$ is

$$\begin{aligned} (\nabla_x < f >_{\nu^x}, \varphi)_\beta &= J\{< f(., \varphi)_\beta >_{\nu^x} - < f >_{\nu^x} < (., \varphi)_\beta >_{\nu^x}\} \\ &= J\mathrm{Cov}_{\nu^x}(f, (., \varphi)_\beta). \end{aligned} \tag{3.14}$$

Our next aim is to find the estimate of this derivative which does not depend on $x \in \mathcal{H}_\beta$. We start with the Schwartz inequality

$$|(\nabla_x < f >_{\nu^x}, \varphi)_\beta| \leq J\sqrt{\mathrm{Var}_{\nu^x} f}\sqrt{\mathrm{Var}_{\nu^x}(., \varphi)_\beta}, \tag{3.15}$$

where

$$\begin{aligned} \mathrm{Var}_{\nu^x} f &= \int_{\mathcal{H}_\beta} f^2(\omega)\nu^x(d\omega) \\ &- \int_{\mathcal{H}_\beta}\int_{\mathcal{H}_\beta} f(\omega)f(\omega')\nu^x(d\omega)\nu^x(d\omega'), \end{aligned} \tag{3.16}$$

$$\begin{aligned} \mathrm{Var}_{\nu^x}(.,\varphi)_\beta &= \int_{\mathcal{H}_\beta} (\omega,\varphi)_\beta^2 \nu^x(d\omega) \\ &- \int_{\mathcal{H}_\beta}\int_{\mathcal{H}_\beta} (\omega,\varphi)_\beta(\omega',\varphi)_\beta \nu^x(d\omega)\nu^x(d\omega'), \end{aligned} \tag{3.17}$$

that may also be rewritten as follows

$$\mathrm{Var}_{\nu^x} f = \frac{1}{2}\int_{\mathcal{H}_\beta}\int_{\mathcal{H}_\beta} (f(\omega)-f(\omega'))^2 \nu^x(d\omega)\nu^x(d\omega'), \tag{3.18}$$

$$\mathrm{Var}_{\nu^x}(.,\varphi)_\beta = \frac{1}{2}\int_{\mathcal{H}_\beta}\int_{\mathcal{H}_\beta} (\omega-\omega',\varphi)_\beta^2 \ \nu^x(d\omega)\nu^x(d\omega'). \tag{3.19}$$

Suppose now that we have estimated the first variance by a bound, which continuously depends on β, on the parameters a, b_l of the function (2.3), and on the value of the physical mass m. At the same time let the second variance be bounded by a function of a, b_l, and m, multiplied by the square of the norm of φ. It is very important that both estimates would hold uniformly for all $x \in \mathcal{H}_\beta$. Then the mean–value theorem together with the representation of the Dobrushin coefficients (3.12) imply that the conditions (3.7), (3.10) may be satisfied provided the product of the mentioned bounds is sufficiently small. Below we shall implement this idea.

It can be observed that the expression (3.19) defines a bounded positive quadratic form on $\mathcal{H}_\beta$. That is

$$\begin{aligned} \mathrm{Var}_{\nu^x}(.,\varphi)_\beta &\stackrel{\mathrm{def}}{=} (\varphi, K_x\varphi)_\beta \\ &\leq \frac{1}{2}\|\varphi\|_\beta^2 \int_{\mathcal{H}_\beta}\int_{\mathcal{H}_\beta} \|\omega-\omega'\|_\beta^2 \ \nu^x(d\omega)\nu^x(d\omega'). \end{aligned} \tag{3.20}$$

The operator K_x which appears here has the integral representation

$$(K_x\varphi)(\tau) = \int_0^\beta K_x(\tau,\tau')\varphi(\tau')d\tau', \quad \tau \in I_\beta, \tag{3.21}$$

with the kernel $K_x(\tau,\tau')$ given by

$$K_x(\tau,\tau') = \frac{1}{2}\int_{\mathcal{H}_\beta}\int_{\mathcal{H}_\beta} [\omega(\tau)-\omega'(\tau)][\omega(\tau')-\omega'(\tau')]\nu^x(d\omega)\nu^x(d\omega'). \tag{3.22}$$

Clearly, for every $x \in \mathcal{H}_\beta$, the operator K_x is symmetric and positive, its kernel (3.22) is a continuous function of τ, $\tau' \in I_\beta$, and

$$\mathrm{trace}(K_x) = \frac{1}{2}\int_{\mathcal{H}_\beta}\int_{\mathcal{H}_\beta} \|\omega-\omega'\|_\beta^2 \ \nu^x(d\omega)\nu^x(d\omega') < \infty. \tag{3.23}$$

