CARTAN-DECOMPOSITION SUBGROUPS OF $\mathrm{SO}(2,n)$

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Abstract. For $G = \mathrm{SL}(3,\mathbb{R})$ and $G = \mathrm{SO}(2,n)$, we give explicit, practical conditions that determine whether or not a closed, connected subgroup H of G has the property that there exists a compact subset C of G with $CHC = G$. To do this, we fix a Cartan decomposition $G = KA^+K$ of G, and then carry out an approximate calculation of $(KHK) \cap A^+$ for each closed, connected subgroup H of G.

1. Introduction

1.1. Notation. Throughout this paper, G is a Zariski-connected, almost simple, linear, real Lie group. ("Almost simple" means that every proper normal subgroup of G either is finite or has finite index.) In almost all of the main results, G is assumed to be either $\mathrm{SL}(3,\mathbb{R})$ or $\mathrm{SO}(2,n)$ (with $n \geq 3$). There would be no essential loss of generality if one were to require G to be connected, instead of only Zariski connected (see 3.21). However, $\mathrm{SO}(2,n)$ is not connected (it has two components) and the authors prefer to state results for $\mathrm{SO}(2,n)$, instead of for the identity component of $\mathrm{SO}(2,n)$.

Fix an Iwasawa decomposition $G = KAN$ and a corresponding Cartan decomposition $G = KA^+K$, where A^+ is the (closed) positive Weyl chamber of A in which the roots occurring in the Lie algebra of N are positive. Thus, K is a maximal compact subgroup, A is the identity component of a maximal split torus, and N is a maximal unipotent subgroup.

The terminology introduced in the following definition is new, but the underlying concept is well known (see, for example, Proposition 1.4 and Theorem 1.6 below).

1.2. Definition. Let H be a closed subgroup of G. We say that H is a *Cartan-decomposition subgroup* of G if

- H is connected, and
- there is a compact subset C of G, such that $CHC = G$.

(Note that C is only assumed to be a sub*set* of G; it need not be a sub*group*.)

1.3. Example. The Cartan decomposition $G = KAK$ shows that the maximal split torus A is a Cartan-decomposition subgroup of G.

It is known that $G = KNK$ [Kos, Thm. 5.1], so the maximal unipotent subgroup N is also a Cartan-decomposition subgroup.

If G is compact (that is, if $\mathbb{R}\text{-rank}\, G = 0$), then all subgroups of G are Cartan-decomposition subgroups. On the other hand, if G is noncompact, then not all subgroups are Cartan-decomposition subgroups, because it is obvious that every Cartan-decomposition subgroup of G is noncompact. It is somewhat less obvious that if H is a Cartan-decomposition subgroup of G, then $\dim H \geq \mathbb{R}\text{-rank}\, G$ (see 3.13).

Our interest in Cartan-decomposition subgroups is largely motivated by the following basic observation that, to construct nicely behaved actions on homogeneous spaces, one must find subgroups that are *not* Cartan-decomposition subgroups. (See [Kb3, §3] for some historical background on this result.)

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1.4. Proposition (Calabi-Markus phenomenon, cf. [Kul, pf. of Thm. A.1.2]). *If H is a Cartan-decomposition subgroup of G, then no closed, noncompact subgroup of G acts properly on G/H.*

Our goal is to determine which closed, connected subgroups of G are Cartan-decomposition subgroups, and which are not. Our main tool is the Cartan projection.

1.5. Definition (Cartan projection). For each element g of G, the Cartan decomposition $G = KA^+K$ implies that there is an element a of A^+ with $g \in KaK$. In fact, the element a is unique, so there is a well-defined function $\mu\colon G \to A^+$ given by $g \in K\,\mu(g)\,K$. The function μ is continuous and proper (that is, the inverse image of any compact set is compact). Some properties of the Cartan projection are discussed in [Ben] and [Kb3].

We have $\mu(H) = A^+$ if and only if $KHK = G$. This immediately implies that if $\mu(H) = A^+$, then H is a Cartan-decomposition subgroup. Y. Benoist and T. Kobayashi proved the deeper statement that, in the general case, H is a Cartan-decomposition subgroup if and only if $\mu(H)$ comes within a bounded distance of every point in A^+.

1.6. Theorem (Benoist [Ben, Prop. 5.1], Kobayashi [Kb2, Thm. 1.1]). *A closed, connected subgroup H of G is a Cartan-decomposition subgroup if and only if there is a compact subset C of A, such that $\mu(H)C \supset A^+$.*

We noted above that every subgroup is a Cartan-decomposition subgroup if $\mathbb{R}\text{-rank}\, G = 0$. Therefore, the characterization of Cartan-decomposition subgroups of G is trivial if $\mathbb{R}\text{-rank}\, G = 0$. The following simple proposition shows that the characterization is again very easy if $\mathbb{R}\text{-rank}\, G = 1$.

1.7. Proposition (cf. [Kb1, Lem. 3.2]). *Assume that $\mathbb{R}\text{-rank}\, G = 1$. A closed, connected subgroup H of G is a Cartan-decomposition subgroup if and only if H is noncompact.*

Proof. ($\Leftarrow$) We have $\mu(e) = e$, and, because μ is a proper map, we have $\mu(h) \to \infty$ as $h \to \infty$ in H. Because $\mathbb{R}\text{-rank}\, G = 1$, we know that A^+ is homeomorphic to the half-line $[0, \infty)$ (with the point e in A^+ corresponding to the endpoint 0 of the half-line), so, by continuity, it must be the case that $\mu(H) = A^+$. Therefore $KHK = G$, so H is a Cartan-decomposition subgroup. $\square$

It seems to be much more difficult to characterize the Cartan-decomposition subgroups when $\mathbb{R}\text{-rank}\, G = 2$, so these are the first interesting cases. In this paper, we study two examples in detail. Namely, we describe all the Cartan-decomposition subgroups of $\mathrm{SL}(3,\mathbb{R})$ and of $\mathrm{SO}(2,n)$. We also explicitly describe the closed, connected subgroups that are *not* Cartan-decomposition subgroups, and approximately calculate the image of each of these subgroups under the Cartan projection.

Obviously, any connected, closed subgroup that contains a Cartan-decomposition subgroup is itself a Cartan-decomposition subgroup. Therefore the minimal Cartan-decomposition subgroups are the most interesting ones. As a simple example of our results, we state the following theorem.

1.8. Theorem. *Assume that $G = \mathrm{SL}(3,\mathbb{R})$. Up to conjugation by automorphisms of G, the only minimal Cartan-decomposition subgroups of G are:*

$$A, \qquad \left\{ \begin{pmatrix} 1 & r & s \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; r,s \in \mathbb{R} \right\}, \qquad \left\{ \begin{pmatrix} e^t & te^t & s \\ 0 & e^t & r \\ 0 & 0 & e^{-2t} \end{pmatrix} \;\middle|\; r,s,t \in \mathbb{R} \right\},$$

and subgroups of the form

$$\left\{ \begin{pmatrix} e^{pt} & r & 0 \\ 0 & e^{qt} & 0 \\ 0 & 0 & e^{-(p+q)t} \end{pmatrix} \;\middle|\; r,t \in \mathbb{R} \right\}, \tag{1.9}$$

where p and q are fixed real numbers with $\max\{p, q\} = 1$ *and* $\min\{p, q\} \geq -1/2$*, or of the form*

$$\left\{ \begin{pmatrix} e^t \cos pt & e^t \sin pt & s \\ -e^t \sin pt & e^t \cos pt & r \\ 0 & 0 & e^{-2t} \end{pmatrix} \middle| \; r, s, t \in \mathbb{R} \right\}, \tag{1.10}$$

where p is a fixed nonzero real number.

Note that AN contains uncountably many nonconjugate minimal Cartan-decomposition subgroups of G, because the minimum of the two parameters p and q in (1.9) can be varied continuously. However, up to conjugacy under $\operatorname{Aut} G$, there is only one minimal Cartan-decomposition subgroup contained in A (namely, A itself), and only one contained in N.

1.11. Corollary. *Let H be a closed, connected subgroup of $G = \mathrm{SL}(3, \mathbb{R})$, and let K_H be a maximal compact subgroup of H. If $\dim H - \dim K_H \geq 3$, then H is a Cartan-decomposition subgroup of G.*

For an explicit description of the Cartan-decomposition subgroups of $\mathrm{SL}(3, \mathbb{R})$ (not just up to conjugacy), see Theorem 4.5. The subgroups of $\mathrm{SL}(3, \mathbb{R})$ that are not Cartan-decomposition subgroups are described in Corollary 4.7, and their images under the Cartan projection are described in Proposition 4.12. These results are stated only for subgroups of AN, because the general case reduces to this (see Remark 2.10).

Theorem 1.14 is a sample of our results on Cartan-decomposition subgroups of $\mathrm{SO}(2, n)$. Note that, for simplicity, we restrict here to subgroups of N.

1.12. Notation. We realize $\mathrm{SO}(2, n)$ as isometries of the indefinite form $\langle v \mid v \rangle = v_1 v_{n+2} + v_2 v_{n+1} + \sum_{i=3}^{n} v_i^2$ on $\mathbb{R}^{n+2}$ (for $v = (v_1, v_2, \dots, v_{n+2}) \in \mathbb{R}^{n+2}$). The virtue of this particular realization is that we may choose A to consist of the diagonal matrices in $\mathrm{SO}(2, n)$ (with nonnegative entries) and N to consist of the upper-triangular matrices in $\mathrm{SO}(2, n)$ with only 1's on the diagonal. Thus, the Lie algebra of AN is

$$\mathfrak{a} + \mathfrak{n} = \left\{ \begin{pmatrix} t_1 & \phi & x & \eta & 0 \\ & t_2 & y & 0 & -\eta \\ & & 0 & -y^T & -x^T \\ & & & -t_2 & -\phi \\ & & & & -t_1 \end{pmatrix} \middle| \; t_1, t_2, \phi, \eta \in \mathbb{R}, x, y \in \mathbb{R}^{n-2} \right\}. \tag{1.13}$$

Note that the first two rows of any element of $\mathfrak{a} + \mathfrak{n}$ are sufficient to determine the entire matrix.

Whenever $m < n$, there is an obvious embedding of $\mathrm{SO}(2, m)$ in $\mathrm{SO}(2, n)$, induced by an inclusion $\mathbb{R}^{m+2} \hookrightarrow \mathbb{R}^{n+2}$, so, abusing notation, we speak of $\mathrm{SO}(2, m)$ as a subgroup of $\mathrm{SO}(2, n)$.

1.14. Theorem. *Assume that $G = \mathrm{SO}(2, n)$. If $n \geq 5$, then there are exactly 6 non-conjugate minimal Cartan-decomposition subgroups of G contained in N. Each such subgroup H is conjugate to a subgroup of $\mathrm{SO}(2, 5)$ and, as a subalgebra of $\mathfrak{so}(2, 5)$, the Lie algebra of H is conjugate to one of the following:*

1) $\left\{ \begin{pmatrix} 0 & \phi & 0 & 0 & 0 & \eta & 0 \\ & 0 & \epsilon_1 \phi & 0 & 0 & 0 & -\eta \\ & & & \cdots & & & \end{pmatrix} \middle| \; \phi, \eta \in \mathbb{R} \right\}$, *where* $\epsilon_1 \in \{0, 1\}$

2) $\left\{ \begin{pmatrix} 0 & \phi & x & 0 & 0 & 0 & 0 \\ & 0 & 0 & \epsilon_2 \phi & 0 & 0 & 0 \\ & & & \cdots & & & \end{pmatrix} \middle| \; \phi, x \in \mathbb{R} \right\}$, *where* $\epsilon_2 \in \{0, 1\}$

3) $\left\{ \begin{pmatrix} 0 & 0 & x & 0 & \epsilon_3 y & 0 & 0 \\ & 0 & 0 & y & 0 & 0 & 0 \\ & & & \cdots & & & \end{pmatrix} \middle| \; x, y \in \mathbb{R} \right\}$, *where* $\epsilon_3 \in \{0, 1\}$.

There are 5 non-conjugate minimal Cartan-decomposition subgroups of $\mathrm{SO}(2,4)$ *contained in* N. *The Lie algebra of any such subgroup* H *is conjugate either to one of the two subalgebras of type* (1), *to one of the two subalgebras of type* (2), *or to the subalgebra of type* (3) *with* $\epsilon_3 = 0$. (*These are the five of the above-listed subalgebras that are contained in* $\mathfrak{so}(2,4)$, *namely, the five whose 5th column is all* 0*'s.*)

There are 3 non-conjugate minimal Cartan-decomposition subgroups of $\mathrm{SO}(2,3)$ *contained in* N. *The Lie algebra of any such subgroup* H *is conjugate either to one of the two subalgebras of type* (1), *or to the subalgebra of type* (2) *with* $\epsilon_2 = 0$. (*These are the three of the above-listed subalgebras that are contained in* $\mathfrak{so}(2,3)$, *namely, the three whose 4th and 5th columns are all* 0*'s.*)

The detailed study of Cartan-decomposition subgroups of $\mathrm{SO}(2,n)$ is rather complicated, so we break it up into three parts: subgroups of N (Theorem 5.3), subgroups not in N that can be written as a semidirect product $T \ltimes U$ with $T \subset A$ and $U \subset N$ (Theorem 6.1), and subgroups that cannot be written as such a semidirect product (Theorem 6.3). We also describe the subgroups of $\mathrm{SO}(2,n)$ that are not Cartan-decomposition subgroups (see Theorem 5.5 and Corollaries 6.2 and 6.4), and approximately calculate their Cartan projections (see Proposition 5.8 and Corollaries 6.2 and 6.4).

If H is a Cartan-decomposition subgroup of G, and G/H is not compact, then the Calabi-Markus phenomenon 1.4 implies that G/H does not have a compact Clifford-Klein form. (That is, there does not exist a discrete subgroup Γ of G that acts properly on G/H, such that the quotient space $\Gamma\backslash G/H$ is compact.) Thus, our work on Cartan-decomposition subgroups is a first step toward understanding which homogeneous spaces of G have a compact Clifford-Klein form. Building on this, a sequel [OW] determines exactly which homogeneous spaces of $\mathrm{SO}(2,n)$ have a compact Clifford-Klein form in the case where n is even (and assuming that the isotropy group H is connected), but the results are not quite complete when n is odd. The work leads to new examples of compact Clifford-Klein forms of $\mathrm{SO}(2,n)$, when n is even.

The paper is organized as follows. Section 2 collects some known results on Lie groups and Zariski closures. Section 3 presents some general results on Cartan-decomposition subgroups. Section 4 states and proves our results on Cartan-decomposition subgroups of $\mathrm{SL}(3,\mathbb{R})$. Section 5 contains our results on Cartan-decomposition subgroups of $\mathrm{SO}(2,n)$ that are contained in N, and Section 6 is devoted to the subgroups of $\mathrm{SO}(2,n)$ that are not contained in N.

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2. Preliminaries on Lie groups and Zariski closures

Most of the results in this section are well known, and none are new. The reader is encouraged to skip over this section, and refer back when necessary.

We assume familiarity with the basic theory of Lie groups and Lie algebras (as in, for example, [Hoc]). At some points, we also assume some familiarity with the structure of algebraic groups over $\mathbb{R}$, in the spirit of [Rag, §P.2, pp. 7–11].

2.1. Notation. We use German letters $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{a}$, $\mathfrak{n}$, $\mathfrak{u}$, $\mathfrak{t}$ to denote the Lie algebras of Lie groups G, H, A, N, U, T, etc.

For a linear functional ω on $\mathfrak{a}$, we use $\mathfrak{u}_\omega$ to denote the corresponding weight space of the Lie algebra $\mathfrak{g}$ (so $\mathfrak{u}_\omega = 0$ unless ω is either 0 or a real root of G), and U_ω to denote $\exp \mathfrak{u}_\omega$. Note that $U_\omega U_{2\omega}$ is a subgroup of G (but U_ω may not be a subgroup if both ω and 2ω are real roots of G).

The idea of the following definition is to require that the choice of the Cartan subgroup A be compatible with a particular subgroup H. It is not a severe restriction on H, because Lemma 2.3 shows that it can always be satisfied by replacing H with a conjugate. Then, under the assumption that $\mathbb{R}\text{-rank}\, G = 2$, Lemma 2.4 shows that H has a fairly simple description in terms of (1) an appropriate subgroup of A, (2) the intersection of H with N, and, perhaps, (3) a homomorphism into a root subgroup of G. (If $\mathbb{R}\text{-rank}\, G > 2$, then the description of a typical H would require several root subgroups.)

2.2. Definition. Let us say that a subgroup H of AN is *compatible* with A if $H \subset TUC_N(T)$, where $T = A \cap (HN)$, $U = H \cap N$, and $C_N(T)$ denotes the centralizer of T in N.

2.3. Lemma. *If H is a closed, connected subgroup of AN, then H is conjugate, via an element of N, to a subgroup that is compatible with A.*

Proof. Let $\overline{H}$ be the identity component of the Zariski closure of H, and write $\overline{H} = \overline{T} \ltimes \overline{U}$, where $\overline{U}$ is a subgroup of N and $\overline{T}$ is conjugate, via an element of N, to a subgroup of A. Replacing H by a conjugate, we may assume that $\overline{T} \subset A$. Let $U = H \cap N$. Then $[\overline{H}, \overline{H}] \subset H \cap N$ (cf. [Var, Cor. 3.8.4, p. 207] and [Wit, Lem 3.24]), so we have $[\overline{T}, \overline{\mathfrak{u}}] \subset \mathfrak{u}$. Because $\overline{T} \subset A$ and $\overline{\mathfrak{u}}$ is $(\mathrm{Ad}_G(T))$-invariant, the adjoint action of $\overline{T}$ on $\overline{\mathfrak{u}}$ is completely reducible, so this implies that there is a subspace $\mathfrak{c}$ of $\overline{\mathfrak{u}}$, such that $[\overline{T}, \mathfrak{c}] = 0$ and $\mathfrak{u} + \mathfrak{c} = \overline{\mathfrak{u}}$. Therefore, $U\, C_N(\overline{T}) = \overline{U}$, so $\overline{H} = \overline{T}UC_N(\overline{T})$.

Let $\pi \colon AN \to A$ be the projection with kernel N, and let $T = \pi(H)$. Then $T = \pi(H) \subset \pi(\overline{H}) = \overline{T}$, so $C_N(T) \supset C_N(\overline{T})$. For any $h \in H$, there exist $t \in \overline{T}$, $u \in U$ and $c \in C_N(\overline{T})$, such that $h = tuc$. Because $uc \in N$, we must have $t = \pi(h) \in T$ and, because $C_N(T) \supset C_N(\overline{T})$, we have $c \in C_N(T)$. Therefore, $h \in TUC_N(T)$. We conclude that $H \subset TUC_N(T)$, so H is compatible with A. □

2.4. Lemma. *Assume that $\mathbb{R}\text{-rank}\, G = 2$. Let H be a closed, connected subgroup of AN, and assume that H is compatible with A. Then either*

1) *$H = (H \cap A) \ltimes (H \cap N)$; or*
2) *there is a positive root ω, a nontrivial group homomorphism $\psi \colon \ker \omega \to U_\omega U_{2\omega}$, and a closed, connected subgroup U of N, such that*
   (a) *$H = \{\, a\psi(a) \mid a \in \ker \omega \,\}U$;*
   (b) *$U \cap \psi(\ker \omega) = e$; and*
   (c) *U is normalized by both $\ker \omega$ and $\psi(\ker \omega)$.*

Proof. Because H is compatible with A, we have $H \subset TUC_N(T)$, where $T = A \cap (HN)$ and $U = H \cap N$. We may assume that $H \neq TU$, for otherwise we have $H = (H \cap A) \ltimes (H \cap N)$. Therefore $C_N(T) \neq e$. Because $\mathfrak{n}$ is a sum of root spaces, this implies that there is a positive root ω, such that $T \subset \ker \omega$. Because $\mathbb{R}\text{-rank}\, G = 2$, we must have $T = \ker \omega$, for otherwise we would have $T = e$, so $H = U = TU$. Therefore, $C_N(T) = U_\omega U_{2\omega}$.

