COMPACT CLIFFORD-KLEIN FORMS OF HOMOGENEOUS SPACES OF $\mathrm{SO}(2,n)$

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ABSTRACT. A homogeneous space G/H is said to have a compact Clifford-Klein form if there exists a discrete subgroup Γ of G that acts properly discontinuously on G/H, such that the quotient space $\Gamma\backslash G/H$ is compact. When n is even, we find every closed, connected subgroup H of $G = \mathrm{SO}(2,n)$, such that G/H has a compact Clifford-Klein form, but our classification is not quite complete when n is odd. The work reveals new examples of homogeneous spaces of $\mathrm{SO}(2,n)$ that have compact Clifford-Klein forms, if n is even. Furthermore, we show that if H is a closed, connected subgroup of $G = \mathrm{SL}(3,\mathbb{R})$, and neither H nor G/H is compact, then G/H does not have a compact Clifford-Klein form, and we also study noncompact Clifford-Klein forms of finite volume.

1. INTRODUCTION

1.1. Assumption. Throughout this paper, G is a Zariski-connected, almost simple, linear Lie group ("Almost simple" means that every proper normal subgroup of G either is finite or has finite index.) In the main results, G is assumed to be $\mathrm{SO}(2,n)$ (with $n \geq 3$).

There would be no essential loss of generality if one were to require G to be connected, instead of only Zariski connected (see 3.10). However, $\mathrm{SO}(2,n)$ is not connected (it has two components) and the authors prefer to state results for $\mathrm{SO}(2,n)$, instead of for the identity component of $\mathrm{SO}(2,n)$.

1.2. Definition. Let H be a closed, connected subgroup of G. We say that the homogeneous space G/H has a *compact Clifford-Klein form* if there is a discrete subgroup Γ of G, such that

- Γ acts properly on G/H; and
- $\Gamma\backslash G/H$ is compact.

(Alternative terminology would be to say that G/H has a *tessellation*, because the Γ-translates of a fundamental domain for $\Gamma\backslash G/H$ tessellate G/H, or one could simply say that G/H has a *compact quotient.*) See the surveys [Kb5] and [Lab] for references to some of the previous work on the existence of compact Clifford-Klein forms.

We determine exactly which homogeneous spaces of $\mathrm{SO}(2,n)$ have a compact Clifford-Klein form in the case where n is even (see 1.7), and we have almost complete results in the case where n is odd (see 1.9). (We only consider homogeneous spaces G/H in which H is connected.) The work leads to new examples of homogeneous spaces that have compact Clifford-Klein forms, if n is even (see 1.5). We also show that only the obvious homogeneous spaces of $\mathrm{SL}(3,\mathbb{R})$ have compact Clifford-Klein forms (see 1.10), and we study noncompact Clifford-Klein forms of finite volume (see §6).

1.3. Notation. We realize $\mathrm{SO}(2,n)$ as isometries of the indefinite form $\langle v \mid v\rangle = v_1v_{n+2}+v_2v_{n+1}+\sum_{i=3}^{n} v_i^2$ on $\mathbb{R}^{n+2}$ (for $v = (v_1,v_2,\ldots,v_{n+2}) \in \mathbb{R}^{n+2}$). Let A be the subgroup consisting of the diagonal matrices in $\mathrm{SO}(2,n)$ whose diagonal entries are all positive, and let N be the subgroup consisting of the upper-triangular matrices in $\mathrm{SO}(2,n)$ with only 1's on the diagonal. Thus, the

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Lie algebra of AN is

$$\mathfrak{a}+\mathfrak{n} = \left\{ \left. \begin{pmatrix} t_1 & \phi & x & \eta & 0 \\ & t_2 & y & 0 & -\eta \\ & & 0 & -y^T & -x^T \\ & & & -t_2 & -\phi \\ & & & & -t_1 \end{pmatrix} \right| \, t_1, t_2, \phi, \eta \in \mathbb{R}, x, y \in \mathbb{R}^{n-2} \right\}. \tag{1.4}$$

Note that the first two rows of any element of $\mathfrak{a}+\mathfrak{n}$ are sufficient to determine the entire matrix.

Let us recall a construction of compact Clifford-Klein forms found by Kulkarni [Kul, Thm. 6.1] (see also [Kb1, Prop. 4.9]). The subgroup $\mathrm{SU}(1,m)$, embedded into $\mathrm{SO}(2,2m)$ in a standard way, acts properly and transitively on the homogeneous space $\mathrm{SO}(2,2m)/\mathrm{SO}(1,2m)$. Therefore, any co-compact lattice Γ in $\mathrm{SO}(1,2m)$ acts properly on $\mathrm{SO}(2,2m)/\mathrm{SU}(1,m)$, and the quotient $\Gamma\backslash \mathrm{SO}(2,2m)/\mathrm{SU}(1,m)$ is compact. Now let $H_{\mathrm{SU}} = \mathrm{SU}(1,m) \cap (AN)$. Then the Clifford-Klein form $\Gamma\backslash \mathrm{SO}(2,2m)/H_{\mathrm{SU}}$ is also compact, since $\mathrm{SU}(1,m)/H_{\mathrm{SU}}$ is compact. (Similarly, Kulkarni also constructed compact Clifford-Klein forms $\Lambda\backslash \mathrm{SO}(2,2m)/\mathrm{SO}(1,2m)$, by letting Λ be a co-compact lattice in $\mathrm{SU}(1,m)$.)

The following theorem demonstrates how to construct new examples of compact Clifford-Klein forms $\Gamma\backslash \mathrm{SO}(2,2m)/H_B$. The subgroup H_B of $\mathrm{SO}(2,2m)$ is obtained by deforming H_{SU}, but H_B is almost never contained in any conjugate of $\mathrm{SU}(1,m)$.

1.5. Theorem. *Assume that $G = \mathrm{SO}(2,2m)$. Let $B\colon \mathbb{R}^{2m-2} \to \mathbb{R}^{2m-2}$ be a linear transformation that has no real eigenvalue. Set*

$$\mathfrak{h}_B = \left\{ \left. \begin{pmatrix} t & 0 & x & \eta & 0 \\ & t & B(x) & 0 & -\eta \\ & & \cdots & & \end{pmatrix} \right| \, x \in \mathbb{R}^{2m-2}, t, \eta \in \mathbb{R} \right\}, \tag{1.6}$$

let H_B be the corresponding closed, connected subgroup of G, and let Γ be a co-compact lattice in $\mathrm{SO}(1,2m)$. Then

1) *the subgroup Γ acts properly on $\mathrm{SO}(2,2m)/H_B$;*
2) *the quotient $\Gamma\backslash \mathrm{SO}(2,2m)/H_B$ is compact; and*
3) *H_B is conjugate via $\mathrm{O}(2,2m)$ to a subgroup of $\mathrm{SU}(1,m)$ if and only if for some $a,b \in \mathbb{R}$ (with $b \neq 0$), the matrix of B with respect to some orthonormal basis of $\mathbb{R}^{2m-2}$ is a block diagonal matrix each of whose blocks is $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.*

Furthermore, by varying B, one can obtain uncountably many pairwise nonconjugate subgroups.

We recall that T. Kobayashi [Kb6, Thm. B] showed that a co-compact lattice in $\mathrm{SU}(1,m)$ can be deformed to a discrete subgroup Λ, such that Λ acts properly on $\mathrm{SO}(2,2m)/\mathrm{SO}(1,2m)$ and the quotient space $\Lambda\backslash \mathrm{SO}(2,2m)/\mathrm{SO}(1,2m)$ is compact, but Λ is not contained in any conjugate of $\mathrm{SU}(1,m)$. (This example is part of an extension of work of W. Goldman [Gol].) Note that Kobayashi created new compact Clifford-Klein forms by deforming the discrete group while keeping the homogeneous space $\mathrm{SO}(2,2m)/\mathrm{SO}(1,2m)$ fixed. In contrast, we deform the homogeneous space $\mathrm{SO}(2,2m)/H_{\mathrm{SU}}$ to another homogeneous space $\mathrm{SO}(2,2m)/H_B$ while keeping the discrete group Γ in $\mathrm{SO}(1,2m)$ fixed.

For even n, we show that the Kulkarni examples and our deformations are essentially the only interesting homogeneous spaces of $\mathrm{SO}(2,n)$ that have compact Clifford-Klein forms. We assume that $H \subset AN$, because the general case reduces to this (see 3.8).

1.7. Theorem (cf. Thm. 5.7). *Assume that $G = \mathrm{SO}(2,2m)$. Let H be a closed, connected subgroup of AN, such that neither H nor G/H is compact. The homogeneous space G/H has a compact Clifford-Klein form if and only if either*

1) *H is conjugate to a co-compact subgroup of* $\mathrm{SO}(1,2m)$*; or*
2) *H is conjugate to H_B, for some B, as described in Theorem* 1.5.

It is conjectured [Kb6, 1.4] that if H is reductive and G/H has a compact Clifford-Klein form, then there exists a reductive subgroup L of G, such that L acts properly on G/H, and the double-coset space $L\backslash G/H$ is compact. Because there is no such subgroup L in the case where $G = \mathrm{SO}(2,2m+1)$ and $H = \mathrm{SU}(1,m)$ (see 5.10), the following is a special case of the general conjecture.

1.8. Conjecture. For $m \geq 1$, the homogeneous space $\mathrm{SO}(2,2m+1)/\mathrm{SU}(1,m)$ does not have a compact Clifford-Klein form.

If this conjecture is true, then, for odd n, there is no interesting example of a homogeneous space of $\mathrm{SO}(2,n)$ that has a compact Clifford-Klein form.

1.9. Theorem. *Assume that $G = \mathrm{SO}(2,2m+1)$, and that $G/\mathrm{SU}(1,m)$ does not have a compact Clifford-Klein form. If H is a closed, connected subgroup of G, such that neither H nor G/H is compact, then G/H does not have a compact Clifford-Klein form.*

The main results of [OW] list the homogeneous spaces of $\mathrm{SO}(2,n)$ that admit a proper action of a noncompact subgroup of $\mathrm{SO}(2,n)$ (see §2). Our proofs of Theorems 1.7 and 1.9 consist of case-by-case analysis to decide whether each of these homogeneous spaces has a compact Clifford-Klein form. The following proposition does not require such a detailed analysis, but is obtained easily by combining theorems of Y. Benoist (see 7.1) and G. A. Margulis (see 3.6).

1.10. Proposition. *Let H be a closed, connected subgroup of $G = \mathrm{SL}(3,\mathbb{R})$. If G/H has a compact Clifford-Klein form, then H is either compact or co-compact.*

The paper is organized as follows. Section 2 recalls some definitions and results, mostly from [OW]. Section 3 presents some general results on Clifford-Klein forms. Section 4 proves Theorem 1.5, the new examples of compact Clifford-Klein forms. Section 5 proves Theorems 1.7 and 1.9, the classification of compact Clifford-Klein forms. Section 6 discusses noncompact Clifford-Klein forms of finite volume. Section 7 briefly discusses Clifford-Klein forms of homogeneous spaces of $\mathrm{SL}(3,\mathbb{R})$, proving Proposition 1.10. An appendix presents a short proof of a theorem of Benoist.

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2. Cartan projections of subgroups of $\mathrm{SO}(2,n)$

Our study of the compact Clifford-Klein forms of homogeneous spaces of $\mathrm{SO}(2,n)$ is based on a case-by-case analysis of the subgroups of $\mathrm{SO}(2,n)$ that are not Cartan-decomposition subgroups (see Defn. 2.1). We need to know not only what the subgroups are, but also the image of each subgroup under the Cartan projection (see Defn. 2.3). This information is provided by [OW]. In this section, we recall the relevant results, notation, and definitions. (However, we have omitted a few of the more well-known definitions that appear in [OW].)

2.1. Definition ([OW, Defn. 1.2]). Let H be a closed, connected subgroup of G. We say that H is a *Cartan-decomposition subgroup* of G if there is a compact subset C of G, such that $CHC = G$. (Note that C is only assumed to be a sub*set* of G; it need not be a sub*group*.)

2.2. Notation. Fix an Iwasawa decomposition $G = KAN$ and a corresponding Cartan decomposition $G = KA^+K$, where A^+ is the (closed) positive Weyl chamber of A in which the roots occurring in the Lie algebra of N are positive. Thus, K is a maximal compact subgroup, A is the identity component of a maximal split torus, and N is a maximal unipotent subgroup.

2.3. Definition (Cartan projection)**.** For each element g of G, the Cartan decomposition $G = KA^+K$ implies that there is an element a of A^+ with $g \in KaK$. In fact, the element a is unique, so there is a well-defined function $\mu\colon G \to A^+$ given by $g \in K\,\mu(g)\,K$. The function μ is continuous and proper (that is, the inverse image of any compact set is compact). Some properties of the Cartan projection are discussed in [Ben], [Kb5], and [OW].