We shall find the bound of the quadratic form (3.20), (3.21) comparing it with an auxiliary form which, however, does not depend on x. To construct the latter we modify our notations introducing

$$\tilde{P}_{2p}(x) = \sum_{l=2}^{p} 2^{1-l} b_l x^{2l}, \ p \geq 2 \tag{3.24}$$

$$\tilde{V}(x) = \tilde{P}_{2p}(x) - \frac{1}{2} a x^2, \tag{3.25}$$

$$\tilde{H} = -\frac{1}{2m}\frac{d^2}{dx^2} + \frac{1}{2}(2Jd+1)q^2 + \tilde{V}(q). \tag{3.26}$$

The operator q acts in $L^2(I\!R, dx)$ as follows

$$(q\psi)(x) \stackrel{\text{def}}{=} x\psi(x). \tag{3.27}$$

Further, along with the measures defined by (2.7) and (3.11), we introduce

$$\tilde{\sigma}_\beta(d\omega) = \frac{1}{\tilde{X}_\beta} \exp\left\{ -\int_0^\beta \tilde{V}(\omega(\tau))d\tau \right\} \gamma_\beta(d\omega), \tag{3.28}$$

$$\tilde{X}_\beta = \int_{\mathcal{H}_\beta} \exp\left\{ -\int_0^\beta \tilde{V}(\omega(\tau))d\tau \right\} \gamma_\beta(d\omega),$$

$$\tilde{\nu}^x(d\omega) = \frac{1}{\tilde{Z}_x} \exp\left\{ -Jd\|\omega\|_\beta^2 + J(\omega, x)_\beta \right\} \tilde{\sigma}_\beta(d\omega), \tag{3.29}$$

where the partition function $\tilde{Z}_x$ may be written as in (3.11) but with $\tilde{\sigma}_\beta$ instead of σ_β. Having these new measures we introduce the operator $\tilde{K}_x$ by means of eq.(3.21) with the kernel

$$\tilde{K}_x(\tau, \tau') = \frac{1}{2}\int_{\mathcal{H}_\beta}\int_{\mathcal{H}_\beta} [\omega(\tau) - \omega'(\tau)][\omega(\tau') - \omega'(\tau')]\tilde{\nu}^x(d\omega)\tilde{\nu}^x(d\omega'). \tag{3.30}$$

Its trace is finite and can be calculated just as in (3.23) with the measure $\tilde{\sigma}_\beta$ instead of σ_β.

For a bounded linear operator $T : \mathcal{H}_\beta \to \mathcal{H}_\beta$, let $\mathcal{S}_{pp}(T)$ denote the pure point spectrum of T and let $\|T\|$ be its operator norm. For a positive compact operator T, one has

$$\|T\| = \max \mathcal{S}_{pp}(T). \tag{3.31}$$

On the other hand, the norm of a symmetric positive T may be written as follows (see e.g [28], p.216):

$$\|T\| = \sup\{ \ \frac{(\varphi, T\varphi)_\beta}{\|\varphi\|_\beta^2} \mid \varphi \in \mathcal{H}_\beta \setminus \{0\} \ \}. \tag{3.32}$$

The construction of the bounds mentioned above is based upon the following lemmas, which are proved in Section 4 below.

Lemma 3.1 *For every $x \in \mathcal{H}_\beta$ and all $\varphi \in \mathcal{H}_\beta$*

$$(\varphi, K_x\varphi)_\beta \leq \left\|\tilde{K}_0\right\| \|\varphi\|_\beta^2. \tag{3.33}$$

Lemma 3.2 *For every $x \in \mathcal{H}_\beta$,*

$$\mathrm{trace}(K_x) \leq \mathrm{trace}(\tilde{K}_0). \tag{3.34}$$

It is known (see [10], p.57) that (the closure of) $\tilde{H}$ given by (3.26) is a self–adjoint operator with a discrete spectrum, all its eigenvalues $\tilde{E}_l$, $l \in I\!N$ are simple. We set

$$\tilde{\Delta} \stackrel{\mathrm{def}}{=} \inf\{|\tilde{E}_l - \tilde{E}_{l'}| \mid l, l' \in I\!N, l \neq l\}. \tag{3.35}$$

Lemma 3.3 *For the spectral radius of the operator $\tilde{K}_0$, the following estimate holds*

$$\max \mathcal{S}_{pp}(\tilde{K}_0) \leq \frac{1}{m\tilde{\Delta}^2}. \tag{3.36}$$

Lemma 3.4 *There exists a positive constant χ_0 (independent of m) such that for sufficiently small positive values of the physical mass m, the following estimate holds*

$$\frac{1}{m\tilde{\Delta}^2} \leq \chi_0 m^{\frac{p-1}{p+1}}. \tag{3.37}$$

Now we construct the bounds of $\mathrm{Var}_{\nu^x} f$. The first such a bound has the following form (recall that the function V given by eq. (3.18) is a polynomial and $\deg V = 2p$)

