Finite subcategories and heredity chains

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Abstract

Let Λ be an artin algebra and $\mathcal{S}$ a finite set of indecomposable Λ–modules such that $\operatorname{add}\mathcal{S}$ is closed under submodules or factor modules. We show that the endomorphism algebra $A_{\mathcal{S}} := (\operatorname{End}(\oplus_{X\in\mathcal{S}}X))^{\mathrm{op}}$ has global dimension ≤ 2. This produces a large class of quasi–hereditary algebras. Further, we generalize the notion of a splitting filtration introduced by Dlab and Ringel and define left and right splitting filtrations of $\mathcal{S}$. Heredity chains of $A_{\mathcal{S}}$ associated with these filtrations are obtained.

Auslander algebra $A(\Lambda)$ of a representation–finite artin algebra Λ is by definition the endomorphism algebra $(\operatorname{End}_\Lambda(M_\Lambda))^{\mathrm{op}}$, where M_Λ is the direct sum of all indecomposable Λ–modules. It is well–known that the global dimension of $A(\Lambda)$ is at most 2, thus, by a result of Dlab and Ringel [3], $A(\Lambda)$ is quasi–hereditary. In [4] the authors introduced the notion of a splitting filtration on the class of all indecomposable Λ–modules and obtained in this way a heredity chain of ideals of $A(\Lambda)$.

In the present paper we deal with finite sets $\mathcal{S}$ of indecomposable modules over an arbitrary artin algebra Λ and show that the endomorphism algebra $A_{\mathcal{S}} := (\operatorname{End}_\Lambda(\oplus_{X\in\mathcal{S}}X))^{\mathrm{op}}$ has global dimension ≤ 2 if $\operatorname{add}\mathcal{S}$ is closed under submodules or factor modules. Further, we generalize the notion of a splitting filtration in [4] and define left and right splitting filtrations of $\mathcal{S}$. It turns out that the notions of left and right filtrations are well–adapted to our general set–up and are closely related to notions of preinjective and preprojective partitions of $\mathcal{S}$. Following [4], heredity chains of ideals of $A_{\mathcal{S}}$ associated with these filtrations are obtained.

1. Preliminaries

Throughout the paper, Λ denotes an arbitrary artin algebra, $\operatorname{mod}\Lambda$ the category of finitely generated left modules over Λ, and $\operatorname{ind}\Lambda$ a complete set of representatives of the isomorphism classes of indecomposables in $\mathrm{mod}\,\Lambda$. For a morphism $f : M \to N$, by $\mathrm{Ker}\,(f)$ and $\mathrm{Im}\,(f)$ we denote respectively the kernel and the image of f. For each $\mathcal{M} \subseteq \mathrm{ind}\,\Lambda$, $\mathrm{add}\,\mathcal{M}$ denotes the full additive subcategory of $\mathrm{mod}\,\Lambda$ consisting of all Λ–modules isomorphic to summands of finite sums of modules in $\mathcal{M}$. In the following we recall from [1] some definitions and facts which will be needed in the paper.

Definition 1.1. Let $\mathcal{C}$ be a full additive subcategory of $\mathrm{mod}\,\Lambda$. We say that $\mathcal{C}$ has right almost split morphisms if for each indecomposable object C in $\mathcal{C}$ there is a B in $\mathcal{C}$ and an $f : B \to C$ which is a right almost split morphism in $\mathcal{C}$, that is, f is not a splittable epimorphism, and whenever there is a nonsplittable epimorphism $g : C' \to C$ in $\mathcal{C}$, there eaxits a $g' : C' \to B$ such that $fg' = g$. The category $\mathcal{C}$ having left almost split morphisms is defined dually. We say that $\mathcal{C}$ has almost split morphisms if $\mathcal{C}$ has both left and right almost split morphisms.

Definition 1.2. Let $\mathcal{D}$ be a subset of $\mathrm{ind}\,\Lambda$.