2.4. Notation. For subsets $X, Y \subset A^+$, we write $X \approx Y$ if there is a compact subset C of A with $X \subset YC$ and $Y \subset XC$.

2.5. Notation. Define a representation

$$\rho\colon\ \mathrm{SO}(2,n) \to \mathrm{SL}\big(\mathbb{R}^{n+2} \wedge \mathbb{R}^{n+2}\big) \text{ by } \rho(g) = g \wedge g.$$

For functions $f_1, f_2\colon \mathbb{R}^+ \to \mathbb{R}^+$, and a subgroup H of $\mathrm{SO}(2,n)$, we write $\mu(H) \approx \big[f_1(\|h\|), f_2(\|h\|)\big]$ if, for every sufficiently large $C > 1$, we have

$$\mu(H) \approx \big\{\, a \in A^+ \;\big|\; C^{-1} f_1\big(\|a\|\big) \le \|\rho(a)\| \le C f_2\big(\|a\|\big) \,\big\}\,,$$

where $\|h\|$ denotes the norm of the linear transformation h. (If f_1 and f_2 are monomials, or other very tame functions, then it does not matter which particular norm is used, because all norms are equivalent up to a bounded factor.) In particular, H is a Cartan-decomposition subgroup of $\mathrm{SO}(2,n)$ if and only if $\mu(H) \approx \big[\|h\|, \|h\|^2\big]$ (see [OW, Lem. 5.7]).

2.6. Remark. Let H and H' be closed, connected subgroups of $\mathrm{SO}(2,n)$. Suppose that

$$f_1, f_2, f_1', f_2'\colon \mathbb{R}^+ \to \mathbb{R}^+$$

satisfy

$$\mu(H) \approx \big[f_1(\|h\|), f_2(\|h\|)\big] \qquad \text{and} \qquad \mu(H') \approx \big[f_1'(\|h\|), f_2'(\|h\|)\big].$$

If, for all large $t \in \mathbb{R}^+$, we have $f_1(t) \le f_1'(t) \le f_2'(t) \le f_2(t)$, then there is a compact subset C of G, such that $H' \subset CHC$.

2.7. Notation (cf. 1.4)**.** Assume that $G = \mathrm{SO}(2,n)$. For every $h \in \mathfrak{n}$, there exist unique $\phi_h, \eta_h \in \mathbb{R}$ and $x_h, y_h \in \mathbb{R}^{n-2}$, such that

$$h = \begin{pmatrix} 0 & \phi_h & x_h & \eta_h & 0 \\ & 0 & y_h & 0 & -\eta_h \\ & & \cdots & & \end{pmatrix}.$$

2.8. Notation. We let α and β be the simple real roots of $\mathrm{SO}(2,n)$, defined by $\alpha(a) = a_1/a_2$ and $\beta(a) = a_2$, for an element a of A of the form

$$a = \mathrm{diag}(a_1, a_2, 1, 1, \ldots, 1, 1, a_2^{-1}, a_1^{-1}).$$

Thus,

- the root space $\mathfrak{u}_\alpha$ is the ϕ-axis in $\mathfrak{n}$,
- the root space $\mathfrak{u}_\beta$ is the y-subspace in $\mathfrak{n}$,
- the root space $\mathfrak{u}_{\alpha+\beta}$ is the x-subspace in $\mathfrak{n}$, and
- the root space $\mathfrak{u}_{\alpha+2\beta}$ is the η-axis in $\mathfrak{n}$.

We now reproduce a string of results from [OW] that describe the subgroups of $\mathrm{SO}(2,n)$ that are not Cartan-decomposition subgroups, and also describe the image of each subgroup under the Cartan projection.

2.9. Theorem ([OW, Thm. 5.5 and Prop. 5.8]). *Assume that $G = \mathrm{SO}(2,n)$. A closed, connected subgroup H of N is not a Cartan-decomposition subgroup of G if and only if either*

1) $\dim H \le 1$, *in which case $\mu(H) \approx \big[\|h\|^p, \|h\|^p\big]$, for some $p \in \{0, 1, 3/2, 2\}$; or*
2) *for every nonzero element h of $\mathfrak{h}$, we have $\phi_h = 0$ and $\dim\langle x_h, y_h\rangle \neq 1$, in which case $\mu(H) \approx \big[\|h\|^2, \|h\|^2\big]$; or*
3) *for every nonzero element h of $\mathfrak{h}$, we have $\phi_h = 0$ and $\dim\langle x_h, y_h\rangle = 1$, in which case $\mu(H) \approx \big[\|h\|, \|h\|\big]$; or*
4) *there exists a subspace X_0 of $\mathbb{R}^{n-2}$, $b \in X_0$, $c \in X_0^\perp$, and $p \in \mathbb{R}$ with $\|b\|^2 - \|c\|^2 - 2p < 0$, such that for every element of $\mathfrak{h}$, we have $y_h = 0$, $x_h \in \phi_h c + X_0$, and $\eta_h = p\phi_h + b \cdot x_h$ (where $b \cdot x_h$ denotes the Euclidean dot product of the vectors b and x_h in $\mathbb{R}^{n-2}$), in which case $\mu(H) \approx \big[\|h\|, \|h\|\big]$.*

2.10. Theorem ([OW, Cor. 6.2]). *Assume that $G = \mathrm{SO}(2,n)$. Let H be a closed, connected subgroup of AN, such that $H = (H \cap A) \ltimes (H \cap N)$, and $H \not\subset N$. The subgroup H is not a Cartan-decomposition subgroup of G if and only if either*

1) *$H = H \cap A$ is a one-dimensional subgroup of A; or*
2) *$H \cap A = \ker \alpha$, and we have $\phi_h = 0$ and $\dim\langle x_h, y_h\rangle \neq 1$ for every nonzero element h of $\mathfrak{h} \cap \mathfrak{n}$, in which case $\mu(H) \approx \big[\|h\|^2, \|h\|^2\big]$; or*
3) *$H \cap A = \ker \beta$, and we have $\phi_h = 0$, $y_h = 0$, and $x_h \neq 0$ for every nonzero element h of $\mathfrak{h} \cap \mathfrak{n}$, in which case $\mu(H) \approx \big[\|h\|, \|h\|\big]$; or*
4) *$H \cap A = \ker(\alpha + \beta)$, and we have $\phi_h = 0$, $x_h = 0$, and $y_h \neq 0$, for every nonzero element h of $\mathfrak{h} \cap \mathfrak{n}$, in which case $\mu(H) \approx \big[\|h\|, \|h\|\big]$; or*
5) *$H \cap A = \ker \beta$, and there exist a subspace X_0 of $\mathbb{R}^{n-2}$, $b \in X_0$, $c \in X_0^\perp$, and $p \in \mathbb{R}$, such that $\|b\|^2 - \|c\|^2 - 2p < 0$, and we have $y_h = 0$, $x_h \in \phi_h c + X_0$, and $\eta_h = p\phi + b \cdot x_h$ for every $h \in \mathfrak{h} \cap \mathfrak{n}$, in which case $\mu(H) \approx \big[\|h\|, \|h\|\big]$; or*
6) *$H \cap A = \ker(\alpha - \beta)$, $\dim H = 2$, and there are $\hat\phi \in \mathfrak{u}_\alpha$ and $\hat y \in \mathfrak{u}_\beta$, such that $\hat\phi \neq 0$, $\hat y \neq 0$, and $\mathfrak{h} \cap \mathfrak{n} = \mathbb{R}(\hat\phi + \hat y)$, in which case $\mu(H) \approx \big[\|h\|^{3/2}, \|h\|^{3/2}\big]$; or*
7) *$H \cap A = \ker \beta$, $\dim H = 2$, and we have $y_h = 0$ and $\|x_h\|^2 \neq -2\phi_h\eta_h$ for every $h \in H$, in which case $\mu(H) \approx \big[\|h\|, \|h\|\big]$; or*
8) *there is a positive root ω and a one-dimensional subspace $\mathfrak{t}$ of $\mathfrak{a}$, such that $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_\omega$, $\mathfrak{h} = \mathfrak{t} + (\mathfrak{h} \cap \mathfrak{n})$, and Proposition 2.14 implies that H is not a Cartan-decomposition subgroup.*

Not every connected subgroup of AN is conjugate to a subgroup of the form $T \ltimes U$, where $T \subset A$ and $U \subset N$. The following definition and lemma describe how close we can come to this ideal situation.

2.11. Definition. Let us say that a subgroup H of AN is *compatible* with A if $H \subset TUC_N(T)$, where $T = A \cap (HN)$, $U = H \cap N$, and $C_N(T)$ denotes the centralizer of T in N.

2.12. Lemma ([OW, Lem. 2.3]). *If H is a closed, connected subgroup of AN, then H is conjugate, via an element of N, to a subgroup that is compatible with A.*

2.13. Theorem ([OW, Cor. 6.4]). *Assume that $G = \mathrm{SO}(2,n)$. Let H be a closed, connected subgroup of AN that is compatible with A (see 2.11), and assume that $H \neq (H \cap A) \ltimes (H \cap N)$. Then there is a positive root ω, and a one-dimensional subspace $\mathfrak{x}$ of $(\ker\omega) + \mathfrak{u}_\omega$, such that $\mathfrak{h} = \mathfrak{x} + (\mathfrak{h} \cap \mathfrak{n})$.*

If H is not a Cartan-decomposition subgroup of G, then either:

1) *$\omega = \alpha$ and $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_{\alpha+\beta}$, in which case $\mu(H) \approx \big[\|h\|, \|h\|^2/(\log\|h\|)\big]$; or*
2) *$\omega = \alpha$ and $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_{\alpha+2\beta}$, in which case $\mu(H) \approx \big[\|h\|^2/(\log\|h\|)^2, \|h\|^2\big]$; or*
3) *$\omega = \alpha + 2\beta$ and $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_\alpha$, in which case $\mu(H) \approx \big[\|h\|^2/(\log\|h\|)^2, \|h\|^2\big]$; or*
4) *$\omega = \alpha + 2\beta$ and either $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_\beta$ or $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_{\alpha+\beta}$, in which case $\mu(H) \approx \big[\|h\|, \|h\|^2/(\log\|h\|)\big]$; or*

5) *$\omega \in \{\beta, \alpha+\beta\}$, $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_\omega + \mathfrak{u}_{\alpha+2\beta}$, and $\mathfrak{h} \cap \mathfrak{u}_\omega = \mathfrak{h} \cap \mathfrak{u}_{\alpha+2\beta} = 0$, in which case $\mu(H) \approx \big[\|h\|, \|h\|^{3/2}\big]$; or*
6) *there is a root γ with $\{\omega, \gamma\} = \{\beta, \alpha+\beta\}$, $\mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_\gamma + \mathfrak{u}_{\alpha+2\beta}$, and $\mathfrak{h} \cap \mathfrak{u}_{\alpha+2\beta} = 0$, in which case $\mu(H) \approx \big[\|h\|, \|h\|(\log\|h\|)^2\big]$; or*
7) *$\omega \in \{\beta, \alpha+\beta\}$ and $\mathfrak{h} \cap \mathfrak{n} = \mathfrak{u}_{\alpha+2\beta}$, in which case $\mu(H) \approx \big[\|h\|(\log\|h\|), \|h\|^2\big]$; or*
8) *$\omega = \alpha+\beta$ and $\mathfrak{h} \cap \mathfrak{n} = \mathfrak{u}_\alpha$, in which case $\mu(H) \approx \big[\|h\|(\log\|h\|), \|h\|^2\big]$.*

2.14. Proposition (cf. [OW, Prop. 3.17, Cor. 3.18]). *Assume that $G = \mathrm{SO}(2,n)$. Let H be a closed, connected subgroup of AN, such that there is a positive root ω and a one-dimensional subspace $\mathfrak{t}$ of $\mathfrak{a}$, such that $0 \neq \mathfrak{h} \cap \mathfrak{n} \subset \mathfrak{u}_\omega$, and $\mathfrak{h} = \mathfrak{t} + (\mathfrak{h} \cap \mathfrak{n})$.*

Let γ be the positive root that is perpendicular to ω, so $\{\omega, \gamma\}$ is either $\{\alpha, \alpha+2\beta\}$ or $\{\beta, \alpha+\beta\}$. If $\mathfrak{t} = \ker\omega$, then H is a Cartan-decomposition subgroup of G. Otherwise, there is a real number p, such that $\mathfrak{t} = \ker(p\omega + \gamma)$.