Lemma 3.5 *There exists a positive constant ϕ_0 (independent of m and β) such that for all $x \in \mathcal{H}_\beta$, arbitrary $f \in \mathrm{Lip}_1(\mathcal{H}_\beta)$, and for sufficiently small m, the following estimate holds*

$$\mathrm{Var}_{\nu^x} f \leq \beta\phi_0 m^{-1/(p+1)}. \tag{3.38}$$

This bound, linear in β, is, however, useful only for $p > 2$. In the case where the function V is

$$V(x) = -\frac{1}{2}ax^2 + P_4(x) = -\frac{1}{2}ax^2 + b_2x^4,$$

it remains only to use the following bound, which grows exponentially as a function of β. Such a bound is obtained by means of the logarithmic Sobolev inequality, similarly as it was done in [1].

Lemma 3.6 *For all $x \in \mathcal{H}_\beta$ and arbitrary $f \in \mathrm{Lip}_1(\mathcal{H}_\beta)$,*

$$\mathrm{Var}_{\nu^x} f \le \frac{e^{\beta\delta_0}}{2Jd + 1 + a/4}, \tag{3.39}$$

where the exponent δ_0 is defined by the parameters of the function V

$$\delta_0 = \frac{25}{288}\frac{a^2}{b_2}.$$

Proof of Theorem 3.1. We start with estimating of the variance $\mathrm{Var}_{\nu^x}(., \varphi)_\beta$ given by eqs. (3.17), (3.19), (3.20). By means of Lemma 3.1 and (3.31) one obtains

$$\mathrm{Var}_{\nu^x}(., \varphi)_\beta = (\varphi, K_x\varphi)_\beta \ \le \ \left\|\tilde{K}_0\right\| \|\varphi\|_\beta^2 = \max\mathcal{S}_{pp}(\tilde{K}_0)\|\varphi\|_\beta^2,$$

and further by means of Lemma 3.3, Lemma 3.4

$$\mathrm{Var}_{\nu^x}(., \varphi)_\beta \le \chi_0 m^{\frac{p-1}{p+1}} \|\varphi\|_\beta^2, \tag{3.40}$$

that holds for sufficiently small positive m. In the case $p > 2$ one may use the estimate (3.38) and choose m so small that both latter estimates hold. This yields for the distance (3.4)

$$R(\nu^x, \nu^y) \le \|x - y\|_\beta J\sqrt{\beta\phi_0\chi_0}\, m^{\frac{p-2}{2(p+1)}},$$

which means in view of (3.12), (3.10) that the condition of Dobrushin's criterion (3.7) is satisfied provided

$$\beta m^{\frac{p-2}{p+1}} < \frac{1}{(2Jd)^2\phi_0\chi_0}. \tag{3.41}$$

Therefore, in the case $p > 2$ the upper bound for m is

$$m_*(\beta) = \frac{\beta^{-\frac{p+1}{p-2}}}{[(2dJ)^2\phi_0\chi_0]^{\frac{p+1}{p-2}}}. \tag{3.42}$$

For $p = 2$, we use the estimate (3.39), choose m so small that (3.40) also holds, and obtain

$$R(\nu^x, \nu^y) \le \|x - y\|_\beta J e^{\beta\delta_0/2} m^{\frac{1}{6}} \sqrt{\frac{\chi_0}{2Jd + 1 + a/4}},$$

which implies in turn that Dobrushin's criterion is satisfied provided

$$m < m_*(\beta) \stackrel{\mathrm{def}}{=} e^{-3\beta\delta_0} \left(\frac{2Jd + 1 + a/4}{(2Jd)^2\chi_0}\right)^3, \tag{3.43}$$

■

4 Proof of Lemmas

The proof of the first two lemmas is based upon the fact that the measures (2.9), (2.10), (3.8), (3.11), (3.29) admit the lattice approximations similar to the case of $P(\phi)_2$ models of the Euclidean quantum field theory (see [17], [19], [20], and [30], where some notions used below are introduced). The approximating measures have much in common with the Gibbs measures of general classical (nonquantum) ferromagnets. In particular, their moments obey certain inequalities which can be proved to hold also for the moments of the limiting measures.

The tools for the proof of the first two lemmas formulated in the preceding section are described just below. The proof of the first two propositions is similar to that of the case of the Euclidean fields. We do not give here its technical details referring to our forthcoming paper [6], where a number of inequalities for temperature loop measures are proved.

Proposition 4.1 [FKG inequality] *Let the measure μ on $\mathcal{H}_\beta$ be approximable by general ferromagnets. Then for every $\tau, \tau' \in I_\beta$,*

$$< \omega(\tau)\omega(\tau') >_\mu \geq < \omega(\tau) >_\mu < \omega(\tau') >_\mu .$$

In particular

$$K_x(\tau, \tau') \geq 0, \tag{4.1}$$

for every $x \in \mathcal{H}_\beta$, and for all $\tau, \tau' \in I_\beta$.