Because $U \subset H \subset TUC_N(T)$, we have $H = U\big[H \cap \big(TC_N(T)\big)\big]$, so there is a nontrivial one-parameter subgroup $\{x^t\}$ in $H \cap \big(TC_N(T)\big)$ that is not contained in U. Because T centralizes $C_N(T)$, we may write $x^t = a^t u^t$ where $\{a^t\}$ is a one-parameter subgroup of T and $\{u^t\}$ is a one-parameter subgroup of $C_N(T)$. Furthermore, this Jordan decomposition is unique, because $T \cap C_N(T) = e$. Replacing H by a conjugate subgroup, we may assume that $a^t \in A$. Define $\psi \colon \ker \omega \to U_\omega U_{2\omega}$ by $\psi(a^t) = u^t$ for all $t \in \mathbb{R}$.

(2a) For all $t \in \mathbb{R}$, we have $a^t \psi(a^t) = a^t u^t = x^t \in H$, which establishes one inclusion of (2a). The other will follow if we show that $\dim H - \dim U = 1$, so suppose $\dim H - \dim U \geq 2$. Then Lemma 2.8 implies that $A \subset H$, so it follows from Lemma 2.5 that $H = A \ltimes (H \cap N)$, contradicting our assumption that $H \neq TU$.

(2b) Suppose $U \cap \psi(\ker \omega) \neq e$. Because the exponential map from $\mathfrak{n}$ to N is bijective, we know that the intersection is connected. Because $\dim(\ker \omega) = 1$, this implies that $\psi(\ker \omega) \subset U$. Therefore $a^t = x^t u^{-t} \in HU = H$, so $T \subset H$. This contradicts the fact that $H \neq TU$.

(2c) Because $x^t \in H$, we know that the Jordan components a^t and u^t of x^t belong to the Zariski closure of H [Hm2, Thm. 15.3, p. 99]. Therefore, both of a^t and u^t normalize H (see 2.6). Being in AN, they also normalize N. Therefore, they normalize $H \cap N = U$. □

2.5. Lemma (cf. [Hm1, pf. of Thm. 20.2(d), pp. 108–109]). *If H is a closed connected subgroup of AN that is normalized by A, and ω is a weight of the adjoint representation of A on $\mathfrak{a} + \mathfrak{n}$, then $\pi_\omega(\mathfrak{h}) \subset \mathfrak{h}$, where $\pi_\omega \colon \mathfrak{a} + \mathfrak{n} \to \mathfrak{u}_\omega$ is the A-equivariant projection.*

In particular, letting $\omega = 0$, we see that $H = (H \cap A) \ltimes (H \cap N)$.

2.6. Lemma (cf. [Zim, pf. of Thm. 3.2.5, p. 42]). *If H is a closed, connected subgroup of G, then the Zariski closure of H normalizes H.*

2.7. Lemma (cf. [Bor, Thm. 10.6, pp. 137–138]). *If H is a closed connected subgroup of AN, such that H has finite index in its Zariski closure, then H can be written as a semidirect product $H = T \ltimes U$, where U is a subgroup of N and T is conjugate, via an element of N, to a subgroup of A.*

2.8. Lemma. *Let H be a closed, connected subgroup of AN. If $\dim H - \dim(H \cap N) \geq \mathbb{R}\text{-rank}\, G$, then H contains a conjugate of A, so H is a Cartan-decomposition subgroup.*

Proof. Let $\pi \colon AN \to A$ be the projection with kernel N. Since

$$\mathbb{R}\text{-rank}\, G \leq (\dim H) - \dim(H \cap N) = \dim\big(\pi(H)\big) \leq \dim A = \mathbb{R}\text{-rank}\, G,$$

we must have $\pi(H) = A$, so $HN = AN$. Therefore, letting $\overline{H}$ be the Zariski closure of H, we may assume that $\overline{H}$ contains A (see 2.7), by replacing H with a conjugate subgroup. So H is normalized by A (see 2.6). Therefore, since $\pi(H) = A$, we conclude that $A \subset H$ (see 2.5). □

All maximal compact subgroups of any connected Lie group are conjugate [Hoc, Thm. XV.3.1, p. 180–181], so the quantity $\dim H - \dim K_H$ in the statement of the following lemma is independent of the choice of K_H.

2.9. Lemma. *Let H be a closed, connected subgroup of G. Then there is a closed, connected subgroup H' of G and a compact subgroup C of G, such that $CH = CH'$, and H' is conjugate to a subgroup of AN. Furthermore, there is a continuous function $f \colon H \to H'$ with $f(h) \in Ch$ for every $h \in H$, and we have $\dim H' = \dim H - \dim K_H$, where K_H is a maximal compact subgroup of H.*

Sketch of proof. Let L be a maximal connected semisimple subgroup of H, and let T be a maximal compact torus of the Zariski closure of $\operatorname{Rad} H$, the solvable radical of H. Replacing T by a conjugate torus, we may assume that L centralizes T. Let $L = K_L A_L N_L$ be the Iwasawa decomposition of L. From L. Auslander's nilshadow construction (cf. [Wit, §4]), we know that there is a unique connected, closed subgroup R of G, such that R is conjugate to a subgroup of AN, and $RT = (\operatorname{Rad} H)T$. Let $H' = A_L N_L R$, and note that the uniqueness of R implies that H normalizes R, so H' is a subgroup of G. Then $K_L T H' = TH = K_L T H$.

Define $f \colon H \to H'$ by specifying that $h \in K_L T \cdot f(h)$. Because H' is conjugate to a subgroup of AN, which has no nontrivial compact subgroups, we have $(K_L T) \cap H' = e$, so $f(h)$ is well defined. □

2.10. Remark. The proof of Lemma 2.9 is constructive. Furthermore, given a subgroup H and the corresponding subgroup H', it clear that H is a Cartan-decomposition subgroup if and only if H' is a Cartan-decomposition subgroup. Thus, to characterize all the Cartan-decomposition subgroups of G, it suffices to find all the Cartan-decomposition subgroups that are contained in AN.

2.11. Remark. Our restriction to closed subgroups in the definition of Cartan-decomposition subgroups is not very important. Namely, if one were to allow non-closed subgroups, then one would prove that a subgroup is a Cartan-decomposition subgroup if and only if its closure is a Cartan-decomposition subgroup. This follows from the theorem of M. Goto and A. Malcev (independently) that if H is a connected Lie subgroup of G, then there is a compact subgroup C of G, such that CH is the closure of H [Pog, Thm. 1.3]. (This theorem can be derived from the proof of Lemma 2.9, because every connected subgroup of AN is closed.)

3. General results on Cartan-decomposition subgroups

3.1. Notation. We employ the usual Big Oh and little oh notation: for functions f_1, f_2 on H, and a subset Z of H, we say $f_1 = O(f_2)$ *for* $z \in Z$ if there is a constant C, such that, for all large $z \in Z$, we have $\|f_1(z)\| \le C\|f_2(z)\|$. (The values of each f_i are assumed to belong to some finite-dimensional normed vector space, typically either $\mathbb{R}$ or a space of real matrices. Which particular norm is used does not matter, because all norms are equivalent up to a bounded factor.) We say $f_1 = o(f_2)$ *for* $z \in Z$ if $\|f_1(z)\|/\|f_2(z)\| \to 0$ as $z \to \infty$. Also, we write $f_1 \asymp f_2$ if $f_1 = O(f_2)$ and $f_2 = O(f_1)$.

3.2. Notation. It is known (cf. [Ben, Lem. 2.3]) that there exist irreducible real finite-dimensional representations ρ_i, $i = 1, \dots, k$ of (a finite cover of) G, such that the highest weight space of each ρ_i is one-dimensional and if χ_i is the highest weight of ρ_i, then $\{\, \chi_i \mid i = 1, \cdots, k \,\}$ is a basis of the vector space A^* of all continuous group homomorphisms from A to $\mathbb{R}^+$ (where the vector-space structure on A^* is defined by $(s\alpha + t\beta)(a) = \alpha(a)^s \beta(a)^t$ for $s, t \in \mathbb{R}$, $\alpha, \beta \in A^*$, and $a \in A$). In particular, we have $k = \mathbb{R}\text{-rank}\, G$.

When we have fixed a particular choice of ρ_i, $i = 1, \dots, k$, we may refer to $\rho_1, \dots, \rho_k$ as the *fundamental* representations of G.

3.3. Proposition (Benoist [Ben, Lem. 2.4]). *For each $i = 1, \dots, \mathbb{R}\text{-rank}\, G$, we have $\chi_i\big(\mu(g)\big) \asymp \rho_i(g)$, for $g \in G$.*

Because $\mu(g)$ is determined by the values $\chi_1\big(\mu(g)\big), \dots, \chi_k\big(\mu(g)\big)$ (and $\chi_i(a) = |\chi_i(a)|$ for each $a \in A$), it follows from the preceding proposition that the Cartan projection $\mu(g)$ can be calculated with bounded error by finding the norms of $\rho_1(g), \dots, \rho_k(g)$. (The error bound depends only on G; it is independent of g). This is theoretically useful (see, for example, Corollaries 3.4, 3.5, and 3.7, and note that Theorem 1.6 is a special case of Corollary 3.4) and is also the method we use in practice in Sections 4, 5 and 6 to calculate the image under μ of subgroups of $\mathrm{SL}(3,\mathbb{R})$ and $\mathrm{SO}(2,n)$.

3.4. Corollary (Benoist [Ben, Prop. 5.1]). *For any compact set C_1 in G, there is a compact set C_2 in A, such that $\mu(C_1 g C_1) \subset \mu(g) C_2$, for all $g \in G$.*

3.5. Corollary (Benoist [Ben, Prop. 1.5], Kobayashi [Kb2, Cor. 3.5]). *Let H_1 and H_2 be closed subgroups of G. The subgroup H_1 acts properly on H_2 if and only if, for every compact subset C of A, the intersection $\big(\mu(H_1)C\big) \cap \mu(H_2)$ is compact.*

3.6. Notation. For subsets $X, Y \subset A^+$, we write $X \approx Y$ if there is a compact subset C of A with $X \subset YC$ and $Y \subset XC$.

It is obvious from the definition that if H is a Cartan-decomposition subgroup, then every conjugate of H is also a Cartan-decomposition subgroup. In other words, if $\mu(H) \approx A^+$, then $\mu(g^{-1}Hg) \approx A^+$. The following corollary is a generalization of this observation.

3.7. Corollary. *If H is a subgroup of G, then $\mu(g^{-1}Hg) \approx \mu(H)$, for every $g \in G$.*

In particular, every conjugate of a Cartan-decomposition subgroup is a Cartan-decomposition subgroup.

3.8. Notation. Suppose that $\mathbb{R}$-rank $G = 2$ and let $\{\alpha_1, \alpha_2\}$ be the set of simple roots with respect to A^+ in G. We denote by L_i the wall of A^+ defined by $\alpha_i = 1$ for each $i = 1, 2$. Since, by definition, $\{\chi_1, \chi_2\}$ is a basis of A^*, there exists some real number k_i such that

$$L_i = \{\, a \in A^+ \mid \chi_1(a)^{k_i} = \chi_2(a) \,\}.$$

(Note that α_i cannot be a scalar multiple of χ_1, because χ_1, being a highest weight, is in A^+, but α, being a simple root, is not.) Although we do not need this general fact, we mention that k_i is always a rational number.

We now introduce convenient notation for describing the image of a subgroup under the Cartan projection μ.

3.9. Notation. Suppose that ρ is a representation of G and that G is a matrix group (that is, $G \subset \mathrm{GL}(\ell, \mathbb{R})$, for some ℓ), and assume that the two fundamental representations of G are $\rho_1(g) = g$ and $\rho_2(g) = \rho(g)$ (cf. Notation 3.2). For functions $f_1, f_2 \colon \mathbb{R}^+ \to \mathbb{R}^+$, and a subgroup H of G, we write $\mu(H) \approx \big[f_1(\|h\|), f_2(\|h\|) \big]$ if, for every sufficiently large $C > 1$, we have

$$\mu(H) \approx \big\{\, a \in A^+ \;\big|\; C^{-1} f_1\big(\|a\|\big) \le \|\rho(a)\| \le C f_2\big(\|a\|\big) \,\big\}.$$

(If f_1 and f_2 are monomials, or other very tame functions, then it does not matter which particular norm is used.) In particular, from Proposition 3.3, we see that H is a Cartan-decomposition subgroup of G if and only if $\mu(H) \approx \big[\|h\|^{k_1}, \|h\|^{k_2} \big]$, where k_1 and k_2 are as described in Notation 3.8.

If $\mathbb{R}$-rank $G = 2$, Proposition 3.14 provides a simple way to determine whether or not a subgroup H of AN is a Cartan-decomposition subgroup. One simply needs to determine whether or not there are arbitrarily large elements h_1 and h_2 of H, such that $\|\rho_2(h_i)\|$ is approximately $\|\rho_1(h_i)\|^{k_i}$.

The proof of Proposition 3.14 uses some basic properties of polynomials of exponentials. The class of exp-definable functions (see 3.10) is much more general, and the theory of the real exponential function is o-minimal [W1, W2], so general properties of o-minimal structures [PS, KPS, vdD] tell us that every exp-definable function behaves very much like an ordinary polynomial.

The following definition assumes some familiarity with first-order logic.

3.10. Definition (cf. [vdD, (5.3)]). A subset X of $\mathbb{R}^n$ is exp-*definable* if there are real numbers $a_1, \ldots, a_m$, and a first-order sentence $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ in the language of ordered fields augmented with the exponential function exp, such that

$$X = \big\{\, (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \phi(x_1, \ldots, x_n, a_1, \ldots, a_m) \,\big\}.$$

A map $f \colon X \to Y$ is exp-*definable* if its graph is an exp-definable subset of $X \times Y$.

We remark that every exp-definable set X has a well-defined dimension [vdD, p. 5 and p. 63]. Namely, X can be written as the union of finitely many cells, and the dimension of X is the maximum of the dimensions of these cells.

3.11. Lemma. *If H is a closed, connected subgroup of AN, then $\mu(H)$ is an* exp-*definable subset of A, and* $\dim \mu(H) \le \dim H$.

Proof. Because $H = \exp \mathfrak{h}$, it is easy to see that H is an exp-definable subset of G. Furthermore, the map μ is obviously exp-definable. Therefore, $\mu(H)$ is an exp-definable subset of A, and $\dim \mu(H) \le \dim H$ [vdD, Cor. 4.1.6(ii), p. 66]. $\square$

3.12. Lemma ([vdD, Props. 3.2.8 and 6.3.2, pp. 57, 100]). *Any* exp-*definable set has only finitely many path-connected components.*

3.13. Proposition. *Let H be a closed, connected subgroup of G, and let K_H be a maximal compact subgroup of H. If H is a Cartan-decomposition subgroup of G, or, more generally, if no closed, noncompact subgroup of G acts properly on G/H, then* $\dim H - \dim K_H \ge \mathbb{R}$-rank G.

Proof. We may assume that $H \subset AN$ (see 2.9), so $\dim H - \dim K_H = \dim H$. We prove the contrapositive: suppose that $\dim H < \mathbb{R}\text{-rank}\, G$. Let $\mathfrak{s} = \{\, a \in \mathfrak{a}^+ \mid \exp(a) \in \mu(H) \,\}$. Then $\mathfrak{s}$ is exp-definable and $\dim \mathfrak{s} \leq \dim H$ (see 3.11). Let $\mathfrak{a}_\infty$ be the sphere at infinity, and let $\mathfrak{s}_\infty$ be the set of accumulation points of $\mathfrak{s}$ in $\mathfrak{a}_\infty$. For any nonempty exp-definable set S, we have $\dim \partial S < \dim S$, where ∂S is the complement of S in its closure [vdD, Thm. 4.1.8, p. 67], so we know that $\dim \mathfrak{s}_\infty < \dim \mathfrak{s}$. Therefore, $\dim \mathfrak{s}_\infty < \dim \mathfrak{a} - 1 = \dim \mathfrak{a}_\infty$, so $\mathfrak{s}_\infty$ does not contain any nonempty open subset of $\mathfrak{a}_\infty$. On the other hand, it is obvious that $\mathfrak{a}^+_\infty$ does contain an open subset of $\mathfrak{a}_\infty$, so we conclude that some point $r \in \mathfrak{a}_\infty$ does not belong to $\mathfrak{s}_\infty$. Let $\mathfrak{r}^+$ be the ray in $\mathfrak{a}^+$ from 0 in the direction r. Because $r \notin \mathfrak{s}_\infty$, we know that, for every compact subset $\mathfrak{c}$ of $\mathfrak{a}$, the intersection $\mathfrak{r}^+ \cap (\mathfrak{s} + \mathfrak{c})$ is bounded. Thus, Corollary 3.4 implies that, for every compact subset C of G, the intersection $\exp(\mathfrak{r}^+) \cap CHC$ is bounded. Therefore, the one-parameter subgroup $\exp(\mathfrak{r}^+ \cup -\mathfrak{r}^+)$ acts properly on G/H, so H is not a Cartan-decomposition subgroup (see 1.4). □

The converse of Proposition 1.4 is not known to hold in general. However, the following proposition (combined with Proposition 1.7) shows that the converse does hold if $\mathbb{R}\text{-rank}\, G \leq 2$. Condition 3.14(3) is used throughout Sections 4, 5, and 6 to determine whether or not a subgroup of G is a Cartan-decomposition subgroup.

3.14. Proposition. *Assume that $\mathbb{R}\text{-rank}\, G = 2$. If H is a connected subgroup of AN, then the following are equivalent:*

1) *H is a Cartan-decomposition subgroup of G.*
2) *No closed, noncompact subgroup of G acts properly on G/H.*
3) *We have $\dim H \geq 2$ and, for each $i = 1, 2$, there exists a sequence $h_i(n)$ in H, such that $h_i(n) \to \infty$ as $n \to \infty$, and $\rho_2\big(h_i(n)\big) \asymp \|\rho_1\big(h_i(n)\big)\|^{k_i}$ for $n \in \mathbb{Z}^+$.*

Proof. ($1 \Rightarrow 2$) See Proposition 1.4.

($2 \Rightarrow 3$) This follows from Proposition 3.13, Theorem 1.6, and Proposition 3.3.

($3 \Rightarrow 1$) We begin by showing that, instead of only sequences $h_1(n)$ and $h_2(n)$, there are two continuous curves $h_1'(t)$ and $h_2'(t)$, $t \in [0, \infty)$, in H, such that $h_i'(t) \to \infty$ as $t \to \infty$, and, for each $i = 1, 2$, we have $\rho_2\big(h_i'(t)\big) \asymp \|\rho_1\big(h_i'(t)\big)\|^{k_i}$, for $t \in [0, \infty)$. Fix $i \in \{1, 2\}$. For each n, and each $C > 0$ let

$$A(n, C) = \{\, a \in \mu(H) \mid \|a\| > n \text{ and } C^{-1}\|\rho_1(a)\|^{k_i} < \|\rho_2(a)\| < C\|\rho_1(a)\|^{k_i} \,\}.$$

The existence of the sequence $\{h_i(n)\}$ implies that there is some $C_0 > 0$, such that, for every n, the set $A(n, C_0)$ is nonempty. Because $A(n, C_0)$ is exp-definable (c.f. 3.11), Lemma 3.12 implies that we may choose an unbounded path-connected component A_n of $A(n, C_0)$, for each n. Furthermore, we may assume that $A_{n+1} \subset A_n$. Now, for each natural number n, choose a point $a_n \in A_n$, and let $h_i'(t)_{n \leq t \leq n+1}$ be any path in A_n from a_n to a_{n+1}.

Since H is homeomorphic to some Euclidean space $\mathbb{R}^m$, with $m \geq 2$, it is easy to find a continuous and proper map $\Phi\colon [1, 2] \times \mathbb{R}^+ \to H$ such that $\Phi(i, t) = h_i'(t)$ for $i = 1, 2$ and for all $t \in \mathbb{R}^+$. From Proposition 3.3 and the definition of k_i, we know that the curve $\mu\big(h_i'(t)\big)$ stays within a bounded distance from the wall L_i; say $\operatorname{dist}\big[\big(\Phi(i, t)\big), L_i\big] < C$ for all t. We may assume that C is large enough that $\operatorname{dist}\big(\Phi(s, 1), e\big) < C$ for all $s \in [1, 2]$. Then an elementary homotopy argument shows that $\mu\big[\Phi\big([1, 2] \times \mathbb{R}^+\big)\big]$ contains $\{\, a \in A^+ \mid \operatorname{dist}(a, L_1 \cup L_2) > C \,\}$, so $\mu\big[\Phi\big([1, 2] \times \mathbb{R}^+\big)\big] \approx A^+$. Because $\mu(H) \supset \mu\big[\Phi\big([1, 2] \times \mathbb{R}^+\big)\big]$, we conclude from Theorem 1.6 that H is a Cartan-decomposition subgroup. □

3.15. Remark. When verifying Condition 3.14(3) for a specific example, one may use any matrix norm, because any two norms are equivalent up to a bounded factor. In practice, the authors use the maximum absolute value of the matrix entries, but the reader is free to make another choice, because we calculate our results only to within a bounded factor. In our applications to $\mathrm{SL}(3, \mathbb{R})$

and $\mathrm{SO}(2,n)$, we have $\rho_2 = \rho_1 \wedge \rho_1$, the second exterior power of ρ_1. So we calculate $\|\rho_2(h)\|$ as the maximum absolute value among the determinants of all the 2×2 submatrices of $\rho_1(h)$.