- *If $|p| \geq 1$, then H is a Cartan-decomposition subgroup of G.*
- *If $|p| < 1$, and $\omega \in \{\alpha, \alpha+2\beta\}$, then $\mu(H) \approx \big[\|h\|^{2/(1+|p|)}, \|h\|^2\big]$.*
- *If $|p| < 1$, and $\omega \in \{\beta, \alpha+\beta\}$, then $\mu(H) \approx \big[\|h\|, \|h\|^{1+|p|}\big]$.*

For ease of reference, we now collect a few miscellaneous facts.

2.15. Lemma ([OW, Lem. 2.8]). *Let H be a closed, connected subgroup of AN. If $\dim H - \dim(H \cap N) \geq \mathbb{R}\text{-rank}\, G$, then H contains a conjugate of A, so H is a Cartan-decomposition subgroup.*

2.16. Remark. We realize $\mathrm{SO}(1,n)$ as the stabilizer of the vector

$$(0,1,0,0,\ldots,0,-1,0).$$

Thus, the Lie algebra of $\mathrm{SO}(1,n) \cap AN$ is

$$\left\{ \left. \begin{pmatrix} t & \phi & x & \phi & 0 \\ & 0 & 0 & 0 & -\phi \\ & & \cdots & & \end{pmatrix} \right| t, \phi \in \mathbb{R} x \in \mathbb{R}^{n-2} \right\},$$

that is, it is of type 2.10(5), with $X_0 = \mathbb{R}^{n-2}$, $b = 0$, $c = 0$, and $p = 1$. Therefore, we see that $\mu\big(\mathrm{SO}(1,n)\big) \approx \big[\|h\|, \|h\|\big]$.

2.17. Proposition (cf. [OW, Case 3 of the pf. of Thm. 6.1]). *Assume that $G = \mathrm{SO}(2,n)$. Suppose that H is a closed, connected subgroup of AN, such that $H = (H \cap A) \ltimes (H \cap N)$. Assume that there exist a subspace X_0 of $\mathbb{R}^{n-2}$, vectors $b \in X_0$ and $c \in X_0^\perp$, and a real number p, such that, for every $h \in H \cap N$, we have $y_h = 0$, $x_h \in \phi_h c + X_0$, and $\eta_h = p\phi_h + b \cdot x_h$. If $\|b\|^2 - \|c\|^2 - 2p < 0$, then H is conjugate to a subgroup of $\mathrm{SO}(1,n)$.*

3. General results on compact Clifford-Klein forms

Before beginning our study of the specific group $\mathrm{SO}(2,n)$, let us state some general results on compact Clifford-Klein forms. Recall that G is a Zariski-connected, almost simple, linear Lie group.

The Calabi-Markus phenomenon asserts that if H is a Cartan-decomposition subgroup of G, then no closed, noncompact subgroup of G acts properly on G/H (cf. [Kul, pf. of Thm. A.1.2]). The following well-known fact is a direct consequence of this observation.

3.1. Lemma. *Let H be a Cartan-decomposition subgroup of G. Then G/H does not have a compact Clifford-Klein form, unless G/H itself is compact.*

By combining this lemma with the following proposition, we see that if G/H has a compact Clifford-Klein form, and $\mathbb{R}\text{-rank}\, G = 1$, then either H or G/H is compact.

3.2. Proposition ([OW, Prop. 1.8], [Kb3, Lem. 3.2])**.** *Assume that* $\mathbb{R}$-rank $G = 1$*. A closed, connected subgroup* H *of* G *is a Cartan-decomposition subgroup if and only if* H *is noncompact.*

Lemma 3.1 can be obtained as a special case of Theorem 3.4(1) by letting $L = G$. The generalization is very useful.

3.3. Notation (cf. [Kb1, (2.5), §5])**.** For any connected Lie group H, let $d(H) = \dim H - \dim K_H$, where K_H is a maximal compact subgroup of H. This is well defined, because all the maximal compact subgroups of H are conjugate [Hoc, Thm. XV.3.1, p. 180–181]. (This concept originated with K. Iwasawa [Iwa, p. 533], who called $d(H)$ the "characteristic index" of H.) Note that if $H \subset AN$, then $d(H) = \dim H$, because AN has no nontrivial compact subgroups.

3.4. Theorem (Kobayashi, cf. [Kb1, Cor. 5.5] and [Kb2, Thm. 1.5])**.** *Let* H *and* L *be closed, connected subgroups of* G*, and assume that there is a compact subset* C *of* G*, such that* $L \subset CHC$*.*

1) *If* $d(L) > d(H)$*, then* G/H *does not have a compact Clifford-Klein form.*
2) *If* $d(L) = d(H)$*, and* G/H *has a compact Clifford-Klein form, then* G/L *also has a compact Clifford-Klein form.*
3) *If there is a closed subgroup* L' *of* G*, such that* L' *acts properly on* G/H*,* $d(H)+d(L') = d(G)$*, and there is a co-compact lattice* Γ *in* L'*, then* G/H *has a compact Clifford-Klein form, namely, the quotient* $\Gamma\backslash G/H$ *is compact.*

Comments on the proof. Kobayashi assumed that H is reductive, but the same proof works with only minor changes. From Lemma 3.8, we see that we may assume that $H \subset AN$. Then, from the Iwasawa decomposition $G = KAN$, it is immediate that the homogeneous space G/H is homeomorphic to the cartesian product $K \times (AN/H)$. Because AN is a simply connected, solvable Lie group and H is a connected subgroup, the homogeneous space AN/H is homeomorphic to a Euclidean space $\mathbb{R}^M$ (cf. [Mo1, Prop. 11.2]). Therefore, we see that G/H has the same homotopy type as K ($= K/(H \cap K)$, because $H \cap K$ is trivial). Thus, the proof of [Kb1, Lem. 5.3] goes through essentially unchanged in the general setting. This yields a general version of [Kb1, Cor. 5.5], from which general versions of [Kb2, Thm. 1.5] and [Kb1, Thm. 4.7] follow. Our conclusion (1) is the natural generalization of [Kb2, Thm. 1.5]. Conclusion (3) is the natural generalization of [Kb1, Thm. 4.7], and conclusion (2) is proved similarly. □

The following useful theorem of G. A. Margulis is used in the proof of Proposition 3.7.

3.5. Definition (cf. [Mar, Defn. 2.2, Rmk. 2.2])**.** A closed subgroup H of G is (G,K)*-tempered* if there exists a (positive) function $q \in L^1(H)$, such that, for every non-trivial, irreducible, unitary representation π of G with a K-fixed unit vector v, we have $|\langle \pi(h)v, v\rangle| \leq q(h)$ for all $h \in H$.

(We remark that, because π is irreducible, the K-invariant vector v is unique, up to a scalar multiple [Kna, Thm. 8.1].)

3.6. Theorem (Margulis [Mar, Thm. 3.1])**.** *If* H *is a closed, noncompact,* (G,K)*-tempered subgroup of* G*, then* G/H *does not have a compact Clifford-Klein form.*

3.7. Proposition. *If* H *is a closed, noncompact one-parameter subgroup of a connected, simple, linear Lie group* G*, then* G/H *does not have a compact Clifford-Klein form.*

Proof. Suppose that G/H does have a compact Clifford-Klein form.

Assume for the moment that $\mathbb{R}$-rank $G = 1$. Then H is a Cartan-decomposition subgroup of G (see 3.2), so we see from Lemma 3.1 that G/H must be compact. But the dimension of every connected, co-compact subgroup of G is at least $d(G)$ [GW, (1.2), p. 263], and $d(G) \geq 2$. This contradicts the fact that $\dim H = 1$. Thus, we now know that $\mathbb{R}$-rank $G \geq 2$.

Case 1. Assume that H *is unipotent.* The Jacobson-Morosov Lemma [Jac, Thm. 17(1), p. 100] implies that there exists a connected, closed subgroup L of G that is locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$

(and has finite center), and contains H. Then H is a Cartan-decomposition subgroup of L (see 3.2), and $d(L) = 2 > 1 = d(H)$, so Theorem 3.4(1) applies.

Case 2. Assume that H is not unipotent. We may assume that $H = \{\, a_t u_t \mid t \in \mathbb{R} \}$ where $a_t \in A$ is a semisimple one parameter subgroup and $u_t \in N$ is a unipotent one parameter subgroup such that a_t commutes with u_t (cf. 3.8 and 2.12). It is well known (cf. [KS, §3, p. 140], where a stronger result is obtained by combining [How, Cor. 7.2 and §7] with [Cow, Thm. 2.4.2]) that there are constants $C > 0$ and $p > 0$ such that, for any non-trivial irreducible unitary representation ρ with a K-fixed unit vector, say v, we have

$$|\langle \rho(g)v, v\rangle| \leq C \exp\bigl(-pd(e,g)\bigr) \qquad \text{for } g \in G$$

where d is some bi-K-invariant Riemannian metric on G. We may assume that $d(e, a_t) = |t|$ with a suitable parameterization. Since the growth of a unipotent one parameter subgroup is logarithmic while that of semisimple one parameter subgroup is linear, we can find a large T such that $d(e, a_t u_t) \geq \frac{1}{2} d(e, a_t) = |t|/2$ for all $t > T$. Hence

$$|\langle \rho(a_t u_t)v, v\rangle| \leq C \exp\left(-\frac{p|t|}{2}\right) \qquad \text{for } t > T.$$

Since the function $\exp(-p|t|/2)$ is in $L^1(\mathbb{R})$, it follows that H is (G,K)-tempered. Therefore, Theorem 3.6 implies that G/H does not have compact Clifford-Klein forms. □

We use the following well-known lemma to reduce the study of compact Clifford-Klein forms of G/H to the case where $H \subset AN$. We remark that the proof of the lemma is constructive. For example, replace H by a conjugate, so that $\overline{H} \cap AN$ is co-compact in $\overline{H}$, where $\overline{H}$ is the Zariski closure of $\overline{H}$, and choose a maximal compact subgroup $\overline{C}$ of $\overline{H}^\circ$. Then write $\overline{C} = C_1 C_2$, where C_1 is a maximal compact subgroup of H, and C_2 is contained in the Zariski closure of $\operatorname{Rad} H$. Finally, let $H' = (HC_2) \cap (AN)$.

3.8. Lemma (cf. [OW, Lem. 2.9]). *Let H be a closed, connected subgroup of G. Then there is a closed, connected subgroup H' of G and compact, connected subgroups C_1 and C_2 of G, such that*

1) *H' is conjugate to a subgroup of AN;*
2) *$\dim H' = d(H)$ (see Notation 3.3);*
3) *$C_2 H = C_1 C_2 H'$;*
4) *$C_1 \subset H$, C_2 is abelian, C_1 centralizes C_2, and C_2 normalizes both H and H'; and*
5) *if $H/\operatorname{Rad} H$ is compact, then C_1 normalizes H'.*

Moreover, from 3.4(2), we know that the homogeneous space G/H has a compact Clifford-Klein form if and only if G/H' has a compact Clifford-Klein form.

We now recall a fundamental result of Benoist and Kobayashi.

3.9. Theorem (Benoist [Ben, Prop. 1.5], Kobayashi [Kb4, Cor. 3.5]). *Let H_1 and H_2 be closed subgroups of G. The subgroup H_1 acts properly on G/H_2 if and only if, for every compact subset C of A, the intersection $\bigl(\mu(H_1)C\bigr) \cap \mu(H_2)$ is compact.*

3.10. Lemma. *Let G° be the identity component of G, let H be a closed, connected subgroup of G, and let Γ be a discrete subgroup of G. Then:*

1) *Γ acts properly on G/H if and only if $\Gamma \cap G^\circ$ acts properly on G°/H.*
2) *$\Gamma\backslash G/H$ is compact if and only if $(\Gamma \cap G^\circ)\backslash G^\circ/H$ is compact.*

Proof. (1) Because every element of the Weyl group of G has a representative in G° [BT1, Cor. 14.6], we see that G and G° have the same positive Weyl chamber A^+, and the Cartan projection $G^\circ \to A^+$ is the restriction of the Cartan projection $G \to A^+$. Thus, the desired conclusion is immediate from Corollary 3.9.

(2) This is an easy consequence of the fact that G/G° is finite [Mo2, Appendix]. □

4. New examples of compact Clifford-Klein forms

Proof of Theorem 1.5. From Remark 2.16, we have $\mu\big(\mathrm{SO}(1,2m)\big) \approx \big[\|h\|, \|h\|\big]$. From Theorem 2.10(2), we have $\mu(H_B) \approx \big[\|h\|^2, \|h\|^2\big]$. Thus, Theorem 3.9 implies that $\mathrm{SO}(1,2m)$ acts properly on $\mathrm{SO}(2,2m)/H_B$. We have $d\big(\mathrm{SO}(2,2m)\big) = 4m$, $d\big(\mathrm{SO}(1,2m)\big) = 2m$, and $\dim(H_B) = 2m$, so $d\big(\mathrm{SO}(2,2m)\big) = d\big(\mathrm{SO}(1,2m)\big) + \dim(H_B)$. Therefore, conclusions (1) and (2) follow from Theorem 3.4(3).