This statement is analogous to Theorem VIII.18 of book [30], see also [19].

Proposition 4.2 [GKS inequality] *Let the measure μ on $\mathcal{H}_\beta$ be approximable by even ferromagnets. Then for every $k \in \mathbb{Z}_+$ and for all t, $\tau, \tau' \in I_\beta$,*

$$< \xi^{2k}(t)\xi(\tau)\xi(\tau') >_\mu \geq < \xi^{2k}(t) >_\mu < \xi(\tau)\xi(\tau') >_\mu$$

This statement is analogous to Theorem VIII.15 of book [30], see also [19].

Lemma 4.1 *For every $x \in \mathcal{H}_\beta$, and for every $\tau, \tau' \in I_\beta$,*

$$K_x(\tau, \tau') \leq \tilde{K}_0(\tau, \tau'). \tag{4.2}$$

Proof. For the kernels $K_x(\tau, \tau')$, $\tilde{K}_0(\tau, \tau')$, one has the representations (3.22), (3.30). By means of (3.11), (2.7), and (2.3), we rewrite the former

one as follows

$$\begin{aligned} K_x(\tau,\tau') &= <\omega(\tau)\omega(\tau')>_{\nu^x} - <\omega(\tau)>_{\nu^x}<\omega(\tau')>_{\nu^x} \qquad (4.3)\\ &= \frac{1}{(Z_x X_\beta)^2}\int\int_{\mathcal{C}_\beta\times\mathcal{C}_\beta}\frac{\omega(\tau)-\omega'(\tau)}{\sqrt{2}}\frac{\omega(\tau')-\omega'(\tau')}{\sqrt{2}}\\ &\quad \exp\{-\frac{2Jd-a}{2}(\|\omega\|_\beta^2+\|\omega'\|_\beta^2)+J(\omega+\omega',x)_\beta\\ &- \int_0^\beta [P_{2p}(\omega(\tau))+P_{2p}(\omega'(\tau))]d\tau\}(\gamma_\beta\otimes\gamma_\beta)(d\omega,d\omega'). \end{aligned}$$

Similarly by means of (3.28), (3.29), and (3.24), (3.25), one obtains

$$\begin{aligned} \tilde{K}_0(\tau,\tau') &= <\omega(\tau)\omega(\tau')>_{\tilde{\nu}^0} - <\omega(\tau)>_{\tilde{\nu}^0}<\omega(\tau')>_{\tilde{\nu}^0} \qquad (4.4)\\ &= <\omega(\tau)\omega(\tau')>_{\tilde{\nu}^0}\\ &= \frac{1}{\tilde{Z}_x\tilde{X}_\beta}\int_{\mathcal{C}_\beta}\omega(\tau)\omega(\tau')\exp\{-\frac{1}{2}(2Jd-a)\|\omega\|_\beta^2-\int_0^\beta\tilde{P}_{2p}(\omega(\tau))\}\gamma_\beta(d\omega), \end{aligned}$$

For the polynomials P_{2p} and $\tilde{P}_{2p}$ defined by (2.3), (3.24), one has

$$P_{2p}(\frac{x+y}{\sqrt{2}})+P_{2p}(\frac{x-y}{\sqrt{2}})=\tilde{P}_{2p}(x)+\tilde{P}_{2p}(y)+\Pi_{2p}(x|y),\ x,y\in I\!R, \qquad (4.5)$$

where

$$\Pi_{2p}(x|y) = \sum_{k=1}^{p-1}\frac{x^{2k}}{(2k)!}\frac{d^{2k}\tilde{P}_{2p}}{dy^{2k}}(y) \stackrel{\text{def}}{=} \sum_{k=1}^{p-1}\pi_k(y)x^{2k}, \qquad (4.6)$$

Since $b_k\geq 0$, all coefficients $\pi_k(y)$ are nonnegative for every $y\in I\!R$.

To separate the variables in the integral in (4.3) we use the following orthogonal transformation of $\mathcal{C}_\beta\times\mathcal{C}_\beta$

$$\begin{aligned} \xi_(\tau) &= \frac{1}{\sqrt{2}}(\omega(\tau)-\omega'(\tau)), \quad \eta(\tau) = \frac{1}{\sqrt{2}}(\omega(\tau)+\omega'(\tau)), \qquad (4.7)\\ \omega(\tau) &= \frac{1}{\sqrt{2}}(\xi(\tau)+\eta(\tau)), \quad \omega'(\tau) = \frac{1}{\sqrt{2}}(\eta(\tau)-\xi(\tau)). \end{aligned}$$