(a) A partition $I_0(\mathcal{D}), I_1(\mathcal{D}), \cdots, I_n(\mathcal{D}), \cdots, I_\infty(\mathcal{D})$ of $\mathcal{D}$ is called a preinjective partition of $\mathcal{D}$ if

i) $\cup_{i=0}^{\infty} I_i(\mathcal{D}) = \mathcal{D}$;

ii) $I_i(\mathcal{D}) \cap I_j(\mathcal{D}) = \emptyset$ if $i \neq j$;

iii) For each $i < \infty$, $I_i(\mathcal{D})$ is finite, and an indecomposable object D of $\mathcal{D}$ is in $I_i(\mathcal{D})$ if and only if every monomorphism $g : D \to C$, with $C \in \mathrm{add}\,(\cup_{k \geq i} I_k(\mathcal{D}))$, splits.

(b) A partition $P_0(\mathcal{D}), P_1(\mathcal{D}), \cdots, P_n(\mathcal{D}), \cdots, P_\infty(\mathcal{D})$ of $\mathcal{D}$ is called a preprojective partition of $\mathcal{D}$ if

i) $\cup_{i=0}^{\infty} P_i(\mathcal{D}) = \mathcal{D}$;

ii) $P_i(\mathcal{D}) \cap P_j(\mathcal{D}) = \emptyset$ is $i \neq j$;

iii) For each $i < \infty$, $P_i(\mathcal{D})$ is finite, and an indecomposable object D of $\mathcal{D}$ is in $P_i(\mathcal{D})$ if and only if every epimorphism $f : C \to D$, with $C \in \mathrm{add}\,(\cup_{k \geq i} P_k(\mathcal{D}))$, splits.

Lemma 1.3. (see [1]) *Let $\mathcal{S}$ be a finite subset of indΛ. Then*

(1) add$\mathcal{C}$ has right and left almost split morphisms.

(2) $\mathcal{C}$ has the preinjective partition $\{I_i(\mathcal{S})\}_{i=0}^{p}$ and the preprojective partition $\{P_i(\mathcal{S})\}_{i=0}^{q}$, where p and q are nonnegative integers.

2. Finite subcategories and global dimensions

Let Λ be an artin algebra and $\mathcal{S}$ a finite subset of $\mathrm{ind}\,\Lambda$. We set $M_{\mathcal{S}} = \oplus_{X \in \mathcal{S}} X$ and denote by $A_{\mathcal{S}}$ the endomorphism algebra $(\mathrm{End}_\Lambda(M_{\mathcal{S}}))^{\mathrm{op}}$. For each Y, by $l_Y : Y \to \oplus_{X \in \mathcal{S}} X$ and $\pi_Y : \oplus_{X \in \mathcal{S}} X \to Y$ we denote respectively the canonical injection and projection. Thus, $e_Y := l_Y \pi_Y$ is an idempotent of $\mathcal{A}_{\mathcal{S}}$. Finally, we set $P_Y = A_{\mathcal{S}} e_Y$, the projective $A_{\mathcal{S}}$–module, and denote by

S_Y the corresponding simple $A_{\mathcal{S}}$–module. Note that P_Y can be identified with $\mathrm{Hom}_\Lambda(M_{\mathcal{S}}, Y)$.

Theorem 2.1. *Let Λ be an artin algebra and $\mathcal{S}$ a finite subset of ind$\,\Lambda$ such that add$\,\mathcal{S}$ is closed under submodules or factor modules. Then gl.dim. $A_{\mathcal{S}} \leq 2$.*