If $\dim H = 1$ and $\mathbb{R}\text{-rank}\, G \geq 2$, then we know, from Proposition 3.13, that H is not a Cartan-decomposition subgroup of G. For completeness, the following simple proposition describes $\mu(H)$ fairly explicitly. Namely, $\mu(H)$ consists of either one or two rays, or one or two logarithmic curves.

3.16. Proposition. *Assume that $\mathbb{R}\text{-rank}(G) \geq 2$, and let H be a nontrivial one-parameter subgroup of AN.*

1) *If H is conjugate to a subgroup of A, then there is a ray R in A^+, such that $\mu(H) \approx R \cup i(R)$, where $i\colon A^+ \to A^+$ is the opposition involution (see 4.10).*
2) *If $H \subset N$, then there is a ray R in A^+, such that $\mu(H) \approx R$.*
3) *If neither (1) nor (2) applies, then there is a ray R in A^+, a ray R' in A that is perpendicular to R, and a positive number k, such that $\mu(H) \approx X \cup i(X)$, where $X = \{\, rs \mid r \in R,\ s \in R',\ \|s\| = (\log\|r\|)^k \,\}$.*

Proof. Let $H = \{h^t\}$. Replacing H by a conjugate, we may assume that $h^t = a^t u^t$, where $\{a^t\}$ is a one-parameter subgroup of A and $\{u^t\}$ is a one-parameter subgroup of N, such that $\{a^t\}$ centralizes $\{u^t\}$ (c.f. 2.4). The subgroup $\{u^t\}$ is contained in a subgroup S of G that is locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$ [Jac, Thm. 17(1), p. 100]. Replacing H by a conjugate, we may assume that the intersection $A_S = A \cap S$ is nontrivial.

(1) We have $\{h^t\} = \{a^t\}$. Replacing H by a conjugate under the Weyl group, we may assume that $a^t \in A^+$ for all $t \geq 0$. Let $R = \{\, a^t \mid t \geq 0 \,\}$.

(2) We have $\{h^t\} = \{u^t\}$, so H is contained in S. Then, because both H and A_S are Cartan-decomposition subgroups of S (see 1.7), we have $\mu(H) \asymp \mu(S) \asymp \mu(A_S^+)$. Let $R = \mu(A_S^+)$.

(3) Both a^t and u^t are nontrivial. Because $\{a^t\}$ centralizes $\{u^t\}$, and hence centralizes all of S, we know that A_S is perpendicular to $\{a^t\}$ (with respect to the Killing form). Let $\mu_S\colon S \to A_S^+$ be the Cartan projection of S. (We may assume that $K \cap S$ is a maximal compact subgroup of S.) Assuming, for simplicity, that $\{a^t \mid t \geq 0\} \subset A^+$, we see from the proof of Proposition 3.17 that $\mu(h^t) = a^t \mu_S(u^t)$. Then, because $\|a^t\|$ grows exponentially and $\|u^t\|$ grows polynomially, the desired conclusion follows, with $R = \{a^t \mid t \geq 0\}$ and $R' = A_S^+$. ☐

3.17. Proposition. *Assume that $\mathbb{R}\text{-rank}(G) = 2$, and that ω is a real root of G. Let H be a closed, connected subgroup of G, such that $\mathfrak{h} = \mathfrak{t} + \mathfrak{u}$, for some one-dimensional subspace $\mathfrak{t}$ of $\mathfrak{a}$ and some nontrivial subalgebra $\mathfrak{u}$ of $\mathfrak{u}_\omega + \mathfrak{u}_{2\omega}$.*

1) *Let $\mathfrak{a}_\omega = [\mathfrak{u}_\omega, \mathfrak{u}_{-\omega}] \cap \mathfrak{a}$, a one-dimensional subspace of $\mathfrak{a}$.*
2) *Choose rays $\mathfrak{t}^+$ in $\mathfrak{t}$ and $\mathfrak{a}_\omega^+$ in $\mathfrak{a}_\omega$, such that $\langle t \mid a\rangle \geq 0$ for every $t \in \mathfrak{t}^+$ and every $a \in \mathfrak{a}_\omega^+$, where $\langle \cdot \mid \cdot \rangle$ is the inner product on $\mathfrak{a}$ defined by the Killing form.*

The subgroup H is a Cartan-decomposition subgroup if and only if $\mathfrak{t}^+ - \{0\}$ is **not** *contained in the interior of the region $\mathcal{C}$ defined as follows:*

a) *if $\mathfrak{a}_\omega^+$ is contained in the interior of a Weyl chamber, let $\mathcal{C}$ be the interior of that Weyl chamber;*
b) *if $\mathfrak{a}_\omega^+$ is the wall between two Weyl chambers $\mathcal{C}_1$ and $\mathcal{C}_2$, let $\mathcal{C}$ be the interior of $\mathcal{C}_1 \cup \mathfrak{a}_\omega^+ \cup \mathcal{C}_2$.*

Proof. Let $M = \langle U_\omega, U_{2\omega}, U_{-\omega}, U_{-2\omega}\rangle$, so M is semisimple of real rank one. (The only real roots of M are $\pm\omega$ and, possibly, $\pm 2\omega$.) Let $A_\omega = M \cap A$, so A_ω is a maximal split torus of M, and note that the Lie algebra of A_ω is $\mathfrak{a}_\omega$. The choice of the ray $\mathfrak{a}_\omega^+$ determines a corresponding Weyl chamber A_ω^+ in A_ω.

We may assume that $K_M = K \cap M$ is a maximal compact subgroup of M, and let $\mu_M\colon M \to A_\omega^+$ be the corresponding Cartan projection.

Let $S = C_A(M)$. The subgroup MA is reductive, so we have $MA = MS$. In particular, any element of A may be written uniquely in the form as with $a \in A_\omega$ and $s \in S$. We extend μ_M to a map $\mu_{MA}\colon MA \to A$ by defining $\mu_{MA}(ms) = \mu_M(m)s$ for $m \in M$ and $s \in S$.

Because $H \cap N \subset U_\omega U_{2\omega} \subset M$ and $T \subset A$, we have $H \subset MA$.

Claim. We have $\mu_{MA}(H) = A_\omega^+ \, \mu_{MA}(T)$. Given $h \in H$, we may write $h = ut$, with $u \in H \cap N$ and $t \in T$. Furthermore, we may write $t = as$, with $a \in A_\omega$ and $s \in S$. Choose $k, k' \in K_M$ with $kuak' = \mu_M(ua) \in A_\omega^+$.

Assume for simplicity that G is a matrix group, that is, $G \subset \mathrm{GL}(\ell, \mathbb{R})$, for some ℓ. Then, for $g \in G$, we may let $\|g\|^2$ be the sum of the squares of the matrix entries of g. Assume, furthermore, that $K = G \cap \mathrm{SO}(\ell)$, that A is the group of diagonal matrices in G, and that N is the group of unipotent upper-triangular matrices in G. Then it is clear that $\|ua\| \geq \|a\|$ and, because $\|\cdot\|$ is bi-K-invariant, that $\|\mu(g)\| = \|g\|$ for all $g \in G$.

Because

$$\|\mu_M(ua)\| = \|ua\| \geq \|a\| = \|\mu_M(a)\|,$$

and both of $\mu_M(ua)$ and $\mu_M(a)$ are in A_ω^+, there is some $a^+ \in A_\omega^+$ with $\mu_M(ua) = a^+ \mu_M(a)$. Therefore

$$\mu_{MA}(h) = \mu_{MA}(uas) = \mu_M(ua)s = a^+ \mu_M(a)s = a^+ \mu_{MA}(as) \in A_\omega^+ \mu_{MA}(T).$$

Therefore $\mu_{MA}(H) \subset A_\omega^+ \, \mu_{MA}(T)$.

Conversely, given $a^+ \in A_\omega^+$ and $t \in T$, write $t = as$ with $a \in A_\omega$ and $s \in S$. Because $\|a^+ \mu_M(a)\| \geq \|\mu_M(a)\| = \|a\|$, there is some $u \in H \cap N$ with $\|ua\| = \|a^+ \mu_M(a)\|$. Then $\mu_M(ua) = a^+ \mu_M(a)$, because there is only one element of A_ω^+ with any given norm, so

$$\mu_{MA}(ut) = \mu_M(ua)s = a^+ \mu_M(a)s = a^+ \mu_{MA}(t).$$

This completes the proof of the claim.

Note that $\mu_{MA}(g) \in K_M g K_M \subset KgK$, so $\mu(g) = \mu\big(\mu_{MA}(g)\big)$, for all $g \in MA$.

Now, to clarify the situation, let us define $\mu_{\mathfrak{m}+\mathfrak{a}} \colon \mathfrak{m} + \mathfrak{a} \to \mathfrak{a}$ by $\mu_{MA}(\exp z) = \exp\big(\mu_{\mathfrak{m}+\mathfrak{a}}(z)\big)$ for $z \in \mathfrak{m} + \mathfrak{a}$, and let us introduce a convenient coordinate system on the Lie algebra $\mathfrak{a} \cong \mathbb{R}^2$. Let the x-axis be $\mathfrak{s}$, the Lie algebra of S, and let the positive y-axis be $\mathfrak{a}_\omega^+$. In these coordinates, the restriction of $\mu_{\mathfrak{m}+\mathfrak{a}}$ to $\mathfrak{a}$ is given by $\mu_{\mathfrak{m}+\mathfrak{a}}(x,y) = (x, |y|)$. The line $\mathfrak{t}$ has an equation of the form $y = m_{\mathfrak{t}} x$, so $\mu_{\mathfrak{m}+\mathfrak{a}}(T)$ has the equation $y = |m_{\mathfrak{t}} x|$. Thus, the claim asserts that

$$\mu_{\mathfrak{m}+\mathfrak{a}}(H) = \{\, (x,y) \mid y \geq |m_{\mathfrak{t}} x| \,\}.$$

For some constant $m_{\mathcal{C}}$, we have

$$\mathcal{C} = \{\, (x,y) \mid y > |m_{\mathcal{C}} x| \,\}.$$

Thus, $\mathfrak{t}^+ - \{0\}$ is in the interior of $\mathcal{C}$ if and only if $|m_{\mathfrak{t}}| > |m_{\mathcal{C}}|$.

($\Leftarrow$) If $\mathfrak{t}^+ - \{0\}$ is not in the interior of $\mathcal{C}$, then, because $|m_{\mathfrak{t}}| \leq |m_{\mathcal{C}}|$, we see that $\mu_{\mathfrak{m}+\mathfrak{a}}(H)$ contains the closure of $\mathcal{C}$. By definition, we know that the closure of $\mathcal{C}$ contains a Weyl chamber, so we conclude that $\mu(H) \supset A^+$, so H is a Cartan-decomposition subgroup.

($\Rightarrow$) If $\mathfrak{t}^+ - \{0\}$ is in the interior of $\mathcal{C}$, then, because $|m_{\mathfrak{t}}| < |m_{\mathcal{C}}|$, we see that $\mu_{\mathfrak{m}+\mathfrak{a}}(H)$ is a proper subcone of $\mathcal{C}$. If $\mathcal{C}$ is a single Weyl chamber, then this immediately implies that H is not a Cartan-decomposition subgroup. Now assume the other possibility, namely, that the y-axis $\mathfrak{a}_\omega$ is the wall between two Weyl chambers $\mathcal{C}_1$ and $\mathcal{C}_2$. Assume for simplicity that $\mathcal{C}_2 = A^+$. The reflection across the y-axis is the Weyl reflection that maps $\mathcal{C}_1$ to $\mathcal{C}_2$. Because $\mu_{\mathfrak{m}+\mathfrak{a}}(H)$ is invariant under this reflection (that is, the inequality defining $\mu_{\mathfrak{m}+\mathfrak{a}}(H)$ depends only on $|x|$, not on x itself), we see immediately that $\mu(H) = \mathcal{C}_2 \cap \mu_{\mathfrak{m}+\mathfrak{a}}(H)$ is a proper subcone of $\mathcal{C}_2$. So H is not a Cartan-decomposition subgroup. □

3.18. Corollary (of proof). *Let G, ω, H, T, M, and $\mathfrak{a}_\omega^+$ be as in the statement and proof of Proposition* 3.17. *If H is not a Cartan-decomposition subgroup, then the Cartan projection $\mu(H)$ is the image under a Weyl element of either the closed, convex cone bounded by $\mu_{MA}(T)$* (*if $\mathfrak{a}_\omega^+$ is in the interior of a Weyl chamber*) *or the closed, convex cone bounded by A_ω^+ and a ray of T* (*if $\mathfrak{a}_\omega^+$ is a wall of a Weyl chamber*).

3.19. Example. Suppose that H is of the form (1.9) for some $p,q \in \mathbb{R}$, not both zero. As an illustration of the application of Proposition 3.17, we show that H is a Cartan-decomposition subgroup of $\mathrm{SL}(3,\mathbb{R})$ if and only if either $p+q \leq -\max\{p,q\}$ or $p+q \geq -\min\{p,q\}$. Let $\mathfrak{a}_\omega = \{\, \mathrm{diag}(a,-a,0) \mid a \geq 0 \,\}$. Replacing p and q by their negatives results in the same group, so we may assume that $p \geq q$. Then we may choose $\mathfrak{t}^+ = \{\, \mathrm{diag}(pt, qt, -(p+q)t \mid t \geq 0 \,\}$. The Weyl chamber

$$\mathcal{C} = \bigl\{\, \mathrm{diag}\bigl(a, b, -(a+b)\bigr) \mid a \geq -(a+b) \geq b \,\bigr\}$$

contains $\mathfrak{a}_\omega^+$ in its interior. It is clear that $\mathfrak{t}^+$ is in the interior of $\mathcal{C}$ if and only if $p > -(p+q) > q$. So $\mathfrak{t}^+$ is a Cartan-decomposition subgroup if and only if this condition fails, that is, if and only if either $-(p+q) \geq p$ or $-(p+q) \leq q$. Because we have assumed that $p \geq q$, which means that $p = \max\{p,q\}$ and $q = \min\{p,q\}$, this is the desired conclusion. (The condition in Theorem 1.8 that $\max\{p,q\} = 1$ and $\min\{p,q\} \geq -1/2$ is obtained by assuming that $\max\{p,q\} = 1$, instead of assuming that $p \geq q$, as we have here. Replacing the pair (p,q) with any nonzero scalar multiple results in the same subgroup, so assuming that $\max\{p,q\} = 1$ results in no loss of generality.)

The following proposition shows that characterizing the Cartan-decomposition subgroups of a reductive group reduces to the problem of characterizing the Cartan-decomposition subgroups of its almost simple factors. Thus, our standing assumption that G is almost simple is not as restrictive as it might seem.

3.20. Proposition. *Suppose that $\tilde{G}$ is a connected, reductive, linear Lie group, let $\tilde{G} = \tilde{K}\tilde{A}\tilde{N}$ be an Iwasawa decomposition of $\tilde{G}$, and write $\tilde{G} = G_1 G_2 \cdots G_m Z$, where Z is the center of $\tilde{G}$, and each G_i is a connected, almost simple, normal subgroup of $\tilde{G}$. Let H be a closed, connected subgroup of $\tilde{A}\tilde{N}$.*

1) *H is a Cartan-decomposition subgroup of $\tilde{G}$ if and only if* (a) *H contains $Z \cap \tilde{A}$, and* (b) *for each i, the intersection $H \cap G_i$ is a Cartan-decomposition subgroup of G_i.*
2) *No closed, noncompact subgroup of $\tilde{G}$ acts properly on $\tilde{G}/H$ if and only if* (a) *H contains $Z \cap \tilde{A}$, and* (b) *for each i, no closed, noncompact subgroup of G_i acts properly on $G_i/(H \cap G_i)$.*

Proof. We prove only the nontrivial direction of each conclusion.

(1) Suppose that H is a Cartan-decomposition subgroup of $\tilde{G}$. (a) The proof of the claim in Proposition 3.17, with $G_1 G_2 \cdots G_m$ in the role of M, shows that H contains $Z \cap \tilde{A}$. (b) From Theorem 1.6, we know that there is a compact set $C \subset A$, such that, for each $a \in A_1^+$, there is some $h(a) \in H$, such that $\mu\bigl(h(a)\bigr) \in aC$. The Cartan projection of $\tilde{G}/G_1$ is a proper map, and, because HG_1 is closed (see [Hoc, Thm. XII.2.2, p. 137]), the natural map $H/(H \cap G_1) \to \tilde{G}/G_1$ is a proper map. Hence, there is a compact subset C_1 of H, such that $h(A_1^+) \subset C_1(H \cap G_1)$. Then Corollary 3.4 implies that there is a compact subset C_2 of A_1, such that $\mu(H \cap G_1)C_2 \supset \mu\bigl(h(A_1^+)\bigr) \approx A_1^+$. Therefore, $H \cap G_1$ is a Cartan-decomposition subgroup of G_1.

(2) Proof by contradiction. Assume that (a) and (b) hold, and let L be a closed, noncompact subgroup of $\tilde{G}$ that acts properly on $\tilde{G}/H$. By passing to a subgroup, we may assume that L is cyclic. Then there is a one-parameter subgroup L' of $\tilde{G}$, such that $L' \approx L$, and, from Lemma 2.9, we may assume that $L' \subset AN$. From (a), we see that we may assume that $L' \subset G_1 \cdots G_m$, so we may write $L' = \{l^t\}$ and $l^t = l_1^t \cdots l_m^t$, where $\{l_i^t\}$ is a one-parameter subgroup of G_i. Every nontrivial, connected subgroup of $\tilde{A}\tilde{N}$ is closed and noncompact, so, for each i, (b) implies that either $\{l_i^t\}$ is trivial or $\{l_i^t\}$ does not act properly on $G_i/(H \cap G_i)$. Then, from Corollary 3.5, we see that, for each i, there is a compact subset C_i of $A \cap G_i$, such that $\{\, t \geq 0 \mid l_i^t \in \mu_i(H \cap G_i)C_i \,\}$ is unbounded. (By taking C_i to be a large ball, we may assume that C_i is exp-definable.) Because this set is exp-definable (c.f. 3.11), Lemma 3.12 implies that there is some $T > 0$, such that, for each i and for all $t \geq T$, we have $l_i^t \in \mu_i(H \cap G_i)C_i$. Therefore, for all $t \geq T$, we have $l^t \in \mu(H)C_1C_2 \cdots C_m$, so $L' = \{l^t\}$ does not act properly on $\tilde{G}/H$. This is a contradiction. $\square$

3.21. Lemma. *Let H be a closed, connected subgroup of G. The subgroup H is a Cartan-decomposition subgroup of G if and only if H is a Cartan-decomposition subgroup of the identity component of G.*

Proof. Let G° be the identity component of G. Because every element of the Weyl group of G has a representative in G° [BT, Cor. 14.6], we see that G and G° have the same positive Weyl chamber A^+, and the Cartan projection $G^\circ \to A^+$ is the restriction of the Cartan projection $G \to A^+$. Thus, the desired conclusion is immediate from Theorem 1.6. □

4. Cartan-decomposition subgroups of $\mathrm{SL}(3,\mathbb{R})$

In this section, we find all the Cartan-decomposition subgroups of $\mathrm{SL}(3,\mathbb{R})$ (see 4.5). With Remark 2.10 in mind, we restrict our attention to finding the Cartan-decomposition subgroups contained in AN.