For the Gaussian measure $\gamma_\beta\otimes\gamma_\beta$, the orthogonality implies

$$(\gamma_\beta\otimes\gamma_\beta)(d\omega,d\omega')=(\gamma_\beta\otimes\gamma_\beta)(d\xi,d\eta),$$

which, together with (4.5), yields in (4.3)

$$\begin{aligned} K_x(\tau,\tau') &= \frac{1}{(Z_x X_\beta)^2}\int\int_{\mathcal{C}_\beta\times\mathcal{C}_\beta}\xi(\tau)\xi(\tau')\exp\{-\frac{2Jd-a}{2}(\|\xi\|_\beta^2+\|\eta\|_\beta^2) \\ &+ \sqrt{2}J(\eta,x)_\beta - \int_0^\beta \Pi_{2p}(\xi(\tau)|\eta(\tau))d\tau \\ &- \int_0^\beta [\tilde{P}_{2p}(\xi(\tau))+\tilde{P}_{2p}(\eta(\tau))]d\tau\}(\gamma_\beta\otimes\gamma_\beta)(d\xi,d\eta). \end{aligned} \tag{4.8}$$

For $\vartheta\in[0,1]$, we define the following functions

$$\begin{aligned} \Xi(\vartheta|\tau,\tau') &= \frac{1}{Y(\vartheta)}\int_{\mathcal{C}_\beta}\xi(\tau)\xi(\tau')\exp\{-\frac{1}{2}(2Jd-a)\|\xi\|_\beta^2 \\ &- \int_0^\beta \tilde{P}_{2p}(\xi(\tau))d\tau - \vartheta\int_0^\beta \Pi_{2p}(\xi(\tau)|\eta(\tau))d\tau)\}\gamma_\beta(d\xi), \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} Y(\vartheta) &= \int_{\mathcal{C}_\beta}\exp\{-\frac{1}{2}(2Jd-a)\|\xi\|_\beta^2 - \int_0^\beta \tilde{P}_{2p}(\xi(\tau))d\tau \\ &- \vartheta\int_0^\beta \Pi_{2p}(\xi(\tau)|\eta(\tau))d\tau)\}\gamma_\beta(d\xi). \end{aligned} \tag{4.10}$$

Using them in (4.8) we bring it into the following form

$$\begin{aligned} K_x(\tau,\tau') &= \frac{1}{(Z_x X_\beta)^2}\int_{\mathcal{C}_\beta}\Xi(1|\tau,\tau')Y(1)\exp\{-\frac{2Jd-a}{2}\|\eta\|_\beta^2 \\ &+ \sqrt{2}J(\eta,x)_\beta - \int_0^\beta \tilde{P}_{2p}(\eta(\tau))d\tau\}\gamma_\beta(d\eta). \end{aligned} \tag{4.11}$$

For every $\tau,\tau'\in I_\beta$, $\Xi(\vartheta|\tau,\tau')$ as well as $Y(\vartheta)$, as functions of ϑ, are continuous on $[0,1]$ and differentiable on $(0,1)$. Furthermore,

$$\begin{aligned} &\frac{\partial}{\partial\vartheta}\Xi(\vartheta|\tau,\tau') = -\frac{1}{Y(\vartheta)}\int_{\mathcal{C}_\beta}\left(\sum_{k=1}^{p-1}\int_0^\beta \pi_k(\eta(t))\xi^{2k}(t)dt\right)\xi(\tau)\xi(\tau') \\ &\exp\{-\frac{1}{2}(2Jd-a)\|\xi\|_\beta^2 - \int_0^\beta \tilde{P}_{2p}(\xi(t))dt - \vartheta\int_0^\beta \Pi_{2p}(\xi(t)|\eta(t))dt\}\gamma_\beta(d\xi) \\ +\ &\Xi(\vartheta|\tau,\tau')\frac{1}{Y(\vartheta)}\int_{\mathcal{C}_\beta}\left(\sum_{k=1}^{p-1}\int_0^\beta \pi_k(\eta(t))\xi^{2k}(t)dt\right) \\ &\exp\{-\frac{1}{2}(2Jd-a)\|\xi\|_\beta^2 - \int_0^\beta \tilde{P}_{2p}(\xi(t))dt - \vartheta\int_0^\beta \Pi_{2p}(\xi(t)|\eta(t))dt\}\gamma_\beta(d\xi). \end{aligned}$$