Proof. Up to duality, we may suppose that add $\mathcal{S}$ is closed under submodules. Since $\mathcal{S}$ is finite, by Lemma 1.3, add $\mathcal{S}$ has almost split morphisms. Thus, for each $X \in \mathcal{S}$, there is a right almost split morphism $f_X : C_X \to X$ with $C_X \in \mathrm{add}\,\mathcal{S}$. By applying $\mathrm{Hom}_\Lambda(M_{\mathcal{S}}, -)$, we obtain the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_\Lambda(M_{\mathcal{S}}, \mathrm{Ker}\,(f_X)) \longrightarrow \mathrm{Hom}_\Lambda(M_{\mathcal{S}}, C_X) \xrightarrow{\phi_X} \mathrm{Hom}_\Lambda(M_{\mathcal{S}}, X)$$

where $\phi_X = \mathrm{Hom}_\Lambda(M_{\mathcal{S}}, f_X)$. Because f_X is right almost split, we infer that the cokernel of ϕ_X is isomorphic to the simpe $A_{\mathcal{S}}$–module S_X, thus proj.dim.$S_X \leq 2$ since $\mathrm{Ker}\,(f_X) \in \mathrm{add}\,\mathcal{S}$. Hence, gl.dim. $A_{\mathcal{S}} \leq 2$.

Corollary 2.2. Let Λ be an artin algebra and $\mathcal{S}$ a finite subset of ind Λ such that add $\mathcal{S}$ is closed under submodules or factor modules. Then $A_{\mathcal{S}}$ is quasi–hereditary.

Proof. This follows directly from Theorem 2.1 and [1, Theorem 2].

Remark 2.3. Let $\mathcal{S}$ be a finite subset of ind Λ such that add $\mathcal{S}$ is closed under submodules. If $\{I_i(\mathcal{S})\}_{i=0}^{p}$ is the preinjective partition of $\mathcal{S}$. Then, for each $0 \leq j \leq p$, $\cup_{i=j}^{p} I_i(\mathcal{S})$ is finite such that add $(\cup_{i=j}^{p} I_i(\mathcal{S}))$ is closed under submodules.

Dually, let $\mathcal{S}$ be a finite subset of ind Λ such that add $\mathcal{S}$ is closed under factor modules. If $\{P_i(\mathcal{S})\}_{i=0}^{q}$ is the preprojective partition of $\mathcal{S}$. Then, for each $0 \leq l \leq q$, $\cup_{i=l}^{q} P_i(\mathcal{S})$ is finite such that add $(\cup_{i=j}^{q} P_i(\mathcal{S}))$ is closed under factor modules.

In particular, if Λ is of finite representation type and $\mathcal{S} = \mathrm{ind}\,\Lambda$, then $\mathrm{add}\,\mathcal{S} = \mathrm{mod}\,\Lambda$ is closed under both submodules and factor modules. The construction above thus yields subsets of ind Λ which are either closed under submodules, or closed under factor modules.

3. Heredity chains

The notion of a quasi–hereditary algebra has been introduced by E.Cline, B.Parshall and L.Scott [2, 6] in order to study the highest weight categories in representation theory of semisimple Lie algebras and algebraic groups. We recall now the definition of a quasi–hereditary algebra by the existence of a heredity chain of ideals. Let Λ be an artin algebra and rad Λ its Jacobson radical. An ideal J of A is called a heredity ideal if $J^2 = J$, $J \mathrm{rad}\,\Lambda J = 0$, and J considered as a left or right ideal is a projective A–module. The algebra Λ is called quasi–hereditary if there exists a chain

$$0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_m = A$$

of ideals of A such that, for each $1 \leq t \leq m$, the ideal J_t/J_{t-1} is a heredity ideal of A/J_{t-1}. Such a sequence of ideals is called a heredity chain.

Let Λ be an artin algebra and $\mathcal{S}$ a finite subset of $\mathrm{ind}\,\Lambda$. Left and right splitting filtrations of $\mathcal{S}$ are defined as follows (compare with [4]).

Definition 3.1. A chain

$$\emptyset = \mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \cdots \subseteq \mathcal{S}_m = \mathcal{S}$$

of subsets of $\mathcal{S}$ is called a left splitting filtration if the following two properties are satisfied for all $1 \leq t \leq m$.

(1) If $M, M' \in \mathcal{S}_t$, and $\phi : M \to M'$ is not invertible, then ϕ factors through a module in $\mathrm{add}\,\mathcal{S}_{t-1}$.

