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Geometry of tame tubular families of modules

Abstract. Let $\mathbf{d}$ be the dimension-vector of a direct sum of modules from a separating tubular family of modules over a tame algebra A. We prove that the affine variety $\mathrm{mod}_A(\mathbf{d})$ of all A-modules of dimension-vector $\mathbf{d}$ is irreducible, normal and a complete intersection.

Introduction

The class of finite dimensional algebras (associative, with an identity) over an algebraically closed field K may be divided into two disjoint classes (see [9], [11]). One class consists of tame algebras for which the indecomposable modules occur, in each dimension, in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras whose representation theory is as complicated as the study of finite dimensional vector spaces together with two noncommuting endomorphisms, for which the classification up to isomorphism is a well-known unsolved problem. Hence, we can realistically hope to describe modules only for tame algebras. Given a finite dimensional algebra A over K and a nonnegative vector $\mathbf{d}$ in the Grothendieck group $K_0(A)$ of A, it is an interesting task to study the affine variety $\mathrm{mod}_A(\mathbf{d})$ of all A-modules of dimension-vector $\mathbf{d}$ and the action of the corresponding product $G(\mathbf{d})$ of general linear groups (see [1], [2], [4], [5], [6], [7], [8], [13], [16], [17], [19], [20], [22] for some results in this direction). For example, we may ask when these affine varieties are irreducible, smooth, complete intersections, Gorenstein, Cohen-Macaulay, normal, ... ?

The aim of this paper is to describe the geometry of module varieties $\mathrm{mod}_A(\mathbf{d})$ given by the dimension-vectors $\mathbf{d}$ of modules from the additive categories of separating $\mathbb{P}_1(K)$-families of stable tubes over tame concealed and tubular algebras A (in the sense of [23]), playing a crucial role in recent investigations of tame algebras (see [23], [26], [27], [28]). It is also known that the Auslander–Reiten quiver of a tame algebra A contains a sincere separating family of stable tubes if and only if A is Morita equivalent to a tame concealed or tubular algebra (see [18], [25], [28]). Moreover, the Tits (geometric) integral quadratic form q_A of a tame concealed or tubular algebra A coincides with its (homological) Euler form χ_A, and is weakly nonnegative, that is, $q_A(\mathbf{d}) \geq 0$ for any nonnegative vector $\mathbf{d}$ of $K_0(A)$.

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Mathematics Subject Classification (1991): 14L30, 14M10, 16G20, 16G60, 16G70

The following theorem is the main result of the paper.

Theorem. *Let A be a tame concealed or tubular algebra, $\mathcal{T}$ a separating $\mathbb{P}_1(K)$-family of stable tubes in the Auslander–Reiten quiver of A, and $\mathbf{d}$ the dimension-vector of a module from the additive category of $\mathcal{T}$. Then the affine variety $\operatorname{mod}_A(\mathbf{d})$ is irreducible, normal, a complete intersection and $\dim \operatorname{mod}_A(\mathbf{d}) = \dim G(\mathbf{d}) - q_A(\mathbf{d})$. Moreover, the maximal $G(\mathbf{d})$-orbits in $\operatorname{mod}_A(\mathbf{d})$ form the open sheet consisting of nonsingular points.*

The paper is organized as follows. Section 1 is devoted to some geometric and homological results on the affine varieties of modules, applied in the proof of our main result. In Section 2 we recall basic facts on the module categories of tame concealed and tubular algebras. In Section 3 we describe all maximal orbits in the considered varieties $\operatorname{mod}_A(\mathbf{d})$ and prove that they are irreducible and complete intersections of the required dimension, while in Section 4 we prove that they are normal.

For basic background on the topics considered here we refer to [12], [16], [17], [23] and [24].

The research has been supported by the Polish Scientific Grant KBN No. 2PO3A 012 14 (both authors), SFB 343 Bielefeld (first named author) and the Alexander von Humboldt Foundation (second named author).