4.1. Notation. To provide a convenient way to refer to specific elements of AN, we define $h\colon (\mathbb{R}^+)^3 \times \mathbb{R}^3 \to \mathrm{GL}(3,\mathbb{R})$ by

$$h(a_1,a_2,a_3,u_1,u_2,u_3) = \begin{pmatrix} a_1 & u_1 & u_3 \\ 0 & a_2 & u_2 \\ 0 & 0 & a_3 \end{pmatrix}.$$

4.2. Notation. Let

$$A = \{\, a = h(a_1,a_2,a_3,0,0,0) \mid a_i \in \mathbb{R}^+,\ a_1a_2a_3 = 1 \,\}, \tag{4.3}$$
$$A^+ = \{\, h(a_1,a_2,a_3,0,0,0) \mid a_1 \ge a_2 \ge a_3 > 0,\ a_1a_2a_3 = 1 \,\},$$
$$N = \{\, h(1,1,1,u,v,w) \mid u,v,w \in \mathbb{R} \,\},$$

and $K = \mathrm{SO}(3)$.

4.4. Notation. We let α and β be the simple roots of $\mathrm{SL}(3,\mathbb{R})$, defined by $\alpha(a) = a_1/a_2$ and $\beta(a) = a_2/a_3$, for an element a of A of the form (4.3). Thus,

- the root space $\mathfrak{u}_\alpha$ consists of the matrices in which all entries except $h_{1,2}$ are 0;
- the root space $\mathfrak{u}_\beta$ consists of the matrices in which all entries except $h_{2,3}$ are 0; and
- the root space $\mathfrak{u}_{\alpha+\beta}$ consists of the matrices in which all entries except $h_{1,3}$ are 0.

4.5. Theorem. *Assume that $G = \mathrm{SL}(3,\mathbb{R})$. Let H be a closed, connected subgroup of AN, and assume that H is compatible with A (see 2.2).*

The subgroup H is a Cartan-decomposition subgroup of G if and only if either:

1) $\dim H \ge 3$*; or*
2) $H = A$*; or*
3) *for some $p \ne 0$, we have $H = \{\, h(1,1,1,s,ps,t) \mid s,t \in \mathbb{R} \,\} \subset N$; or*
4) *for some $p \ne 0$, we have $H = \{\, h(e^t,e^t,e^{-2t},0,s,ps) \mid s,t \in \mathbb{R} \,\}$; or*
5) *for some $p \ne 0$, we have $H = \{\, h(e^{-2t},e^t,e^t,s,0,ps) \mid s,t \in \mathbb{R} \,\}$; or*
6) *$H = T \ltimes U_\omega$, for some one-parameter subgroup T of A and some positive root ω, and H satisfies the conditions of Proposition 3.17.*

4.6. Remark. If $p = 0$ in 4.5(4) or 4.5(5), then H is a Cartan-decomposition subgroup, but H is of the type considered in 4.5(6). However, if $p = 0$ in 4.5(3), then H is *not* a Cartan-decomposition subgroup.

Theorem 4.5 describes the Cartan-decomposition subgroups of $\mathrm{SL}(3,\mathbb{R})$. We now describe the subgroups that are *not* Cartan-decomposition subgroups.

4.7. Corollary. *Assume that $G = \mathrm{SL}(3,\mathbb{R})$. Let H be a closed, connected subgroup of AN, and assume that H is compatible with A (see 2.2).*

The subgroup H fails to be a Cartan-decomposition subgroup of G if and only if either:

1) $\dim H \le 1$; *or*
2) $\mathfrak{h} = \mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+\beta}$ *or* $\mathfrak{h} = \mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta}$; *or*
3) $\mathfrak{h}$ *is of the form* $\mathfrak{h} = \big(\ker(\alpha-\beta)\big) + \mathfrak{u}$, *for some one-dimensional subspace* $\mathfrak{u}$ *of* $\mathfrak{u}_\alpha + \mathfrak{u}_\beta$, *with* $\mathfrak{u} \notin \{\mathfrak{u}_\alpha, \mathfrak{u}_\beta\}$; *or*
4) $\dim H = 2$ *and* $H \ne (H \cap A) \ltimes (H \cap N)$; *or*
5) $H = T \ltimes U_\omega$, *for some one-parameter subgroup* T *of* A *and some positive root* ω, *and Proposition* 3.17 *implies that* H *is not a Cartan-decomposition subgroup.*

In the course of the proof of Theorem 4.5, we calculate the Cartan projection of each subgroup that is not a Cartan-decomposition subgroup. Proposition 4.12 collects these results (see also Corollary 3.18 and Proposition 3.16). The statement of this proposition and the proof of Theorem 4.5 are based on Proposition 3.14, so we describe the required representations ρ_1 and ρ_2 of G.

4.8. Notation. Define a representation

$$\rho\colon \operatorname{SL}(3,\mathbb{R}) \to \operatorname{SL}(\mathbb{R}^3 \wedge \mathbb{R}^3) \text{ by } \rho(g) = g \wedge g.$$

In the notation of Proposition 3.3, we have

$$\begin{aligned} \rho_1 &= \text{the standard representation on } \mathbb{R}^3 \qquad (\rho_1(g) = g), \\ \rho_2 &= \rho \qquad (= \rho_1 \wedge \rho_1), \\ k_1 &= 1/2 \qquad \text{and} \qquad k_2 = 2. \end{aligned}$$

Thus, Proposition 3.3 yields the following fundamental lemma.

4.9. Lemma. *Assume that* $G = \operatorname{SL}(3,\mathbb{R})$. *A subgroup* H *of* G *is a Cartan-decomposition subgroup if and only if* $\mu(H) \approx \big[\|h\|^{1/2}, \|h\|^2\big]$, *where the representation* ρ *is defined in Notation* 4.8.

4.10. Notation. Let i be the opposition involution in A^+, that is, for $a \in A^+$, $i(a)$ is the unique element of A^+ that is conjugate to a^{-1}, and set $B^+ = \{\, a \in A^+ \mid i(a) = a \,\}$.

4.11. Corollary. *Assume that* $G = \operatorname{SL}(3,\mathbb{R})$. *Let* H *be a closed, connected subgroup of* G *with* $\dim H - \dim K_H \ge 2$, *where* K_H *is a maximal compact subgroup of* H. *The subgroup* H *is a Cartan-decomposition subgroup if and only if there is a sequence* $h_n \to \infty$ *in* H *with* $\rho(h_n) \asymp \|h_n\|^2$.

Proof. ($\Leftarrow$) We may assume that $H \subset AN$ (see 2.9). If we identify the Lie algebra $\mathfrak{a}$ of A with the connected component of A containing e, then A^+ is a convex cone in $\mathfrak{a}$ and the opposition involution i is the reflection in A^+ across the ray B^+. If L_1 and L_2 are the two walls of the Weyl chamber A^+, then $i(L_1) = L_2$. Therefore, because $\mu(h_n)$ is a bounded distance from one of the walls, we know that $i\big(\mu(h_n)\big)$ is a bounded distance from the other wall. That is, $\rho\big[i\big(\mu(h_n)\big)\big] \asymp \big\|i\big(\mu(h_n)\big)\big\|^{1/2}$. In other words, we have $\rho(h_n^{-1}) \asymp \|h_n^{-1}\|^{1/2}$. Therefore, using the sequences h_n and h_n^{-1}, we conclude from Proposition 3.14 that H is a Cartan-decomposition subgroup. ∎

We now describe the Cartan projections of the subgroups that are not Cartan-decomposition subgroups.

4.12. Proposition. *Assume that* $G = \operatorname{SL}(3,\mathbb{R})$.
1) *If* H *is of type* 4.7(2), *then* $\rho(h) \asymp h$ *for every* $h \in H$.
2) *If* H *is of type* 4.7(3), *then* $\rho(h) \asymp h$ *for every* $h \in H$.
3) *If* H *is of type* 4.7(4), *then* $\mu(H) \asymp \big[(\|h\| \log \|h\|)^{1/2}, \|h\|^2/(\log \|h\|)\big]$.

Proof of Theorem 4.5. If $A \subset H$, then H is a Cartan-decomposition subgroup (because A is a Cartan-decomposition subgroup) and we have either $\dim H \ge 3$ or $H = A$. Thus, we may henceforth assume that $A \not\subset H$. Then, from the proof of Lemma 2.8, we see that A is not contained in the Zariski closure $\overline{H}$ of H. Therefore $\dim(A \cap \overline{H}) \le 1$, so there are $a, b, c \in \mathbb{R}$ with

$$\overline{H} \cap A = \{\, h(e^{at}, e^{bt}, e^{ct}, 0, 0, 0) \mid t \in \mathbb{R} \,\}. \tag{4.13}$$

and $a + b + c = 0$. (If $A \cap \overline{H} = e$, then $a = b = c = 0$.) Because a 1-dimensional subgroup cannot be a Cartan-decomposition subgroup (see 3.13), we may assume that $\dim H \geq 2$.

Case 1. Assume that $H \subset N$ *and that* $\dim H = 2$. We show that H is a Cartan-decomposition subgroup if and only if H is not normalized by A. (That is, if and only if H is of type (3).) In the case where H is not a Cartan-decomposition subgroup, we show that $\rho(h) \asymp h$ for every $h \in H$.

Every 2-dimensional subalgebra of $\mathfrak{n}$ contains $\mathfrak{u}_{\alpha+\beta}$, so we must have $\mathfrak{u}_{\alpha+\beta} \subset \mathfrak{h}$.

($\Rightarrow$) We prove the contrapositive. Thus, we suppose that H is normalized by A, so $\mathfrak{h}$ is a sum of root spaces. Therefore, $\mathfrak{h}$ is either $\mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+\beta}$ or $\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta}$. In either case, every element of H is conjugate, via K, to an element of $U_{\alpha+\beta}$, so $\mu(H) = \mu(U_{\alpha+\beta})$. But $U_{\alpha+\beta}$, being one-dimensional, cannot be a Cartan-decomposition subgroup (see 3.13), so we conclude that H is not a Cartan-decomposition subgroup.

($\Leftarrow$) Because H is not normalized by A, and contains $\mathfrak{u}_{\alpha+\beta}$, we must have $H = \{ h \in N \mid h_{1,2} = p h_{2,3} \}$, for some nonzero $p \in \mathbb{R}$. For each $t \in \mathbb{R}$, let $h_t = h(1, 1, 1, pt, t, 0)$. Then $\rho(h_t) \asymp t^2 \asymp \|h_t\|^2$, so H is a Cartan-decomposition subgroup.

Case 2. Assume that $\dim H \geq 3$. We show that H is a Cartan-decomposition subgroup.

If $\dim H \geq 4$, then H must contain either A or N (c.f. 2.8), so H is a Cartan-decomposition subgroup. Thus, we may assume that $\dim H = 3$. Furthermore, from Lemma 2.8, we may assume that $\dim(H \cap N) = 2$.

We may assume that $H \cap N$ is normalized by A, for, otherwise, Case 1 implies that $H \cap N$ (and, hence, H) is a Cartan-decomposition subgroup. So $\mathfrak{h} \cap \mathfrak{n}$ must be either $\mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+\beta}$ or $\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta}$. For definiteness, let us assume that $\mathfrak{h} \cap \mathfrak{n} = \mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+\beta}$. (The calculations are essentially the same in the other case. In fact, $\mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+\beta}$ and $\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta}$ are conjugate under an outer automorphism of $\mathfrak{g}$.)

Subcase 2.1. Assume that $H = (H \cap A) \ltimes (H \cap N)$. We may assume that $a \leq 0$. (Recall that a, b, c are defined in (4.13).) For each $t \in \mathbb{R}^+$, define $h^t = h(e^{at}, e^{bt}, e^{ct}, e^{dt}, 0, e^{dt}) \in H$, where $d = \max\{b, c\} > 0$. Then $\rho(h^t) \asymp e^{2dt} \asymp \|h^t\|^2$, so H is a Cartan-decomposition subgroup.

Subcase 2.2. Assume that $H \neq (H \cap A) \ltimes (H \cap N)$. Let ω be a positive root as described in Lemma 2.4. Since $\mathfrak{u}_\alpha, \mathfrak{u}_{\alpha+\beta} \subset \mathfrak{h}$, we must have $\omega = \beta$. Thus, for some nonzero p, we have

$$H = \{ h(e^t, e^t, e^{-2t}, u, pte^t, v) \mid t, u, v \in \mathbb{R} \}.$$

For each $t \in \mathbb{R}^+$, let $h_t = h(e^t, e^t, e^{-2t}, pte^t, pte^t, 0) \in H$. Then $\rho(h_t) \asymp t^2 e^{2t} \asymp \|h_t\|^2$, so H is a Cartan-decomposition subgroup.

Case 3. Assume that $\dim H = 2$*, that* $H \not\subset N$*, and that* $H = (H \cap A) \ltimes (H \cap N)$. We may assume that $H \cap N$ is not a root group, for otherwise Proposition 3.17 applies. We show that H is a Cartan-decomposition subgroup if and only if $H \cap A = \ker \alpha$ or $H \cap A = \ker \beta$. (That is, if and only if H is of type (4) or (5), respectively.) When H is not a Cartan-decomposition subgroup, we have $H \cap A = \ker(\alpha - \beta)$, in which case we show that $\rho(h) \asymp h$ for every $h \in H$.

Because $H \cap N$ is normalized by $H \cap A$, and is one-dimensional, every element of $\mathfrak{h} \cap \mathfrak{n}$ is an eigenvector for each element of $H \cap A$. Because, by assumption, $\mathfrak{h} \cap \mathfrak{n}$ is not a root space, this implies that two positive roots must agree on $H \cap A$. So $H \cap A$ is the kernel of α, β, or $\alpha - \beta$.

If $H \cap A = \ker \alpha$ or $H \cap A = \ker \beta$, then we have $\rho(a) \asymp \|a\|^2$ or $\rho(a^{-1}) \asymp \|a^{-1}\|^2$, for every $a \in H \cap A$. Thus, H is a Cartan-decomposition subgroup.

Assume now that $H \cap A = \ker(\alpha - \beta)$. Then

$$H = \{ h(e^t, 1, e^{-t}, e^t u, pu, e^t pu^2/2) \mid t, u \in \mathbb{R} \},$$

for some nonzero $p \in \mathbb{R}$. Thus, we have $\rho(h) \asymp \max\{e^t u^2, e^{-t}\} \asymp h$ for every $h \in H$ with $|t|, |u| > 1$. Any element of H is within a bounded distance of some point with $|t|, |u| > 1$, so we conclude from Lemma 3.4 that $\mu(h) \asymp h$ for every $h \in H$. Therefore, H is not a Cartan-decomposition subgroup.

Case 4. Assume that $\dim H = 2$ *and that* $H \neq (H \cap A) \ltimes (H \cap N)$. We show that H is not a Cartan-decomposition subgroup, and that

$$\mu(H) \asymp \bigl[\|h\|^{1/2}(\log\|h\|), \|h\|^2/(\log\|h\|)\bigr].$$

Let the positive root ω and the subgroup U of N be as described in Lemma 2.4. Assume for definiteness that $\omega = \alpha$. (The calculations are similar in the other cases. Indeed, the groups in the other cases are conjugate to these under an automorphism of G.) Because U_α is one-dimensional, we must have $\psi(\ker\alpha) = U_\alpha$.

Because the restrictions of β and $\alpha+\beta$ to $\ker\alpha$ are nontrivial, unlike the restriction of α, and $\ker\alpha$ normalizes U, but $U_\alpha \not\subset U$, we see that $U \subset U_\beta U_{\alpha+\beta}$. Then, because $U = H \cap N$ is one-dimensional and is normalized by U_α, we conclude that $H \cap N = U_{\alpha+\beta}$. Thus, we have $H = \{h(e^t, e^t, e^{-2t}, pte^t, 0, s) \mid s,t \in \mathbb{R}\}$, for some nonzero $p \in \mathbb{R}$. For $|s|, |t| \geq 1$, we have $h \asymp \max\{|t|e^t, e^{-2t}, |s|\}$ and $\rho \asymp \max\{e^{2t}, |t|e^{-t}, |s|e^t\}$.

Letting $s = te^t \gg 0$ yields $\rho(h) \asymp te^{2t} \asymp \|h\|^2/(\log\|h\|)$. We now show that this is (approximately) the largest possible size of $\rho(h)$ relative to h. Because $\mu(H)$ is invariant under the opposition involution (see 4.10), this implies that $\mu(H) \asymp \bigl[(\|h\|\log\|h\|)^{1/2}, \|h\|^2/(\log\|h\|)\bigr]$, so H is not a Cartan-decomposition subgroup. For $t < -1$, we have $e^{2t} < 1$, $|t|e^{-t} < e^{-2t}$, and $|s|e^t < |s|$, so $\|\rho(h)\| = O(\|h\|)$ is much smaller than $\|h\|^2/(\log\|h\|)$. Now suppose that $t > 1$. If $|s| < |t|e^t$, then

$$\rho(h) = O(te^{2t}) = O\bigl((te^t)^2/t\bigr) = O(\|h\|^2/\log\|h\|).$$

If $|s| \geq |t|e^t$, then

$$\rho(h) = O(|s|e^t) = O\left(\frac{s^2}{|s|/e^t}\right) = O\left(\frac{s^2}{\log s}\right) = O\left(\frac{\|h\|^2}{\log\|h\|}\right),$$

as desired. □

Proof of Theorem 1.8. Let H be a minimal Cartan-decomposition subgroup of $G = \mathrm{SL}(3,\mathbb{R})$.

Case 1. Assume that $H \subset AN$ *and that* $\dim H < 3$. Consider the possibilities given by Theorem 4.5. If H is of type 4.5(2), then $H = A$ is listed in Theorem 1.8. If H is of type 4.5(3), then we may assume that $p = 1$, by replacing H with a conjugate via an element of A; thus, H is the subgroup of N that is listed in Theorem 1.8. If H is of type 4.5(4) or 4.5(5), then we may assume that $p = 0$, by replacing H with a conjugate via an element of $\mathrm{SO}(2) \times \mathrm{Id}$ or $\mathrm{Id} \times \mathrm{SO}(2)$, respectively (a maximal compact subgroup of the centralizer of $H \cap A$); thus, H is of type 4.5(6), discussed below.

If H is of type 4.5(6), then, because all roots are conjugate under the Weyl group, we may assume that $\omega = \alpha$; thus, H is of the form (1.9). Example 3.19 shows that we may assume that $\max\{p,q\} = 1$ and that $\min\{p,q\} \geq -1/2$, so H is listed in Theorem 1.8.

Case 2. Assume that $H \subset AN$ *and that* $\dim H \geq 3$. Because H is minimal, we know that H does not contain any conjugate of A, so $\dim(H \cap N) \geq 2$ (see 2.8). Then, because $H \cap N$ is not a Cartan-decomposition subgroup, we see from Corollary 4.7 that $\dim(H \cap N) = 2$ and that $\mathfrak{h}$ is either $\mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+\beta}$ or $\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta}$. There is an automorphism ϕ of G that normalizes A and $U_{\alpha+\beta}$, but interchanges U_α with U_β (namely, ϕ is the composition of the Cartan involution ("transpose-inverse") with the Weyl reflection corresponding to the root $\alpha+\beta$), so we may assume that $H = U_\beta + U_{\alpha+\beta}$.

If $H \cap A$ is nontrivial, then, because the Weyl chamber containing A_β^+ and the Weyl chamber containing $A_{\alpha+\beta}^+$ have no common interior, we see from Proposition 3.17 that either $(H \cap A) \ltimes U_\alpha$ or $(H \cap A) \ltimes U_{\alpha+\beta}$ is a Cartan-decomposition subgroup, which contradicts the minimality of H.

Thus, $H \cap A$ must be trivial, and we may assume that H is compatible with A (see 2.3), so, from Lemma 2.4, we see that H is conjugate (via an element of A) to the three-dimensional subgroup of AN that is listed in Theorem 1.8. (We must have $\omega = \alpha$, because H contains U_β and $U_{\alpha+\beta}$.)

Case 3. Assume that H is not conjugate to a subgroup of AN. Any nontrivial, connected, semisimple Lie group has a connected, proper, cocompact subgroup (namely, either the trivial subgroup or a parabolic subgroup, depending on whether H is compact or not), so the minimality implies that H is solvable. From the nilshadow construction (cf. proof of Lemma 2.9), we may assume, after replacing H by a conjugate subgroup, that there is a connected subgroup H' of AN and a compact, connected, abelian subgroup T of G, such that T normalizes both H and H', and we have $HT = H'T$. We may assume that H' is compatible with A (see 2.3). Because $H' \cap N$ is a nontrivial, connected subgroup of N that is normalized by a nontrivial, connected, compact subgroup of G, we must have either $H' \cap N = U_\alpha U_{\alpha+\beta}$ or $H' \cap N = U_\beta U_{\alpha+\beta}$. Replacing H by a conjugate via an automorphism of G (cf. the automorphism ϕ described in Case 2), we may assume that $H' \cap N = U_\beta U_{\alpha+\beta}$. Then $H' \cap N$ is not a Cartan-decomposition subgroup, so $H' \not\subset N$. Let $T_H = (HN) \cap A$. From Lemma 2.4, we know that T_H is contained in the Zariski closure of H', so, by replacing H with a conjugate, we may assume that T centralizes T_H. Thus, $T_H = \ker\omega$, for some root ω, and $T \subset \langle U_\omega, U_{-\omega}\rangle$. Because $U_\beta U_{\alpha+\beta} = H' \cap N$ is normalized by T, we must have $\omega = \alpha$.