(2) $\mathrm{add}\,\mathcal{S}_t$ is closed under submodules, and each monomorphism $U \xrightarrow{\phi} W$, with $W \in \mathrm{add}\,\mathcal{S}_t$ and $U \in \mathrm{add}\,(\mathcal{S}_t \backslash \mathcal{S}_{t-1})$, splits.

Definition 3.2. A chain

$$\emptyset = \mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \cdots \subseteq \mathcal{S}_m = \mathcal{S}$$

of subsets of $\mathcal{S}$ is called a right splitting filtration if the following two properties are satisfied for all $1 \leq t \leq m$.

(1) If $M, M' \in \mathcal{S}_t$, and $\phi : M \to M'$ is not invertible, then ϕ factors through a module in $\mathrm{add}\,\mathcal{S}_{t-1}$.

(2) $\mathrm{add}\,\mathcal{S}_t$ is closed under factor modules, and each epimorphism $W \xrightarrow{\psi} V$, with $W \in \mathrm{add}\,\mathcal{S}_t$ and $V \in \mathrm{add}\,(\mathcal{S}_t \backslash \mathcal{S}_{t-1})$, splits.

Following the ideas in [4], we obtain the following results.

Proposition 3.3. *Let $\mathcal{S}$ be finite with $\mathrm{add}\,\mathcal{S}$ closed under submodules. If $\emptyset = \mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \cdots \subseteq \mathcal{S}_m = \mathcal{S}$ is a chain of subsets of $\mathcal{S}$ such that the following two properties are satisfied for all $1 \leq t \leq m$.*

(i) The modules in $\mathcal{S}_t \backslash \mathcal{S}_{t-1}$ are of equal length.

(ii) Every monomorphism $X \to Y$, where $Y \in \mathrm{add}\,\mathcal{S}_t$ and $X \in \mathcal{S}_t \backslash \mathcal{S}_{t-1}$, splits.

Then $\emptyset = \mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \cdots \subseteq \mathcal{S}_m = \mathcal{S}$ is a left splitting filtration.

Proof. We first show that for each $1 \leq t \leq m$, $\mathrm{add}\,\mathcal{S}_t$ is closed under submodules. Indeed, let $M \in \mathrm{add}\,\mathcal{S}_t$ and N be an indecomposable submodule of M. Then N lies again in $\mathcal{S}$, say $N \in \mathcal{S}_s \backslash \mathcal{S}_{s-1}$ for some $1 \leq s \leq m$. If $s > t$, it then follows from (ii) that the inclusion $N \to M$ splits since $\mathcal{S}_t \subseteq \mathcal{S}_s$. This implies that N belongs to $\mathcal{S}_t \subseteq \mathcal{S}_{s-1}$, a contradiction. Hence, $s \leq t$ and $N \in \mathcal{S}_t$. As a result, each submodule of M belongs to $\operatorname{add} \mathcal{S}_t$.

Let $M, M' \in \mathcal{S}_t \backslash \mathcal{S}_{t-1}$ and $\phi : M \to M'$ be a non–invertible map. Since M and M' have equal length, the image of ϕ, $\operatorname{Im}(\phi)$, is a proper submodule of M' and belongs to $\operatorname{add} \mathcal{S}_t$. If I is an indecomposable direct summand of $\operatorname{Im}(\phi)$, then $I \in \mathcal{S}_{t-1}$. Otherwise, the inclusion map $I \to M'$ would split according to (ii). This shows that $\operatorname{Im} \phi \in \operatorname{add} \mathcal{S}_{t-1}$, that is, the property (1) in Definition 3.1 is satisfied.

To complete the proof of the proposition, it remains to show that each monomorphism $U \xrightarrow{\phi} W$, with $W \in \operatorname{add} \mathcal{S}_t$ and $U \in \operatorname{add}(\mathcal{S}_t \backslash \mathcal{S}_{t-1})$, splits. This can be easily shown by using the property (ii) and induction on the number of indecomposable direct summands of U.