To show conclusion (3), suppose that $g\mathfrak{h}_B g^{-1} \subset \mathfrak{su}(1,m)$ for some $g \in SO(2,2m)$. Because all maximal split tori in $\mathrm{SU}(1,m)$ are conjugate, we may assume that g normalizes $\ker\alpha$. In fact, because $\mathrm{SU}(1,m)$ has an element (namely, the nontrivial element of the Weyl group) that inverts $\ker\alpha$, we may assume that g centralizes $\ker\alpha$. Thus, g is a block diagonal matrix with R, S and $J(R^T)^{-1}J$ on the diagonal, where $R \in \mathrm{GL}(2,\mathbb{R})$, $S \in \mathrm{O}(2m-2)$, and $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Conjugating by $I_2 \times S \times I_2$ amounts to choosing a different orthonormal basis for $\mathbb{R}^{2m-2}$, so we may assume that S is trivial. Now, the assumption that $g\mathfrak{h}_B g^{-1} \subset \mathfrak{su}(1,m)$ implies that $R\begin{pmatrix} x \\ B(x) \end{pmatrix}$ is of the form

$$\begin{pmatrix} z_1 & -z_2 & z_3 & -z_4 & \cdots & z_{2m-3} & -z_{2m-2} \\ z_2 & z_1 & z_4 & z_3 & \cdots & z_{2m-2} & z_{2m-3} \end{pmatrix}.$$

A direct calculation, setting

$$R_{1,1}x_{2i-1} + R_{1,2}B(x)_{2i-1} = R_{2,1}x_{2i} + R_{2,2}B(x)_{2i}$$

and

$$R_{1,1}x_{2i} + R_{1,2}B(x)_{2i} = -\big(R_{2,1}x_{2i-1} + R_{2,2}B(x)_{2i-1}\big)$$

for all $i \in \{1, \ldots, m-1\}$, now establishes that B is block diagonal as described in conclusion (3).

All that remains is to show that there are uncountably many nonconjugate subgroups of the form H_B. Let L be the group of all block diagonal matrices with R, S and $J(R^T)^{-1}J$ on the diagonal, where $R \in \mathrm{GL}(2,\mathbb{R})$, $S \in \mathrm{O}(2m-2)$, and $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and let $\mathcal{B}$ be the set of all matrices in $\mathrm{GL}_{2m-2}(\mathbb{R})$ that have no real eigenvalues, so $\mathcal{B}$ is an open subset of $\mathrm{GL}_{2m-2}(\mathbb{R})$. Because $B \mapsto H_B$ is injective, the action of L by conjugation on $\{\, H_B \mid B \in \mathcal{B} \,\}$ yields an action of L on $\mathcal{B}$. By arguing as in the proof of (3), we see that if B_1 and B_2 are in different L-orbits in $\mathcal{B}$, then H_{B_1} is not conjugate to H_{B_2}. Thus, it suffices to show that there are uncountably many L-orbits on $\mathcal{B}$.

If $2m \geq 6$, then

$$\dim L = 4 + \frac{1}{2}(2m-2)(2m-3) < (2m-2)^2 = \dim \mathcal{B},$$

so each L-orbit on $\mathcal{B}$ has measure zero. Thus, obviously, there are uncountably many L-orbits.

Now assume that $m = 2$. In this case, for each $B \in \mathcal{B}$, there exists $B' \in \mathcal{B}$, such that

$$\left\{ \begin{pmatrix} x \\ B(x) \end{pmatrix} \;\middle|\; x \in \mathbb{R}^2 \right\} = \left\{ \begin{pmatrix} v & B'(v) \end{pmatrix} \;\middle|\; v \in \mathbb{R}^2 \right\},$$

where v and $B'(v)$ are considered as column vectors. Note that the centralizer of B' in $\mathrm{GL}(2,\mathbb{R})$ contains a 2-dimensional connected subgroup. Thus, B' is centralized by a nontrivial connected subgroup of $\mathrm{PGL}(2,\mathbb{R})$, so we see that the normalizer $N_L(H_B)$ contains a 2-dimensional subgroup (consisting of block diagonal matrices with R, Id and $J(R^T)^{-1}J$ on the diagonal, for R centralizing B'), so $\dim L - \dim\big(N_L(H_B)\big) \leq 3 < 4 = \dim \mathcal{B}$. Therefore, as in the previous case, each L-orbit on $\mathcal{B}$ has measure zero, so there are uncountably many L-orbits. $\square$

4.1. Remark. The subgroups H_B of Theorem 1.5 are not all isomorphic (unless $m = 2$). For example, let $m = 3$ and let

$$B = \begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial of B is $\det(\lambda - B) = \lambda^4 - \lambda^2 + 1$, which has no real zeros, so B has no real eigenvalues. Let $v = (0,0,0,1)$. We have $B^T v = Bv$, so, for every $x \in \mathbb{R}^4$, we have $x \cdot Bv - v \cdot Bx = 0$. Thus, if h is any element of $\mathfrak{h}_B \cap \mathfrak{n}$ with $x_h = v$, then h is in the center of $\mathfrak{h}_B \cap \mathfrak{n}$. Therefore, the center of $\mathfrak{h}_B \cap \mathfrak{n}$ contains $\langle h, \mathfrak{u}_{\alpha+2\beta} \rangle$, so the dimension of the center is at least 2. (In fact, the center is 3-dimensional, as will be explained below.) Because the center of $\mathfrak{h}_{\mathrm{SU}} \cap \mathfrak{n}$ is $\mathfrak{u}_{\alpha+2\beta}$, which is one-dimensional, we conclude that $\mathfrak{h}_B$ is not isomorphic to $\mathfrak{h}_{\mathrm{SU}}$.

4.2. Remark. Almost every H_B is isomorphic to H_{SU}. Namely, H_B is isomorphic to H_{SU} if B belongs to the dense, open set where $\det(B^T - B) \neq 0$. (Perturb B by adding almost any skew-symmetric matrix to make the determinant nonzero. If the skew-symmetric matrix is small enough, the perturbation will not have any real eigenvalues.) To see this, note that, for every B, the torus $A \cap H_B$ has only two weights on the unipotent radical (one on $\mathfrak{h}_B \cap (\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta})$, and 2 times that on $\mathfrak{u}_{\alpha+2\beta} = [\mathfrak{h}_B \cap \mathfrak{n}, \mathfrak{h}_B \cap \mathfrak{n}]$). So two H_B's are isomorphic if and only if their unipotent radicals are isomorphic. We show below that each unipotent radical is the direct product of an abelian group with a Heisenberg group, so the unipotent radicals are isomorphic if and only if their centers have the same dimension. Finally, the dimension of the center of H_B is $1 +$ the dimension of the kernel of $B^T - B$. Therefore, it is easy to see which H_B's are isomorphic. (Also, there are only finitely many different H_B's, up to isomorphism.)

We now show that the unipotent radical of H_B is the direct product of an abelian group with a Heisenberg group. Let $\mathfrak{q}_0$ be the kernel of $B^T - B$, and let $\mathfrak{w}_0$ be a subspace of R^{2m-2} that is complementary to $\mathfrak{q}$. Define

$$\mathfrak{q} = \left\{ \left. \begin{pmatrix} 0 & 0 & v & 0 & 0 \\ & 0 & B(v) & 0 & 0 \\ & & \cdots & & \end{pmatrix} \right| v \in \mathfrak{q}_0 \right\} \subset \mathfrak{h}_B$$

and

$$\mathfrak{w} = \left\{ \left. \begin{pmatrix} 0 & 0 & w & 0 & 0 \\ & 0 & B(w) & 0 & 0 \\ & & \cdots & & \end{pmatrix} \right| w \in \mathfrak{w}_0 \right\} \subset \mathfrak{h}_B,$$

and let $\mathfrak{z}$ be the center of $\mathfrak{n}$. Then the Lie algebra of the unipotent radical of H_B is $(\mathfrak{w} + \mathfrak{z}) + \mathfrak{q}$. Let us see that $\mathfrak{w} + \mathfrak{z}$ is a Heisenberg Lie algebra. Choose some nonzero $z_0 \in \mathfrak{z}$. For any $v, w \in \mathfrak{w}$, there is some scalar $\langle v \mid w \rangle$, such that

$$[v, w] = \langle v \mid w \rangle z_0 \tag{4.3}$$

(because $[\mathfrak{w}, \mathfrak{w}] \subset \mathfrak{z}$ and $\mathfrak{z}$ is one-dimensional). Clearly, $\langle \cdot \mid \cdot \rangle$ is skew symmetric, because $[v, w] = -[w, v]$. Also, for every $v \in \mathfrak{w}$, we have $\langle v \mid \mathfrak{w} \rangle \neq 0$, because $\mathfrak{w} \cap \mathfrak{q} = 0$; so the form is nondegenerate on $\mathfrak{w}$. Thus, $\langle \cdot \mid \cdot \rangle$ is a symplectic form on $\mathfrak{w}$, so (4.3) is the definition of a Heisenberg Lie algebra.

5. Non-existence results on compact Clifford-Klein forms of $\mathrm{SO}(2,n)/H$

The following lemma is obtained by combining Theorem 3.4 with some of our results on Cartan projections.

5.1. Lemma. *Assume that $G = \mathrm{SO}(2,n)$. Let H be a closed, connected subgroup of AN.*

1) *If $\mu(H) \approx \big[\|h\|, \cdot\big]$ and $\dim H < n$, then G/H does not have a compact Clifford-Klein form.*

2) *If* $\mu(H) \approx \left[\cdot, \|h\|^2\right]$ *and* $\dim H < 2\lfloor n/2 \rfloor$, *then* G/H *does not have a compact Clifford-Klein form.*

Proof. By Theorem 3.4(1), it is enough to find a subgroup L of AN, such that $\dim L > \dim H$ and $L \subset CHC$ for some compact set C.

(1) We take L to be $\mathrm{SO}(1,n) \cap AN$. We have $\dim L = n > \dim H$. Furthermore, we know from Remark 2.16 that $\mu(L) \approx \left[\|h\|, \|h\|\right]$. Thus, from the assumption on the form of $\mu(H)$, we know that there is a compact subset C of G with $L \subset CHC$ (see 2.6).

(2) Let $n' = 2\lfloor n/2 \rfloor$. Because $n' - 2$ is even, there is a linear transformation $B \colon \mathbb{R}^{n'-2} \to \mathbb{R}^{n'-2}$ that has no real eigenvalues. In other words, $\dim\langle x, Bx\rangle = 2$ for all nonzero $x \in \mathbb{R}^{n'-2}$. Let

$$\mathfrak{h}_B = (\ker \alpha) + \{\, h \in \mathfrak{n} \mid \phi_h = 0, x_h \in \mathbb{R}^{n'-2}, y_h = B(x_h) \,\},$$

where we identify $\mathbb{R}^{n'-2}$ with a codimension-one subspace of $\mathbb{R}^{n-2}$ in the case where n is odd (so $n' = n - 1$), and let H_B be the corresponding connected, closed subgroup of AN.

Let $L = H_B$. (Thus, for the appropriate choice of B, we could take $L = \mathrm{SU}(1, n'/2) \cap AN$.) Then $\dim L = n' > \dim H$. We know from Corollary 2.10(2) that $\mu(L) \approx \left[\|h\|^2, \|h\|^2\right]$. Thus, from the assumption on the form of $\mu(H)$, we know that there is a compact subset C of G with $L \subset CHC$ (see 2.6), as desired. □

5.2. Proposition (see 6.7). *Assume that* $G = \mathrm{SO}(2,n)$. *If* H *is a nontrivial, connected, unipotent subgroup of* G, *then* G/H *does not have a compact Clifford-Klein form.*

5.3. Notation. Let $L_5 \subset \mathrm{SL}_5(\mathbb{R})$ be the image of $\mathrm{SL}_2(\mathbb{R})$ under an irreducible 5-dimensional representation of $\mathrm{SL}_2(\mathbb{R})$. There is an L_5-invariant, symmetric, bilinear form of signature $(2,3)$ on $\mathbb{R}^5$, so we may view L_5 as a subgroup of $\mathrm{SO}(2,3)$. More concretely, we may take the Lie algebra of L_5 to be the image of the homomorphism $\pi \colon \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{so}(2,3)$ given by

$$\pi\begin{pmatrix} t & u \\ v & -t \end{pmatrix} = \left\{ \begin{pmatrix} 4t & 2u & 0 & 0 & 0 \\ 2v & 2t & \sqrt{6}u & 0 & 0 \\ 0 & \sqrt{6}v & 0 & -\sqrt{6}u & 0 \\ 0 & 0 & -\sqrt{6}v & -2t & -2u \\ 0 & 0 & 0 & -2v & -4t \end{pmatrix} \;\middle|\; t, u, v \in \mathbb{R} \right\}.$$

Via the embedding $\mathbb{R}^5 \hookrightarrow \mathbb{R}^{n+2}$ given by

$$(x_1, x_2, x_3, x_4, x_5) \mapsto (x_1, x_2, x_3, 0, 0, \ldots, 0, 0, x_4, x_5),$$

we may realize $\mathrm{SO}(2,3)$ as a subgroup of $\mathrm{SO}(2,n)$, so we may view L_5 as a subgroup of $\mathrm{SO}(2,n)$ (for any $n \geq 3$).