Now TT_H is a maximal connected subgroup of the centralizer $C_G(T_H)$ (because $\mathrm{SO}(2)$ is maximal in $\mathrm{SL}(2,\mathbb{R})$), so

$$H' \cap C_G(T_H) = (H'TT_H) \cap C_G(T_H) \cap H' = (TT_H) \cap H' \subset T_H.$$

Thus, H' is not of type 2.4(2), so Lemma 2.4 implies that $H' = (H' \cap A) \ltimes (H' \cap N)$. Therefore, $H \subset (T \ker\alpha) \ltimes (U_\beta U_{\alpha+\beta})$, so, after conjugating T to a subgroup of $\mathrm{SO}(2) \times \mathrm{Id}$, it is easy to see that H is of the form (1.10). □

5. Subgroups of SO(2, n) contained in N

We now study $G = \mathrm{SO}(2,n)$. In this section, we determine which subgroups of N are Cartan-decomposition subgroups of G. Theorem 5.3 gives simple conditions to check whether a subgroup of N is a Cartan-decomposition subgroup, but the statement of the result requires some notation. Theorem 5.5 describes all the subgroups of N that are *not* Cartan-decomposition subgroups, and Proposition 5.8 describes the image of each of these subgroups under the Cartan projection. Analogous results for subgroups not contained in N appear in Section 6.

5.1. Notation. Recall that we realize $\mathrm{SO}(2,n)$ as isometries of the indefinite form $\langle v \mid v\rangle = v_1 v_{n+2} + v_2 v_{n+1} + \sum_{i=3}^n v_i^2$ on $\mathbb{R}^{n+2}$, that A consists of the group of diagonal matrices, and that N consists of the upper-triangular unipotent matrices (see 1.12). Let

$$A^+ = \{\, \mathrm{diag}(a_1, a_2, 1, 1, \dots, 1, 1, a_2^{-1}, a_1^{-1}) \mid a_1 \geq a_2 \geq 1 \,\}$$

and $K = \mathrm{SO}(2,n) \cap \mathrm{SO}(n+2)$.

5.2. Notation. Given $\phi, \eta \in \mathbb{R}$ and $x, y \in \mathbb{R}^{n-2}$ (and letting $t_1 = t_2 = 0$), there is a corresponding element of $\mathfrak{n}$ (see 1.13). The exponential of this is an element $h = h(\phi, x, y, \eta)$ of N. Namely,

$$h = \begin{pmatrix} 1 & \phi & x + \frac{1}{2}\phi y & \eta - \frac{1}{2}(x \cdot y) - \frac{1}{6}\phi\|y\|^2 & -\phi\eta - \frac{1}{2}\|x\|^2 + \frac{1}{24}\phi^2\|y\|^2 \\ & 1 & y & -\frac{1}{2}\|y\|^2 & -\eta - \frac{1}{2}(x \cdot y) + \frac{1}{6}\phi\|y\|^2 \\ & & \mathrm{Id} & -y^T & -x^T + \frac{1}{2}\phi y^T \\ & & & 1 & -\phi \\ & & & & 1 \end{pmatrix}.$$

Because the exponential map is a diffeomorphism from $\mathfrak{n}$ onto N, each element of N has a unique representation in this form. Thus, each element of N determines corresponding values of ϕ, x, y, and η. We sometimes write ϕ_h, x_h, y_h, and η_h for these values, to emphasize the element h of N that is under consideration.

The following description of the Cartan-decomposition subgroups contained in N is obtained by combining Lemmas 5.11 and 5.14 with Proposition 3.13.

5.3. Theorem. *Assume that $G = \mathrm{SO}(2,n)$. A closed, connected subgroup H of N is a Cartan-decomposition subgroup of G if and only if either*

1) $\dim H = 2$, $\mathfrak{u}_{\alpha+2\beta} \subset \mathfrak{h}$, *and there is an element of $\mathfrak{h}$ such that $\phi \neq 0$ and $y \neq 0$; or*
2) $\dim H \geq 2$ *and*
   (a) *there is an element of $\mathfrak{h}$ such that $\langle(\phi,x),(0,y)\rangle$ is one-dimensional in $\mathbb{R}^{n-1}$; and*
   (b) *there is a nonzero element of $\mathfrak{h}$ such that either*
      (i) *$\langle(\phi,x),(0,y)\rangle$ is two-dimensional in $\mathbb{R}^{n-1}$; or*
      (ii) *$y = 0$ and $\|x\|^2 = -2\phi\eta$.*

5.4. Notation. We let α and β be the simple real roots of $\mathrm{SO}(2,n)$, defined by $\alpha(a) = a_1/a_2$ and $\beta(a) = a_2$, for an element a of A of the form

$$a = \mathrm{diag}(a_1, a_2, 1, 1, \ldots, 1, 1, a_2^{-1}, a_1^{-1}).$$

Thus,

- the root space $\mathfrak{u}_\alpha$ is the ϕ-axis in $\mathfrak{n}$,
- the root space $\mathfrak{u}_\beta$ is the y-subspace in $\mathfrak{n}$,
- the root space $\mathfrak{u}_{\alpha+\beta}$ is the x-subspace in $\mathfrak{n}$, and
- the root space $\mathfrak{u}_{\alpha+2\beta}$ is the η-axis in $\mathfrak{n}$.

Theorem 5.3 describes Cartan-decomposition subgroups. The following result describes the subgroups of N that are *not* Cartan-decomposition subgroups. It is obtained by combining Corollaries 5.13 and 5.15 with Proposition 3.13.

5.5. Theorem. *Assume that $G = \mathrm{SO}(2,n)$. A closed, connected subgroup H of N is not a Cartan-decomposition subgroup of G if and only if either*

1) $\dim H \leq 1$; *or*
2) *for every nonzero element of $\mathfrak{h}$, we have $\phi = 0$ and $\dim\langle x,y\rangle \neq 1$ (i.e., $\dim\langle x,y\rangle \in \{0,2\}$); or*
3) *for every nonzero element of $\mathfrak{h}$, we have $\phi = 0$ and $\dim\langle x,y\rangle = 1$; or*
4) *there exists a subspace X_0 of $\mathbb{R}^{n-2}$, $b \in X_0$, $c \in X_0^\perp$, and $p \in \mathbb{R}$ with $\|b\|^2 - \|c\|^2 - 2p < 0$, such that for every element of $\mathfrak{h}$, we have $y = 0$, $x \in \phi c + X_0$, and $\eta = p\phi + b \cdot x$ (where $b \cdot x$ denotes the Euclidean dot product of the vectors b and x in $\mathbb{R}^{n-2}$).*

In the course of the proof of Theorem 5.5, we calculate the Cartan projection of each subgroup that is not a Cartan-decomposition subgroup. Proposition 5.8 collects these results. The statement (and all of the proofs in this section) is based on Proposition 3.14, so we describe the required representations ρ_1 and ρ_2 of G.

5.6. Notation. Define a representation

$$\rho\colon \mathrm{SO}(2,n) \to \mathrm{SL}(\mathbb{R}^{n+2} \wedge \mathbb{R}^{n+2}) \text{ by } \rho(g) = g \wedge g.$$

In the notation of Proposition 3.3, we have

$$\begin{aligned}
\rho_1 &= \text{the standard representation on } \mathbb{R}^{n+2} \qquad (\rho_1(g) = g),\\
\rho_2 &= \rho \qquad (= \rho_1 \wedge \rho_1),\\
k_1 &= 1 \qquad \text{and} \qquad k_2 = 2.
\end{aligned}$$

Thus, Proposition 3.3 yields the following fundamental lemma.

5.7. Lemma. *Assume that $G = \mathrm{SO}(2,n)$. A subgroup H of G is a Cartan-decomposition subgroup if and only if $\mu(H) \approx \big[\|h\|, \|h\|^2\big]$, where the representation ρ is defined in Notation 5.6.*

5.8. Proposition. *Assume that $G = \mathrm{SO}(2,n)$ and let H be a closed, connected subgroup of N.*

1) *If H is of type* 5.5(1) (*i.e., if* $\dim H \le 1$), *then either* $\rho(h) \asymp h$ *for all* $h \in H$, *or* $\rho(h) \asymp \|h\|^{3/2}$ *for all* $h \in H$, *or* $\rho(h) \asymp \|h\|^2$ *for all* $h \in H$.
2) *If H is of type* 5.5(2), *then* $\rho(h) \asymp \|h\|^2$ *for all* $h \in H$.
3) *If H is of type* 5.5(3), *then* $\rho(h) \asymp h$ *for all* $h \in H$.
4) *If H is of type* 5.5(4), *then* $\rho(h) \asymp h$ *for all* $h \in H$.

Note that if either $\rho(h) \asymp h$ for all $h \in H$, or $\rho(h) \asymp \|h\|^2$ for all $h \in H$, then $\mu(H)$ is within a bounded distance of one of the walls of A^+. On the other hand, if H is a Cartan-decomposition subgroup, then $\mu(H) \approx A^+$. Thus, certain one-dimensional subgroups (described in Lemma 5.20) are the only connected subgroups of N for which $\mu(H)$ is neither a wall of the Weyl chamber nor all of A^+ (up to bounded distance). In Section 6, where H is not assumed to be contained in N, we will see several more examples of this.

Proof of Theorem 1.14. It is easy to see from Theorem 5.3 that the subgroup of G corresponding to each of the subalgebras listed in Theorem 1.14 is a Cartan-decomposition subgroup. Each is minimal, because each is 2-dimensional (see 3.13). Lemma 5.10 shows that no two of the listed subalgebras are conjugate.

Thus, given a minimal Cartan-decomposition subgroup of G that is contained in N, all that remains is to show that the Lie algebra of H is conjugate to one on the list.

The Weyl reflection corresponding to the root β fixes $\alpha+\beta$, but interchanges α and $\alpha+2\beta$. Therefore, the subalgebra

$$\left\{ \begin{pmatrix} 0 & 0 & x & 0 & 0 & \eta & 0 \\ & 0 & 0 & 0 & 0 & 0 & -\eta \\ & & & \cdots & & & \end{pmatrix} \;\middle|\; x, \eta \in \mathbb{R} \right\} \tag{5.9}$$

is conjugate to the subalgebra of type 1.14(2) with $\epsilon_2 = 0$.

Case 1. Assume that $\mathfrak{h} \subset \mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta}$. From Theorem 5.3, we know that there is an element u of $\mathfrak{h}$ with $\dim\langle x_u, y_u\rangle = 1$. By replacing H with a conjugate under $\langle U_\alpha, U_{-\alpha}\rangle$, we may assume that $y_u = 0$. Then, by replacing H with a conjugate under U_β, we may assume that $\eta_u = 0$. If $\mathfrak{u}_{\alpha+2\beta} \subset \mathfrak{h}$, then $\mathfrak{h}$ contains a conjugate of the subalgebra (5.9). Thus, we may now assume that $\mathfrak{u}_{\alpha+2\beta} \not\subset \mathfrak{h}$. Therefore, $\mathfrak{h}$ has no nonzero elements of type 5.3(2(b)ii), so there must be some $v \in H$ with $\dim\langle x_v, y_v\rangle = 2$. Because $\mathfrak{u}_{\alpha+2\beta} \not\subset \mathfrak{h}$, we know that $\mathfrak{h}$ is abelian, so v commutes with u, which means that x_u is perpendicular to y_v. Thus, $\langle u, v\rangle$ is conjugate to one of the two subalgebras of type 1.14(3).

Case 2. Assume that there is an element u of $\mathfrak{h}$, *such that* $u \notin \mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta}$.

Subcase 2.1. Assume that there exists $v \in \big(\mathfrak{h} \cap (\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta})\big) \setminus (\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta})$. We have $[u, v] \in (\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta}) \setminus \mathfrak{u}_{\alpha+2\beta}$ and $[u, v, v] \in \mathfrak{u}_{\alpha+2\beta} \setminus \{0\}$, so $\langle [u, v], [u, v, v]\rangle$ is conjugate to the subalgebra (5.9).

Subcase 2.2. Assume that $\mathfrak{h} \subset \mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta}$. No element of $\mathfrak{h}$ is of type 5.3(2(b)i), so there must be a nonzero element v of $\mathfrak{h}$ with $\|x_v\|^2 = -2\phi_v\eta_v$. Replacing H by a conjugate under U_β, we may assume that $x_v = 0$; then either ϕ_v or η_v must also be 0. (Even after conjugation, we must have $\rho\big(\exp(tv)\big) \asymp \|\exp(tv)\|^2$.)

If $\phi_v = 0$, then $\mathfrak{u}_{\alpha+2\beta} \subset \mathfrak{h}$, so $\mathfrak{h}$ contains a conjugate of either the subalgebra (5.9) or the subalgebra 1.14(1).

If $\eta_v = 0$, then $\mathfrak{u}_\alpha \subset \mathfrak{h}$. Let w be a nonzero element of $\mathfrak{h} \cap (\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta})$. If $\eta_w = 0$, then $\langle v, w\rangle$ is conjugate to the subalgebra of type 1.14(2) with $\epsilon_2 = 0$. If $\eta_2 \neq 0$, then, by replacing $\mathfrak{h}$ with a conjugate under $U_{-\beta}$, we may assume that $x_w = 0$, so $\langle v, w\rangle$ is the subalgebra 1.14(1).

Subcase 2.3. The general case. If H is of type 5.3(1), then $\mathfrak{h}$ is conjugate to the subalgebra of type 1.14(1) with $\epsilon_1 = 1$. Thus, we henceforth assume that H is of type 5.3(2).

Let v be an element of $\mathfrak{h}$ with $\dim\langle(\phi_v, x_v), (0, y_v)\rangle = 1$. We may assume that Subcase 2.2 does not apply, so we may assume that $y_u \neq 0$.

Suppose that $\phi_v \neq 0$. Then y_v must be 0. Because $\phi_v \neq 0$, $y_v = 0$ and $y_u \neq 0$, there is some linear combination of v and u, such that $\phi = 0$ and $y \neq 0$, so Subcase 2.1 applies.

We may now assume that $\phi_v = 0$. We may assume that y_v is also 0, for, otherwise, Subcase 2.1 applies. Thus, $v \in (\mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta}) \setminus \mathfrak{u}_{\alpha+2\beta}$. We may assume that $[u, v] = 0$; otherwise, $\langle v, [u, v]\rangle$ is conjugate to the subalgebra (5.9), because $[u, v] \in \mathfrak{u}_{\alpha+2\beta}$.

Conjugating by an element of U_β, we may assume that x_u is a scalar multiple of x_v, and that $\eta_v = 0$. Then, by replacing u with $u + \lambda v$, for an appropriate scalar λ, we may assume that $x_u = 0$. Recall that $y_u \neq 0$. Therefore, conjugating by an element of $U_{\alpha+\beta}$, we may assume that $\eta_u = 0$. Because $[u, v] = 0$, we know that x_v is perpendicular to y_u. Thus, $\langle u, v\rangle$ is conjugate to one of the two subalgebras of type 1.14(2). □

5.10. Lemma. *Assume that $G = \mathrm{SO}(2, n)$. No two of the subalgebras of $\mathfrak{n}$ listed in Theorem 1.14 are conjugate under G. (In fact, they are not even conjugate under $\mathrm{GL}(n + 2, \mathbb{R})$.*

Proof. For each subalgebra $\mathfrak{h}$, we look at the restriction of the exponential map to $\mathfrak{h}$. This is a polynomial function from $\mathfrak{h}$ into N. For convenience, we use (ϕ, η), (x, y), or (ϕ, x) as coordinates on $\mathfrak{h}$, and use the matrix entries as coordinates on N.

The subalgebra of type 1.14(1) with $\epsilon_1 = 1$ and the subalgebra of type 1.14(2) with $\epsilon_2 = 1$ are the only cases where the exponential has a 4th degree term (namely, $\phi^4/24$), so they cannot be conjugate to any of the others. They are not conjugate to each other, because in the subalgebra of type 1.14(1) with $\epsilon_1 = 1$, there is a nontrivial subspace $\mathfrak{u}$ such that $\mu(\exp \mathfrak{u}) \approx \big[\|h\|^2, \|h\|^2\big]$ (namely, $\mathfrak{u} = \mathfrak{u}_{\alpha+2\beta}$), but there is no such subspace in the subalgebra of type 1.14(2) with $\epsilon_2 = 1$.

Of the others, it is only for the two subalgebras of type 1.14(3) that there are two independent quadratic terms (so the polynomial $\rho\big(\exp(u)\big)$ is of degree 4). For the subalgebra with $\epsilon_3 = 0$, one or the other of the quadratic terms vanishes if either x or y is 0 (that is, on two lines); whereas, for $\epsilon_3 = 1$, neither quadratic term vanishes unless $y = 0$ (a single line).

For each of two remaining subalgebras, the exponential function has a single quadratic term. For the subalgebra of type 1.14(2) with $\epsilon_2 = 0$, the quadratic term does not vanish unless $x = 0$ (a single line). Whereas, for the subalgebra of type 1.14(1) with $\epsilon_1 = 0$, the quadratic term vanishes if either $\phi = 0$ or $\eta = 0$ (two lines). □

The remainder of this section has three parts. In §5A, we determine when $\mu(H)$ contains elements close to the wall given by $\rho(h) \asymp h$. In §5B, we determine when $\mu(H)$ contains elements close to the other wall, given by $\rho(h) \asymp \|h\|^2$. In §5C, we calculate $\mu(H)$ for some subgroups that are not Cartan-decomposition subgroups.

5A. **When is the size of $\rho(h)$ linear?**

5.11. Lemma. *Assume that $G = \mathrm{SO}(2, n)$. Suppose that H is a closed, connected subgroup of N, and that $\dim H \neq 1$. Then the following are equivalent:*

1) *There is a sequence $\{h_n\}$ of elements of H with $h_n \to \infty$, such that $\rho(h_n) \asymp h_n$.*
2) *There is a continuous curve $\{h_t\}_{t\in\mathbb{R}^+}$ of elements of H with $h_t \to \infty$, such that $\rho(h_t) \asymp h_t$.*
3) *Either*
   (a) *there is an element of $\mathfrak{h}$ such that $\langle(\phi, x), (0, y)\rangle$ is one-dimensional in $\mathbb{R}^{n-1}$; or*
   (b) $\dim H = 2$, $\mathfrak{u}_{\alpha+2\beta} \subset \mathfrak{h}$, *and there is an element of $\mathfrak{h}$ such that $\phi \neq 0$ and $y \neq 0$.*

We prove the following more general (but slightly more complicated) version that does not assume that $\dim H \neq 1$.

5.12. Lemma. *Assume that $G = \mathrm{SO}(2, n)$, and let H be a closed, connected subgroup of N. Then the following are equivalent:*

1) *There is a sequence $\{h_m\}$ of elements of H with $h_m \to \infty$, such that $\rho(h_m) \asymp h_m$.*
2) *There is a continuous curve $\{h_t\}_{t\in\mathbb{R}^+}$ of elements of H with $h_t \to \infty$, such that $\rho(h_t) \asymp h_t$.*
3) *Either*
   (a) *There is an element of $\mathfrak{h}$ such that $\langle(\phi, x), (0, y)\rangle$ is one-dimensional in $\mathbb{R}^{n-1}$, and either $y \neq 0$ or $\|x\|^2 \neq -2\phi\eta$; or*
   (b) $\dim H = 2$, $\mathfrak{u}_{\alpha+2\beta} \subset \mathfrak{h}$, *and there is an element of $\mathfrak{h}$ such that $\phi \neq 0$ and $y \neq 0$.*

Proof. ($1 \Rightarrow 3$)

Case 1. Assume that $\phi = O(1)$ in h_m. We may assume that $\phi = 0$ (see 3.4). Let $a_m = k'_m h_m k_m$ be the Cartan projection of h_m, with $k'_m, k_m \in K$, and write $a_m = \operatorname{diag}(a_m^{(1)}, \dots, a_m^{(n+2)})$, with $a_m^{(1)} \geq a_m^{(2)} \geq \cdots \geq a_m^{(n+2)}$ (because $a_m \in A^+$). Let

$$V = \{\, v \in \mathbb{R}^{n+2} \mid \forall i \leq n,\ v_i = 0 \,\} \text{ and } W = \{\, w \in \mathbb{R}^{n+2} \mid w_1 = 0 \,\},$$

so $\dim V = 2$ and $\dim W = n+1$. Then $\dim V + \dim(k_m W) = n+3 > n+2$, so there is a nonzero vector $v_m \in V \cap (k_m W)$; we may assume that $\|v_m\| = 1$.