Corollary 3.4. *Let $\mathcal{S}$ be finite such that $\operatorname{add} \mathcal{S}$ is closed under submodules. Let $\{I_i(\mathcal{S})\}_{i=0}^p$ be the preinjective partition of $\mathcal{S}$. Given an $X \in \mathcal{S}$, let $u(X)$ be defined by $X \in I_{u(X)}(\mathcal{S})$. If*

$$\emptyset = \mathcal{S}_0^i \subseteq \mathcal{S}_1^i \subseteq \cdots \subseteq \mathcal{S}_m^i = \mathcal{S}$$

is a chian of subsets of $\mathcal{S}$ such that for $1 \leq t \leq m$, the modules in $\mathcal{S}_t^i \backslash \mathcal{S}_{t-1}^i$ have the same length, and such that for $X \in \mathcal{S}_s^i \backslash \mathcal{S}_{s-1}^i$, $Y \in \mathcal{S}_t^i \backslash \mathcal{S}_{t-1}^i$, with $s \leq t$, we have $u(X) \geq u(Y)$. Then $\emptyset = \mathcal{S}_0^i \subseteq \mathcal{S}_1^i \subseteq \cdots \subseteq \mathcal{S}_m^i = \mathcal{S}$ is a left splitting filtration of $\mathcal{S}$.

Proof. The corollary follows easily from Definition 1.1 and Proposition 3.3.

Dually, we have the following proposition and corollary.

Proposition 3.5. *Let $\mathcal{S}$ be finite with $\operatorname{add} \mathcal{S}$ closed under factor modules. If $\emptyset = \mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \cdots \subseteq \mathcal{S}_m = \mathcal{S}$ is a chain of subsets of $\mathcal{S}$ such that the following two properties are satisfied for all $1 \leq t \leq m$.*

(i) The modules in $\mathcal{S}_t \backslash \mathcal{S}_{t-1}$ are of equal length.

(ii) Every epimorphism $Y \to Z$, where $Y \in \operatorname{add} \mathcal{S}_t$ and $Z \in \mathcal{S}_t \backslash \mathcal{S}_{t-1}$, splits.

Then $\emptyset = \mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \cdots \subseteq \mathcal{S}_m = \mathcal{S}$ is a right splitting filtration.

Corollary 3.6. *Let $\mathcal{S}$ be finite such that $\operatorname{add} \mathcal{S}$ is closed under factor modules. Let $\{P_i(\mathcal{S})\}_{i=0}^q$ be the preprojective partition of $\mathcal{S}$. Given an $X \in \mathcal{S}$, let $v(X)$ be defined by $X \in P_{v(X)}(\mathcal{S})$. If*

$$\emptyset = \mathcal{S}_0^p \subseteq \mathcal{S}_1^p \subseteq \cdots \subseteq \mathcal{S}_n^p = \mathcal{S}$$

is a chian of subsets of $\mathcal{S}$ such that for $1 \leq t \leq n$, the modules in $\mathcal{S}_t^p \backslash \mathcal{S}_{t-1}^p$ have the same length, and such that for $X \in \mathcal{S}_s^p \backslash \mathcal{S}_{s-1}^p$, $Y \in \mathcal{S}_t^p \backslash \mathcal{S}_{t-1}^p$, with $s \leq t$, we *have* $v(X) \geq v(Y)$*. Then* $\emptyset = \mathcal{S}_0^p \subseteq \mathcal{S}_1^p \subseteq \cdots \subseteq \mathcal{S}_n^p = \mathcal{S}$ *is a right splitting filtration of* $\mathcal{S}$*.*