1. Geometric preliminaries

Throughout the paper K will denote a fixed algebraically closed field. By an algebra is meant an associative finite dimensional K-algebra with an identity, which we shall assume (without loss of generality) to be basic and connected. For such an algebra A, there exists an isomorphism $A \simeq KQ/I$, where KQ is the path algebra of the Gabriel quiver $Q = Q_A$ of A and I is an admissible ideal of KQ, generated by a (finite) system of forms $\sum_{1\le j\le t} \lambda_j \alpha_{m_j,j} \cdots \alpha_{1,j}$ (called K-linear relations), where $\lambda_1, \dots, \lambda_t$ are elements of K and $\alpha_{m_j,j} \cdots \alpha_{1,j}$, $1 \le j \le t$, are paths of length ≥ 2 in Q having a common source and a common end. Denote by Q_0 the set of vertices of Q, by Q_1 the set of arrows of Q, and by $s, e : Q_1 \to Q_0$ the maps which assign to each arrow $\alpha \in Q_1$ its source $s(\alpha)$ and its end $e(\alpha)$, respectively. The category mod_A of all finite dimensional (over K) left A-modules is equivalent to the category $\operatorname{rep}_K(Q, I)$ of all finite dimensional K-linear representations $V = (V_i, \varphi_\alpha)_{i\in Q_0, \alpha\in Q_1}$ of Q, where V_i, $i \in Q_0$, are finite dimensional K-vector spaces and $\varphi_\alpha : V_{s(\alpha)} \to V_{e(\alpha)}$, $\alpha \in Q_1$, are K-linear maps, satisfying the equations $\sum_{1\le j\le t} \lambda_j \varphi_{\alpha_{m_j,j}} \cdots \varphi_{\alpha_{1,j}} = 0$ for all K-linear relations $\sum_{1\le j\le t} \lambda_j \alpha_{m_j,j} \cdots \alpha_{1,j} \in I$ (see [14, Section 4] for details). We shall identify mod_A with $\operatorname{rep}_K(Q, I)$ and call the finite dimensional left A-modules shortly A-modules. The Grothendieck group $K_0(A)$ of A is then identified with the group $\mathbb{Z}^{Q_0}$, and we may assign to each A-module $V = (V_i, \varphi_\alpha)$ its dimension-vector $\mathbf{dim}\, V = (\dim_K V_i)_{i \in Q_0}$. Moreover, we denote by Γ_A the Auslander-Reiten quiver of A, and by τ_A and τ_A^- the Auslander-Reiten translations $D\,\mathrm{Tr}$ and $\mathrm{Tr}\, D$, respectively. We shall not distinguish between an indecomposable A-module and the vertex of Γ_A corresponding to it. A component in Γ_A of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, $r \geq 1$, is said to be a stable tube (of rank r). For a family $\mathcal{C}$ of components in Γ_A we denote by $\operatorname{add} \mathcal{C}$ the additive category of $\mathcal{C}$ formed by finite direct sums of modules from $\mathcal{C}$.

Fix now a vector $\mathbf{d} = (d_i) \in K_0(A) = \mathbb{Z}^{Q_0}$ with nonnegative coordinates. Denote by $\mathrm{mod}_A(\mathbf{d})$ the set of all representations $V = (V_i, \varphi_i)$ in $\mathrm{rep}_K(Q, I)$ with $V_i = K^{d_i}$ for all $i \in Q_0$. A representation V in $\mathrm{mod}_A(\mathbf{d})$ is given by $d_{e(\alpha)} \times d_{s(\alpha)}$-matrices $V(\alpha)$ determining the maps $\varphi_\alpha : K^{s(\alpha)} \to K^{e(\alpha)}$, $\alpha \in Q_1$, in the canonical bases of K^{d_i}, $i \in Q_0$. Moreover, the matrices $V(\alpha)$, $\alpha \in Q_1$, satisfy the equations

$$\sum_{1 \leq j \leq t} \lambda_j V(\alpha_{m_j,j}) \cdots V(\alpha_{1,j}) = 0$$

