Quasi–hereditary algebras and quadratic forms

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The fundamental theorem on representation–finite quivers in [6] indicates a close connection between the representation type of a quiver and the definiteness of a certain quadratic form. Later on a similar connection has been discovered in other classification problems of representation theory. It turns out that there is a strong interaction of quadratic forms and the reperesentation theory of finite–dimensional algebras.

Given a quasi–hereditary algebra A, instead of the complete module category, one studies the Δ–good module category $\mathcal{F}(\Delta)$ consisting of A–modules which have a filtration by standard modules. Analogously to a complete module category, one can associate quadratic forms (may also called Euler form and Tits form) with the Δ–good module category $\mathcal{F}(\Delta)$. The aim of this paper is to study Δ–good module categories in term of these forms.

First, we study Δ–directing and Δ–omnipresent modules in the Δ–good module category $\mathcal{F}(\Delta)$ over a quasi–hereditary algebra A and show that the existence of a Δ–directing and Δ–omnipresent module in $\mathcal{F}(\Delta)$ implies that all standard modules have projective dimension at most 2. By using the process of standardization introduced in [5], the study of Δ–directing modules in $\mathcal{F}(\Delta)$ can be reduced to the study of those over certain quasi–hereditary algebras which admit a Δ–directing and Δ–omnipresent module. For these quasi–hereditary algebras the quadratic forms associated with their Δ–good module categories are well behaved. By applying this reduction, we show that $\mathcal{F}(\Delta)$ is finite (i.e. there are only finitely many isomorphism classes of indecomposables in $\mathcal{F}(\Delta)$) if all indecomposables in $\mathcal{F}(\Delta)$ are Δ–directing. Note that this statement has been proved in [4] in a more general situation in connection with certain vector space category, but the proof presented here is more straightforward. Further, if A is connected and $\mathcal{F}(\Delta)$ has a preprojective component, then the weak positivity of the Tits form of $\mathcal{F}(\Delta)$ implies the finiteness of $\mathcal{F}(\Delta)$.

Secondly, we study Δ–good module categories over hereditary algebras. Note that hereditary algebras as a particular class of quasi–hereditary algebras admit the following description: An algebra is hereditary if and only if it is quasi-hereditary

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with respect to any ordering of the simple modules (see [5]). In [15] it is shown that the Δ–good module category $\mathcal{F}(\Delta)$ over a hereditary algebra is finite if and only if the Tits form is weakly positive. By using the well-developed tilting theory, we obtain easily that the Δ–good module category over a hereditary algebra is tame if and only if the Tits form is weakly nonnegative. In this case, we describe compeltely the dimension vectors of all indecomposable Δ–good modules in terms of the Tits form. Further, by applying the well-known structure of the Auslander–Reiten quiver of a tilted algebra, a description of the shape of the Auslander–Reiten quiver of the Δ–good module category over a hereditary algebra is given.

1. Preliminaries

Throughout the paper, A denotes a finite–dimensional algebra over an algebraically closed field k, $\operatorname{mod} A$ denotes the category of finite–dimesional left A–modules, and the composition of maps $f : M_1 \longrightarrow M_2$ and $g : M_2 \longrightarrow M_3$ will be denoted by gf. For each map $f : M \to N$, we denote by $\operatorname{Ker}(f)$ and $\operatorname{Im}(f)$ the kernel and the image of f, respectively. Given a class Θ of A–modules, we denote by $\mathcal{F}(\Theta)$ the full subcategory of all A–modules which have a Θ–filtration, that is, a filtration

$$0 = M_t \subset M_{t-1} \subset \cdots \subset M_1 \subset M_0 = M$$

such that each factor M_{i-1}/M_i is isomorphic to one object in Θ for $1 \leq i \leq t$. The modules in $\mathcal{F}(\Theta)$ are called Θ-good modules, and the category $\mathcal{F}(\Theta)$ is called the Θ-good module category.

We now recall the definition of a quasi–hereditary algebra. Let $E(1), \cdots E(n)$ be a complete set of non–isomorphic simple A–modules. Note that we fix a particular ordering for labelling the simple A–modules. For each i, let $P(i)$ be a projective cover of $E(i)$ and denote by $\Delta(i)$ (or $\Delta_A(i)$ in order to indicate for the algebra A) the maximal factor module of $P(i)$ with composition factors of the form $E(j)$, $j \leq i$. Let Δ be the full subcategory consisting of all $\Delta(i)$, $1 \leq i \leq n$. We call modules in Δ standard modules. The category $\mathcal{F}(\Delta) = \mathcal{F}(\Delta_A)$ is called the Δ-good module category. The cosatndard modules $\nabla(i)$ and the ∇–good module category $\mathcal{F}(\nabla)$ are defined dually.

The algebra A is called quasi–hereditary if $\operatorname{End}_A(\Delta(i)) \cong k$, for each $1 \leq i \leq n$, and ${}_AA$ belongs to $\mathcal{F}(\Delta)$.

In case A is quasi–hereditary, Ringel [21] shows that there are indecomposable modules $T(i)$, $1 \leq i \leq n$, and exact sequences

$$0 \longrightarrow \Delta(i) \longrightarrow T(i) \longrightarrow X(i) \longrightarrow 0$$

and

$$0 \longrightarrow Y(i) \longrightarrow T(i) \longrightarrow \nabla(i) \longrightarrow 0$$

where $X(i)$ is filtered with factors $\Delta(j)$, $j < i$, and $Y(i)$ is filtered with factors $\nabla(j)$, $j < i$, such that $T = \oplus_{i=1}^n T(i)$ satisfies

$$\operatorname{add} T = \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla).$$

Moreover, the module T is a generalized tilting and cotilting module and called the characteristic tilting module of A.

For each module M over a quasi–hereditary algebra A, we denote by $\underline{\dim} M$ the usual dimension vector of M. If M belongs to $\mathcal{F}(\Delta)$, we denote by $\underline{\dim}_\Delta M$ the vector whose i-th component $(\underline{\dim}_\Delta M)(i)$ is the multiplicity of $\Delta(i)$ in a Δ–filtration of M, called Δ–dimension vector of M.

For each vector $x = (x(i))_i \in \mathbb{N}^n$, we define x^Δ to be the vector $\sum_{i=1}^n x(i) \underline{\dim} \Delta(i)$.

Definition 1.1 Let A be quasi–hereditary with standard modules $\{\Delta(i) : 1 \leq i \leq n\}$. The Euler form χ_{Δ_A} (or simply χ_Δ) of $\mathcal{F}(\Delta)$ is defined to be

$$\chi_{\Delta_A}(x) = \sum_{l \geq 0} \sum_{i,j} (-1)^l \dim_k \operatorname{Ext}^l(\Delta(i), \Delta(j)) x(i) x(j),$$

where $x = (x(1), \cdots, x(n)) \in \mathbb{Z}^n$, and the Tits form q_{Δ_A} (or simply q_Δ) of $\mathcal{F}(\Delta)$ is defined to be

$$q_{\Delta_A}(x) = \sum_{l=0}^{2} \sum_{i,j} (-1)^l \dim_k \operatorname{Ext}^l(\Delta(i), \Delta(j)) x(i) x(j).$$

The quadratic form χ_Δ has been defined in [15] (Note that there always holds $\operatorname{Ext}^l(\Delta(i), \Delta(j)) = 0$ for $i > j$ and $l \geq 0$). Clearly, if $\operatorname{proj.dim} \Delta(i) \leq 2$ for all i, the forms χ_Δ and q_Δ then coincide.

By χ_A and q_A we denote respectively the usual Euler form and Tits form of A (see [17] and [1]). If M lies in $\mathcal{F}(\Delta)$, then

$$\chi_\Delta(\underline{\dim}_\Delta M) = \sum_{l \geq 0} (-1)^l \dim_k \operatorname{Ext}^l(M, M) = \chi_A(\underline{\dim} M).$$

But, in general, we do not have the equality $q_\Delta(\underline{\dim}_\Delta M) = q_A(\underline{\dim} M)$.

Further, by the BGG–reciprocity law (see [5, Lemma 2.5]) we have

$$\chi_\Delta(x) = x (\underline{\dim} \nabla)^{-t} (\underline{\dim} \Delta)^t x^t,$$

where $\underline{\dim} \Delta$ and $\underline{\dim} \nabla$ denote $n \times n$–matrices whose i–th rows are $\underline{\dim} \Delta(i)$ and $\underline{\dim} \nabla(i)$, respectively.