Because $v_m \in V$, we may write $v_m = (0, 0, \dots, 0, q_m, p_m)$, so

$$h_m v_m = \big(*, *, -(p_m x_m + q_m y_m), *, *\big).$$

Because $\rho(a_m) \asymp \rho(h_m) \asymp h_m \asymp a_m$, we must have $a_m^{(i)} = O(1)$ for $i \geq 2$, so $a_m w = O(w)$ for all $w \in W$. Then, since $k_m^{-1} v_m \in W$, we have

$$\begin{aligned} \|p_m x_m + q_m y_m\| &\leq \|h_m v_m\| = \|(k'_m)^{-1} a_m k_m^{-1} v_m\| \\ &= \|a_m k_m^{-1} v_m\| = O(k_m^{-1} v_m) = O(v_m) = O(1). \end{aligned}$$

Passing to a subsequence, we may assume that $\{v_m\}$ converges, so $\{p_m\}$ and $\{q_m\}$ converge. Clearly, $\|x_m\| + \|y_m\|$ tends to infinity, because $\rho(h) \asymp \|h\|^2$ if ϕ, x, and y are all zero (so the same is true if $\phi, x, y = O(1)$ (see 3.4)). Thus, letting $x'_m = x_m/(\|x_m\| + \|y_m\|)$ and $y'_m = y_m/(\|x_m\| + \|y_m\|)$, we have $p_m x'_m + q_m y'_m = o(1)$. Because $\|x'_m\|, \|y'_m\| \leq 1$, we may pass to a convergent subsequence. Then $p_\infty x'_\infty + q_\infty y'_\infty = 0$, so x'_∞ and y'_∞ are linearly dependent. Because $\mathfrak{h}$ is a closed subset of $\mathfrak{n}$, there is some element of $\mathfrak{h}$ with $\phi = 0$, $x = x'_\infty$, and $y = y'_\infty$, so conclusion (3a) holds.

Case 2. Assume that ϕ is unbounded in $\{h_m\}$. By passing to a subsequence, we may assume that $|\phi_m| \to \infty$.

Assume for the moment that some element u of $\mathfrak{h} \cap (\mathfrak{u}_\beta \oplus \mathfrak{u}_{\alpha+\beta} \oplus \mathfrak{u}_{\alpha+2\beta})$ does not belong to $\mathfrak{u}_{\alpha+2\beta}$. If $u \notin \mathfrak{u}_{\alpha+\beta} \oplus \mathfrak{u}_{\alpha+2\beta}$, then $[u, \log h]$ belongs to $\mathfrak{u}_{\alpha+\beta} \oplus \mathfrak{u}_{\alpha+2\beta}$, but is not in $\mathfrak{u}_{\alpha+2\beta}$. Thus, by replacing u with $[u, \log h]$ if necessary, we may assume, without loss of generality, that $u \in \mathfrak{u}_{\alpha+\beta} \oplus \mathfrak{u}_{\alpha+2\beta}$. This implies that $\phi_u = 0$, $y_u = 0$ and $x_u \neq 0$, so $\dim\langle(\phi_u, x_u), (0, y_u)\rangle = 1$, as desired.

We may henceforth assume that every element of $\mathfrak{h} \cap (\mathfrak{u}_\beta \oplus \mathfrak{u}_{\alpha+\beta} \oplus \mathfrak{u}_{\alpha+2\beta})$ belongs to $\mathfrak{u}_{\alpha+2\beta}$. This means that x and y are uniquely determined by ϕ, so $\dim H \leq 2$, and there are $\hat\phi \in \mathfrak{u}_\alpha$, $\hat x \in \mathfrak{u}_{\alpha+\beta}$, and $\hat y \in \mathfrak{u}_\beta$, such that $\hat\phi \neq 0$, and $\mathfrak{h} \subset \mathbb{R}(\hat\phi + \hat x + \hat y) + \mathfrak{u}_{\alpha+2\beta}$.

Suppose that $\hat y \neq 0$. If $\dim H = 1$, then Lemma 5.20 implies that $\rho(h) \asymp \|h\|^{3/2}$ for $h \in H$, so $\rho(h) \not\asymp h$, which is a contradiction. Thus, we must have $\dim H = 2$, so conclusion (3b) holds.

We now know $\hat y = 0$. If $\dim \mathfrak{h} = 1$ and $\|x\|^2 = -2\phi\eta$ for some (and, hence, all) elements of $\mathfrak{h}$, then Lemma 5.14(3b) implies that $\rho(h) \asymp \|h\|^2 \not\asymp h$, a contradiction. If $\dim \mathfrak{h} = 2$, then $\mathfrak{u}_{\alpha+2\beta} \subset \mathfrak{h}$, so it is easy to choose an element of $\mathfrak{h}$ with $\|x\|^2 \neq -2\phi\eta$.

($3 \Rightarrow 2$)

Case 1. Assume that there is an element h of H, such that $\dim\langle(\phi, x), (0, y)\rangle = 1$, *and* $y \neq 0$. Because $\dim\langle(\phi, x), (0, y)\rangle = 1$, we know that (ϕ, x) is a scalar multiple of $(0, y)$, so $\phi = 0$. Furthermore, we may assume, after replacing H (and, hence, h) with a conjugate under U_α, that $x = 0$

(see 3.7). Then, letting $h^t = \exp(t \log h)$, we have

$$h^t_{ij} = \begin{cases} O(1) & \text{if } i \neq 2 \text{ and } j \neq n+1 \\ O(t) & \text{if } (i,j) \neq (2, n+1) \end{cases}$$

and $h^t_{2,n+1} \asymp t^2$. Therefore $\rho(h^t) \asymp t^2 \asymp h^t$, as desired.

Case 2. Assume that there is an element h of H, such that $y = 0$ and $\|x\|^2 \neq -2\phi\eta$. Letting $h^t = \exp(t \log h)$, we have

$$h^t_{ij} = \begin{cases} O(1) & \text{if } i \neq 1 \text{ and } j \neq n+2 \\ O(t) & \text{if } (i,j) \neq (1, n+2) \end{cases}$$

and, because $\|x\|^2 \neq -2\phi\eta$, we have $h^t_{1,n+2} \asymp t^2$. Therefore it is not difficult to see that $\rho(h^t) \asymp t^2 \asymp h^t$, as desired.

Case 3. Assume that $\dim H = 2$, $\mathfrak{u}_{\alpha+2\beta} \subset \mathfrak{h}$, *and there is an element u of $\mathfrak{h}$ such that $\phi_u \neq 0$ and $y_u \neq 0$.* Replacing H by a conjugate under U_β, we may assume that $x_u = 0$. For any large real number t, let $h = h^t$ be the element of $\exp(tu + \mathfrak{u}_{\alpha+2\beta})$ that satisfies $\eta = -\phi\|y\|^2/12$. Then

$$h = \begin{pmatrix} 1 & \phi & \frac{1}{2}\phi y & -\frac{1}{4}\phi\|y\|^2 & \frac{1}{8}\phi^2\|y\|^2 \\ & 1 & y & -\frac{1}{2}\|y\|^2 & \frac{1}{4}\phi\|y\|^2 \\ & & 1 & -y^T & \frac{1}{2}\phi y^T \\ & & & 1 & -\phi \\ & & & & 1 \end{pmatrix}.$$

Because t is large, we know that ϕ and $\|y\|$ are large, so it is clear that $h \asymp \phi^2\|y\|^2$.

Let h' be the matrix obtained from h by deleting the first two columns. Then the first two rows of h' are linearly dependent (because the first row of h' is $\phi/2$ times the remainder of the second row). Therefore, we have

$$\det\begin{pmatrix} h_{1,i} & h_{1,j} \\ h_{2,i} & h_{2,j} \end{pmatrix} = 0, \qquad \text{whenever } i,j > 2.$$

Similarly, we have

$$\det\begin{pmatrix} h_{i,n+1} & h_{i,n+2} \\ h_{j,n+1} & h_{j,n+2} \end{pmatrix} = 0, \qquad \text{whenever } i,j \leq n.$$

It is easy to see that the determinant of any other 2×2 submatrix of h is $O(\phi^2\|y\|^2)$. Thus, we conclude that $\rho(h) = O(\phi^2\|y\|^2) = O(h)$, as desired. □

5.13. Corollary (of proof)**.** *Assume that $G = \mathrm{SO}(2,n)$, and let H be a closed, connected subgroup of N. The following are equivalent:*

1) *There is no sequence $\{h_m\}$ of elements of H with $h_m \to \infty$ and $\rho(h_m) \asymp h_m$.*
2) *There is no curve $\{h_t\}_{t\in\mathbb{R}^+}$ in H with $h_t \to \infty$ and $\rho(h_t) \asymp h_t$.*
3) *Either*
   (a) *for every element of $\mathfrak{h}$, we have $\phi = 0$ and $\dim\langle x, y\rangle \neq 1$; or*
   (b) $\dim H = 1$, *and every nonzero element of $\mathfrak{h}$ satisfies $\phi \neq 0$ and $y \neq 0$; or*
   (c) $\dim H = 1$, *and every element of H satisfies $y = 0$ and $\|x\|^2 = -2\phi\eta$.*

5B. When is the size of $\rho(h)$ quadratic?

5.14. Lemma. *Assume that $G = \mathrm{SO}(2,n)$, and let H be a closed, connected subgroup of N. The following are equivalent:*

1) *There is a sequence $\{h_m\}$ of elements of H with $h_m \to \infty$ and $\rho(h_m) \asymp \|h_m\|^2$.*
2) *There is a curve $\{h_t\}_{t\in\mathbb{R}^+}$ in H with $h_t \to \infty$ and $\rho(h_t) \asymp \|h_t\|^2$.*
3) *there is a nonzero element of $\mathfrak{h}$ such that either*

(a) $\langle(\phi, x),(0, y)\rangle$ *is two-dimensional in* $\mathbb{R}^{n-1}$, *and either* $\dim \mathfrak{h} \neq 1$ *or* $\phi=0$*; or*
(b) $y=0$ *and* $\|x\|^2=-2\phi\eta$.

Proof. ($1 \Rightarrow 3$) Assume that neither (3a) nor (3b) holds. We have $\mathfrak{h} \cap \mathfrak{u}_{\alpha+2\beta}=0$, for otherwise there is an element of $\mathfrak{h}$ for which x, y, and ϕ are all zero, so (3b) holds.

Case 1. Assume that $y=0$ *for every element of* $\mathfrak{h}$. Because $\mathfrak{h} \cap \mathfrak{u}_{\alpha+2\beta}=0$, we know that η is determined by ϕ and x, so there exist $p \in \mathbb{R}$ and $b \in \mathbb{R}^{n-2}$, such that, for every element of $\mathfrak{h}$, we have $\eta=p\phi+b \cdot x$.

Because $y=0$ and $\eta=O\big((\phi, x)\big)$, we have $h_{ij}=O\big((\phi, x)\big)$ for all $(i, j) \neq(1, n+2)$. Therefore, if $\rho(h) \asymp\|h\|^2$, then we must have $h_{1, n+2}=O\big((\phi, x)\big)$. However, $h_{1, n+2}=-\phi\big(p\phi+b \cdot x\big)-\frac{1}{2}\|x\|^2$ is a quadratic form on $\mathfrak{h}$. Therefore, because there are vectors (ϕ, x) with $h_{1, n+2}=O\big((\phi, x)\big) \ll \|(\phi, x)\|^2$, this quadratic form must represent 0 nontrivially, so there is an element of H with $h_{1, n+2}=0$; that is, $\phi\eta+\|x\|^2/2=0$, so conclusion (3b) holds.

Case 2. Assume that $\phi=0$ *for every element of* $\mathfrak{h}$. From the negation of (3a), we see that x and y are linearly dependent, for every element of $\mathfrak{h}$. Also, we may assume that there is an element of $\mathfrak{h}$ with $y \neq 0$, for, otherwise, Case 1 applies. Thus, conjugating by an element of U_α, we may assume that $x=0$ for every element of $\mathfrak{h}$ (see 3.7). Because $\mathfrak{h} \cap \mathfrak{u}_{\alpha+2\beta}$, ϕ, and x are all zero, we know that $\eta=O(y)$, so $h_{ij}=O(y)$ for all $(i, j) \neq(2, n+1)$. On the other hand, $h_{2, n+1} \asymp\|y\|^2$. Therefore, $\rho(h)=O(\|y\|^3) \ll\|y\|^4 \asymp\|h\|^2$.

Case 3. The remaining cases. If (ϕ, x) and $(0, y)$ are linearly dependent, for every element of $\mathfrak{h}$, then either Case 1 or Case 2 applies. Thus, we may assume that there is some element of $\mathfrak{h}$, such that $\langle(\phi, x),(0, y)\rangle$ is two-dimensional. (In particular, we must have $y \neq 0$.) Then, from the negation of (3a), we see that $\dim H=1$, and $\phi \neq 0$. Then Lemma 5.20 implies that $\rho(h) \asymp\|h\|^{3/2} \ll\|h\|^2$.

($3 \Rightarrow 2$) For any element h of $U_{\alpha+2\beta}$, we have $h_{ij}=O(1)$ for all $(i, j) \notin\{(1, n+1),(2, n+2)\}$, so it is obvious that $\rho(h) \asymp\|h\|^2$. Thus, we may assume that $\mathfrak{h} \cap \mathfrak{u}_{\alpha+2\beta}=0$.

Let $h \in H$, and let $h^t=\exp(t \log h)$.

Case 1. Assume that $\phi=0$. Because $\phi=0$, it is clear that $h_{ij}^t=O(t)$ for all $(i, j) \notin\{1,2\} \times\{n+1, n+2\}$. On the other hand, by hypothesis, we may assume that $\langle(0, x),(0, y)\rangle$ is two-dimensional; that is, x and y are linearly independent. Thus, $\det \begin{pmatrix} x \cdot y & \|x\|^2 \\ \|y\|^2 & x \cdot y \end{pmatrix} \neq 0$, so it is easy to see that $\det \begin{pmatrix} h_{1, n+1}^t & h_{1, n+2}^t \\ h_{2, n+1}^t & h_{2, n+2}^t \end{pmatrix} \asymp t^4 \asymp\|h^t\|^2$. Therefore, $\rho(h^t) \asymp\|h^t\|^2$.

Case 2. Assume that $y=0$. In this case, it is clear that

$$h_{ij}^t= \begin{cases}O(1) & \text{if } i \neq 1 \text{ and } j \neq n+2 \\ O(t) & \text{if }(i, j) \neq(1, n+2)\end{cases}.$$

By hypothesis, we may assume that $\|x\|^2=-2\phi\eta$, so $h_{1, n+2}^t=0$. Thus, because ϕ, x, and η cannot all be zero, it is clear that $\rho(h^t) \asymp t^2 \asymp\|h^t\|^2$.

Case 3. Assume that both of ϕ *and* y *are nonzero.* We see from (3) that $\dim H \neq 1$, so there is an element u of H with $\phi_u=0$. We know that $y_u=0$, for, otherwise, $[\log u,[\log u, \log h]]$ would be a nonzero element of $\mathfrak{h} \cap \mathfrak{u}_{\alpha+2\beta}$. Then, because $\mathfrak{h} \cap \mathfrak{u}_{\alpha+2\beta}=0$, we must have $x_u \neq 0$.

For each large $t \in \mathbb{R}$, define $f_t \colon \mathbb{R} \rightarrow \mathbb{R}$ by

$$f_t(s)=-t\phi_h(t\eta_h+s\eta_u)-\frac{1}{2}\|t x_h+s x_u\|^2+\frac{1}{24} t^4 \phi_h^2\|y_h\|^2 .$$

Because t is large, we know that $f_t(0)>0$, so there is some $s=s(t) \in \mathbb{R}^{+}$ with $f_t(s)=0$. Note that s is approximately $\frac{1}{\sqrt{12}} t^2|\phi_h|\|y_h\|/\|x_u\|$.

Let $h^t = \exp(t \log h + s \log u)$. Then $h^t_{1,n+2} = f_t(s) = 0$, so we see that $h \asymp t^3$. Also, we have

$$\det \begin{pmatrix} h^t_{1,n+1} & h^t_{1,n+2} \\ h^t_{2,n+1} & h^t_{2,n+2} \end{pmatrix} = h^t_{1,n+1} h^t_{2,n+2} \asymp t^6 \asymp \|h^t\|^2,$$

as desired. □

In the statement of the following corollary, we use Proposition 5.19 to convert the condition that $\|x\|^2 \neq -2\phi\eta$ to the condition that $\|b\|^2 - \|c\|^2 - 2p < 0$.

5.15. Corollary (of proof)**.** *Assume that $G = \mathrm{SO}(2,n)$, and let H be a closed, connected subgroup of N. The following are equivalent:*

1) *There is no sequence $\{h_m\}$ of elements of H with $h_m \to \infty$ and $\rho(h_m) \asymp \|h_m\|^2$.*
2) *There is no curve $\{h_t\}_{t \in \mathbb{R}^+}$ in H with $h_t \to \infty$ and $\rho(h_t) \asymp \|h_t\|^2$.*
3) *Either*
   (a) *for every nonzero element of $\mathfrak{h}$, we have $\phi = 0$ and $\dim\langle x, y\rangle = 1$; or*
   (b) *there exists a subspace X_0 of $\mathbb{R}^{n-2}$, $b \in X_0$, $c \in X_0^\perp$, and $p \in \mathbb{R}$ with $\|b\|^2 - \|c\|^2 - 2p < 0$, such that for every element of $\mathfrak{h}$, we have $y = 0$, $x \in \phi c + X_0$, and $\eta = p\phi + b \cdot x$; or*
   (c) *$\dim H = 1$, and we have $\phi \neq 0$ and $y \neq 0$ for every element of H; or*
   (d) *$\dim H = 1$ and we have $y = 0$ and $\|x\|^2 \neq -2\phi\eta$ for every element of H.*

5C. **Some calculations of $\mu(H)$.**

5.16. Lemma. *Assume that $G = \mathrm{SO}(2,n)$. Then $\mu(g^{-1}) = \mu(g)$ for all $g \in G$.*

Proof. This follows from the fact that there is an element of the Weyl group of $\mathrm{SO}(2,n)$ that sends each element of A to its inverse, so each element of A is conjugate to its inverse under K. □

5.17. Lemma. *Assume that $G = \mathrm{SO}(2,n)$, and let H be a connected, closed subgroup of N. Assume that for every nonzero element of $\mathfrak{h}$, we have $\phi = 0$ and $\dim\langle x, y\rangle \neq 1$. Then $\rho(h) \asymp \|h\|^2$, for every $h \in H$.*

Proof. For future reference we prove the stronger fact that for every real number $t \geq 1$ and every $h \in H$, we have $\rho(ah) \asymp \|ah\|^2$, where

$$a = \mathrm{diag}(t, t, 1, 1, \dots, 1, 1, t^{-1}, t^{-1}).$$

Let $g = ah$. We have $\det \begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix} = t^2$ and

$$\begin{aligned} 4\left|\det \begin{pmatrix} g_{1,n+1} & g_{1,n+2} \\ g_{2,n+1} & g_{2,n+2} \end{pmatrix}\right| &= 4t^2 \left|\det \begin{pmatrix} h_{1,n+1} & h_{1,n+2} \\ h_{2,n+1} & h_{2,n+2} \end{pmatrix}\right| \\ &= 4t^2 \Big(\eta^2 + \big(\|x\|^2\,\|y\|^2 - (x \cdot y)^2\big)\Big). \end{aligned}$$

Because x and y are linearly independent for all nontrivial $h \in H$, we have

$$\|x\|^2\,\|y\|^2 - (x \cdot y)^2 \asymp \|x\|^2\,\|y\|^2 \asymp \|x\|^4,$$

so we conclude that $\rho(ah) \asymp \max(t^2, t^2\eta^2, t^2\|x\|^4) \asymp \|ah\|^2$. □

5.18. Notation. We realize $\mathrm{SO}(1,n)$ as the stabilizer in $\mathrm{SO}(2,n)$ of the vector

$$(0, 1, 0, \dots, 0, 0, -1, 0) \in \mathbb{R}^{n+2}.$$

5.19. Proposition. *Assume that $G = \mathrm{SO}(2,n)$. Suppose that H is a closed, connected subgroup of N and that there exist a subspace X_0 of $\mathbb{R}^{n-2}$, vectors $b \in X_0$ and $c \in X_0^\perp$, and a real number p, such that*

$$H = \{\, h \in N \mid x \in \phi c + X_0,\ \eta = p\phi + b \cdot x,\ y = 0 \,\}.$$

If $\dim H \geq 2$, then the following are equivalent:

1) *H is conjugate to a subgroup of* $\mathrm{SO}(1,n)$.
2) *We have* $\|b\|^2 - \|c\|^2 - 2p < 0$.
3) $\rho(h) \asymp h$ *for every* $h \in H$.
4) *We have* $\|x\|^2 \neq -2\phi\eta$, *for every nonzero element of* $\mathfrak{h}$.