5.4. Proposition (Oh [Oh, Ex. 5.6]). *Assume that* $G = \mathrm{SO}(2,3)$. *Then* L_5 *is* (G,K)*-tempered* (*see* 3.5).

5.5. Corollary. *Assume that* $G = \mathrm{SO}(2,n)$. *Then* G/L_5 *does not have a compact Clifford-Klein form.*

Proof. From Proposition 5.4, we know that L_5 is $\big(\mathrm{SO}(2,3), K'\big)$-tempered, for any maximal compact subgroup K' of $\mathrm{SO}(2,3)$. From the definition, it is clear that this implies that L_5 is (G,K)-tempered. (More generally, a tempered subgroup of any closed subgroup of G is a tempered subgroup of G.) Therefore, Theorem 3.6 implies that G/L_5 does not have a compact Clifford-Klein form. □

5.6. Proposition (Kulkarni [Kul, Cor. 2.10]). *If* n *is odd, then* $\mathrm{SO}(2,n)/SO(1,n)$ *does not have a compact Clifford-Klein form.*

One direction of Theorem 1.7 in the introduction is contained in the following theorem. The converse is obtained by combining Theorem 1.5 with Kulkarni's construction [Kul, Thm. 6.1] of a compact Clifford-Klein form of $\mathrm{SO}(2,n)/\mathrm{SO}(1,n)$ when n is even.

5.7. Theorem. *Assume that $G = \mathrm{SO}(2,n)$, with $n \geq 3$. Let H be a closed, connected subgroup of AN, and such that H is compatible with A (see 2.11). Assume that neither H nor G/H is compact, and that G/H has a compact Clifford-Klein form. If n is even, then H is one of the two types described in Theorem 1.7. If n is odd, then either*

1) $\dim H = n-1$, *and H is of type 2.10(2); or*
2) $n = 3$, $\dim H = 2$, *and either*
   (a) *H is of type 2.13(2); or*
   (b) *H is of type 2.13(3); or*
   (c) *H is of type 2.10(8), and there is a positive root γ and a real number $p \in (-1/3, 1/3)$, such that $\{\omega, \gamma\} = \{\alpha, \alpha + 2\beta\}$ and $T = \ker(p\omega + \gamma)$.*

Proof. Because of Proposition 6.7 and Lemma 3.1, we know that $H \not\subset N$ and that H is not a Cartan-decomposition subgroup. Thus, H must be one of the subgroups described in either Theorem 2.10 or Theorem 2.13. From Proposition 3.7, we know that H cannot be of type 2.10(1).

Let us first consider the cases where $\mu(H)$ is of the form $\mu(H) \approx \left[\|h\|, \cdot\right]$.

- If H is of type 2.10(3), then $\dim H \leq n-1$.
- If H is of type 2.10(4), then $\dim H \leq n-1$.
- If H is of type 2.10(5), then $\dim H \leq n$.
- If H is of type 2.10(7), then $\dim H = 2$.
- If H is of type 2.10(8) and $\omega \in \{\beta, \alpha+\beta\}$, then $\dim H \leq n-1$.
- If H is of type 2.13(1), 2.13(4), 2.13(5), or 2.13(6), then $\dim H \leq n-1$.

Thus $\dim H < n$ except possibly when H is of type 2.10(5). Thus, in all cases except 2.10(5), we conclude from Lemma 5.1(1) that G/H does not have a compact Clifford-Klein form. Now suppose that $\dim H = n$ and that H is of type 2.10(5). From Proposition 2.17 (and comparing dimensions), we see that H is conjugate to $\mathrm{SO}(1,n) \cap (AN)$. If n is even, then H is listed in Theorem 1.7(1); if n is odd, then Proposition 5.6 implies that G/H does not have a compact Clifford-Klein form.

We now consider the one case where $\mu(H) \approx \left[\|h\|^{3/2}, \|h\|^{3/2}\right]$; namely, we assume that H is of type 2.10(6). Then H is conjugate to a co-compact subgroup of the subgroup L_5 (see 5.3), so Corollary 5.5 implies that G/H does not have a compact Clifford-Klein form.

Finally, we consider the cases where $\mu(H)$ is of the form $\mu(H) \approx \left[\cdot, \|h\|^2\right]$.

- If H is of type 2.10(2), then $\dim H \leq 2\lfloor n/2 \rfloor$ (see 5.8).
- If H is of type 2.10(8) and $\omega \in \{\alpha, \alpha+2\beta\}$, then $\dim H = 2$.
- If H is of type 2.13(2), 2.13(3), 2.13(7), or 2.13(8), then $\dim H = 2$.

Assume for the moment that $n \geq 4$. Then $2\lfloor n/2 \rfloor > 3$, so $\dim H < 2\lfloor n/2 \rfloor$ except possibly when H is of type 2.10(2), in which case H is either listed in Theorem 1.7(2) (if n is even; see 5.9) or listed in Theorem 5.7(1) (if n is odd). In all other cases with $n > 3$, we conclude from Lemma 5.1(2) that G/H does not have a compact Clifford-Klein form.

We assume, henceforth, that $n = 3$. Let H_5 be a subgroup of AN of type 2.10(6). We know, from above, that G/H_5 does not have a compact Clifford-Klein form.

If H is of type 2.10(2), 2.13(2), or 2.13(3), then H is listed in Theorem 5.7.

If H is of type 2.13(7) or 2.13(8), then there is a compact subset C of G with $H_5 \subset CHC$. Because $\dim H_5 = \dim H$, we conclude from Lemma 3.4(2) that G/H does not have a compact Clifford-Klein form.

We may now assume that H is of type 2.10(8), and that $\omega \in \{\alpha, \alpha+2\beta\}$. If T is not of the form described in Theorem 5.7(2c), then Proposition 2.14 implies that there is a compact subset C of G with $H_3 \subset CHC$, so we conclude from Lemma 3.4(2) that G/H does not have a compact Clifford-Klein form. □

We now prove a lemma used in the proof of the preceding theorem.

5.8. Lemma. *Assume that $G = \mathrm{SO}(2,n)$ and that n is odd. If H is of type 2.10(2), then $\dim H \leq n-1$.*

Proof. Suppose that $\dim H \geq n$. Let $X = \{\, x_h \mid h \in \mathfrak{h} \cap \mathfrak{n} \,\}$. For any $x \in X$, there is a unique $B(x) \in \mathbb{R}^{n-2}$ such that there is some $h \in \mathfrak{h} \cap \mathfrak{n}$ with $x_h = x$ and $y_h = B(x)$. (The element $B(x)$ is unique because of the assumption that $\dim\langle x, y\rangle \neq 1$.) Because $n - 1 \leq \dim(\mathfrak{h} \cap \mathfrak{n}) \leq 1 + \dim X$ (with equality on the right if $\mathfrak{u}_{\alpha+2\beta} \subset \mathfrak{h}$) and $X \subset \mathbb{R}^{n-2}$, we must have $X = \mathbb{R}^{n-2}$.

Thus, $B \colon \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$ is a linear transformation, and $\{x, Bx\}$ is linearly independent for all nonzero $x \in \mathbb{R}^{n-2}$. This is impossible, because any linear transformation on a real vector space of odd dimension has an eigenvector. □

5.9. Corollary (of proof). *Assume that $G = \mathrm{SO}(2,n)$, and that n is even. If H is of type 2.10(2), and $\dim H = n$, then there is a linear transformation $B \colon \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$ without any real eigenvalue, such that $H = H_B$ is the corresponding subgroup described in Theorem 1.5.*

Proof of Theorem 1.9. Assume that G/H has a compact Clifford-Klein form. Replacing H by a conjugate subgroup, we may assume that there is a closed, connected subgroup H' of AN and a compact subgroup C of G, such that $CH = CH'$ (see 3.8). Furthermore, we may assume that H' is compatible with A (see 2.12). Then H' must be one of the types listed in Theorem 5.7 (with $n = 2m+1$). Therefore, noting that $\mu\big(\mathrm{SU}(1,m)\big) \approx \big[\|h\|^2, \|h\|^2\big]$ and using the calculation of $\mu(H')$ (see 2.10(2), 2.13(2), 2.13(3), or 2.14), we see that there is a compact subset C_1 of G, such that $\mathrm{SU}(1,m) \subset C_1 H' C_1$. Then, because $d\big(\mathrm{SU}(1,m)\big) = 2m = \dim H'$, we conclude from Theorem 3.4(2) that $G/\mathrm{SU}(1,m)$ has a compact Clifford-Klein form. This is a contradiction. □

The following lemma is used in the above proof of Theorem 1.9. Although the result is known, we are unable to locate a proof in the literature. The proof here is based on our classification of possible compact Clifford-Klein forms (Theorem 5.7), but the result can also be derived from the classification of simple Lie groups of real rank one.

5.10. Lemma. *Assume that $G = \mathrm{SO}(2, 2m+1)$. There does not exist a connected, reductive subgroup L of G, such that L acts properly on $G/\mathrm{SU}(1,m)$, and $L\backslash G/\mathrm{SU}(1,m)$ is compact.*

Proof. Suppose there does exist such a subgroup L. Let $L = K_L A_L N_L$ be an Iwasawa decomposition of L, and let $H = A_L N_L \subset AN$. For any co-compact lattice Γ in $\mathrm{SU}(1,m)$, we see that the Clifford-Klein form $\Gamma\backslash G/H$ is compact, so H must be one of the subgroups described in Theorem 5.7 (or 1.7). Because $2m+1$ is odd, we know that (1.7) does not apply. Thus, we see, from Theorem 2.10 (or Proposition 2.14 in Case 5.7(2c)), that $\mu(H) \approx \big[\cdot, \|h\|^2\big]$. Because $\mu\big(\mathrm{SU}(1,m)\big) \approx \big[\|h\|^2, \|h\|^2\big]$, we conclude (e.g., from 2.6) that H does not act properly on $G/\mathrm{SU}(1,m)$. This contradicts the fact that L acts properly on $G/\mathrm{SU}(1,m)$. □

6. Finite-volume Clifford-Klein forms

6.1. Definition (cf. [Kb1, Def. 2.2]). Let H be a closed, connected subgroup of G, such that G/H has a G-invariant regular Borel measure. (Because G is unimodular, this means that H is unimodular [Rag, Lem. 1.4, p. 18].) We say that G/H has a *finite-volume Clifford-Klein form* if there is a discrete subgroup Γ of G, such that

- Γ acts properly on G/H; and
- there is a Borel subset $\mathcal{F}$ of G/H, such that $\mathcal{F}$ has finite measure, and $\Gamma\mathcal{F} = G/H$.

Unfortunately, the study of finite-volume Clifford-Klein forms does not usually reduce to the case where $H \subset AN$, because the subgroup H' of Proposition 3.8 is usually not unimodular.

6.2. Theorem. *Assume that $G = \mathrm{SO}(2,n)$. Let H be a closed, connected subgroup of G. If G/H has a finite-volume Clifford-Klein form, then either*

1) *H has a co-compact, normal subgroup that is conjugate under* $\mathrm{O}(2,n)$ *to the identity component of either* $\mathrm{SO}(1,n)$, $\mathrm{SU}(1,\lfloor n/2 \rfloor)$, *or* L_5 *(see 5.3); or*
2) $d(H) \leq 1$ *(see 3.3); or*
3) $H = G$.

It is natural to conjecture that $\mathrm{SO}(2,2m+1)/\mathrm{SU}(1,m)$ and $\mathrm{SO}(2,n)/L_5$ do not have finite-volume Clifford-Klein forms, and that G/H does not have a finite-volume Clifford-Klein form when $d(H) = 1$, either.

To prepare for the proof of Theorem 6.2, we present some preliminary results.

Unfortunately, we do not have an analogue of Theorem 3.4(1) for finite-volume Clifford-Klein forms. The following lemma is a weak substitute.