Note that $\mathcal{F}(\Delta)$ defines an integral bilinear form

$$< x, y >_\Delta = \sum_{l \geq 0} \sum_{i,j} (-1)^l \dim_k \operatorname{Ext}^l(\Delta(i), \Delta(j)) x(i) y(j)$$

over $\mathbb{Z}^n$ such that

$$< \underline{\dim}_\Delta M, \underline{\dim}_\Delta N >_\Delta = \sum_{l\geq 0}(-1)^l \dim_k \mathrm{Ext}^l(M,N) \qquad \text{for } M,N \in \mathcal{F}(\Delta).$$

Definition 1.2. Let A be a finite–dimensional algebra, $\mathcal{C}$ a full subcategory of $\mathrm{mod}\, A$ which is closed under extensions.

(a) We say $\mathcal{C}$ to be of finite type (or simply finite) if there are only finitely many isoclasses (=isomorphism classes) of indecomposables in $\mathcal{C}$.

(b) We say $\mathcal{C}$ to be of tame type (or simply tame) if for every d in $\mathbb{N}$, there are finitely many $A - k[x]$–bimodules $M_1, \cdots, M_s$ which are free of rank d over $k[x]$ such that, up to isomorphism, almost every indecomposable module in $\mathcal{F}(\Delta)$ of dimension d is of the form $M_i \otimes_{k[x]} k[x]/(x-\lambda)$ for some $1 \leq i \leq s$ and some $\lambda \in k$. We write $\mu_{\mathcal{C}}(d)$ for the minimal possible s in this definition. $\mathcal{C}$ is said to be domestic if there is an $N > 0$ such that $\mu_{\mathcal{C}}(d) \leq N$ for all $d \in \mathbb{N}$.

(c) We say $\mathcal{C}$ to be of wild type (or simply wild) if there is an $A - k<x,y>$–bimodule M which is finitely free over $k<x,y>$ and induces a functor

$$F : \mathrm{mod}\, k<x,y> \longrightarrow \mathcal{C},\ N \longmapsto M \otimes_{k<x,y>} N$$

such that F preserves indecomposability and respects isoclasses. $\mathcal{C}$ is said to be strictly wild if, in addition, the functor F is full.

Finally, for a quasi–hereditary algebra A, we denote by $\Gamma_{\mathcal{F}(\Delta)}$ the Auslander–Reiten quiver of $\mathcal{F}(\Delta)$ (see the definition in [22]). For a non–projective A–module in $\mathcal{F}(\Delta)$, by $\tau_\Delta X = \tau_{\Delta_A} X$ we denote the left hand term in an almost split sequence $0 \to \tau_\Delta X \to E \to X \to 0$ in $\mathcal{F}(\Delta)$ (Note that by Γ_A we denote the Auslander–Reiten quiver of the complete module category mod A, and by τ_A the usual Auslander–Reiten translate DTr).

A component $\mathcal{C}$ of Γ_A (or $\Gamma_{\mathcal{F}(\Delta)}$) is called preprojective if $\mathcal{C}$ contains no oriented cycle and consists of τ_A–(or τ_Δ–) orbits of projectives only (see [9]). Such a component $\mathcal{C}$ is called complete if it consists of all indecomposable projectives.

2. Δ–good module categories

Throughout this section, A is a quasi–herediatry algebra and $\mathcal{F}(\Delta) = \mathcal{F}(\Delta_A)$ denotes the Δ–good module category of A. A path in $\mathcal{F}(\Delta)$ is a sequence $(X_0, \cdots, X_s)$ of indecomposable modules $X_i \in \mathcal{F}(\Delta)$, $0 \leq i \leq s$ such that $\mathrm{rad}(X_{i-1}, X_i) \neq 0$ for all $1 \leq i \leq s$. We write $X \preceq_\Delta Y$ to indicate that a path from X to Y exists. If $s \geq 1$, and $X_0 = X_s$, then the path $(X_0, \cdots, X_s)$ is called a cycle in $\mathcal{F}(\Delta)$. An indecomposable module X in $\mathcal{F}(\Delta)$ is called Δ–directing if X does not occur in a cycle in $\mathcal{F}(\Delta)$. Obviously, every module in a preprojective component of $\Gamma_{\mathcal{F}(\Delta)}$ is Δ–directing.

Lemma 2.1. *Let $X \in \mathcal{F}(\Delta)$ be Δ–directing. Then $End_A(X) = k$ and $Ext^i_A(X,X) = 0$ for all $i \geq 1$. In particular, $\underline{dim}_\Delta X$ is a root of χ_Δ.*

This lemma follows from similar arguments in [19, 2.5(8)].

A module $X \in \mathcal{F}(\Delta)$ is called Δ–omnipresent if $(\underline{\dim}_\Delta X)(i) \neq 0$ for all $1 \leq i \leq n$, that is, each $\Delta(i)$ appears as a factor in a Δ–filtration of X.

Lemma 2.2. *Let $X \in \mathcal{F}(\Delta)$ be Δ–omnipresent. Then $Hom_A(P(i), X) \neq 0$ and $Hom_A(X, T(i)) \neq 0$ for all $1 \leq i \leq n$.*

Proof. Let X be Δ–omnipresent. It is clear that $\mathrm{Hom}_A(P(i), X) \neq 0$ for each $1 \leq i \leq n$. Since $\Delta(i)$ appears as a factor in a Δ–filtration of X, there are submodules $X_1 \subset X_2 \subseteq X$ such that $X_2/X_1 \cong \Delta(i)$. Then there is a monomorphism $f : X_2/X_1 \to T(i)$. Because $T(i)$ is Ext–injective in $\mathcal{F}(\Delta)$, f can be extended to a morphism $0 \neq g : X/X_1 \to T(i)$. Then the composition of g and the canonical projection $X \to X/X_1$ is a nonzero morphism from X to $T(i)$, that is, $\mathrm{Hom}_A(X, T(i)) \neq 0$. This finishes the proof.

Note that, in general, neither $\mathrm{Hom}_A(P(i), X) \neq 0$ nor $\mathrm{Hom}_A(X, T(i)) \neq 0$ implies that $(\underline{\dim}_\Delta X)(i) \neq 0$.

The following proposition is an analogue to [19, 2.4(7)] (see also [9] and [2]). However, the proof given here uses similar arguments in [2].

Proposition 2.3. *Let X be Δ–directing and Δ–omnipresent in $\mathcal{F}(\Delta)$. Then proj.dim.$\Delta(i) \leq 2$ for all $1 \leq i \leq n$. Moreover, there hold that $Ext^2_A(X, Z) = 0$ and $Ext^2_A(Z, X) = 0$ for all $Z \in \mathcal{F}(\Delta)$.*

Proof. Let X be Δ–directing and Δ–omnipresent. First we show that an indecomposable Y in $\mathcal{F}(\Delta)$ satisfying $Y \preceq_\Delta X$ has projective dimension ≤ 1. By [7, Corollary 9.4], there is an isomorphism $\mathrm{Ext}^1_A(Y, Z) \cong D\overline{\mathrm{Hom}}_A(Z, \tau_\Delta Y)$ for all Z in $\mathcal{F}(\Delta)$, where $\overline{\mathrm{Hom}}_A(Z, \tau_\Delta Y)$ denotes the factor group of $\mathrm{Hom}_A(Z, \tau_\Delta Y)$ modulo morphisms factorizing the Ext–injective objects in $\mathcal{F}(\Delta)$, that is, modules in $\mathrm{add}\, T$, where T is the charateristic tilting module of A. Since X is Δ–directing and Δ–omnipresent, one has that $\mathrm{Hom}_A(M, \tau_\Delta Y) = 0$ for all $M \in \mathrm{add}\, T$. Thus $\mathrm{Ext}^1_A(Y, -) \cong D\mathrm{Hom}_A(-, \tau_\Delta Y)$ is right exact in $\mathcal{F}(\Delta)$, that is, $\mathrm{Ext}^2_A(Y, Z) = 0$ for all $Z \in \mathcal{F}(\Delta)$. Since $X \preceq_\Delta X$, this implies that $\mathrm{Ext}^2_A(X, Z) = 0$.

Consider a short exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow Y \longrightarrow 0$$

with P projective. The kernel K lies also in $\mathcal{F}(\Delta)$ since $\mathcal{F}(\Delta)$ is closed under kernels of surjective maps. It then follows that $\mathrm{Ext}^1_A(K, Z) = \mathrm{Ext}^2_A(Y, Z) = 0$ for each $Z \in \mathcal{F}(\Delta)$, that is, K is Ext–projective in $\mathcal{F}(\Delta)$, thus by [5] a projective A–module. Hence proj.dim. $Y \leq 1$.

For each $1 \leq i \leq n$, there is an exact sequence

$$0 \longrightarrow K(i) \longrightarrow P(i) \longrightarrow \Delta(i) \longrightarrow 0.$$

Since X is Δ–omnipresent, each indecomposable summand K of $K(i)$ satisfies $K \preceq_\Delta X$, thus has projective dimension ≤ 1. As a result, proj.dim $\Delta(i) \leq 2$.