Theorem 3.7. *Let* Λ *be an artin algebra and* $\mathcal{S}$ *a finite subset of* $ind\,\Lambda$ *such that* $add\,\mathcal{S}$ *is closed under submodules (resp. factor modules). Assume that* $\emptyset = \mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \cdots \subseteq \mathcal{S}_m = \mathcal{S}$ *is a left (resp. right) splitting filtration of* $\mathcal{S}$*. Then*

$$0 = J_o \subseteq J_1 \subseteq \cdots \subseteq J_m = A_{\mathcal{S}}$$

is a heredity chain of A*, where* J_t *denotes the set of all endomorphisms in* $A_{\mathcal{S}}$ *which factor through a module in* $add\,\mathcal{S}_t$*,* $1 \leq t \leq m$*.*

Proof. The theorem can be proved by using similar arguments in [4, Theorem]. However, we persent here a proof by directly applying [5, Proposition].

Up to duality, we assume that $\mathrm{add}\,\mathcal{S}$ is closed under submodules and that $\emptyset = \mathcal{S}_0 \subseteq \mathcal{S}_1 \subseteq \cdots \subseteq \mathcal{S}_m = \mathcal{S}$ is a left splitting filtration of $\mathcal{S}$. According to the proposition in [5], to prove that J_t/J_{t-1}, $1 \leq t \leq m$, is a heredity ideal of A/J_{t-1}, it suffices to show the following (that is, the condition (b) in the proposition in [5] is satisfied for $M' = \oplus_{X \in \mathcal{S}_{t-1}} X$, $M = \oplus_{X \in \mathcal{S}_t \setminus \mathcal{S}_{t-1}} X$ and $M'' = \oplus_{X \in \mathcal{S} \setminus \mathcal{S}_t} X$):

Let $X, Y \in \mathcal{S} \setminus \mathcal{S}_{t-1}$. If $\delta : Y \to C$, $\gamma : C \to X$ are morphisms with $C \in \mathrm{add}\,\mathcal{S}_t$ such that $\gamma\delta$ factors through $\mathrm{add}\,\mathcal{S}_{t-1}$, then there exists a decomposition

$$C = C_1 \oplus C_2,\ \delta = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix},\ \gamma = \begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix}$$

such that both δ_1 and γ_2 factor through $\mathrm{add}\,\mathcal{S}_{t-1}$.

Indeed, let $\gamma\delta = \gamma'\delta'$, where $\delta' : Y \to C'$, $\gamma' : C' \to X$, and $C' \in \mathrm{add}\,\mathcal{S}_{t-1}$. By forming the pullback of γ, γ', we obtain the following diagram

$$\begin{tikzcd}
 & & C \arrow[drr, "\gamma"] & & \\
Y \arrow[urr, "\delta"] \arrow[r, "\alpha"] \arrow[drr, "\delta'"'] & U \arrow[ur, "\beta"] \arrow[dr, "\beta'"] & & & X \\
 & & C' \arrow[urr, "\gamma'"'] & &
\end{tikzcd}$$

with $\delta = \beta\alpha$, $\delta' = \beta'\alpha$ and $\gamma\beta = \gamma'\beta'$. Since $\mathrm{add}\,\mathcal{S}_t$ is closed under submodules, it holds that $U = \mathrm{Ker}\,([\gamma\,\gamma'] : C \oplus C' \to X)$ lies again in $\mathrm{add}\,\mathcal{S}_t$. Thus, there is a decomposition $U = U_1 \oplus U_2$ with $U_1 \in \mathrm{add}\,\mathcal{S}_{t-1}$ and $U_2 \in \mathrm{add}\,(\mathcal{S}_t \setminus \mathcal{S}_{t-1})$. By the definition of a left splitting filtration, the composition of the morphisms

$$U_2 \xrightarrow{i_2} U_1 \oplus U_2 = U \xrightarrow{[\beta\,\beta']} C \oplus C'$$

splits, where i_2 denotes the canonical inclusion. This implies that $\beta i_2 : U_2 \to C$ splits since $C' \in \operatorname{add} \mathcal{S}_{t-1}$. Hence, C admits a decomposition $C = C_1 \oplus C_2$ such that

$$\beta = \begin{bmatrix} \beta_{11} & 0 \\ \beta_{21} & \beta_{22} \end{bmatrix} : U = U_1 \oplus U_2 \longrightarrow C_1 \oplus C_2 = C$$