In what follows

$$\Xi(\vartheta|\tau,\tau') =< \xi(\tau)\xi(\tau') >_{\rho_\vartheta},$$

and

$$\begin{aligned}\frac{\partial}{\partial\vartheta}\Xi(\vartheta|\tau,\tau') &= -\frac{1}{Y(\vartheta)}\int_0^\beta \sum_{k=1}^{p-1}\pi_k(\eta(t)) \qquad (4.12)\\ &\Big(< \xi^{2k}(t)\xi(\tau)\xi(\tau') >_{\rho_\vartheta} - < \xi^{2k}(t) >_{\rho_\vartheta} < \xi(\tau)\xi(\tau') >_{\rho_\vartheta}\Big)\, dt\end{aligned}$$

where the expectations $< . >_{\rho_\vartheta}$ are taken with respect to the measure

$$\begin{aligned}\rho_\vartheta(d\xi|\eta) &\stackrel{\text{def}}{=} \frac{1}{Y(\vartheta)}\exp\{-\frac{1}{2}(2Jd-a)\|\xi\|_\beta^2 - \int_0^\beta \tilde{P}_{2p}(\xi(t))dt\\ &- \vartheta\int_0^\beta \Pi_{2p}(\xi(t)|\eta(t))dt\}\gamma_\beta(d\xi). \qquad (4.13)\end{aligned}$$

For every $\eta \in \mathcal{C}_\beta$ and $\vartheta \in [0,1]$, the measure $\rho_\vartheta(d\xi|\eta)$ satisfies the conditions of Proposition 4.1 and Proposition 4.2, therefore

$$\begin{aligned} < \xi(\tau)\xi(\tau') >_{\rho_\vartheta} &\geq 0,\\ < \xi^{2k}(t)\xi(\tau)\xi(\tau') >_{\rho_\vartheta} &\geq < \xi^{2k}(t) >_{\rho_\vartheta} < \xi(\tau)\xi(\tau') >_{\rho_\vartheta}, \quad k \in I\!N.\end{aligned}$$

The latter inequality yields that the right–hand side of (4.12) is nonpositive, that implies together with the former one

$$0 \leq \Xi(\vartheta|\tau,\tau') \leq \Xi(0|\tau,\tau'), \quad \forall\vartheta \in [0,1]. \qquad (4.14)$$

Now we employ these bounds in (4.11) and obtain

$$\begin{aligned}K_x(\tau,\tau') &\leq \frac{1}{(Z_x X_\beta)^2}\Xi(0|\tau,\tau')\int_{\mathcal{C}_\beta} Y(1)\exp\{-\frac{2Jd-a}{2}\|\eta\|_\beta^2\\ &+ \sqrt{2}J(\eta,x)_\beta - \int_0^\beta \tilde{P}_{2p}(\eta(\tau))d\tau\}\gamma_\beta(d\eta)\\ &= \frac{1}{(Z_x X_\beta)^2}\Xi(0|\tau,\tau')\\ &\int\int_{\mathcal{C}_\beta\times\mathcal{C}_\beta}\exp\{-\frac{2Jd-a}{2}(\|\xi\|_\beta^2+\|\eta\|_\beta^2)\\ &+ \sqrt{2}J(\eta,x)_\beta - \int_0^\beta \Pi_{2p}(\xi(\tau)|\eta(\tau))d\tau\\ &- \int_0^\beta [\tilde{P}_{2p}(\xi(\tau)) + \tilde{P}_{2p}(\eta(\tau))]d\tau\}(\gamma_\beta\otimes\gamma_\beta)(d\xi,d\eta).\end{aligned}$$

Returning in the latter integral to the initial variables by means of (4.7) one obtains

$$\begin{aligned} K_x(\tau,\tau') &\leq \frac{1}{(Z_x X_\beta)^2}\Xi(0|\tau,\tau') \\ &\int\int_{\mathcal{C}_\beta\times\mathcal{C}_\beta} \exp\{-\frac{2Jd-a}{2}(\|\omega\|_\beta^2+\|\omega'\|_\beta^2)+J(\omega+\omega',x)_\beta \\ &- \int_0^\beta [P_{2p}(\omega(\tau))+P_{2p}(\omega'(\tau))]d\tau\}(\gamma_\beta\otimes\gamma_\beta)(d\omega,d\omega') \\ &= \Xi(0|\tau,\tau')\int_{\mathcal{C}_\beta}\int_{\mathcal{C}_\beta}\nu^x(d\omega)\nu^x(d\omega') = \Xi(0|\tau,\tau'). \end{aligned} \tag{4.15}$$

Comparing (4.9), (4.13), and (3.28), (3.29) one gets $\rho_0=\tilde{\nu}^0$ and

$$\Xi(0|\tau,\tau') = <\omega(\tau)\omega(\tau')>_{\rho_0} = <\omega(\tau)\omega(\tau')>_{\tilde{\nu}^0},$$

that yields in (4.15)

$$K_x(\tau,\tau') \leq <\omega(\tau)\omega(\tau')>_{\tilde{\nu}^0} = \tilde{K}_0(\tau,\tau'),$$

where (4.4) has also been used. ■

Now we may prove the lemmas stated in the preceding section.