Dually, for each indecomposable Y' in $\mathcal{F}(\Delta)$ with $X \preceq_\Delta Y'$, one obtains that $\mathrm{Ext}_A^2(Z, Y') = 0$ for all $Z \in \mathcal{F}(\Delta)$. Thus $\mathrm{Ext}_A^2(Z, X) = 0$. This finishes the proof.

In the following we recall a construction given in [5] and reduce each module in $\mathcal{F}(\Delta_A)$ to a Δ–omnipresent module over another quasi–hereditary algebra. Let X be a module in $\mathcal{F}(\Delta_A)$. We denote by $\{i_1 < \cdots < i_m\}$ the set of all $1 \le i \le n$ with $(\underline{\dim}_\Delta X)(i) \neq 0$. Obviousely, $\{\Delta_A(i_s) : 1 \le s \le m\}$ is a standardizable set in $\mathrm{mod}\, A$. By the proof of [5, Theorem 2] and the dual arguments, for each $1 \le s \le m$, there are exact sequences

$$0 \longrightarrow R(s) \longrightarrow Q(s) \longrightarrow \Delta_A(i_s) \longrightarrow 0$$

$$0 \longrightarrow \Delta_A(i_s) \longrightarrow U(s) \longrightarrow V(s) \longrightarrow 0$$

such that $Q(s)$ (resp. $U(s)$) is indecomposable and Ext–projective (resp. Ext–injective) in $\mathcal{F}(\Delta_A(i_s) : 1 \le s \le m)$ and that $R(s)$ (resp. $V(s)$) is filtered by $\Delta_A(i_t)$ with $t > s$ (resp. $t < s$). Moreover, the endomorphism algebra $B_X := (\mathrm{End}_A(Q))^{\mathrm{op}}$, where $Q = \oplus_{s=1}^m Q(s)$, is again quasi–hereditary (with respect to the natural ordering $1 < \cdots < m$). The functor $F_X := \mathrm{Hom}_A(Q, -) : \mathrm{mod}\, A \to \mathrm{mod}\, B_X$ induces an exact equivalence from $\mathcal{F}(\Delta_A(i_s) : 1 \le s \le m)$ to $\mathcal{F}(\Delta_{B_X})$ such that $F_X(\Delta_A(i_s)) = \Delta_{B_X}(s)$, $1 \le s \le m$. Thus, for each $Y \in \mathcal{F}(\Delta_A(i_s) : 1 \le s \le m)$, it holds that $\underline{\dim}_\Delta Y = \underline{\dim}_\Delta F_X(Y)$. In particular, $F_X(X)$ is Δ–omnipresent in $\mathcal{F}(\Delta_{B_X})$. Moreover, if X is Δ–directing in $\mathcal{F}(\Delta_A)$, then $F_X(X)$ is Δ–directing in $\mathcal{F}(\Delta_{B_X})$. Further, we have the following lemma.

Lemma 2.4 *Let X be Δ–directing in $\mathcal{F}(\Delta_A)$. Then, for $1 \le s, t \le m$, there holds that*

$$dim_k Ext_A^l(\Delta_A(i_s), \Delta_A(i_t)) = dim_k Ext_{B_X}^l(\Delta_{B_X}(s), \Delta_{B_X}(t))$$

for $1 \le s, t \le m$ and $l = 0, 1, 2$. In particular, for each $Y \in \mathcal{F}(\Delta_A(i_s) : 1 \le s \le m)$, it holds that

$$q_{\Delta_A}(\underline{dim}_\Delta Y) = q_{\Delta_{B_X}}(\underline{dim}_\Delta F_X(Y)).$$

Proof. Since $\mathcal{F}(\Delta_A(i_s) : 1 \le s \le m)$ and $\mathcal{F}(\Delta_{B_X})$ are exact equivalent, we have obviously that

$$\dim_k \mathrm{Ext}_A^l(Y, Z) = \dim_k \mathrm{Ext}_{B_X}^l(F_X(Y), F_X(Z))$$

for all $Y, Z \in \mathcal{F}(\Delta_A(i_s) : 1 \le s \le m)$ and $l = 0, 1$. It holds particularly that

$$\dim_k \mathrm{Ext}_A^l(\Delta_A(i_s), \Delta_A(i_t)) = \dim_k \mathrm{Ext}_{B_X}^l(\Delta_{B_X}(s), \Delta_{B_X}(t))$$

for $1 \le s, t \le m$ and $l = 0, 1$. Further, by applying $\mathrm{Hom}_A(-, \Delta_A(i_t))$ to the exact sequence

$$0 \longrightarrow R(s) \longrightarrow Q(s) \longrightarrow \Delta_A(i_s) \longrightarrow 0,$$

we obtain the following exact sequence

$$\begin{aligned} 0 = \ & \mathrm{Ext}^1_A(Q(s), \Delta_A(i_t)) \longrightarrow \mathrm{Ext}^1_A(R(s), \Delta_A(i_t)) \longrightarrow \mathrm{Ext}^2_A(\Delta_A(i_s), \Delta_A(i_t)) \\ \longrightarrow \ & \mathrm{Ext}^2_A(Q(s), \Delta_A(i_t)). \end{aligned}$$

We claim that $\mathrm{Ext}^2_A(Q(s), \Delta_A(i_t)) = 0$. If it is not the case, then this would imply that

$$\mathrm{Ext}^2_A(Q(s), U(t)) \neq 0$$

since we have the following exact sequence

$$0 = \mathrm{Ext}^1_A(Q(s), V(t)) \longrightarrow \mathrm{Ext}^2_A(Q(s), \Delta_A(i_t)) \longrightarrow \mathrm{Ext}^2_A(Q(s), U(t)).$$

Furhter, consider the exact sequence

$$0 \longrightarrow K \longrightarrow P \xrightarrow{f} Q(s) \longrightarrow 0$$

with $f : P \to Q(s)$ a projective cover of $Q(s)$ and $K = \mathrm{Ker}(f)$. Then we have that

$$\mathrm{Ext}^1_A(K, U(t)) \cong \mathrm{Ext}^2_A(Q(s), U(t)) \neq 0.$$

Hence, there is an indecomposable summand K' of K such that $\mathrm{Ext}^1_A(K', U(t)) \neq 0$, thus $U(t) \preceq_\Delta K' \preceq_\Delta Q(s)$ in $\mathcal{F}(\Delta_A)$. Because $Q(s)$ and $U(t)$ are respectively Ext–projective and Ext–injective in $\mathcal{F}(\Delta_A(i_s) : 1 \leq s \leq m)$, we obtain in $\mathcal{F}(\Delta_A)$ a cycle

$$X \preceq_\Delta U(t) \preceq_\Delta K' \preceq_\Delta Q(s) \preceq_\Delta X$$

This contradicts the Δ–directness of X. Thus $\mathrm{Ext}^2_A(Q(s), \Delta_A(i_t)) = 0$. This imples that

$$\mathrm{Ext}^2_A(\Delta_A(i_s), \Delta_A(i_t)) \cong \mathrm{Ext}^1_A(R(s), \Delta_A(i_t)).$$

On the other hand, by applying $\mathrm{Hom}_{B_X}(-, F_X(\Delta(i_t)))$ to the exact sequence

$$0 \longrightarrow F_X(R(s)) \longrightarrow F_X(Q(s)) \longrightarrow F_X(\Delta_A(i_s)) \longrightarrow 0,$$

we get that

$$\mathrm{Ext}^2_{B_X}(F_X(\Delta_A(i_s)), F_X(\Delta_A(i_t))) \cong \mathrm{Ext}^1_{B_X}(F_X(R(s)), F_X(\Delta_A(i_t)))$$

since $F_X(Q(s))$ is a projective B_X–module. As a result, we have that

$$\dim_k \mathrm{Ext}^2_A(\Delta_A(i_s), \Delta_A(i_t)) = \dim_k \mathrm{Ext}^2_{B_X}(\Delta_{B_X}(s)), \Delta_{B_X}(t)).$$

This finishes the proof.

The following two lemmas have analogues in a complete module category (see [1, Lemma 3.1] and [19, 2.4(8)]). Their proofs use the reduction given above.

Lemma 2.5. *Let $\Gamma_{\mathcal{F}(\Delta_A)}$ have a preprojective component $\mathcal{C}$ and X be indecomposable in $\mathcal{C}$. Then the connected component $\mathcal{D}$ of $F_X(X)$ in $\Gamma_{\mathcal{F}(\Delta_{B_X})}$ is again preprojective. In particular, proj.dim.$\Delta_{B_X}(s) \leq 2$ for $1 \leq s \leq m$.*

Proof. Since $\mathcal{C}$ is preprojective, each indecomposable in $\mathcal{C}$ is Δ–directing. We have seen that $F_X(X)$ is Δ–directing and Δ–omnipresent in $\mathcal{F}(\Delta_{B_X})$.