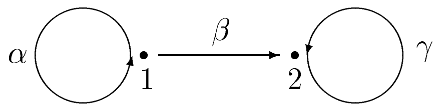
with β_{22} an isomorphism. Finally, we set

$$\delta = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} : Y \to C_1 \oplus C_2 = C, \quad \gamma = \begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix} : C = C_1 \oplus C_2 \to X,$$

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} : Y \to U_1 \oplus U_2 = U, \quad \beta' = \begin{bmatrix} \beta'_1 & \beta'_2 \end{bmatrix} : U = U_1 \oplus U_2 \to X.$$

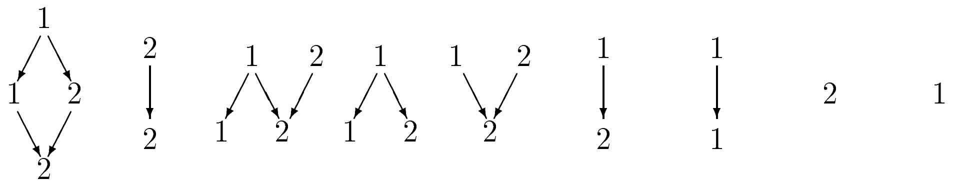
It is then clear that $\delta_1 = \beta_{11}\alpha$ and $\gamma_2 = \gamma'\beta'_2(\beta_{22})^{-1}$ both factor through $\operatorname{add} \mathcal{S}_{t-1}$. This finishes the proof.

4. Example. We consider the example given in [4]. Let Λ be the quotient of the path algebra of the quiver



modulo the ideal generated by $\{\alpha^2, \beta\alpha - \gamma\beta, \gamma^2\}$. Then, up to isomorphism, there are nine indecomposable Λ–modules which are listed as follows by their Loewy factors:

$a \quad b \quad c \quad d \quad e \quad f \quad g \quad h \quad i$



One easily observes that there are 34 non–empty subsets $\mathcal{S}$ of $\operatorname{ind} \Lambda$ with $\operatorname{add} \mathcal{S}$ closed under submodules. For example,

$$\{b, e, f, h\},\ \{b, c, d, g, h, i\},\ \{a, b, c, d, e, f, h, i\}.$$

There are also 34 non–empty subsets $\mathcal{T}$ of $\operatorname{ind} \Lambda$ with $\operatorname{add} \mathcal{T}$ closed under factor modules. For example,

$$\{d, f, g, i\},\ \{b, c, e, g, h, i\},\ \{a, c, d, e, f, g, h, i\}.$$

Furhter, the preinjective partition of $\mathcal{S} = \{a, b, c, d, e, f, h, i\}$ is given by

$$I_0(\mathcal{S}) = \{a, c\},\ I_1(\mathcal{S}) = \{d, e\},\ I_2(\mathcal{S}) = \{b, f\},\ I_3(\mathcal{S}) = \{h, i\}.$$

A left splitting filtration of $\mathcal{S}$ corresponding to the preinjective partition is:

$$\emptyset \subset \{h, i\} \subset \{b, f, h, i\} \subset \{b, d, e, f, h, i\} \subset \mathcal{S}.$$

The preprojective partition of $\mathcal{T} = \{a, c, d, e, f, g, h, i\}$ is given by

$$P_0(\mathcal{T}) = \{a, c\},\ P_1(\mathcal{T}) = \{d, e\},\ P_2(\mathcal{T}) = \{f, g\},\ P_3(\mathcal{T}) = \{h, i\}.$$

A right splitting filtration of $\mathcal{T}$ corresponding to the preprojective partition is:

$$\emptyset \subset \{h, i\} \subset \{f, g, h, i\} \subset \{d, e, f, g, h, i\} \subset \mathcal{T}.$$

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