for all K-linear relations $\sum_{1 \leq j \leq t} \lambda_j \alpha_{m_j,j} \cdots \alpha_{1,j} \in I$. Therefore, $\mathrm{mod}_A(\mathbf{d})$ is a closed subset of $\prod_{\alpha \in Q_1} K^{d_{e(\alpha)} \times d_{s(\alpha)}}$ in the Zariski topology, and so $\mathrm{mod}_A(\mathbf{d})$ is an affine variety. We note that $\mathrm{mod}_A(\mathbf{d})$ is not necessarily irreducible. The affine (reductive) algebraic group $G(\mathbf{d}) = \prod_{i \in Q_0} \mathrm{Gl}_{d_i}(K)$ acts on the variety $\mathrm{mod}_A(\mathbf{d})$ by conjugation:

$$(gV)(\alpha) = g_{e(\alpha)} V(\alpha) g_{s(\alpha)}^{-1}$$

for $g = (g_i) \in G(\mathbf{d})$, $V \in \mathrm{mod}_A(\mathbf{d})$, $\alpha \in Q_1$. We shall identify an A-module V of dimension-vector $\mathbf{d}$ with the corresponding point of the variety $\mathrm{mod}_A(\mathbf{d})$. The $G(\mathbf{d})$-orbit $G(\mathbf{d})M$ of a module M in $\mathrm{mod}_A(\mathbf{d})$ will be denoted by $\mathcal{O}(M)$. Observe that two A-modules M and N are isomorphic if and only if $\mathcal{O}(M) = \mathcal{O}(N)$. For $M, N \in \mathrm{mod}_A(\mathbf{d})$, we say that N is a degeneration of M if N belongs to the Zariski closure $\overline{\mathcal{O}(M)}$ of $\mathcal{O}(M)$ in $\mathrm{mod}_A(\mathbf{d})$. If $N \in \overline{\mathcal{O}(M)}$ implies $\mathcal{O}(N) = \mathcal{O}(M)$, then the orbit $\mathcal{O}(N)$ is said to be maximal. Clearly, an orbit in $\mathrm{mod}_A(\mathbf{d})$ of maximal dimension is maximal, but the converse is not true in general. It is known that the union of all $G(\mathbf{d})$-orbits in $\mathrm{mod}_A(\mathbf{d})$ of maximal dimension is an open subset of $\mathrm{mod}_A(\mathbf{d})$, called its open sheet (see [16], [17]). Moreover, we have $\dim \mathcal{O}(M) = \dim G(\mathbf{d}) - \dim_K \mathrm{End}_A(M)$ (see [17]).

Assume now that $A = KQ/I$ is triangular (Q has no oriented cycles), and hence of finite global dimension. Then there is a (nonsymmetric) bilinear form $\langle -, - \rangle_A$ on $K_0(A)$ such that

$$\langle \mathbf{dim}\, M, \mathbf{dim}\, N \rangle_A = \sum_{i=0}^{\infty} (-1)^i \dim_K \mathrm{Ext}_A^i(M, N)$$

for all A-modules M, N (see [23, (2.4)]). Then the corresponding quadratic form $\chi_A(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x} \rangle_A$, $\mathbf{x} \in K_0(A)$, is called the Euler form of A. We may also consider the Tits form q_A of A, defined for $\mathbf{x} = (x_i) \in \mathbb{Z}^{Q_0} = K_0(A)$ as follows

$$q_A(\mathbf{x}) = \sum_{i\in Q_0} x_i^2 - \sum_{\alpha\in Q_1} x_{s(\alpha)}x_{e(\alpha)} + \sum_{i,j\in Q_0} r_{ij}x_ix_j,$$

where r_{ij} is the number of K-linear relations with source i and end j in a minimal set $\mathcal{R}$ of K-linear relations generating the ideal I. If $\operatorname{gl.dim} A \leq 2$ then q_A coincides with χ_A [3]. Moreover, we set

$$a(\mathbf{x}) = \sum_{\alpha\in Q_1} x_{s(\alpha)}x_{e(\alpha)} - \sum_{i,j\in Q_0} r_{ij}x_ix_j$$

for $\mathbf{x} = (x_i) \in \mathbb{Z}^{Q_0}$. We note that if A is a tame concealed or tubular algebra then A is triangular and quasi-tilted, that is, of global dimension at most 2 and each indecomposable A-module has projective dimension at most 1 or injective dimension at most 1, and so χ_A and q_A are defined, and in fact they coincide.