First we show that $\mathcal{D}$ consists of $\tau_{\Delta_{B_X}}$–orbits of projectives only. If Y is a predecessor of $F_X(X)$ (that is, there is a path from Y to $F_X(X)$ in $\mathcal{D}$), by applying $\tau_{\Delta_{B_X}}$ to Y again and again, we get a projective. Otherwise, this would imply that X has infintely many predecessors in $\mathcal{C}$ or that $\mathcal{C}$ contains a cycle. Let $Y \in \mathcal{D}$ be arbitrary. Then there is a walk

$$F_X(X) \longrightarrow Y_1 \longrightarrow \cdots \longrightarrow Y_{n-1} \longrightarrow Y_n = Y$$

By induction on n, we may suppose that $\tau^p_{\Delta_{B_X}} Y_{n-1}$ is projective for some p. Then either $\tau^q_{\Delta_{B_X}} Y_n$ is projective for some $0 \leq q \leq p$, or $\tau^{p+1}_{\Delta_{B_X}} Y_n$ is a predecessor of the projective $\tau^p_{\Delta_{B_X}} Y_{n-1}$. The second case implies that $\tau^{p+1}_{\Delta_{B_X}} Y_n$ is aslo a predecessor of $F_X(X)$ since $\mathrm{Hom}_{B_X}(\tau^p_{\Delta_{B_X}} Y_{n-1}, F_X(X)) \neq 0$. Thus $\tau^{p+1}_{\Delta_{B_X}} Y_n$ belongs to a $\tau_{\Delta_{B_X}}$–orbit of a projective, hence so does $Y_n = Y$.

A cycle in $\mathcal{D}$ can be translated to a cycle containing a projective. Thus the modules in this cycle are predecessors of $F_X(X)$. This gives rise to a cycle in $\mathcal{C}$, a contradiction. This finishes the proof.

Lemma 2.6. *Let X be Δ–directing and Y indecomposable in $\mathcal{F}(\Delta)$ satisfying $\underline{dim}_\Delta X = \underline{dim}_\Delta Y$. Then $X \cong Y$.*

Proof. Let X be Δ–directing in $\mathcal{F}(\Delta)$. Then $F_X(X) := X'$ is Δ–directing and Δ–omnipresent in $\mathcal{F}(\Delta_{B_X})$. Thus, by Proposition 2.3, we have that proj.dim.$\Delta_{B_X}(s) \leq 2$ for $1 \leq s \leq m$, and that $\mathrm{Ext}^2_{B_X}(X', Z) = 0 = \mathrm{Ext}^2_{B_X}(Z, X') = 0$ for all $Z \in \mathcal{F}(\Delta_{B_X})$. Note that $\underline{\dim}_\Delta X = \underline{\dim}_\Delta Y$ implies that $\underline{\dim}_\Delta X' = \underline{\dim}_\Delta Y'$, where $Y' = F_X(Y)$. We then have

$$\begin{aligned} 1 \ &= \chi_{\Delta_{B_X}}(\underline{\dim}_\Delta X') = < \underline{\dim}_\Delta X', \underline{\dim}_\Delta X' >_{\Delta_{B_X}} = < \underline{\dim}_\Delta X', \underline{\dim}_\Delta Y' >_{\Delta_{B_X}} \\ &= \dim_k \mathrm{Hom}_{B_X}(X', Y') - \dim_k \mathrm{Ext}^1_{B_X}(X', Y'), \end{aligned}$$

thus $\mathrm{Hom}_{B_X}(X', Y') \neq 0$. Similarly, using that $\mathrm{Ext}^2_{B_M}(Y', X') = 0$, we infer that $\mathrm{Hom}_{B_X}(Y', X') \neq 0$. By the Δ–directness of X', we conclude that $X' \cong Y'$, that is, $X \cong Y$. The proof is completed.

The following two theorems are analogous to [19, 2.4(9), (9′)] and [1, Theorem 3.3].

Theorem 2.7. *If every indecomposable module in $\mathcal{F}(\Delta)$ is Δ–directing. Then $\mathcal{F}(\Delta)$ is finite. If, moreover, proj.dim.$\Delta(i) \leq 2$ for all $1 \leq i \leq n$, then q_Δ is weakly positive, and the correspondence*

$$X \longmapsto \underline{dim}_\Delta X$$

induces a bijection between the set of isoclasses of indecomposables in $\mathcal{F}(\Delta)$ and the set of positive roots of q_Δ.

Proof. The first assertion of the proposition has been proved in [4] in a more general situation. However, we shall present here a direct proof.

We first assume that proj.dim.$\Delta(i) \leq 2$ for all $1 \leq i \leq n$. In this case, for each indecomposable $X \in \mathcal{F}(\Delta)$, we have that

$$q_\Delta(\underline{\dim}_\Delta X) = \chi_\Delta(\underline{\dim}_\Delta X) = 1.$$

For any $0 \neq z \in \mathbb{N}^n$, we choose a module $X \in \mathcal{F}(\Delta)$ with smallest possible $\dim_k \mathrm{End}_A(X)$ and with $\underline{\dim}_\Delta X = z$. Let $X = \oplus_i X_i$ with all X_i indecomposable. Then, by [19, 2.3(Lemma 1)], there holds $\mathrm{Ext}^1(X_i, X_j) = 0$ for $i \neq j$. Since all X_i are Δ–directing, we have by Lemma 2.1 that $\mathrm{Ext}^1(X_i, X_i) = 0$. We then infer that

$$q_\Delta(z) = \chi_\Delta(\underline{\dim}_\Delta X) = \dim_k \mathrm{End}_A(X) + \dim_k \mathrm{Ext}^2_A(X, X) > 0,$$

that is, q_Δ is weakly positive. According to Lemma 2.6, it remains to show that the map $M \longmapsto \underline{\dim}_\Delta M$ is surjective. For each positive root z of q_Δ, choose a module $X \in \mathcal{F}(\Delta)$ with $\underline{\dim}_\Delta X = z$ as above. Then

$$1 = q_\Delta(z) = \dim \mathrm{End}_A(X) + \dim \mathrm{Ext}^2_A(X, X).$$

It implies that $\mathrm{End}_A(X) = k$, thus X is indecomposable. Therefore, the correspondence $X \longmapsto \underline{\dim}_\Delta X$ induces a bijection between the set of isoclasses of indecomposables in $\mathcal{F}(\Delta)$ and the set of positive roots of q_Δ.

In general, for each indecomposable $X \in \mathcal{F}(\Delta_A)$, we consider the module $F_X(X)$ in $\mathcal{F}(\Delta_{B_X})$. By Proposition 2.3, we have that proj.dim$\Delta_{B_X}(s) \leq 2$ for $1 \leq s \leq m$. Thus $q_{\Delta_{B_X}}$ is weakly positive and $\underline{\dim}_\Delta X = \underline{\dim}_\Delta F_X(X)$ is a root of $q_{\Delta_{B_X}}$. Since there are only finitely many such choices for B_X (up to isomorphism) and each $q_{\Delta_{B_X}}$ has only finitely many positive roots, we conclude that $\mathcal{F}(\Delta_A)$ is finite. This finishes the proof.

Theorem 2.8. *Let A be a connected quasi–hereditary algebra and $\Gamma_{\mathcal{F}(\Delta)}$ admit a preprojective component $\mathcal{C}$. If q_Δ is weakly positive, then $\mathcal{F}(\Delta)$ is finite.*

Proof. Since each $X \in \mathcal{C}$ is Δ–directing, we obtain by Lemmas 2.4 and 2.1 that

$$q_{\Delta_A}(\underline{\dim}_\Delta X) = q_{\Delta_{B_X}}(\underline{\dim}_\Delta F_X(X)) = 1,$$

that is, $\underline{\dim}_\Delta X$ is a root of q_{Δ_A}. Since q_{Δ_A} is weakly positive, it has only finitely many positive roots. Thus, by Lemma 2.6, $\mathcal{C}$ is finite. According to [22], $\mathcal{C}$ must be $\Gamma_{\mathcal{F}(\Delta_A)}$. Hence, $\mathcal{F}(\Delta)$ is finite.

The following proposition gives a sufficicent condition for the existence of a preprojective component in the Δ–good module category $\mathcal{F}(\Delta)$.