Proof. ($2 \Rightarrow 1$) Suppose that $\|b\|^2 - \|c\|^2 - 2p < 0$. Then $t := \sqrt{-(\|b\|^2 - \|c\|^2 - 2p)}$ is a positive real number. Set

$$g = \begin{pmatrix} t & 0 & 0 & 0 & 0 \\ & t^{-1} & t^{-1}(c-b) & -\frac{1}{t}t^{-1}\|c-b\|^2 & 0 \\ & & 0 & (c-b)^T & 0 \\ & & & t^{-1} & 0 \\ & & & & t \end{pmatrix}.$$

Then a matrix calculation, using the facts that $c \cdot b = 0$ and $c \cdot X_0 = 0$, verifies that $y = 0$ and $\eta = \phi$ for every element of gHg^{-1}, so $gHg^{-1} \subset \mathrm{SO}(1,n)$.

($1 \Rightarrow 3$) Because $\mathrm{SO}(1,n)$ is reductive and $A^+ \cap \mathrm{SO}(1,n)$ is the positive Weyl chamber of a maximal split torus of $\mathrm{SO}(1,n)$, we have $\mu\big(\mathrm{SO}(1,n)\big) \subset A^+ \cap \mathrm{SO}(1,n)$. Then, because

$$A \cap \mathrm{SO}(1,n) = \{\, \mathrm{diag}(a, 1, 1, \dots, 1, 1, a^{-1}) \mid a > 0 \,\},$$

it is clear that $\rho(g) \asymp g$ for every $g \in \mathrm{SO}(1,n)$.

($3 \Rightarrow 4$) This is immediate from Lemma 5.14(3b).

($4 \Rightarrow 2$) We prove the contrapositive.

Assume for the moment that $b \neq 0$. The equation $\|sc + b\|^2 = -2s(ps + \|b\|^2)$ is quadratic (or linear) in s. Because the discriminant $4\|b\|^2(\|b\|^2 - \|c\|^2 - 2p)$ of this quadratic equation is nonnegative, the equation has a real solution $s = \phi_0$. Then, for the element of $\mathfrak{h}$ with $\phi = \phi_0$, $x = \phi_0 c + b$, $y = 0$, and $\eta = p\phi_0 + \|b\|^2$, we have $\|x\|^2 = -2\phi\eta$. Because $b \neq 0$, not both of ϕ and x can be 0, so we have the desired conclusion.

We may now assume that $b = 0$. Because $\dim H \geq 2$, there is a nonzero element x_0 of X_0. Because $\|c\|^2 + 2p = -(\|b\|^2 - \|c\|^2 - 2p) > 0$, the equation $\|sc + x_0\|^2 = -2s(ps)$ has a solution $s = \phi_0$. Because $x_0 \neq 0$, we know that $\phi_0 \neq 0$. For the element of $\mathfrak{h}$ with $\phi = \phi_0$, $x = \phi_0 c + x_0$, and $\eta = p\phi_0$, we have $\|x\|^2 = -2\phi\eta$. □

5.20. Lemma. *Assume that $G = \mathrm{SO}(2,n)$. Let H be a closed, connected subgroup of N, such that $\dim H = 1$, and we have $\phi \neq 0$ and $y \neq 0$ for every element of H. Then $\rho(h) \asymp \|h\|^{3/2}$, for every $h \in H$.*

Proof. Let $h^t = \exp(tu)$, where u is some nonzero element of $\mathfrak{h}$. For any large real number t, we see that $h^t_{1,n+2} \asymp t^4$, but $h^t_{i,j} = O(t^3)$ for $(i,j) \neq (1, n+2)$. Thus, $h^t \asymp t^4$. Furthermore, we have $\det \begin{pmatrix} h^t_{1,n+1} & h^t_{1,n+2} \\ h^t_{2,n+1} & h^t_{2,n+2} \end{pmatrix} \asymp t^6$, and we have $h^t_{i,j} = O(t^2)$ whenever $i \neq 1$ and $j \neq n+2$, so we conclude that $\rho(h^t) \asymp t^6 \asymp \|h^t\|^{3/2}$, as desired.

For future reference, we note the stronger fact that for every real number $s \geq 1$ and every $h \in H$, we have $\rho(ah) \asymp \|ah\|^{3/2}$, where

$$a = \mathrm{diag}(s^2, s, 1, 1, \dots, 1, 1, s^{-1}, s^{-2}).$$

□

6. Subgroups of SO$(2,n)$ that are not contained in N

6.1. Theorem. *Assume that $G = \mathrm{SO}(2,n)$. Let H be a closed, connected subgroup of AN, such that $H = (H \cap A) \ltimes (H \cap N)$. The subgroup H is a Cartan-decomposition subgroup of G if and only if either*

1) $A \subset H$*; or*

2) *$H \cap N$ is a Cartan-decomposition subgroup of G (see 5.3); or*
3) *$H \cap A = \ker\alpha$, and we have $\phi = \eta = 0$ and $\dim\langle x,y\rangle = 1$ for every $h \in H \cap N$; or*
4) *$H \cap A = \ker\beta$ and we have $y = 0$ and $\|x\|^2 = -2\phi\eta$ for every element of $H \cap N$; or*
5) *there is a positive root ω such that $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_\omega$, and H satisfies the conditions of Proposition 3.17.*

6.2. Corollary (of proof). *Assume that $G = \mathrm{SO}(2,n)$. Let H be a closed, connected subgroup of AN, such that $H = (H \cap A) \ltimes (H \cap N)$, and $H \not\subset N$. The subgroup H is not a Cartan-decomposition subgroup of G if and only if either*

1) *$H = H \cap A$ is a one-dimensional subgroup of A; or*
2) *$H \cap A = \ker\alpha$, and we have $\phi = 0$ and $\dim\langle x,y\rangle \neq 1$, for every nonzero element of $\mathfrak{h} \cap \mathfrak{n}$, in which case $\rho(h) \asymp \|h\|^2$ for all $h \in H$; or*
3) *$H \cap A = \ker\beta$, and we have $\phi = 0$, $y = 0$, and $x \neq 0$, for every nonzero element of $\mathfrak{h} \cap \mathfrak{n}$, in which case $\rho(h) \asymp h$ for all $h \in H$; or*
4) *$H \cap A = \ker(\alpha+\beta)$, and we have $\phi = 0$, $x = 0$, and $y \neq 0$, for every nonzero element of $\mathfrak{h} \cap \mathfrak{n}$, in which case $\rho(h) \asymp h$ for all $h \in H$; or*
5) *$H \cap A = \ker\beta$, and there exist a subspace X_0 of $\mathbb{R}^{n-2}$, $b \in X_0$, $c \in X_0^\perp$, and $p \in \mathbb{R}$, such that $\|b\|^2 - \|c\|^2 - 2p < 0$, and we have $y = 0$, $x \in \phi c + X_0$, and $\eta = p\phi + b \cdot x$ for every $h \in H$, in which case $\rho(h) \asymp h$ for every $h \in H$; or*
6) *$H \cap A = \ker(\alpha-\beta)$, $\dim H = 2$, and there exist $\hat\phi \in \mathfrak{u}_\alpha$ and $\hat y \in \mathfrak{u}_\beta$, such that $\hat\phi \neq 0$, $\hat y \neq 0$, and $\mathfrak{h} \cap \mathfrak{n} = \mathbb{R}(\hat\phi + \hat y)$, in which case $\rho(h) \asymp \|h\|^{3/2}$ for every $h \in H$; or*
7) *$H \cap A = \ker\beta$, $\dim H = 2$, and we have $y = 0$ and $\|x\|^2 \neq -2\phi\eta$ for every $h \in H$, in which case $\rho(h) \asymp h$ for every $h \in H$; or*
8) *there is a positive root ω and a one-dimensional subspace $\mathfrak{t}$ of $\mathfrak{a}$, such that $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_\omega$, $\mathfrak{h} = \mathfrak{t} + (\mathfrak{h} \cap \mathfrak{n})$, and Proposition 3.17 implies that H is not a Cartan-decomposition subgroup.*

Proof of Theorem 6.1. We may assume that $\mathfrak{h}\cap\mathfrak{n}$ is not contained in any root space, for otherwise Proposition 3.17 applies. We may also assume that $H \not\subset A$, and that $H \cap N$ is not a Cartan-decomposition subgroup. The proof proceeds in cases, suggested by Theorem 5.5.

Case 1. Assume that $\phi = 0$ and that $\dim\langle x,y\rangle \neq 1$, for every nonzero element of $\mathfrak{h} \cap \mathfrak{n}$. We show that $\rho(h) \asymp \|h\|^2$, for every $h \in H$, so H is not a Cartan-decomposition subgroup. Let $h \in H$, and write $h = au$ with $a \in H \cap A$ and $u \in H \cap N$. Because $\mathfrak{h} \cap \mathfrak{n} \not\subset \mathfrak{u}_{\alpha+2\beta}$ (recall that $\mathfrak{h} \cap \mathfrak{n}$ is not contained in any root space), we know that $\mathfrak{h} \cap \mathfrak{n}$ projects nontrivially into $\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta}$. This projection is normalized by $H \cap A$, but, from the assumption on $\dim\langle x,y\rangle$, we know that this projection intersects neither $\mathfrak{u}_\beta$ nor $\mathfrak{u}_{\alpha+\beta}$, so we must have $\alpha(H \cap A) = 1$. Thus, a is of the form $a = \mathrm{diag}(t,t,1,\ldots,1,t^{-1},t^{-1})$. We may assume that $t \geq 1$, by replacing h with h^{-1} if necessary (see 5.16). Then the proof of Proposition 5.17 implies that $\rho(h) \asymp \|h\|^2$.

Case 2. Assume that $\phi = 0$ and that $\dim\langle x,y\rangle = 1$, for every element of $\mathfrak{h}\cap\mathfrak{n}$. We show that H is a Cartan-decomposition subgroup if and only if $\alpha(H \cap A) = 1$ (and $\mathfrak{h}\cap\mathfrak{n} \subset \mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta}$). Furthermore, when H is not a Cartan-decomposition subgroup, we show that $\rho(h) \asymp h$ for every $h \in H$.

($\Leftarrow$) Because $\alpha(H \cap A) = 1$, we know that every element of $H \cap A$ is of the form

$$a = \mathrm{diag}(t,t,1,\ldots,1,t^{-1},t^{-1}),$$

so $\rho(a) \asymp t^{\pm 2} \asymp \|a\|^2$. From Lemma 5.12 we also know that there is a path $h^t \to \infty$ in $H \cap N$ with $\rho(h^t) \asymp h^t$. Thus, H is a Cartan-decomposition subgroup.

($\Rightarrow$) Suppose that $\alpha(H \cap A) \neq 1$. Then, because $\dim\langle x,y\rangle = 1$ for every $h \in H \cap N$, we see that $\mathfrak{h} \cap \mathfrak{u}_{\alpha+2\beta} = 0$ and either

i) $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_\beta + \mathfrak{u}_{\alpha+2\beta}$; or
ii) $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta}$.

Because the Weyl reflection corresponding to the root α fixes $\mathfrak{u}_{\alpha+2\beta}$ and interchanges $\mathfrak{u}_\beta$ with $\mathfrak{u}_{\alpha+\beta}$, we may assume that (ii) holds. Then, by choosing a large negative value for p, we see that H is a subgroup of a group of the type considered in Case 3 below, so $\rho(h) \asymp h$ for every $h \in H$.

Case 3. Assume that there exist a subspace X_0 of $\mathbb{R}^{n-2}$, $b \in X_0$, $c \in X_0^\perp$, and $p \in \mathbb{R}$, with $\|b\|^2 - \|c\|^2 - 2p < 0$, such that we have $y = 0$, $x \in \phi c + X_0$, and $\eta = p\phi + b \cdot x$ for every $h \in H$. We show that H is conjugate to a subgroup of $\mathrm{SO}(1,n)$, so $\rho(h) \asymp h$ for every $h \in H$. Therefore, H is not a Cartan-decomposition subgroup.

Let $h \in H$, and write $h = au$ with $a \in H \cap A$ and $u \in H \cap N$. Because $y = 0$ for every element of $\mathfrak{h} \cap \mathfrak{n}$, we know that $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta}$. Also, because $\|b\|^2 - \|c\|^2 - 2p < 0$, we know that p and c cannot both be 0, so $\mathfrak{h} \cap \mathfrak{n}$ does not intersect $\mathfrak{u}_\alpha$, and, because $\eta = p\phi + b \cdot x$, we know that $\mathfrak{h} \cap \mathfrak{n}$ does not intersect $\mathfrak{u}_{\alpha+2\beta}$. Thus, because $\mathfrak{h} \cap \mathfrak{n} \not\subset \mathfrak{u}_{\alpha+\beta}$, and $H \cap A$ normalizes $\mathfrak{h} \cap \mathfrak{n}$, we see that $\beta(H \cap A) = 1$, so $H \cap A \subset \mathrm{SO}(1,n)$. Furthermore, $H \cap A$ centralizes $\mathfrak{u}_\beta$, so $H \cap A$ is normalized by the element g in the proof of Lemma 5.19 that conjugates $H \cap N$ to a subgroup of $\mathrm{SO}(1,n)$. Thus, we conclude that g conjugates all of H to a subgroup of $\mathrm{SO}(1,n)$.

Case 4. Assume that $\dim(H \cap N) = 1$, and that $\phi \neq 0$ and $y \neq 0$ for every nontrivial element of $H \cap N$. We show that $\rho(h) \asymp \|h\|^{3/2}$ for every $h \in H$, so H is not a Cartan-decomposition subgroup.

Because $H \cap N$ projects nontrivially into both $\mathfrak{u}_\alpha$ and $\mathfrak{u}_\beta$, but does not intersect $\mathfrak{u}_\beta$, we must have $\alpha(a) = \beta(a)$ for every $a \in H \cap A$; that is, a is of the form $a = \operatorname{diag}(t^2, t, 1, \dots, 1, t^{-1}, t^{-2})$. Because $\dim(H \cap N) = 1$, there exist $\hat\phi \in \mathfrak{u}_\alpha$, $\hat x \in \mathfrak{u}_{\alpha+\beta}$, $\hat y \in \mathfrak{u}_\beta$, and $\hat\eta \in \mathfrak{u}_{\alpha+2\beta}$, such that $\mathfrak{h} \cap \mathfrak{n} = \mathbb{R}(\hat\phi + \hat x + \hat y + \hat\eta)$. However, because $\hat\phi + \hat x + \hat y + \hat\eta$ is an eigenvector for every element of $\ker(\alpha - \beta)$, but the restriction of $\ker(\alpha-\beta)$ to $\mathfrak{u}_{\alpha+\beta}$ or $\mathfrak{u}_{\alpha+2\beta}$ is different from the restriction to each of the other root spaces $\mathfrak{u}_\alpha$ and $\mathfrak{u}_\beta$, we see that $\hat x$ and $\hat\eta$ must both be zero. Thus, $\mathfrak{h} \cap \mathfrak{n} = \mathbb{R}(\hat\phi + \hat y)$. ¿From the proof of Lemma 5.20, we conclude that $\rho(h) \asymp \|h\|^{3/2}$, as desired.

Case 5. Assume that $\dim(H \cap N) = 1$ and that $\mathfrak{h} \cap \mathfrak{n}$ is not contained in any root space. We show that H is a Cartan-decomposition subgroup if and only if either

1) we have $\phi = 0$, $\eta = 0$, and $\dim\langle x, y\rangle = 1$ for every element of $\mathfrak{h} \cap \mathfrak{n}$; or
2) we have $y = 0$ and $\|x\|^2 = -2\phi\eta$ for every element of $H \cap N$.

We also show that if H is not a Cartan-decomposition subgroup, and neither Case 1, Case 2, nor Case 4 applies to H, then we have $\rho(h) \asymp h$ for every $h \in H$.

We may assume that $\phi \neq 0$, for some (and hence every) nonzero element of $\mathfrak{h} \cap \mathfrak{n}$, for otherwise Case 1 or 2 applies. Then we may assume that $y = 0$, for otherwise Case 4 applies.

If $\|x\|^2 = -2\phi\eta$, then Lemma 5.14 implies that $\rho(u) \asymp \|u\|^2$ for every $u \in H \cap N$. Furthermore, because $\beta(H \cap A) = 1$, we know that $\rho(a) \asymp a$ for every $a \in H \cap A$. We conclude that H is a Cartan-decomposition subgroup.

On the other hand, if $\|x\|^2 \neq -2\phi\eta$, it is not difficult to see that $\rho(h) \asymp h$ for every $h \in H$ (cf. Case 2 of ($3 \Rightarrow 2$) in the proof of Lemma 5.12). □

We now consider the case where H is not a semidirect product of the form $T \ltimes U$, with $T \subset A$ and $U \subset N$.

6.3. Theorem. *Assume that $G = \mathrm{SO}(2,n)$. Let H be a closed, connected subgroup of AN that is compatible with A (see 2.2). Assume that $H \neq (H \cap A) \ltimes (H \cap N)$ and that $H \cap N$ is not a Cartan-decomposition subgroup. The subgroup H is a Cartan-decomposition subgroup if and only if H is abelian and there exists $\omega \in \{\beta, \alpha+\beta\}$ such that $A \cap (HN) = \ker\omega$, $\mathfrak{h} \cap \mathfrak{u}_\omega \neq 0$, and $\mathfrak{h} \subset \mathfrak{t} + \mathfrak{u}_\omega + \mathfrak{u}_{\alpha+2\beta}$, where $\mathfrak{t}$ is the Lie algebra of $A \cap (HN)$.*

We now describe the Cartan projections of those subgroups that are not Cartan-decomposition subgroups.