6.3. Lemma. *Let H be a closed, connected, unimodular subgroup of G. Suppose that there is a closed subgroup L of G containing H, such that $L \subset CHC$, for some compact subset C. If L/H does not have finite (L-invariant) volume, then G/H does not have a finite-volume Clifford-Klein form.*

Proof. Suppose that $\Gamma\backslash G/H$ is a finite-volume Clifford-Klein form of G/H. Because $L \subset CHC$, we know that Γ is proper on G/L. (In particular, $\Gamma \cap L$ must be finite.) We have a quotient map $\Gamma\backslash G/H \to \Gamma\backslash G/L$, and, because $\Gamma\backslash G/H$ has finite volume, (almost) every fiber has finite measure (cf. [Roh, §3, p. 26–33]). Thus (perhaps after replacing Γ by a conjugate subgroup), $(\Gamma \cap L)\backslash L/H$ has finite volume. Because $\Gamma \cap L$ is finite, this implies that L/H has finite volume, which is a contradiction. □

We now state two special cases of the Borel Density Theorem.

6.4. Lemma (cf. [Rag, Thm. 5.5, p. 79]). *Let H be a closed subgroup of G. If H is connected, and G/H has finite (G-invariant) volume, then $H = G$.*

6.5. Lemma (Mostow [Mo1, Prop. 11.2], [Rag, Thm. 3.1, p. 43]). *Suppose that R is a simply connected, solvable Lie group. If H is a closed, connected subgroup of R, such that R/H either is compact or has finite (R-invariant) volume, then $H = R$.*

The following lemma is analogous to the fact [Rag, Rmk. 1.9, p. 21] that a locally compact group that admits a lattice must be unimodular.

6.6. Lemma (cf. [Zim, Prop. 2.1]). *Let H be a unimodular subgroup of G, and assume that there is an element a of the normalizer of H such that the action of a by conjugation on H does not preserve the Haar measure on H. Then G/H has neither compact nor finite-volume Clifford-Klein forms.*

6.7. Proposition. *Assume that $G = \mathrm{SO}(2,n)$. If H is a nontrivial, connected, unipotent subgroup of G, then G/H has neither a compact nor a finite-volume Clifford-Klein form.*

Proof. Replacing H by a conjugate, we may assume that H is contained in N. Furthermore, because Lemma 6.4 implies that G/H does not have finite volume, we see from Lemma 6.3 that we may assume that H is not a Cartan-decomposition subgroup. The proof now breaks up into cases, determined by Proposition 2.9. Because H is unimodular, any compact Clifford-Klein form would also have finite volume, so we need only consider the more general finite-volume case.

Case 1. Assume that $\dim H \leq 1$. Because H is nontrivial and connected, we must have $\dim H = 1$. Then H is contained in a subgroup S of G that is locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$, and we have $H = N_S$, where $S = K_S A_S N_S$ is an Iwasawa decomposition of S [Jac, Thm. 17(1), p. 100]. The subgroup A_S normalizes H, but does not preserve the Haar measure on H, so we conclude from Lemma 6.6 that G/H does not have a finite-volume Clifford-Klein form.

Henceforth, we assume that $\dim X \geq 2$.

Case 2. Assume, for every nonzero element h of $\mathfrak{h}$, that we have $\phi_h = 0$ and $\dim\langle x_h, y_h\rangle \neq 1$. From Lemma 2.9(2), we have $\mu(H) \approx \left[\|h\|^2, \|h\|^2\right] \approx \mu(U_{\alpha+2\beta}H)$. Thus, if G/H has a finite-volume Clifford-Klein form, then Lemma 6.3 implies that $U_{\alpha+2\beta}H/H$ must have finite volume. Therefore, Lemma 6.5 implies that $U_{\alpha+2\beta}H/H$ must be trivial, so $U_{\alpha+2\beta} \subset H$. Then $\ker\alpha$ normalizes H, but the action of $\ker\alpha$ by conjugation on H does not preserve the Haar measure on H, so Lemma 6.6 implies that G/H does not have a finite-volume Clifford-Klein form.

Case 3. Assume, for every nonzero element h of $\mathfrak{h}$, that we have $\phi_h = 0$ and $\dim\langle x_h, y_h\rangle = 1$. Because $\dim\mathfrak{h} \geq 2$, we know that $\mathfrak{h} \not\subset \mathfrak{u}_{\alpha+2\beta}$, so it cannot be the case that both of x_h and y_h are 0, for every $h \in \mathfrak{h}$. Therefore, by perhaps replacing H with its conjugate under the Weyl reflection corresponding to the root α, we may assume that $x_{h_0} \neq 0$ for some $h_0 \in \mathfrak{h}$. Then, because $\dim\langle x_h, y_h\rangle = 1$ for every $h \in \mathfrak{h}$, it follows that there is a real number p, such that for all $h \in \mathfrak{h}$, we have $y_h = p x_h$. Therefore, by replacing H with a conjugate under $U_{-\alpha}$, we may assume that $y_h = 0$ for every $h \in \mathfrak{h}$. So $\mathfrak{h} \subset \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta}$, so $\ker\beta$ normalizes H. But the action of $\ker\beta$ by conjugation on H does not preserve the Haar measure on H, so Lemma 6.6 implies that G/H does not have a finite-volume Clifford-Klein form.

Case 4. Assume that there exists a subspace X_0 of $\mathbb{R}^{n-2}$, $b \in X_0$, $c \in X_0^{\perp}$, and $p \in \mathbb{R}$ with $\|b\|^2 - \|c\|^2 - 2p < 0$, such that for every element of $\mathfrak{h}$, we have $y = 0$, $x \in \phi c + X_0$, and $\eta = p\phi + b \cdot x$. Replacing H by a conjugate, we may assume that H is contained in $G' = \mathrm{SO}(1,n)$ (see 2.17). Then H is a Cartan-decomposition subgroup of G' (see 3.2), but Lemma 6.4 implies that G'/H does not have finite volume, so we conclude from Lemma 6.3 that G/H does not have a finite-volume Clifford-Klein form. □

Proof of Theorem 6.2. By replacing H with a conjugate, we may assume that there is a compact subgroup C of G and a closed, connected subgroup H' of AN, such that $CH = CH'$ (see 3.8). (Note that if H' is unimodular, then it is easy to see that G/H', like G/H, has a finite-volume Clifford-Klein form.) We may assume that H is not a Cartan-decomposition subgroup (see 6.3 and 6.4), so H' is not a Cartan-decomposition subgroup. From Proposition 6.7, we see that $H' \not\subset N$. Then, assuming, as we may, that H' is compatible with A (see 2.12), the subgroup H' must be one of the subgroups described in either Corollary 2.10 or Corollary 2.13. Furthermore, we may assume that $\dim H' \geq 2$, for otherwise, conclusion (2) holds.

The only unimodular subgroups listed in either Corollary 2.10 or Corollary 2.13 are the subgroups of type 2.10(1), which are one-dimensional. Therefore, H' is not unimodular. Because $CH' = CH$ is unimodular, this implies that C does not normalize H', so we see from 3.8 that $H/\operatorname{Rad} H$ is not compact. In other words, we may assume that there is a connected, noncompact, semisimple Lie group L with no compact factors, and a closed, connected subgroup R of AN, such that $H = L \ltimes R$. Because H is not a Cartan-decomposition subgroup, we know that $A \not\subset L$, so $\mathbb{R}$-rank $L = 1$. Thus, $L \cap A$ is one-dimensional, so $H' \cap A$ is nontrivial. Thus, H' cannot be any of the subgroups listed in Corollary 2.13. In other words, we have $H' = (H' \cap A) \ltimes (H' \cap N)$. Also, because $A \not\subset H$, we know that that $R \subset N$ (see 2.15).

Case 1. Assume that R is nontrivial. Because R is a unipotent, normal subgroup of H, a fundamental theorem of Borel and Tits [BT2, Prop. 3.1] implies that there is a parabolic subgroup P of G, such that H is contained in P, and R is contained in the unipotent radical of P. Now, because $\mathbb{R}$-rank $G = 2$, there are, up to conjugacy, only two parabolic subgroups of G whose maximal connected semisimple subgroups have real rank one.

Subcase 1.1. Assume that $L = \langle U_\alpha, U_{-\alpha}\rangle$ and that $R \subset U_\beta U_{\alpha+\beta} U_{\alpha+2\beta}$. Every nontrivial L-invariant subalgebra of $\mathfrak{u}_\beta + \mathfrak{u}_{\alpha+\beta} + \mathfrak{u}_{\alpha+2\beta}$ contains $\mathfrak{u}_{\alpha+2\beta}$, so $\mathfrak{r}$ must contain $\mathfrak{u}_{\alpha+2\beta}$. Then $\ker\alpha$ normalizes L and R but does not preserve the Haar measure on LR, so Lemma 6.6 implies that G/H does not have a finite-volume Clifford-Klein form.

Subcase 1.2. Assume that $L = \langle U_\beta, U_{-\beta}\rangle$ and that $R \subset U_\alpha U_{\alpha+\beta} U_{\alpha+2\beta}$. Then $\ker\beta$ normalizes L and R, but does not preserve the Haar measure on LR, so Lemma 6.6 implies that G/H does not have a finite-volume Clifford-Klein form.

Case 2. Assume that R is trivial. This means that H is reductive. Then, because $\mathbb{R}\text{-rank}\, H = 1$, we may assume that H is almost simple, by removing the compact factors of H. Thus, we see from Lemma 6.8 (and 6.3) that the identity component of H is conjugate under $\mathrm{O}(2,n)$ to the identity component of either $\mathrm{SO}(1,n)$, $\mathrm{SU}(1,\lfloor n/2\rfloor)$, or L_5, as desired. □

We now prove a classification theorem we used in the above proof. Although this result is known, the authors are not aware of any convenient reference.

6.8. Lemma. *Assume that $G = \mathrm{O}(2,n)$. If L is any connected, almost-simple subgroup of G, such that $\mathbb{R}\text{-rank}\, L = 1$, then L is conjugate to a subgroup of either $\mathrm{SO}(1,n)$, $\mathrm{SU}(1,\lfloor n/2\rfloor)$, or L_5.*

Sketch of proof. Let $L = K_L A_L N_L$ be an Iwasawa decomposition of L, and let $H = A_L N_L$. We may assume that $A_L \subset A$ and that $N_L \subset N$. Because $\mu(L) \approx \mu(A_L) \not\approx A^+$, we know that H is not a Cartan-decomposition subgroup of G, so H must be one of the subgroups described in Theorem 2.10. Because H is an epimorphic subgroup of L [BB, §2], it suffices to show that H is conjugate to a subgroup of either $\mathrm{SO}(1,n)$, $\mathrm{SU}(1,\lfloor n/2\rfloor)$, or L_5.

Because N_L is nontrivial, we know that H is not of type 2.10(1).

If H is of type 2.10(3), 2.10(4), 2.10(5), or 2.10(7), then H is conjugate to a subgroup of $\mathrm{SO}(1,n)$ (cf. 2.17).

If H is of type 2.10(6), then H is conjugate to a subgroup of L_5.

If H is of type 2.10(8), then, because $\mu(H) \approx \mu(A_L)$, we see from Proposition 2.14 that $p = 0$. Replacing H by a conjugate under the Weyl group, we may assume that $\omega \in \{\alpha+\beta, \alpha+2\beta\}$. Therefore, H is contained in either $\mathrm{SO}(1,n)$ or $\mathrm{SU}(1,\lfloor n/2\rfloor)$.

We may now assume that H is of type 2.10(2).

Case 1. Assume that $\mathfrak{u}_{\alpha+2\beta} \subset \mathfrak{n}_L$. There is a subspace V of $\mathbb{R}^{n-2}$ and a linear transformation $B\colon V \to \mathbb{R}^{n-2}$, such that

$$\mathfrak{n}_L = \{\, h \in \mathfrak{n} \mid x_h \in V,\ y_h = Bx_h,\ \phi_h = 0 \,\}.$$

Because $[\mathfrak{n}_L, \mathfrak{n}_L] = \mathfrak{u}_{\alpha+2\beta}$ is one-dimensional, the classification of simple Lie groups of real rank one ($\mathrm{SO}(1,k)$, $\mathrm{SU}(1,k)$, $\mathrm{Sp}(1,k)$, F_4^{-20}) implies that L is locally isomorphic to $\mathrm{SU}(1,k)$, where $k-1 = (\dim V)/2$.

For any $h_0 \in \mathfrak{n}_L \setminus \mathfrak{u}_{\alpha+2\beta}$, we have $\langle h_0, \mathfrak{u}_{\alpha+2\beta}, \mathfrak{u}_{-(\alpha+2\beta)}\rangle \cong \mathfrak{su}(1,2)$. In particular, if h_0 is the element of $\mathrm{SO}(2,4)$ with $x_{h_0} = (1,0)$ and $y_{h_0} = (0,1)$, then

$$\langle h_0, \mathfrak{u}_{\alpha+2\beta}, \mathfrak{u}_{-(\alpha+2\beta)}\rangle \cap \mathfrak{n} = \left\{ \begin{pmatrix} 0 & 0 & s & t & \eta & 0 \\ & 0 & -t & s & 0 & -\eta \\ & & & \cdots & & \end{pmatrix} \;\middle|\; s,t,\eta \in \mathbb{R} \right\}.$$

From this, we conclude that $B(V) = V$.