Proposition 2.9. *Let A be a quasi–hereditary algebra such that Γ_A has a complete preprojective component. Then $\Gamma_{\mathcal{F}(\Delta)}$ admits also a complete preprojective one.*

Proof. Let $\mathcal{P}$ be the complete preprojective component of Γ_A. Since $\mathcal{P}$ consists of all indecomposable projectives $P(i)$, $1 \le i \le n$, each $P(i)$ is directing in $\operatorname{mod} A$ and has only finitely many predecessors X in $\mathcal{P}$. Hence, each $P(i)$ is Δ–directing in $\mathcal{F}(\Delta)$ and has only finitely many predecessors in the connected component of $\Gamma_{\mathcal{F}(\Delta)}$ containing $P(i)$.

Fix an $1 \le i_0 \le n$ and let $\mathcal{P}'$ be the connected component of $\Gamma_{\mathcal{F}(\Delta)}$ containing $P(i_0)$. We claim that $\mathcal{P}'$ consists of τ_Δ–orbits of projectives only. Otherwise, there would be an indecomposable X and a projective P in $\mathcal{P}'$ such that $\tau_\Delta^s X$ exsit for all $s \ge 0$ and that there is an arrow from X to P in $\mathcal{P}'$. Then P has infinitely many predecessors in $\mathcal{P}'$, or P is contained in a cycle in $\mathcal{P}'$. This is a contradiction.

By applying τ_Δ, a cycle in $\mathcal{P}'$ can be translated to one containing an indecomposable projective in $\mathcal{P}'$. This is impossible. Therefore, $\mathcal{P}'$ is a preprojective component. Since A is connected, $\mathcal{P}'$ is complete.

Remark 2.10. Let A be the algebra defined by the following quiver

$$\begin{tikzcd}
 & 4 \arrow[dr, "\beta_1"] & & & \\
3 \arrow[ur, "\alpha_1"] \arrow[dr, "\alpha_2"'] & & 6 \arrow[r, "\gamma"] & 2 \arrow[r, shift left, "\delta_1"] \arrow[r, shift right, "\delta_2"'] & 1 \\
 & 5 \arrow[ur, "\beta_2"'] & & &
\end{tikzcd}$$

with relations $\beta_1\alpha_1 = 0$, $\delta_1\gamma = 0$ and $\delta_2\gamma = 0$. It is easy to check that Γ_A has a (non–complete) preprojective component, but $\Gamma_{\mathcal{F}(\Delta)}$ does not contain any.

As an application of the results above, we obtain the main result in [15].

Corollary 2.11. *Let A be hereditary with standard modules $\{\Delta(i) : 1 \le i \le n\}$. Then $\mathcal{F}(\Delta)$ is finite if and only if the Tits form q_Δ is weakly positive. If this is the case, the correspondence $X \longmapsto \underline{dim}_\Delta X$ provides a bijection between the set of isoclasses of indecomposables in $\mathcal{F}(\Delta)$ and the set of positive roots of q_Δ.*

3. Hereditary algebras as quasi–hereditary algebras

This section is devoted to the study of Δ–good module categories over hereditary algebras. We show that the Δ–good module category over a hereditary algebra is tame if and only if the Tits form is weakly nonnegative. In this case, we also describe the dimension vectors of all indecomposable Δ–good modules.

In [21] Ringel shows that the characteristic tilting module T over an arbitrary quasi–hereditary algebra A characterises the Δ–good module category in the following way:

$$\mathcal{F}(\Delta) = \{X \in \operatorname{mod} A : \operatorname{Ext}^i_A(X, T) = 0, \text{ for all } i \ge 1\}.$$

In particular, if A is hereditary, one then has

$$\mathcal{F}(\Delta) = \{X \in \operatorname{mod} A : \ X \text{ is cogenerated by } T\}.$$

In this case, we have that $T^* = \mathrm{Hom}_k(T, k)$ is a tilting module of A^{op} (=the opposite algebra of A). It defines a torsion theory $(\mathcal{F}(T^*), \mathcal{G}(T^*))$ in $\mathrm{mod}\, A^{\mathrm{op}}$ given by

$$\begin{aligned}
\mathcal{F}(T^*) &= \{X \in \mathrm{mod}\, A^{\mathrm{op}} : \mathrm{Hom}_{A^{\mathrm{op}}}(T^*, X) = 0\} \\
\mathcal{G}(T^*) &= \{X \in \mathrm{mod}\, A^{\mathrm{op}} : \mathrm{Ext}^1_{A^{\mathrm{op}}}(T^*, X) = 0\} \\
&= \{X \in \mathrm{mod}\, A^{\mathrm{op}} : \ X \text{ is generated by } T^*\}.
\end{aligned}$$

Obviously, the duality $D = \mathrm{Hom}_k(-, k)$ induces an anti–equivalence from $\mathcal{F}(\Delta)$ to $\mathcal{G}(T^*)$. Thus, the Δ–good module category over a hereditary algebra can be understood by using the well–developed tilting theory. The following proposition is an easy conclusion of the results in [8] and [12].

Proposition 3.1. *Let B be a hereditary algebra, S a tilting module over B, and $(\mathcal{F}(S), \mathcal{G}(S))$ the torsion theory defined by S. Then*

(1) $\mathcal{G}(S)$ is finite $\Leftrightarrow$ each indecomposable X in $\mathcal{G}(S)$ satisfies

$$dim_k Ext^1_B(X, X) < dim_k End_B(X)$$

$\Leftrightarrow$ each indecomposable X in $\mathcal{G}(S)$ satisfies

$$End_B(X) = k \quad and \quad Ext^1_B(X, X) = 0.$$

(2) $\mathcal{G}(S)$ is tame $\Leftrightarrow$ each indecomposable X in $\mathcal{G}(S)$ satisfies

$$dim_k Ext^1_B(X, X) \leq dim_k End_B(X).$$

In this case, $\mathcal{G}(S)$ is domestic.

(3) $\mathcal{G}(S)$ is wild $\Leftrightarrow$ there exists an indecomposable X in $\mathcal{G}(S)$ satisfying

$$dim_k Ext^1_B(X, X) > dim_k End_B(X).$$

In this case, $\mathcal{G}(S)$ is strictly wild.

Moreover, similar statements of (1)–(3) hold for $\mathcal{F}(S)$.

Proof. By [12, Theorem 4.1] and its proof, there exist a hereditary factor algebra ${}_\infty B$ of B, a tilting module ${}_\infty \hat{S}$ of ${}_\infty B$ without preinjective direct summand, and a functor $\rho : \mathrm{mod}\, {}_\infty B \to \mathrm{mod}\, B$ such that the restriction $\rho_{\mathcal{G}} : \mathcal{G}({}_\infty \hat{S}) \to \mathcal{G}(S)$ is fully, faithful, exact, extension closed and cofinite. Moreover, each indecomposable X in $\mathcal{G}(S)$, which is not isomorphic to an object in $\mathrm{Im}\rho_{\mathcal{G}}$, satisfies that $\mathrm{End}_B(X) = k$ and $\mathrm{Ext}^1_B(X, X) = 0$.

(1) Let $\mathcal{G}(S)$ be finite. Take an indecomposable X in $\mathcal{G}(S)$. We claim that X is a brick (that is, $\mathrm{End}_B(X) = k$). Otherwise, by [18, Theorem*], X admits a factor module Z which is a brick and satisfies $\mathrm{Ext}^1_B(Z, Z) \neq 0$. Since $\mathcal{G}(S)$ is closed under factor modules and extensions, by [17], the module Z provides infinitely many pairwise nonisomorphic indecomposables in $\mathcal{G}(S)$. This is a contradiction. Hence, each indecomposable X in $\mathcal{G}(S)$ satisfies that $\mathrm{End}_B(X) = k$ and $\mathrm{Ext}^1_B(X, X) = 0$.

Now suppose $\mathcal{G}(S)$ is not finite. Then $\mathcal{G}({}_\infty\hat{S})$ is not finite since $\mathcal{G}({}_\infty\hat{S})$ is cofinite in $\mathcal{G}(S)$. This implies that ${}_\infty B$ is a tame (but not representation–finite) or wild hereditary algebra. If ${}_\infty B$ is tame, by [8, Proposition 3.2], $\mathcal{G}({}_\infty\hat{S})$ consists of a simple homogeneous module X. Then $\rho_{\mathcal{G}}(X) \in \mathcal{G}(S)$ satisfies $\dim_k\mathrm{Ext}^1_B(\rho_{\mathcal{G}}(X), \rho_{\mathcal{G}}(X)) = 1 = \dim_k\mathrm{End}_B(\rho_{\mathcal{G}}(X))$. If ${}_\infty B$ is wild, by [12, Corollary 2.2], $\mathcal{G}({}_\infty\hat{S})$ is strictly wild. Thus $\mathcal{G}(S)$ is also strictly wild and contains an indecomposable X satisfying that $\dim_k\mathrm{Ext}^1_B(X, X) > \dim_k\mathrm{End}_B(X)$. This proves (1).