Proof of Lemma 3.1. Applying (4.1), (4.2), and the representation of the norm of $\tilde{K}_0$ (3.32) one obtains

$$\begin{aligned} &(\varphi,K_x\varphi)_\beta = |(\varphi,K_x\varphi)_\beta| \leq \int_0^\beta\int_0^\beta K_x(\tau,\tau')|\varphi(\tau)||\varphi(\tau')|d\tau d\tau' \\ \leq &\int_0^\beta\int_0^\beta \tilde{K}_0(\tau,\tau')|\varphi(\tau)||\varphi(\tau')|d\tau d\tau' \leq \left\|\tilde{K}_0\right\|\||\varphi|\|_\beta^2 = \left\|\tilde{K}_0\right\|\|\varphi\|_\beta^2. \end{aligned}$$

■

Proof of Lemma 3.2. For the kernels of K_x and $\tilde{K}_0$, one has

$$\mathrm{trace}(K_x)=\int_0^\beta K_x(\tau,\tau)d\tau, \quad \mathrm{trace}(\tilde{K}_0)=\int_0^\beta \tilde{K}_0(\tau,\tau)d\tau,$$

which gives (3.34) by means of (4.2). ■

A statement similar to Lemma 3.3 has already been proved earlier (see e.g. Lemma 2.3 of [3]), but for the completeness, we repeat it here.

Proof of Lemma 3.3. For the case $x=0$, one has in (3.30)

$$\tilde{K}_0(\tau,\tau') = \int_{\mathcal{H}_\beta}\omega(\tau)\omega(\tau')\tilde{\nu}^0(d\omega). \tag{4.16}$$

The periodicity of the loops $\omega(\tau)$ implies that the kernel of $\tilde{K}_0$ is also periodic

$$\tilde{K}_0(\tau,\tau') = \tilde{K}_0(\tau+\vartheta,\tau'+\vartheta), \quad \forall\vartheta\in[0,\beta] \tag{4.17}$$

where the addition is modulo β. The latter yields in turn

$$\max\mathcal{S}_{pp}(\tilde{K}_0) = \int_0^\beta \tilde{K}_0(0,\tau)d\tau = \frac{1}{Z}\int_0^\beta \mathrm{trace}\left\{qe^{-\tau\tilde{H}}qe^{-(\beta-\tau)\tilde{H}}\right\}d\tau, \tag{4.18}$$

where

$$Z = \mathrm{trace}(e^{-\beta\tilde{H}}), \tag{4.19}$$

and the operators $\tilde{H}$, q are defined by (3.26) and (3.27). Recalling that the eigenvalues of $\tilde{H}$, $\tilde{E}_l$ are simple we denote by ψ_l the corresponding eigenfunction and $q_{ll'} = (\psi_l, q\psi_{l'})_\beta$. For symmetry reasons $q_{ll} = 0$, then (4.18) may be written as follows

$$\max\mathcal{S}_{pp}(\tilde{K}_0) = \frac{1}{Z}\sum_{l,l'\in I\!N, l\neq l'}(q_{ll'})^2\frac{(\tilde{E}_l-\tilde{E}_{l'})(e^{-\beta\tilde{E}_{l'}}-e^{-\beta\tilde{E}_l})}{(\tilde{E}_l-\tilde{E}_{l'})^2},$$

The case of zero denominator is excluded, thus this denominator may be estimated by means of (3.35), which yields

$$\begin{aligned}\max\mathcal{S}_{pp}(\tilde{K}_0) &\leq \frac{1}{Z}\frac{1}{\tilde{\Delta}^2}\sum_{l,l'\in I\!N}(q_{ll'})^2(\tilde{E}_l-\tilde{E}_{l'})(e^{-\beta\tilde{E}_{l'}}-e^{-\beta\tilde{E}_l})\\ &= \frac{1}{Z}\frac{1}{\tilde{\Delta}^2}\mathrm{trace}[q,[\tilde{H},q]e^{-\beta\tilde{H}} = \frac{1}{m\tilde{\Delta}^2},\end{aligned}$$

where $[.,.]$ means commutator. ∎

Proof of Lemma 3.4. Let us return to the expressions (3.24)–(3.26). For some positive α, consider the unitary operator U_α on $\mathcal{H}_\beta$, which acts as follows

$$(U_\alpha\psi)(x) = \alpha^{1/2}\psi(\alpha x).$$

Then we have on a core for d/dx, resp. q

$$U_\alpha(\frac{d}{dx})U_\alpha^{-1} = \alpha^{-1}(\frac{d}{dx}), \qquad U_\alpha qU_\alpha^{-1} = \alpha q. \tag{4.20}$$