The desired conclusion is easy if $k = 1$, so let us assume that $k \geq 2$. ¿From the structure of $\mathrm{SU}(1,k)$, we know that $C_L(A_L)$ contains a subgroup M that is isomorphic to $\mathrm{SU}(k-1)$ and acts transitively on the unit sphere in $\mathfrak{n}_L/\mathfrak{u}_{\alpha+2\beta}$. Then the action of M on V is transitive on the unit sphere in V and, because M normalizes $\mathfrak{n}_L$, we see that B is M-equivariant. Therefore B is a scalar multiple of an orthogonal transformation, so H is conjugate to a subgroup of $\mathrm{SU}(1,\lfloor n/2\rfloor)$.

Case 2. Assume that $\mathfrak{u}_{\alpha+2\beta} \not\subset \mathfrak{n}_L$. There is a subspace V of $\mathbb{R}^{n-2}$ and a linear transformation $B\colon V \to \mathbb{R}^{n-2}$, such that

$$\mathfrak{n}_L = \{\, h \in \mathfrak{n} \mid x_h \in V,\ y_h = Bx_h,\ \phi_h = \eta_h = 0 \,\}.$$

Note that N_L must be abelian (because $\mathfrak{n}_L \cap \mathfrak{u}_{\alpha+2\beta} = 0$), so, from the classification of simple Lie groups of real rank one, we see that L is locally isomorphic to $\mathrm{SO}(1,k)$, where $k-1 = \dim V$.

Assume, for the moment, that $k \geq 4$, so $\mathrm{SO}(k-1)$ is almost simple. ¿From the structure of $\mathrm{SO}(1,k)$, we see that $C_L(A_L)$ contains a subgroup M that is isomorphic to $\mathrm{SO}(k-1)$ and acts transitively on the unit spheres in V and $B(V)$, such that B is M-equivariant. (In particular, the ratio $\|Bv\|/\|v\|$ is constant on $V \setminus \{0\}$.) Therefore, letting $\pi\colon \mathbb{R}^{n-2} \to V$ be the orthogonal projection, we see that the composition $\pi \circ B\colon V \to V$ must be a real scalar. Replacing L by a conjugate under $U_{-\alpha}$, we may assume that this scalar is 0; thus, $B(V)$ is orthogonal to V, so H is conjugate to a subgroup of $\mathrm{SU}(1, \lfloor n/2 \rfloor)$, as desired.

Now assume that $k < 4$. The desired conclusion is easy if $k = 2$, so we may assume that $k = 3$. Let V_1 be an irreducible summand of the L-representation on $\mathbb{R}^{n-2}$, and let $\langle | \rangle$ be the $\mathrm{SO}(2,n)$-invariant bilinear form on $\mathbb{R}^{n+2}$.

Subcase 2.1. Assume that the restriction of $\langle | \rangle$ to V_1 is 0. There must be L-invariant subspaces V_0 and V_{-1}, such that $\mathbb{R}^{n+2} = V_{-1} \oplus V_0 \oplus V_1$, and $\langle V_i \mid V_j \rangle = 0$ unless $i = -j$. Thus, the restriction of $\langle | \rangle$ to $V_{-1} \oplus V_1$ is split. Because $\langle | \rangle$ has signature $(2,n)$, we conclude that $\dim V_{\pm 1} \leq 2$. Because the smallest nontrivial representation of L is 3-dimensional, this implies that L acts trivially on $V_{-1} \oplus V_1$. Therefore, L fixes a vector of norm -1, so L is contained in a conjugate of $\mathrm{SO}(1,n)$.

Subcase 2.2. Assume that the restriction of $\langle | \rangle$ to V_1 has signature $(1,m)$, for some m. Let $V_1 = W_{-r} \oplus \cdots \oplus W_{r-1} \oplus W_r$ be the decomposition of V_1 into weight spaces (with respect to A_L). We must have $\langle W_i \mid W_j \rangle = 0$ unless $i = -j$. From the assumption of this subcase, we conclude that $r = 1$ and $\dim W_1 = 1$. It follows that V_1 is the standard representation of L on $\mathbb{R}^4$ (whose L-invariant bilinear form is unique up to a real scalar).

There must be another irreducible summand V_2, such that the restriction of $\langle | \rangle$ to V_2 has signature $(1,*)$, and $\mathbb{R}^{n+2} = V_1 \oplus V_2$. From the argument of the preceding paragraph, we see that $\dim V_2 = 4$, and that the representations of L on V_1 and V_2 are isomorphic (so L acts diagonally on $\mathbb{R}^4 \oplus \mathbb{R}^4 \cong \mathbb{R}^8$). Then L is conjugate to a subgroup of $\mathrm{SU}(1,3)$.

Subcase 2.3. Assume that $V_1 = \mathbb{R}^{n+2}$. Let $V_1 = W_{-r} \oplus \cdots \oplus W_{r-1} \oplus W_r$ be the decomposition of V_1 into weight spaces (with respect to A_L).

If $\dim W_r = 2$, then V_1 is an irreducible $\mathbb{C}$-representation of $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$, so $\dim W_i = 2$ for all i. From the signature of $\langle | \rangle$, we must have $r = 1$, so V_1 is the 3-dimensional irreducible $\mathbb{C}$-representation of $\mathfrak{sl}(2,\mathbb{C})$, that is, the adjoint representation. However, the invariant bilinear form $\mathrm{Re}\big(\mathrm{Tr}\, X^2\big)$ for this representation has signature $(3,3)$, not $(2,n)$. This is a contradiction.

We now know that $\dim W_r = 1$. Also, we must have $r = 2$ (because the argument of Subcase 2.2 applies if $r = 1$). The signature of $\langle | \rangle$ implies that $\dim W_1 = \dim W_2 = 1$. There is no such irreducible representation of $\mathfrak{so}(1,3)$, so this is a contradiction. $\square$

7. Non-existence of compact Clifford-Klein forms of $\mathrm{SL}(3,\mathbb{R})/H$

Y. Benoist [Ben, Cor. 1] proved that $\mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(2,\mathbb{R})$ does not have a compact Clifford-Klein form. By combining Benoist's method with the fact that if G/H has a compact Clifford-Klein form, then H cannot be one-dimensional (see 3.7), we show, more generally, that no interesting homogeneous space of $\mathrm{SL}(3,\mathbb{R})$ has a compact Clifford-Klein form.

7.1. Theorem (Benoist [Ben, Thms. 3.3 and 4.1] (see also Prop. A.1))**.** *Let H be a closed, connected subgroup of G, such that, for some compact set C in A, we have $B^+ \subset \mu(H)C$, where B^+ is defined in Notation 7.2. Then G/H has neither compact nor finite-volume Clifford-Klein forms, unless G/H is compact.*

7.2. Notation. Let i be the opposition involution in A^+, that is, for $a \in A^+$, $i(a)$ is the unique element of A^+ that is conjugate to a^{-1}, and set $B^+ = \{\, a \in A^+ \mid i(a) = a \,\}$.

7.3. Proposition. *Assume that $G = \mathrm{SL}(3,\mathbb{R})$. Let H be a closed, connected subgroup of G with $d(H) \geq 2$ (see 3.3). Then $B^+ \subset \mu(H)$.*

Proof. Since H is non-compact, there is a curve h_t in H such that $h_0 = e$ and $h_t \to \infty$ as $t \to \infty$. Since $d(H) \geq 2$ (so H/K_H is homeomorphic to $\mathbb{R}^k$, for some $k \geq 2$), it is easy to find a continuous and proper map $\Phi \colon [0,1] \times \mathbb{R}^+ \to H$ such that $\Phi(0,t) = h_t$ and $\Phi(1,t) = h_t^{-1}$, for all $t \in \mathbb{R}^+$.

If we identify the Lie algebra $\mathfrak{a}$ of A with the connected component of A containing e, then A^+ is a convex cone in $\mathfrak{a}$ and the opposition involution i is the reflection in A^+ across the ray B^+. Thus, for any $a \in A^+$, the points a and $i(a)$ are on opposite sides of B^+, so any continuous curve in A^+ from a to $i(a)$ must intersect B^+. In particular, any curve from $\mu(h_t)$ to $\mu(h_t^{-1})$ must intersect B^+. Thus, we see, from an elementary continuity argument, that $\mu\big[\Phi\big([0,1] \times \mathbb{R}^+\big)\big]$ contains B^+. Therefore, B^+ is contained in $\mu(H)$. □

Proof of Proposition 1.10. Suppose that H is non-compact. By Proposition 3.7, we may assume that $d(H) \geq 2$. Then, from Proposition 7.3, we know that $\mu(H) \supset B^+$, so Theorem 7.1 implies that G/H is compact. □

The following is proved similarly.

7.4. Corollary. *Let H be a closed, connected subgroup of $G = \mathrm{SL}(3,\mathbb{R})$. If G/H has a finite-volume Clifford-Klein form, then either $d(H) \leq 1$, or $H = G$.*

Appendix A. A short proof of a theorem of Benoist

Most of Benoist's paper [Ben] is stunningly elegant; the only exception is Section 4, which presents a somewhat lengthy argument to eliminate a troublesome case. We provide an alternative treatment of this one case. Our proof does not match the elegance of the rest of Benoist's paper, but it does get the unpleasantness over fairly quickly.

Our version of the result is not a complete replacement for Benoist's, because we require the action of Γ on G/H to be proper, while Benoist makes no such assumption. There are many situations where one is interested in improper actions (for example, a quotient of a proper action is usually not proper), but, in applications to Clifford-Klein forms, the action is indeed proper, so our weaker version of the proposition does apply in that situation. Our proof has the virtue that, for the real field, it does not require the subgroup H to be Zariski closed. It also applies to finite-volume Clifford-Klein forms, not just compact ones.

A.1. Proposition (Benoist [Ben, Thm. 4.1])**.** *Let $\mathbb{G}$ be a Zariski connected, semisimple, algebraic group over a local field k of characteristic 0, and let $G = \mathbb{G}_k$. Suppose that Γ and H are closed subgroups of G, and assume that*

1) *Γ is nilpotent;*
2) *Γ acts properly on G/H;*
3) *either*
   (a) *$\Gamma \backslash G/H$ is compact, or*
   (b) *Γ is discrete and $\Gamma \backslash G/H$ has finite volume; and*
4) *either*
   (a) *H is Zariski closed, or*
   (b) *$k = \mathbb{R}$ or $\mathbb{C}$, and H is almost connected.*

Then H contains a conjugate of N.

Proof. To clarify the exposition, let us assume that Γ is abelian, that $H = (H \cap A) \ltimes (H \cap N)$, and that $\Gamma \backslash G/H$ is compact. Remark A.3 describes appropriate modifications of the proof to eliminate these assumptions.

Let X be the Zariski closure of Γ. By replacing Γ with a finite-index subgroup, we may assume that X is Zariski connected. Note that X, like Γ, is abelian. Then, by replacing Γ with a conjugate subgroup, we may assume that $X = (X \cap A)(X \cap N)E$, where E is an anisotropic torus, hence compact. Because we may replace Γ with $(\Gamma E) \cap (AN)$, there is no harm in assuming that $\Gamma \subset AN$.

A.2. Notation (and remarks).

- We define a right-invariant metric on G by $d(g,h) = \log \|gh^{-1}\|$.
- Let $\pi\colon AN \to A$ and $\nu\colon AN \to N$ be the canonical projections, so $g = \pi(g)\nu(g)$, for all $g \in AN$. Note that π is a homomorphism, but ν is not. Also note that $\pi(H) = H \cap A$ and $\nu(H) = H \cap N$, because $H = (H \cap A) \ltimes (H \cap N)$. Also note that $\pi(\Gamma) \subset X \cap A$ and $\nu(\Gamma) \subset X \cap N$, so $\pi(\Gamma)$ and $\nu(\Gamma)$ centralize each other. Thus, the restriction of ν to Γ *is* a homomorphism.
- Because $\pi(\Gamma) \subset A$, and the centralizer of any Zariski-connected subgroup of A is a Zariski-connected, reductive k-subgroup of G [Hum, Thm. 22.3, p. 140], we may write $C_G\big(\pi(\Gamma)\big) = LZ$, where L is a Zariski-connected, reductive k-subgroup of G with compact center, and Z is a Zariski closed (abelian) subgroup of A that centralizes L. We have $\Gamma \subset LZ$ and $\pi(\Gamma) \subset Z$. Note that we must have $\nu(\Gamma) \subset L$, because Z, being a subgroup of A, has no nontrivial unipotent elements.
- Let $\mu_L\colon L \to A \cap L$ be the Cartan projection for the reductive group L. We define $\mu_{LZ}\colon LZ \to A$ by $\mu_{LZ}(lz) = \mu_L(l)z$, for $l \in L$ and $z \in Z$. (Unfortunately, μ_{LZ} is not well defined if $L \cap Z \neq e$. On the other hand, at key points of the proof, we only calculate μ_{LZ} up to bounded error, so, because $L \cap Z$ is finite, this is not an important issue.) In particular, for $\gamma \in \Gamma$, we have $\mu_{LZ}(\gamma) = \mu_L\big(\nu(\gamma)\big)\pi(\gamma)$. Note that, because $\mu_{LZ}(g) \in (K \cap L)g(K \cap L)$, we have $\mu(g) = \mu\big(\mu_{LZ}(g)\big)$, for every $g \in LZ$. Therefore, for any $g \in LZ$ and $a \in A$, we have $d\big(\mu(g), \mu(a)\big) \leq d\big(\mu_{LZ}(g), a\big)$. Thus, if we find a sequence $\gamma_n \to \infty$ in Γ, with $d\big(\mu_{LZ}(\gamma_n), H \cap A\big) = O(1)$, then we have obtained a contradiction to the fact that Γ acts properly on G/H.