(2) Let $\mathcal{G}(S)$ be tame. Then $\mathcal{G}({}_\infty\hat{S})$ is tame and ${}_\infty B$ is a tame hereditary algebra. Hence each indecomposable $X \in \mathcal{G}({}_\infty\hat{S}) \subseteq \mathrm{mod}\,{}_\infty B$ satisfies that $\dim_k\mathrm{Ext}^1_B(X, X) \le \dim_k\mathrm{End}_B(X)$, thus so does each indecomposable in $\mathcal{G}(S)$.

Suppose $\mathcal{G}(S)$ is not tame. Then ${}_\infty B$ is a wild hereditary algebra. As seen in the proof of (1), there is an indecomposable $X \in \mathcal{G}(S)$ satisfying $\dim_k\mathrm{Ext}^1_B(X, X) > \dim_k\mathrm{End}_B(X)$. This implies the first assertion of (2).

Since one has in this case that ${}_\infty B$ is a tame hereditary algebra, $\mathcal{G}({}_\infty\hat{S}) \subseteq \mathrm{mod}\,{}_\infty B$ is domestic, as well as $\mathcal{G}(S)$.

(3) The first assertion follows from (1) and (2). The strict wildness of $\mathcal{G}(S)$ follows from the fact that ${}_\infty B$ is a wild hereditary algebra and [12, Corollary 2.2].

Corollary 3.2. *Let A be hereditary and $\mathcal{F}(\Delta)$ the Δ–good module category. Then*

(1) $\mathcal{F}(\Delta)$ is either domestic or strictly wild.

(2) $\mathcal{F}(\Delta)$ is finite $\Leftrightarrow$ q_Δ is weakly positive.

(3) $\mathcal{F}(\Delta)$ is tame $\Leftrightarrow$ q_Δ is weakly nonnegative.

Proof. We have seen that $\mathcal{F}(\Delta)$ is anti–equivalent to $\mathcal{G}(T^*)$. The statement (1) then follows directly from Proposition 3.1.

(2) By Proposition 3.1, it is clear that the weak positivity of q_Δ implies the finiteness of $\mathcal{F}(\Delta)$. Conversely, let $\mathcal{F}(\Delta)$ be finite. Then, by Proposition 3.1(1), each indecomposable Y in $\mathcal{F}(\Delta)$ satisfies that $\mathrm{End}_A(Y) = k$ and $\mathrm{Ext}^1_A(Y, Y) = 0$. For any $0 \neq x \in \mathbb{N}^n$, choose a module X in $\mathcal{F}(\Delta)$ with the smallest possible $\dim_k\mathrm{End}_A(X)$ and with $\underline{\dim}_\Delta X = x$. Set $X = \oplus_{i=1}^s X_i$ with X_i indecomposable. By [19, 2.3(Lemma 1)], $\mathrm{Ext}^1(X_i, X_j) = 0$ for $i \neq j$. Then we have

$$q_\Delta(x) = q_\Delta(\underline{\dim}_\Delta X) = \dim_k\mathrm{End}_A(X) > 0,$$

that is, q_Δ is weakly positive.

(3) Similar to (2).

In the following we are going to describe the Δ–dimension vectors of indecomposables in $\mathcal{F}(\Delta)$ over a hereditary algebra. Form now on, we assume that A is a hereditary algebra. Up to Morita equivalence, we write $A = kQ$, the path algebra of some directed quiver $Q = (Q_0, Q_1)$. Thus A–modules are representations of Q. We simply write the vertex set $Q_0 = \{1, 2, \cdots, n\}$, and denote by $\Delta(i)$, $1 \le i \le n$, the standard modules with respect to the natural ordering $1 < 2 < \cdots < n$.

For each A–module $V = (V(i), V(\alpha))$ and each path $\rho = \alpha_t \cdots \alpha_1$ from i to j in Q, by $V(\rho)$ we denote the linear map $V(\alpha_t) \cdots V(\alpha_1) : V(i) \to V(j)$. For $i, j \in Q_0$

with $i < j$, we let $\rho_1, \cdots, \rho_{s_{ij}}$ be all paths from j to i such that all the vertices s involved in $\rho_1, \cdots, \rho_{s_{ij}}$ satisfy $s = j$ or $1 \le s \le i$, and denote by ϕ_i^j the following map

$$\phi_i^j = \sum_{t=1}^{s_{ij}} V(\rho_t) : V(j)^{s_{ij}} \longrightarrow V(i),$$

where $V(j)^{s_{ij}}$ denotes the direct sum of s_{ij} copies of $V(j)$.

Finally, for each $1 \le i \le n$, we denote by Φ_V^i the following map

$$\Phi_V^i = \sum_{j>i} \phi_i^j : \oplus_{j>i} V(j)^{s_{ij}} \longrightarrow V(i).$$

The following proposition gives a criterion for modules in $\mathcal{F}(\Delta)$.

Proposition 3.3. *$\mathcal{F}(\Delta)$ consists of exactly the A–modules $V = (V(i), V(\alpha))$ such that all Φ_V^i, $1 \le i \le n$, are injective.*

Proof. Let $\mathcal{A}$ denote the full subcategory of A–modules $V = (V(i), V(\alpha))$ with all Φ_V^i injective. Obviously, all $\Delta(i)$ belong to $\mathcal{A}$, and $\mathcal{A}$ is closed under extensions. This implies that each Δ–good module lies in $\mathcal{A}$.

Conversely, choose a module $V = (V(i), V(\alpha))$ in $\mathcal{A}$. For each $1 \le i \le n$, we denote by $\gamma_1, \cdots, \gamma_{t_i}$ all paths from n to i in Q. One then easily see that the map $\sum_p V(\gamma_p) : V(n)^{t_i} \to V(i)$ is injective. Let $V' = (V'(i), V'(\alpha))$ be the A–module such that $V'(i) = \mathrm{Im}(\sum_p V(\gamma_p)) = \oplus_p \mathrm{Im}(V(\gamma_p))$ and that $V'(\alpha)$ are induced by $V(\alpha)$. By the construction, V' is a submodule of V generated by $V(n)$ and isomorphic to $\Delta(n)^{x(n)}$, where $x(n) = \mathrm{dim}V(n)$. The quotient $\overline{V} := V/V'$ still satisfies that each $\Phi_{\overline{V}}^i$ is injective, thus lies in $\mathcal{A}$. Further, $\overline{V}$ can be considered as a module over the path algebra $\overline{A}$ of the quiver $\overline{Q}$ which is obtained from Q by deleting the vertex n and arrows connecting with n. Inductively, we may suppose that $\overline{V}$ lies in $\mathcal{F}(\Delta_{\overline{A}})$, where $\overline{A}$ is considered as a quasi–hereditary algebra with respect to the order $\{1 < 2 < \cdots < n-1\}$, and $\mathcal{F}(\Delta_{\overline{A}})$ the Δ–good module category of $\overline{A}$. Since $\overline{A} \cong A/Ae_nA$, where e_n denotes the idempotent of A corresponding to the vertex n, $\mathcal{F}(\Delta_{\overline{A}})$ is identified with the full subcategory of $\mathcal{F}(\Delta_A)$ containing modules filtered by $\Delta(1), \cdots, \Delta(n-1)$. Therefore, $\overline{V}$ lies again in $\mathcal{F}(\Delta_A)$. This implies that V belongs to $\mathcal{F}(\Delta_A)$. The proof is completed.

For each $y = (y(i))_i \in \mathbb{N}^n$, each A–module $V = (V(i), V(\alpha))$ with $V(i) = k^{y(i)}$ can be identified with a point of the variety

$$\mathcal{M}_A(y) = \prod_{\substack{\alpha \in Q_1 \\ \alpha : i \to j}} k^{y(i) \times y(j)}.$$

The algebraic group $G(y) = \prod_i GL_{y(i)}(k)$ acts on the variety $\mathcal{M}_A(y)$ such that $(g \cdot V)(\alpha) = g(t)V(\alpha)g(s)^{-1}$ for each arrow $\alpha : s \to t$. The $G(y)$–orbits then correspond bijectively to the isoclasses of A–modules with dimension vector y.

Further, if $y = x^\Delta = \sum_i x(i)\underline{\dim}\Delta(i)$ for some $x \in \mathbb{N}^n$, we denote by $\mathcal{M}_\Delta(x)$ the subset of $\mathcal{M}_A(y)$ formed by Δ–good modules with dimension vector y, or Δ–dimension vector x. Obviously, $\mathcal{M}_\Delta(x)$ is non–empty and $G(y)$–stable, and $G(y)$–orbits in $\mathcal{M}_\Delta(x)$ correspond bijectively to the isoclasses of modules in $\mathcal{F}(\Delta)$ with Δ–dimension vector x.