Now we put $\alpha = m^{-1/(2p+2)}$ and conclude that the operators $\tilde{H}$ and $\tilde{H}(m)$

$$\tilde{H}(m) = m^{-\frac{p}{p+1}}T, \quad T \stackrel{\mathrm{def}}{=} T_0 + m^{1/(p+1)}T_1 \tag{4.21}$$

are unitary equivalent. Here

$$T_0 = -\frac{1}{2}\frac{d^2}{dx^2} + 2^{1-p} b_p q^{2p},$$

$$T_1 = \frac{1}{2}(2Jd + 1 - a) m^{\frac{p-2}{p+1}} q^2 + \sum_{l=2}^{p-1} 2^{1-l} m^{\frac{p-l-1}{p+1}} b_l q^{2l}.$$

Let Δ and Δ_0 be defined by the expression (3.35) but with the eigenvalues of the operators T and T_0 respectively. Then

$$\tilde{\Delta} = m^{-\frac{p}{p+1}} \Delta, \tag{4.22}$$

For the eigenvalues $E_n^{(0)}$ of the operator T_0, one may have the following asymptotics

$$E_n^{(0)} = C(p) n^{\frac{2p}{p+1}} (1 + o(n)), \quad n \to \infty,$$

with an explicitly given constant $C(p)$, see e.g Chapter VII of [33]. Then by means of methods of the book [26] (see the remark just after Theorem I.7.1), one may obtain the same asymptotics for the eigenvalues of T, which holds uniformly with respect to m in some neighborhood of zero. Thus both Δ_0 and Δ may be found by means of finite number of corresponding eigenvalues. Now one may employ the results of the book [21] on the perturbation of the spectrum (see p.208– 215). They imply the existence of a constant c_0 such that

$$\Delta \geq c_0 \Delta_0, \tag{4.23}$$

for sufficiently small positive values of m. These arguments yield the estimate (3.37) with $\chi_0 = 1/(c_0 \Delta_0)^2$. ■

Proof of Lemma 3.6. The estimate of the type of (3.39) has been proved in [1] on the base of the logarithmic Sobolev inequality. Its realization in the form of (3.39) is obtained for the choice of $\tilde{V}$ (3.24), (3.25) with $p = 2$. For more details one may see Section 6 of [1]. ■

Proof of Lemma 3.5. Let us return to the definition of the variance of $f \in \mathrm{Lip}_1(\mathcal{H}_\beta)$ (3.18). By means of (3.2), (3.3), (3.23), (3.34), (4.16), (4.17) one obtains

$$\begin{aligned} &\mathrm{Var}_{\nu^x} f \qquad\qquad (4.24)\\ &\leq \frac{1}{2}\int_{\mathcal{H}_\beta}\int_{\mathcal{H}_\beta} \|\omega - \omega'\|_\beta^2 \nu^x(d\omega)\nu^x(d\omega') = \mathrm{trace}(K_x)\\ &\leq \mathrm{trace}(\tilde{K}_0) = \int_{\mathcal{H}_\beta} \tilde{K}_0(\tau,\tau)d\tau = \frac{\beta}{Z}\mathrm{trace}(q^2 e^{-\beta\tilde{H}}) \stackrel{\mathrm{def}}{=} \beta < q^2 >, \end{aligned}$$

where Z is given by (4.19). For $\max\mathcal{S}_{pp}(\tilde{K}_0)$, we have the estimate (3.36). On the other hand, the same quantity may be expressed in terms of the Duhamel two–point function [15] and hence be estimated from below. The estimate comes from Bogolyubov's inequalities, it was obtained in [15] and for our case it reads

$$\beta < q^2 > f\left(\frac{\beta}{4m < q^2 >}\right) \leq \max\mathcal{S}_{pp}(\tilde{K}_0), \tag{4.25}$$

where the function f was defined in [15] implicitly by the relation

$$f(x\tanh x) = \frac{1}{x}\tanh x, \quad x \in (0, +\infty),$$

and was estimated there as

$$\frac{1}{x}(1 - e^{-x}) \leq f(x). \tag{4.26}$$

Applying this estimate together with (3.36) in (4.25) one gets

$$2m < q^2 > (1 - e^{-\beta/4m<q^2>}) \leq \frac{1}{\tilde{\Delta}} \leq \frac{m^{\frac{p}{p+1}}}{c_0\Delta_0},$$

where we have used (4.22), (4.23). It follows from the latter that one may find a constant ϕ_0 such that

$$< q^2 > \leq \phi_0 m^{-\frac{1}{p+1}},$$

holds for sufficiently small positive values of m. Applying this in (4.24) one obtains (3.38) ∎

Acknowledgement Financial support of the INTAS-Project 378 and SFB-256 Bonn are gratefully acknowledged. Yuri Kozitsky is grateful for kind hospitality extended to him at Bonn and Bochum Universities where his part of this work has mainly been performed. His research was supported in part by the Fundamental Research Fund of Ukraine under the grant 2.4/173 that is also acknowledged.

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