Because $\Gamma \backslash AN/H$ is a closed subset of the compact space $\Gamma \backslash G/H$, we know that it is compact, so we see, by modding out N, that $A/\overline{\pi\big(\Gamma H\big)}$ is compact, so there is a free abelian subgroup Γ' of Γ, such that

1) $\pi(\Gamma')$ is a co-compact, discrete subgroup of $A/\pi(H)$; and
2) $\Gamma' \cap (NH) = e$.

We may assume that G/H is not compact (else AN/H is compact, so Lemma 6.5 implies that $H \supset N$, as desired). Then H cannot be a Cartan-decomposition subgroup of G, so $\dim\big(\pi(H)\big) < \dim A$ (see 2.15). Therefore, $A/\pi(H)$ is not compact, so Γ' is infinite.

Suppose, for the moment, that Γ' is co-compact in Γ. (In this case, there is no harm in assuming that $\Gamma = \Gamma'$.) Then, from (1), we see that $\pi(\Gamma H)$ is closed, so the inverse image in $\Gamma \backslash AN/H$ must be compact; that is, $\Gamma \backslash \Gamma NH/H$ is compact. Therefore, from (2), we see that NH/H is compact. Then $N/(H \cap N)$, being homeomorphic to NH/H, is compact. Therefore, $N \subset H$ (see 6.5) as desired.

We may now assume that Γ/Γ' is not compact. Then, for any $R > 0$, there is some $\gamma_0 \in \Gamma$, such that $d(\gamma_0, \Gamma') > R$. (Let γ_0 be any element of Γ that is not $B_R(e)\Gamma'$, where $B_R(e)$ is the closed ball of radius R around e.) Because $\pi(\Gamma')$ is co-compact in $A/\pi(H)$, there is some $\gamma' \in \Gamma'$, such that $d\big(\mu_{LZ}(\gamma_0)\pi(\gamma'), \pi(H)\big) < C$, where $C > 0$ is an appropriate constant that is independent of R, γ_0, and γ'.

Now, to clarify the argument, assume, for the moment, that $\Gamma' \subset Z$. In this case, we see from the definition that $\mu_{LZ}(\gamma_0\gamma') = \mu_{LZ}(\gamma_0)\gamma'$, and we have $\gamma' = \pi(\gamma')$, so $d\big(\mu_{LZ}(\gamma_0\gamma'), \pi(H)\big) < C$. But, because $\gamma_0\gamma' \in \gamma_0\Gamma'$, we have $\log\|\gamma_0\gamma'\| \geq d\big(\gamma_0, \Gamma'\big) > R$. Because R can be made arbitrarily large, while C is fixed, this contradicts the fact that Γ acts properly on G/H.

Now consider the general case, where Γ' is not assumed to be contained in Z. For convenience, define $f\colon \Gamma \to \mathbb{R}^+$ by $f(\gamma) = d\big(\mu_{LZ}(\gamma), \pi(H)\big)$, and, for $\gamma \in \Gamma'$, let $\ell(\gamma)$ be the word length of γ

with respect to some (fixed) finite generating set of Γ'. To a good approximation, the argument of the preceding paragraph holds. Note that $\ell(\gamma')$ is of order $f(\gamma_0)$. The map $\Gamma' \to \mathbb{R}^+ : \gamma \mapsto \|\nu(\gamma)\|$ is bounded above by a polynomial function of $\ell(\gamma)$ (because $\nu(\Gamma) \subset N$ consists of unipotent matrices), so

$$\|\nu(\gamma')\| = \big(\ell(\gamma')\big)^{O(1)} = \big(f(\gamma_0)\big)^{O(1)}.$$

Therefore, from Lemma A.4, we have

$$\begin{aligned} d\Big(\mu_{LZ}\big(\gamma_0\pi(\gamma')\big), \mu_{LZ}(\gamma_0\gamma')\Big) &= O(1) + \log\|\pi(\gamma')^{-1}\gamma'\| + \log\|(\gamma')^{-1}\pi(\gamma')\| \\ &= O(1) + \log\|\nu(\gamma')\| + \log\|\nu(\gamma')^{-1}\| \\ &= O(1) + O\big(\log^+ f(\gamma_0)\big), \end{aligned}$$

so

$$\begin{aligned} f(\gamma_0\gamma') &\leq f\big(\gamma_0\pi(\gamma')\big) + d\Big(\mu_{LZ}\big(\gamma_0\pi(\gamma')\big), \mu_{LZ}(\gamma_0\gamma')\Big) \\ &< C + \Big(O(1) + O\big(\log^+ f(\gamma_0)\big)\Big) < \frac{f(\gamma_0)}{2}, \end{aligned}$$

if $f(\gamma_0) > C_1$, where C_1 is an appropriate constant that is independent of R, γ_0, and γ'.

This is the start of an inductive procedure: given γ_0 with $d(\gamma_0, \Gamma') > R$, construct a sequence $\gamma_0, \gamma_1, \ldots, \gamma_M$ in $\gamma_0\Gamma'$, such that $f(\gamma_{n+1}) < f(\gamma_n)/2$, for each n. Terminate the sequence when $f(\gamma_M) \leq C_1$, which, obviously, must happen after only finitely many steps. However, because $\log\|\gamma_M\| \geq d(\gamma_M, \Gamma') = d(\gamma_0, \Gamma') > R$ is arbitrarily large, this contradicts the fact that Γ acts properly on G/H. $\square$

A.3. Remark. We now describe how to eliminate the simplifying assumptions made at the start of the proof of Theorem A.1. We discuss only one assumption at a time; the general case is handled by employing a combination of the arguments below.

1) *Suppose that* Γ *is not abelian.* Because Γ is nilpotent, its Zariski closure (if connected) is a direct product $T \times U \times E$, where T is a split torus, U is unipotent, and E is a compact torus. Thus, the beginning of the proof remains valid without essential change, up to (but not including) the definition of Γ'.

   Choose a co-compact, discrete, free abelian subgroup $\bar{\Gamma}$ of $A/(H \cap A)$ that is contained in $\pi(\Gamma)$, choose $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma$, such that $\pi(\gamma_1), \pi(\gamma_2), \ldots, \pi(\gamma_n)$ is a basis of $\bar{\Gamma}$, and let Γ' be the subgroup generated by $\gamma_1, \gamma_2, \ldots, \gamma_n$. (We remark that Γ' may not be closed, because $[\Gamma', \Gamma']$ need not be closed.)

   If the closure of Γ' is not co-compact in Γ, then essentially no changes are needed in the proof. Thus, let us suppose that that the closure of Γ' is co-compact in Γ. Then there is no harm in assuming that Γ' is dense in Γ, so Γ' is not abelian.

   If k is nonarchimedean, then the closure of every finitely generated unipotent subgroup is compact; thus, by replacing Γ with its projection into $X \cap A$, we may assume that Γ is abelian, so the original proof is valid.

   We may now assume that k is archimedean. Because the closure of $[\Gamma, \Gamma]$ is a nontrivial unipotent group, we know that it is noncompact. Thus, there is some $\gamma_0 \in [\Gamma, \Gamma]$ with $\|\gamma_0\| > R$.

   Choose $\gamma' \in \Gamma'$, such that $d\big(\mu_{LZ}(\gamma_0)\pi(\gamma'), \pi(H)\big) < C$. By choosing γ' efficiently, we may assume that $\ell(\gamma') = O\big(\log(\|\gamma_0\|)\big)$. This is the inductive step in the construction of $\gamma_0, \gamma_1, \ldots, \gamma_M$. Because $\ell(\gamma_n^{-1}\gamma_{n+1}) = O\big(\log(\|\gamma_n^{-1}\gamma_{n+1}\|)\big)$, we see that

   $$\|\nu(\gamma_0^{-1}\gamma_M)\| = O(M \log\|\gamma_0\|) = O\big((\log\|\gamma_0\|)^2\big) \ll \|\gamma_0\| = \|\nu(\gamma_0)\|.$$

   Therefore $\|\gamma_M\| \geq \|\nu(\gamma_M)\| \approx \|\nu(\gamma_0)\|$ is large. So Γ is not proper on G/H.

2) *Do not assume that* $H = (H \cap A) \ltimes (H \cap N)$. There is no harm in assuming that $H \subset AN$ (see 3.8 or [Ben, Lem. 4.2.2]). If H is Zariski closed (and Zariski connected), then, after replacing H by a conjugate subgroup, we have $H = (H \cap A) \ltimes (H \cap N)$. Thus, the problem only arises when H is not Zariski closed. In this case, H must be almost connected (and k must be archimedean).

The proof remains unchanged up until the choice of γ'. Instead of only choosing an element $\gamma' \in \Gamma'$, we also choose an element $h' \in H$. Namely, let $h_0 = e$, and choose $\gamma' \in \Gamma'$ and $h' \in H$, such that

$$d\big(\mu_{LZ}(\gamma_0)\pi(\gamma'), \mu_{LZ}(h_0)\pi(h')\big) < C.$$

This begins the inductive construction of sequences $\gamma_0, \gamma_1, \dots, \gamma_M$ in Γ and $h_0, h_1, \dots, h_M$ in H, such that $d(\gamma_n, e) > R$ for each n, and

$$d\big(\mu_{LZ}(\gamma_M), \mu_{LZ}(h_M)\big) < C_2,$$

for an appropriate constant C_2 that is independent of R. This contradicts the fact that Γ is proper on G/H.

3) *Suppose that* $\Gamma\backslash G/H$ *is not compact.* The compactness was used only to show that $A/\overline{\pi(\Gamma H)}$ is compact, and that if Γ' is co-compact in Γ, then H contains N. Thus, it suffices to show, after replacing H by a conjugate, that (3a) $A/\overline{\pi\big(\Gamma(H \cap AN)\big)}$ is compact, and that (3b) if Γ' is co-compact in Γ, then H contains a conjugate of N.

The finite measure on $\Gamma\backslash G/H$ pushes to a finite measure on the quotient $AN\backslash G/H$. (We may assume that $\operatorname{Rad} H \subset AN$, so the quotient $AN\backslash G/H$ is countably separated.) By considering a generic fiber of this quotient map, we see that we may assume, after replacing H by a conjugate subgroup, that $\Gamma\backslash AN/(H \cap AN)$ has finite volume.

(a) Because $A/\overline{\pi\big(\Gamma(H \cap AN)\big)}$ is a quotient of $\Gamma\backslash AN/(H \cap AN)$, it must have finite volume. Therefore, it is compact.

(b) Suppose that Γ' is co-compact in Γ. Almost every fiber of the quotient map $\Gamma\backslash AN/(H \cap AN) \to A/\overline{\pi\big(\Gamma(H \cap AN)\big)}$ must have finite measure. Thus, we may assume that $\Gamma\backslash\Gamma N(H \cap AN)/(H \cap AN)$ has finite measure. Because Γ' is co-compact in Γ (and Γ is discrete), this implies that $N(H \cap AN)/(H \cap AN)$ has finite measure, or, equivalently, that $N/(H \cap N)$ has finite measure. Therefore, $H \cap N = N$ (see 6.5), so $N \subset H$, as desired.

A.4. Proposition (Benoist, cf. [Ben, Prop. 5.1]). *There is a constant* $C > 0$ *such that, for all* $g, h \in G$, *we have*

$$\max\big\{\|\mu(g)^{-1}\mu(gh)\|, \|\mu(g)^{-1}\mu(hg)\|\big\} \leq C \max\big\{\|h\|, \|h^{-1}\|\big\}.$$

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