The following corollary is a direct implication of Proposition 3.3.

Corollary 3.4. *For each $x \in \mathbb{N}^n$, the set $\mathcal{M}_\Delta(x)$ is Zariski–open in $\mathcal{M}_A(x^\Delta)$. In particular,*

$$dim\mathcal{M}_\Delta(x) = dim\mathcal{M}_A(x^\Delta) = \sum_{\substack{\alpha\in Q \\ \alpha:i\to j}} x^\Delta(i)x^\Delta(j).$$

By the corollary, for each $x \in \mathbb{N}^n$, there holds that

$$q_\Delta(x) = q_A(x^\Delta) = \dim G(x^\Delta) - \dim\mathcal{M}_A(x^\Delta) = \dim G(x^\Delta) - \dim\mathcal{M}_\Delta(x).$$

Remark. (1) Since the set of Δ–good modules $\mathcal{M}_\Delta(x)$ is again a variety, one can use geometrical arguments to give a direct proof of statements (2) and (3) in Corollary 3.2.

(2) The statement in Corollary 3.4 even holds for an arbitrary quasi–hereditary algebra. Indeed, let A be a finite dimensional algebra and, up to Morita equivalence, write $A = kQ/I$, where Q is a finite quiver and I an admissible ideal of kQ. For each $y \in \mathbb{N}^n$, a representation in $\mathcal{M}_{kQ}(y)$ satisfying the relations in I can be considered as an A–module. Hence, the set of A–modules with dimension vector y can be interpreted as a closed subvariety $\mathcal{M}_A(y)$ of the affine variety $\mathcal{M}_{kQ}(y)$. Further, if M is an A–module, then the set

$$\mathcal{M}_A^M(y) := \{X \in \mathcal{M}_A(y) : \mathrm{Ext}^1_A(X, M) = 0\}$$

is open in $\mathcal{M}_A(y)$ since the function

$$f_M : \mathcal{M}_A(y) \longrightarrow \mathbb{N},\ M \longmapsto \dim_k \mathrm{Ext}^1_A(X, M)$$

is upper semi–continuous. In particular, if A is quasi–hereditary, then, for each $x \in \mathbb{N}^n$, the set of Δ–good modules with Δ–dimension vector x

$$\begin{aligned}\mathcal{M}_\Delta(x) :\ &= \{X \in \mathcal{M}_A(x^\Delta) : X \in \mathcal{F}(\Delta)\} \\ &= \{X \in \mathcal{M}_A(x^\Delta) : \mathrm{Ext}^1_A(X, \nabla) = 0\}\end{aligned}$$

is open in $\mathcal{M}_A(x^\Delta)$, where $\nabla = \oplus_{i=1}^n \nabla(i)$. Thus, the Δ–good module variety $\mathcal{M}_\Delta(x)$ is a locally closed subset in the affine variety $\mathcal{M}_{kQ}(x^\Delta)$. It would be interesting to know when there holds that $\dim\mathcal{M}_\Delta(x) = \dim\mathcal{M}_A(x^\Delta)$.

Let G be an algebraic group acting on a variety Z. If $X \subset Z$ is a G–stable subset, we write $X = \cup X_{(s)}$ with $X_{(s)} = \{a \in X : \dim Ga = s\}$, where Ga denotes the orbit of a. The number of parameters of X is by definition

$$\mu(X) = \max_s\{\dim X_{(s)} - s\},$$

where $\dim X_{(s)}$ denotes the dimension of the closure of $X_{(s)}$ in Z.

Let us return to the case where $A = kQ$ is hereditary. For each $y \in \mathbb{N}^n$ and each $d \in \mathbb{N}$, set

$$\mathcal{M}_A(y)^{(d)} = \{V \in \mathcal{M}_A(y) : \dim_k \mathrm{End}_A(V) = d\}.$$

By [14, Proposition 2.4], each $\mathcal{M}_A(y)^{(d)}$ is locally closed in $\mathcal{M}_A(y)$, $\mathcal{M}_A(y)^{\max}$ (=the union of all orbits of maximal dimension) is open and dense in $\mathcal{M}_A(y)$, and $\mathcal{M}_A(y)^{(d)}_{\mathrm{ind}} = \{V \in \mathcal{M}_A(y)^{(d)} : V \text{ indecomposable}\}$ is closed in $\mathcal{M}_A(y)^{(d)}$. In particular, $\mathcal{M}_A(y)_{\mathrm{ind}} = \cup_d \mathcal{M}_A(y)^{(d)}_{\mathrm{ind}}$ is a constructible set.

If $y = x^\Delta$ for some $x \in \mathbb{N}^n$, then $\mathcal{M}_\Delta(x)_{\mathrm{ind}} := \mathcal{M}_\Delta(x) \cap \mathcal{M}_A(y)_{\mathrm{ind}}$ is a constructible set in $\mathcal{M}_\Delta(x)$, and $\mathcal{M}_\Delta(x)^{\max} := \mathcal{M}_\Delta(x) \cap \mathcal{M}_A(y)^{\max}$ is open in $\mathcal{M}_\Delta(x)$. Further, if y is a root of Q, then, by Kac's theorem (see [10], [11] or [14]), it holds that

$$\mu(\mathcal{M}_\Delta(x)_{\mathrm{ind}}) \leq \mu(\mathcal{M}_A(x^\Delta)_{\mathrm{ind}}) = 1 - q_A(x^\Delta) = 1 - q_\Delta(x).$$

In particular, if $\mathcal{F}(\Delta)$ is tame, $\mu(\mathcal{M}_\Delta(x)_{\mathrm{ind}}) \leq 1$.

Proposition 3.5. *For each $0 \neq x \in \mathbb{N}^n$, there is a unique decomposition $x = x_1 + \cdots + x_s$ such that the set*

$$\mathcal{M}_\Delta(x)_{gen} = \{V \in \mathcal{M}_\Delta(x) : V = V_1 \oplus \cdots \oplus V_s,\ \underline{\dim}_\Delta V_i = x_i,\ V_i \textit{ indecomposable}\}$$

contains an open and dense set of $\mathcal{M}_\Delta(x)$.

Proof. We use the arguments in [14, Proposition 2.7].

For each decomposition $x = u_1 + \cdots + u_t \in \mathbb{N}^n$, the set of representations $V \in \mathcal{F}(\Delta)$ such that $V = V_1 \oplus \cdots V_t$ with $\underline{\dim}_\Delta V_i = u_i$ is constructible, since it can be viewed as the image of the following map

$$\begin{aligned} G(x^\Delta) \times \mathcal{M}_\Delta(u_1) \times \cdots \times \mathcal{M}_\Delta(u_t) &\longrightarrow \mathcal{M}_\Delta(x) \\ (g, V_1, \cdots, V_t) &\longmapsto g \cdot (V_1 \oplus \cdots \oplus V_t). \end{aligned}$$

Thus the set

$$\mathcal{M}_\Delta(x; u_1, \cdots, u_t) = \{V : V = V_1 \oplus \cdots \oplus V_t,\ \underline{\dim}_\Delta V_i = u_i,\ V_i \text{ indecomposable}\}$$

is constructible, and $\mathcal{M}_\Delta(x)$ is the disjoint union of the $\mathcal{M}_\Delta(x; u_1, \cdots, u_t)$ over the finite set of all distinct decomposition of x. So precisely one of them, say $\mathcal{M}_\Delta(x; x_1, \cdots, x_s)$, contains an open dense set of $\mathcal{M}_\Delta(x)$.

Following [14], we call the decomposition $x = x_1 + \cdots + x_s$ in the proposition the Δ–generic decomposition of x, and representations in $\mathcal{M}_\Delta(x)^{\max}_{\mathrm{gen}} = \mathcal{M}_\Delta(x)_{\mathrm{gen}} \cap \mathcal{M}_\Delta(x)^{\max}$ Δ–generic representations of type x. An $0 \neq x \in \mathbb{N}^n$ is called Δ–indecomposable if $s = 1$.

Since $\mathcal{M}_\Delta(x)$ is open in $\mathcal{M}_A(x^\Delta)$, $\mathcal{M}_\Delta(x; u_1, \cdots, u_t)$ contains also an open and dense subset of $\mathcal{M}_A(x^\Delta)$. Thus the Δ–generic decomposition $x = x_1 + \cdots + x_s$ of x gives rise to the generic decomposition $x^\Delta = x_1^\Delta + \cdots + x_s^\Delta$ of x^Δ in $\mathcal{M}_A(x^\Delta)$. Corollary 2.7 in [14] then implies the following.