6.4. Corollary (of proof). *Assume that $G = \mathrm{SO}(2,n)$. Let H be a closed, connected subgroup of G that is compatible with A (see 2.2), and assume that $H \neq (H \cap A) \ltimes (H \cap N)$. Then there is a positive root ω, and a one-dimensional subspace $\mathfrak{x}$ of $(\ker \omega) + \mathfrak{u}_\omega$, such that $\mathfrak{h} = \mathfrak{x} + (\mathfrak{h} \cap \mathfrak{n})$.*

If H is not a Cartan-decomposition subgroup of G, then either:

1) *$\omega = \alpha$ and $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_{\alpha+\beta}$, in which case $\mu(H) \approx \big[\|h\|, \|h\|^2/(\log \|h\|)\big]$; or*
2) *$\omega = \alpha$ and $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_{\alpha+2\beta}$, in which case $\mu(H) \approx \big[\|h\|^2/(\log \|h\|)^2, \|h\|^2\big]$; or*
3) *$\omega = \alpha+2\beta$ and $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_{\alpha}$, in which case $\mu(H) \approx \big[\|h\|^2/(\log \|h\|)^2, \|h\|^2\big]$; or*
4) *$\omega = \alpha+2\beta$ and either $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_{\beta}$ or $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_{\alpha+\beta}$, in which case $\mu(H) \approx \big[\|h\|, \|h\|^2/(\log \|h\|)\big]$; or*
5) *$\omega \in \{\beta, \alpha+\beta\}$, $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_\omega + \mathfrak{u}_{\alpha+2\beta}$, and $\mathfrak{h} \cap \mathfrak{u}_\omega = \mathfrak{h} \cap \mathfrak{u}_{\alpha+2\beta} = 0$, in which case $\mu(H) \approx \big[\|h\|, \|h\|^{3/2}\big]$; or*
6) *there is a root γ with $\{\omega, \gamma\} = \{\beta, \alpha+\beta\}$, $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_\gamma + \mathfrak{u}_{\alpha+2\beta}$, and $\mathfrak{h} \cap \mathfrak{u}_{\alpha+2\beta} = 0$, in which case $\mu(H) \approx \big[\|h\|, \|h\|(\log \|h\|)^2\big]$; or*
7) *$\omega \in \{\beta, \alpha+\beta\}$ and $\mathfrak{h} \cap \mathfrak{n} = \mathfrak{u}_{\alpha+2\beta}$, in which case $\mu(H) \approx \big[\|h\|(\log \|h\|), \|h\|^2\big]$; or*
8) *$\omega = \alpha+\beta$ and $\mathfrak{h} \cap \mathfrak{n} = \mathfrak{u}_{\alpha}$, in which case $\mu(H) \approx \big[\|h\|(\log \|h\|), \|h\|^2\big]$.*

Proof of Theorem 6.3. Let $T = A \cap (HN)$. There is a positive root ω with $\omega(T) = 1$, and a one-parameter subgroup W of U_ω, such that $\mathfrak{h} \subset \mathfrak{t} + \mathfrak{w} + (\mathfrak{h} \cap \mathfrak{n})$, where $\mathfrak{w}$ is the Lie algebra of W (see 2.4). Note that T and W are contained in the Zariski closure of H.

It is not difficult to see that every nonabelian subalgebra of $\mathfrak{n}$ contains $\mathfrak{u}_{\alpha+2\beta}$. Therefore, if $\mathfrak{h}$ does not contain $\mathfrak{u}_{\alpha+2\beta}$, then $\mathfrak{h} \cap \mathfrak{n}$ is abelian, and is centralized by W.

The proof proceeds in cases. If $\dim(H \cap N) \geq 2$, then Theorem 5.5 implies that one of Cases 1, 4, or 5 applies. If $\dim(H \cap N) = 1$, then there are many more possibilities to consider. Let us see that none of them have been overlooked. Because of Case 6, we may assume that $\mathfrak{h} \cap \mathfrak{n} \not\subset \mathfrak{u}_{\alpha+2\beta}$, so if $\phi = 0$ in $\mathfrak{h} \cap \mathfrak{n}$, then either Case 1 or Case 4 applies. Thus, we may assume that $\phi \neq 0$ in $\mathfrak{h} \cap \mathfrak{n}$. Because of Case 7, we may assume that $\mathfrak{h} \cap \mathfrak{n} \neq \mathfrak{u}_\alpha$, so $\mathfrak{h} \cap \mathfrak{n}$ is not normalized by A. This implies that $\omega = \beta$, and $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta}$, so Case 5 applies.

Case 1. Assume that $\mathfrak{h} \cap \mathfrak{n} \not\subset \mathfrak{u}_{\alpha+2\beta}$, and that we have $\phi = 0$ and $\dim\langle x, y\rangle \neq 1$, for every nonzero element of $\mathfrak{h} \cap \mathfrak{n}$. We show that this is impossible. Because $\mathfrak{h} \cap \mathfrak{n} \not\subset \mathfrak{u}_{\alpha+2\beta}$ and $\dim\langle x, y\rangle \neq 1$, there is some $h \in H$ with both x and y nonzero. Since T normalizes $H \cap N$, we must have $\alpha(T) = 1$ (that is, $\omega = \alpha$), so the Zariski closure of H contains U_α. We have $[h, U_\alpha] \subset U_{\alpha+\beta} U_{\alpha+2\beta}$, but, because $y \neq 0$, we know that $[h, U_\alpha] \not\subset U_{\alpha+2\beta}$. Thus, $\mathfrak{h} \cap \mathfrak{n}$ contains an element of the form $\hat{x} + \hat{\eta}$, with $\hat{x} \in \mathfrak{u}_{\alpha+\beta}$ and $\hat{\eta} \in \mathfrak{u}_{\alpha+2\beta}$, such that $\hat{x} \neq 0$. This contradicts the assumption of this case.

Case 2. Assume that $\omega = \alpha$. We show that H is not a Cartan-decomposition subgroup. Furthermore, $\mathfrak{h} \cap \mathfrak{n}$ is contained in either $\mathfrak{u}_{\alpha+\beta}$ or $\mathfrak{u}_{\alpha+2\beta}$, and $\mu(H) \approx \big[\|h\|, \|h\|^2/(\log \|h\|)^2\big]$ or $\mu(H) \approx \big[\|h\|^2/(\log \|h\|)^2, \|h\|^2\big]$, respectively.

Suppose that $\langle \mathfrak{w}, \mathfrak{h} \cap \mathfrak{n}\rangle$ is nonabelian. We must have $\mathfrak{u}_{\alpha+2\beta} \subset \mathfrak{h}$. Then, since $\langle \mathfrak{w}, \mathfrak{u}_{\alpha+2\beta}\rangle$ is abelian, we must have $\dim(\mathfrak{h} \cap \mathfrak{n}) \geq 2$.

- If Condition 5.5(2) holds, then Case 1 applies.
- Conditions 5.5(3) and 5.5(4) cannot hold, because $\mathfrak{u}_{\alpha+2\beta} \subset \mathfrak{h} \cap \mathfrak{n}$.

We conclude that $H \cap N$ is a Cartan-decomposition subgroup, which contradicts a hypothesis of the theorem.

We now know that $\langle \mathfrak{w}, \mathfrak{h} \cap \mathfrak{n}\rangle$ is abelian. We must have $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta}$, the centralizer of $\mathfrak{u}_\alpha$ in $\mathfrak{n}$. We know that $\ker \alpha$ normalizes $\mathfrak{h} \cap \mathfrak{n}$, that the restrictions of α, $\alpha+\beta$, and $\alpha+2\beta$ to $\ker \alpha$ are all distinct, that $\mathfrak{u}_\alpha \not\subset \mathfrak{h} \cap \mathfrak{n}$, and that $H \cap N$ is not a Cartan-decomposition subgroup, so we see from Proposition 5.5 that $\mathfrak{h} \cap \mathfrak{n}$ must be contained in either $\mathfrak{u}_{\alpha+\beta}$ or $\mathfrak{u}_{\alpha+2\beta}$.

Subcase 2.1. Assume that $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_{\alpha+\beta}$. We have

$$H = \left\{ \left. \begin{pmatrix} e^t & te^t\phi_0 & e^t x & 0 & -e^t\|x\|^2/2 \\ & e^t & 0 & 0 & 0 \\ & & \mathrm{Id} & 0 & -x^T \\ & & & e^{-t} & te^{-t}\phi_0 \\ & & & & e^{-t} \end{pmatrix} \right| t \in \mathbb{R}, x \in X_0 \right\},$$

where X_0 is a subspace of $\mathbb{R}^{n-2}$ and ϕ_0 is a nonzero real number.

We may assume that $t, \|x\| \geq 1$ (see 5.16 and 3.4). Then $h \asymp \max\{te^t, e^t\|x\|^2\}$ and $\rho(h) \asymp e^{2t}\|x\|^2$. Letting $t = 1$ yields $\rho(h) \asymp \|x\|^2 \asymp h$. The largest relative value of $\rho(h)$ is obtained by letting $t \asymp \|x\|^2$, which results in $\rho(h) \asymp te^{2t} \asymp \|h\|^2/(\log\|h\|)$.

Subcase 2.2. Assume that $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_{\alpha+2\beta}$. We have

$$H = \left\{ \left. \begin{pmatrix} e^t & te^t\phi_0 & 0 & e^t\eta & -te^t\phi_0\eta \\ & e^t & 0 & 0 & -e^t\eta \\ & & \mathrm{Id} & 0 & 0 \\ & & & e^{-t} & te^{-t}\phi_0 \\ & & & & e^{-t} \end{pmatrix} \right| t, \eta \in \mathbb{R} \right\},$$

where ϕ_0 is a nonzero real number.

Assuming $t, |\eta| \geq 1$, we have $h \asymp te^t|\eta|$ and $\rho(h) \asymp e^{2t}\eta^2$. (In calculating $\|\rho(h)\|$, one must note that $\det\begin{pmatrix} h_{1,2} & h_{1,n+2} \\ h_{2,2} & h_{2,n+2}\end{pmatrix} = 0$.) Letting $t = 1$ yields $\rho(h) \asymp \eta^2 \asymp \|h\|^2$. The smallest relative value of $\rho(h)$ is obtained by letting $\eta = 1$, which results in $\rho(h) \asymp e^{2t} \asymp \|h\|^2/(\log\|h\|)^2$.

Case 3. Assume that $\omega = \alpha + 2\beta$. Since the restrictions of α, β, $\alpha+\beta$, and $\alpha+2\beta$ to $\ker(\alpha+2\beta)$ are all distinct, and $H \cap N$ is not a Cartan-decomposition subgroup, we see from Proposition 5.5 that $\mathfrak{h} \cap \mathfrak{n}$ must be contained in either $\mathfrak{u}_\alpha$, $\mathfrak{u}_\beta$, or $\mathfrak{u}_{\alpha+\beta}$. Because the Weyl reflection w_α interchanges $\mathfrak{u}_\beta$ and $\mathfrak{u}_{\alpha+\beta}$, we may assume that $\mathfrak{h} \cap \mathfrak{n}$ is contained in either $\mathfrak{u}_\alpha$ or $\mathfrak{u}_\beta$.

Subcase 3.1. Assume that $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_\alpha$. We have

$$H = \left\{ \left. \begin{pmatrix} e^t & e^t\phi & 0 & te^t\eta_0 & -te^t\phi\eta_0 \\ & e^{-t} & 0 & 0 & -te^{-t}\eta_0 \\ & & \mathrm{Id} & 0 & 0 \\ & & & e^t & -e^t\phi \\ & & & & e^{-t} \end{pmatrix} \right| t, \phi \in \mathbb{R} \right\},$$

where η_0 is a nonzero real number.

Assuming that $t, |\phi| \geq 1$, we have $h \asymp te^t\phi$ and $\rho(h) \asymp e^{2t}\phi^2$. (Note that $\det\begin{pmatrix} h_{1,2} & h_{1,n+2} \\ h_{2,2} & h_{2,n+2}\end{pmatrix} = 0$.) Letting $t = 1$ yields $\rho(h) \asymp \phi^2 \asymp \|h\|^2$. The smallest relative value of $\rho(h)$ is obtained by letting $\phi \asymp 1$, which results in $\rho(h) \asymp e^{2t} \asymp \|h\|^2/(\log\|h\|)^2$.

Subcase 3.2. Assume that $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_\beta$. We have

$$H = \left\{ \left. \begin{pmatrix} e^t & 0 & e^t x & te^t\eta_0 & -e^t\|x\|^2/2 \\ & e^{-t} & 0 & 0 & -te^{-t}\eta_0 \\ & & \mathrm{Id} & 0 & -x \\ & & & e^t & 0 \\ & & & & e^{-t} \end{pmatrix} \right| t \in \mathbb{R}, x \in X_0 \right\},$$

where X_0 is a subspace of $\mathbb{R}^{n-2}$, and η_0 is a nonzero real number.

Assuming $t, \|x\| \geq 1$, we have $h \asymp \max\{e^t\|x\|^2, te^t\}$ and $\rho(h) \asymp e^{2t}\|x\|^2$. Letting $t = 1$ yields $\rho(h) \asymp \|x\|^2 \asymp h$. The largest relative value of $\rho(h)$ is obtained by letting $t \asymp \|x\|^2$, which results in $\rho(h) \asymp te^{2t} \asymp \|h\|^2/(\log\|h\|)$.

Case 4. Assume that we have $\phi = 0$ and $\dim\langle x, y\rangle = 1$, for every nonzero element of $\mathfrak{h}\cap\mathfrak{n}$. Because $\dim\langle x, y\rangle = 1$, we know that $\mathfrak{h}$ does not contain $\mathfrak{u}_{\alpha+2\beta}$. Therefore, $\mathfrak{h}\cap\mathfrak{n}$ is abelian, and is centralized by the one-parameter subgroup W of $\mathfrak{u}_\omega$.

We may assume that $\omega \in \{\beta, \alpha+\beta\}$, for otherwise Case 2 or 3 applies.

Note that either $x = 0$ for every $h \in H \cap N$, or $y = 0$ for every $h \in H \cap N$. (If there is some $h \in H \cap N$ with both x and y nonzero, then, because $\omega \neq \alpha$, both $\mathfrak{h}\cap\mathfrak{u}_\beta$ and $\mathfrak{h}\cap\mathfrak{u}_{\alpha+\beta}$ must be nonzero. Then, since $\dim\langle x, y\rangle = 1$, we see that $\mathfrak{h}\cap\mathfrak{n}$ is not abelian, which is a contradiction.) Because the Weyl reflection corresponding to the root α fixes $\mathfrak{u}_{\alpha+2\beta}$ and interchanges $\mathfrak{u}_\beta$ with $\mathfrak{u}_{\alpha+\beta}$, we may assume that $y = 0$ for every $h \in H \cap N$. Thus, $\mathfrak{h}\cap\mathfrak{n} \subset \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta}$.

Subcase 4.1. Assume that $\omega = \alpha+\beta$. We have

$$H = \left\{ \left. \begin{pmatrix} 1 & 0 & tx_0 + x & b_0 \cdot x & -\|tx_0 + x\|^2/2 \\ & e^t & 0 & 0 & -e^t b_0 \cdot x \\ & & \mathrm{Id} & 0 & -(tx_0 + x)^T \\ & & & e^{-t} & 0 \\ & & & & 1 \end{pmatrix} \right| \, t \in \mathbb{R}, x \in X_0 \right\},$$

where X_0 is a subspace of $\mathbb{R}^{n-2}$, and b_0 and x_0 are fixed vectors in $\mathbb{R}^{n-2}$ with $x_0 \notin X_0$.

Because $x_0 \notin X_0$, we have $\|tx_0 + x\| \asymp \max\{|t|, \|x\|\}$. Assuming that $t, \|x\| \geq 1$, we have $h \asymp \max\{e^t, \|x\|^2, e^t|b_0 \cdot x|\}$ and $\rho(h) \asymp \max\{t^2e^t, e^t\|x\|^2\}$. Letting $t = 1$ yields $\rho(h) \asymp \|x\|^2 \asymp h$.

If there is some nonzero $x_1 \in X_0$ with $b_0 \cdot x_1 = 0$, then letting $x = x_1$ and $e^t = \|x\|^2$ yields $\rho(h) \asymp \|x\|^4 \asymp \|h\|^2$, so H is a Cartan-decomposition subgroup.

On the other hand, if we have $b_0 \cdot x \neq 0$ for every nonzero $x \in X_0$, then $b_0 \cdot x \asymp x$ for all $x \in X_0$. The largest relative value of $\rho(h)$ is obtained by letting $e^t \asymp \|x\|$, which results in $\rho(h) \asymp e^{3t} \asymp \|h\|^{3/2}$.

Subcase 4.2. Assume that $\omega = \beta$. We have

$$H = \left\{ \left. \begin{pmatrix} e^t & 0 & e^t x & e^t b_0 \cdot x & -e^t\|x\|^2/2 \\ & 1 & ty_0 & -t^2\|y_0\|^2/2 & -b_0 \cdot x \\ & & \mathrm{Id} & -ty_0^T & -x^T \\ & & & 1 & 0 \\ & & & & e^{-t} \end{pmatrix} \right| \, t \in \mathbb{R}, x \in X_0 \right\},$$

where X_0 is a subspace of $\mathbb{R}^{n-2}$, and b_0 and y_0 are fixed vectors in $\mathbb{R}^{n-2}$ with $y_0 \neq 0$ and $y_0 \perp X_0$.

Assuming $t, \|x\|$ are sufficiently large, we have $h \asymp e^t\|x\|^2$ and $\rho(h) \asymp t^2e^t\|x\|^2$. Letting $t = 1$ yields $\rho(h) \asymp \|x\|^2 \asymp h$. The largest relative value of $\rho(h)$ is obtained by letting $\|x\| \asymp 1$, which results in $\rho(h) \asymp t^2e^t \asymp \|h\|(\log\|h\|)^2$.

Case 5. Assume that $\mathfrak{h}\cap\mathfrak{n} \subset \mathfrak{u}_\alpha + \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta}$, that $\mathfrak{h}\cap\mathfrak{n} \not\subset \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta}$, that $\mathfrak{u}_{\alpha+2\beta} \not\subset \mathfrak{h}\cap\mathfrak{n}$, and that $\mathfrak{h}\cap\mathfrak{n}$ is not normalized by A. Because $\mathfrak{h}\cap\mathfrak{n}$ is not normalized by A, but is normalized by T, we must have $\beta(T) = 1$, so $\omega = \beta$. Also, because $\mathfrak{u}_{\alpha+2\beta} \not\subset \mathfrak{h}\cap\mathfrak{n}$, we know that $\mathfrak{h}\cap\mathfrak{n}$ is centralized by $\mathfrak{w}$. Because the centralizer of $\mathfrak{w}$ (or any other nontrivial subspace of $\mathfrak{u}_\beta$) projects trivially into $\mathfrak{u}_\alpha$, this is a contradiction.

Case 6. Assume that $\mathfrak{h}\cap\mathfrak{n} = \mathfrak{u}_{\alpha+2\beta}$. We may assume that $\omega \in \{\beta, \alpha+\beta\}$, for otherwise $\omega = \alpha$, so Case 2 applies. Then, because the Weyl reflection w_α interchanges β and $\alpha+\beta$, but fixes $\alpha+2\beta$, we may assume that $\omega = \alpha + \beta$. We have

$$H = \left\{ \left. \begin{pmatrix} 1 & 0 & tx_0 & \eta & -t^2\|x_0\|^2/2 \\ & e^t & 0 & 0 & -e^t\eta \\ & & \mathrm{Id} & 0 & -tx_0^T \\ & & & e^{-t} & 0 \\ & & & & 1 \end{pmatrix} \,\right|\, t, \eta \in \mathbb{R} \right\},$$

where x_0 is a nonzero vector in $\mathbb{R}^{n-2}$. Assuming $t, |\eta| \geq 1$, we have $h \asymp e^t|\eta|$ and $\rho(h) \asymp \max\{t^2e^t, e^t\eta^2\}$. Letting $t = 1$ yields $\rho(h) \asymp \eta^2 \asymp \|h\|^2$. The smallest relative value of $\rho(h)$ is obtained by letting $\eta \asymp t$, which results in $\rho(h) \asymp t^2e^t \asymp \|h\|(\log \|h\|)$.

Case 7. Assume that $\mathfrak{h} \cap \mathfrak{n} = \mathfrak{u}_\alpha$. We may assume that $\omega \in \{\beta, \alpha+\beta\}$, for otherwise $\omega = \alpha + 2\beta$, so Case 3 applies. Since $\mathfrak{u}_{\alpha+2\beta} \not\subset \mathfrak{h} \cap \mathfrak{n}$, we know that $\mathfrak{w}$ centralizes $\mathfrak{h} \cap \mathfrak{n}$. Then, because $\mathfrak{u}_\alpha$ does not centralize any nontrivial subspace of $\mathfrak{u}_\beta$, we conclude that $\omega = \alpha + \beta$. We have

$$H = \left\{ \left. \begin{pmatrix} 1 & \phi & tx_0 & 0 & -t^2\|x_0\|^2/2 \\ & e^t & 0 & 0 & 0 \\ & & \mathrm{Id} & 0 & -tx_0^T \\ & & & e^{-t} & -e^{-t}\phi \\ & & & & 1 \end{pmatrix} \,\right|\, t, \phi \in \mathbb{R} \right\},$$

where x_0 is a nonzero vector in $\mathbb{R}^{n-2}$. Assuming $t, |\phi| \geq 1$, we have $h \asymp \max\{e^t, |\phi|\}$ and $\rho(h) \asymp \max\{\phi^2/e^t, t^2e^t\}$. Letting $t = 1$ yields $\rho(h) \asymp \phi^2 \asymp \|h\|^2$. The smallest relative value of $\rho(h)$ is obtained by letting $\phi \asymp te^t$, which results in $\rho(h) \asymp t^2e^t \asymp \|h\| \log \|h\|$. □

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