Corollary 3.6. *Let $\mathcal{F}(\Delta) \ni V = V_1 \oplus \cdots \oplus V_s$ be a Δ–generic representation with V_i indecomposable. Then $End_A(V_i) = k$ for all i, and $Ext_A^1(V_i, V_j) = 0$ for $i \neq j$. In particular, if $s = 1$, we have $End_A(V) = k$, $\mathcal{M}_\Delta(x)^{max} \subset \mathcal{M}_\Delta(x)_{ind}$, and*

$$\mu(\mathcal{M}_\Delta(x)_{ind}) = \mu(\mathcal{M}_\Delta(x)^{max}) = 1 - q_\Delta(x).$$

Given a weakly nonnegative quadratic unit form $q : \mathbb{Z}^n \to \mathbb{Z}$, a vector $0 \neq \sigma \in \mathbb{N}^n$ is called a critical vector of q if the restriction of q to $J := \{j : 1 \leq j \leq n, \sigma(j) \neq 0\} = \mathrm{supp}\sigma$, denoted by q^J, is critical such that

$$\{x \in \mathbb{Z}^n : 0 \leq x \text{ and } q^J(x) = 0\} = \mathbb{N}\sigma^J,$$

where σ^J denotes the restriction of σ to $\mathbb{N}^J$.

Theorem 3.7. *Let A be herediatry, $\mathcal{F}(\Delta)$ be tame, and $\sigma_1, \cdots, \sigma_t$ be all critical vectors of q_Δ. Then each σ_i is Δ-indecomposable. Further, for each $x \in \mathbb{N}^n$, if $y = x^\Delta$ is a real root of Q, then, up to isomorphism, there is a unique indecomposable in $\mathcal{F}(\Delta)$ with Δ–dimension vector x; if $y = x^\Delta$ is an imaginary root of Q, then $x = m\sigma_i$ for some $m \geq 1$ and some $1 \leq i \leq t$, and in this case, there are infinitely many pairwise nonisomorphic indecomposables in $\mathcal{F}(\Delta)$ with Δ–dimension vectors x.*

Proof. The statement that each critical vector of q_Δ is Δ–indecomposable follows directly form the corollary 3.6 and the definition of a critical vector.

Let $x \in \mathbb{N}^n$. If $y = x^\Delta$ is a real root of Q, then, by Kac'c Theorem, $\mathcal{M}_A(y)_{\mathrm{ind}}$ contains only one orbit which is open in $\mathcal{M}_A(y)$. This implies that $\mathcal{M}_\Delta(x)_{\mathrm{ind}} = \mathcal{M}_\Delta(x) \cap \mathcal{M}_A(y)_{\mathrm{ind}}$ is nonempty, thus $\mathcal{M}_\Delta(x)_{\mathrm{ind}} = \mathcal{M}_A(y)_{\mathrm{ind}}$ contains only one orbit, that is, up to isomorphism, there is a unique indecomposable in $\mathcal{F}(\Delta)$ with Δ–dimension vector x. If $y = x^\Delta$ is an imaginary root of Q, then $q_\Delta(x) = 0$ since q_Δ is weakly nonnegative. By [3, Proposition 2.4], there are nonnegative rational numbers $a_1, \cdots, a_t$ such that $y = \sum_{i=1}^t a_i\sigma_i$. By multiplying a positive integer b, we obtain $by = \sum_{i=1}^t b_i\sigma_i$ with all $b_i = ba_i$ nonnegative integers. Since y is an imaginary root of Q, there is an element w in the Weyl group W of Q such that $w(y) =: y'$ lies in the fundamental set F_Q of Q. Further, it holds that $q_A(y') = q_A(y) = q_\Delta(x) = 0$. By [14, 3.1(Lemma 1)], the support of y' is a tame subquiver of Q. On the other hand, we have

$$by' = bw(y) = b_1 w(\sigma_1) + \cdots + b_t w(\sigma_t).$$

Since all σ_i are imaginary roots of Q, $w(\sigma_i)$ are also imaginary positive roots. Thus there is only one $1 \leq i \leq t$ such that $b_i \neq 0$. Hence $x = m\sigma_i$ for some $m \geq 1$. By Corollary 3.6, it holds that $\mu(\mathcal{M}_\Delta(\sigma_i)_{\mathrm{ind}}) = 1 - q_\Delta(\sigma_i) = 1$. Thus there are infinitely many pairwise nonisomorphic Δ–generic representations in $\mathcal{M}_\Delta(\sigma_i)_{\mathrm{ind}}$. Again by Corollary 3.6, each $X \in \mathcal{M}_\Delta(\sigma_i)_{\mathrm{ind}}$ satisfies that $\mathrm{End}_A(X) = k = \mathrm{Ext}_A^1(X, X)$. Then, by [17], such an X provides a family of indecomposable modules $\{X[j]\}_{j=1}^\infty$ such that

$$X = X[1] \subset X[2] \subset \cdots\cdots \subset X[m] \subset \cdots\cdots$$

and such that $X[j+1]/X[j] \cong X$ for all $j \geq 1$. Moreover, if $X, X' \in \mathcal{M}_{\Delta}(\sigma_i)_{\mathrm{ind}}$ are not isomorphic, then $X[j]$ and $X'[j]$ are not isomorphic for all $j \geq 1$. Therefore, the modules $X[m]$ with $X \in \mathcal{M}_{\Delta}(\sigma_i)_{\mathrm{ind}}$ provide infinitely many pairwise nonisomorphic indecomposables with Δ–dimension vector $m\sigma_i = x$. This completes the proof.

The following theorem gives a complete description of the shape of each connected component of $\Gamma_{\mathcal{F}(\Delta)}$. The notation used is refered to [9] and [19].

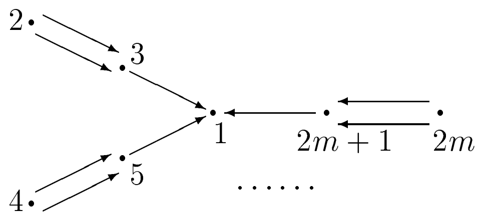
Theorem 3.8. *Let A be a connected hereditary algebra with standard modules $\{\Delta(i) : 1 \leq i \leq n\}$. Then*
(a) $\Gamma_{\mathcal{F}(\Delta)}$ admits a complete preprojective component and a preinjective component.
(b) Each connected component of $\Gamma_{\mathcal{F}(\Delta)}$ which is neither preprojective nor preinjective is quasi–serial or obtained from a quasi–serial translation quiver by coray insertions.

Proof. Let T be the characteristic tilting module of A. Then $T^* = DT := \mathrm{Hom}_k(T, k)$ defines also a torsion theory $(\mathcal{Y}(T^*), \mathcal{X}(T^*))$ in $\mathrm{mod}\, B$, where $B := (\mathrm{End}_{A^{\mathrm{op}}}(T^*))^{\mathrm{op}}$ is the tilted algebra of A^{op},

$$\begin{aligned} & \mathcal{Y}(T^*) = \{{}_BY : \mathrm{Tor}_1^B(T^*_B, Y) = 0\}, \\ \text{and} \quad & \mathcal{X}(T^*) = \{{}_BY : T^* \otimes_B Y = 0\}. \end{aligned}$$

Then $\mathrm{Hom}_A(T^*, D(-))$ defines an anti–equivalence from $\mathcal{F}(\Delta)$ to $\mathcal{Y}(T^*)$. The theorem then follows from the description of componenets of Γ_B lying in $\mathcal{Y}(T^*)$ (see [8], [13], [16], [20], and [23]).

Example. Let A_m, $m > 0$, be the path algebra of the following quiver



It is easy to see that the Tits form of $\mathcal{F}(\Delta_{A_m})$

$$\begin{aligned} q_{\Delta}(x) \; &= \sum_{i=1}^{2m+1} x(i)^2 + \sum_{j=1}^{m} x(1)x(2j+1) - \sum_{j=1}^{m} 2x(2j)x(2j+1) \\ &= x(1)^2 + \sum_{j=1}^{m} \Big(x(2j) - x(2j+1)\Big)^2 + \sum_{j=1}^{m} x(1)x(2j+1) \end{aligned}$$

is weakly nonnegative. Thus, by Corollary 3.2, $\mathcal{F}(\Delta_{A_m})$ is tame. In the Auslander–Reiten quiver of $\mathcal{F}(\Delta_{A_m})$, there are m 1–parameter families of homogeneous tubes. More precisely, $\mathcal{F}(\Delta_{A_m})$ admits one complete preprojective component and m preinjective components. All other components are homogeneous stable tubes.

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