Given a module $M \in \operatorname{mod}_A(\mathbf{d})$ we denote by $T_M(\operatorname{mod}_A(\mathbf{d}))$ the tangent space to $\operatorname{mod}_A(\mathbf{d})$ at M and by $T_M(\mathcal{O}(M))$ the tangent space to $\mathcal{O}(M)$ at M. Then there is a canonical monomorphism of K-vector spaces

$$T_M(\operatorname{mod}_A(\mathbf{d}))/T_M(\mathcal{O}(M)) \hookrightarrow \operatorname{Ext}_A^1(M, M)$$

(see [16, (2.7)]). In particular, if $\operatorname{Ext}_A^1(M, M) = 0$ then $\overline{\mathcal{O}(M)}$ is an irreducible component of $\operatorname{mod}_A(\mathbf{d})$ and $\mathcal{O}(M)$ is an open subset of $\operatorname{mod}_A(\mathbf{d})$. The local dimension $\dim_M \operatorname{mod}_A(\mathbf{d})$ is the maximal dimension of the irreducible components of $\operatorname{mod}_A(\mathbf{d})$ containing M. Then the following inequality $\dim_K T_M(\operatorname{mod}_A(\mathbf{d})) \geq \dim_M \operatorname{mod}_A(\mathbf{d})$ holds. Further, $M \in \operatorname{mod}_A(\mathbf{d})$ is said to be a nonsingular point if $\dim_M \operatorname{mod}_A(\mathbf{d}) = \dim_K T_M(\operatorname{mod}_A(\mathbf{d}))$. If M is a nonsingular point of $\operatorname{mod}_A(\mathbf{d})$ then M belongs to exactly one irreducible component of $\operatorname{mod}_A(\mathbf{d})$ [24, (II.2.6)]. Further, the nonsingular points in $\operatorname{mod}_A(\mathbf{d})$ form an open nonempty subset. Finally, $\operatorname{mod}_A(\mathbf{d})$ is said to be a (set theoretic) complete intersection provided $\operatorname{mod}_A(\mathbf{d})$ is the intersection of $\dim \mathbb{A}(\mathbf{d}) - \dim \operatorname{mod}_A(\mathbf{d})$ hypersurfaces in the affine space $\mathbb{A}(\mathbf{d}) = \prod_{\alpha\in Q_1} K^{d_{e(\alpha)}\times d_{s\alpha)}}$. Observe that it is the case when $\dim \operatorname{mod}_A(\mathbf{d}) = a(\mathbf{d})$. We note also that the irreducible components of a complete intersection have the same dimension. We shall need the following fact proved in [2, Proposition 3.2].

Proposition 1.1. *Let A be a triangular algebra of global dimension at most 2, M an A-module with $\operatorname{Ext}_A^2(M, M) = 0$, and $\mathbf{d} = \mathbf{dim}\, M$. Then M is a nonsingular point of $\operatorname{mod}_A(\mathbf{d})$, $\dim_M \operatorname{mod}_A(\mathbf{d}) = a(\mathbf{d})$, and $\dim \mathcal{O}(M) = a(\mathbf{d}) - \dim_K \operatorname{Ext}_A^1(M, M)$.*

We get the following consequences of the above proposition.

Corollary 1.2. *Let A be a triangular algebra with $\operatorname{gl.dim} A \leq 2$, and $\mathbf{d} \in K_0(A)$ be a positive vector. Assume that for any maximal $G(\mathbf{d})$-orbit $\mathcal{O}(M)$ in $\operatorname{mod}_A(\mathbf{d})$ we have $\operatorname{Ext}_A^2(M, M) = 0$. Then $\operatorname{mod}_A(\mathbf{d})$ is a complete intersection and $\dim \operatorname{mod}_A(\mathbf{d}) = a(\mathbf{d})$.*

Proof. See [2, Corollary 3.3]. □

A module variety $\mathrm{mod}_A(\mathbf{d})$ is said to be normal if the local ring $\mathcal{O}_M$ of any module $M \in \mathrm{mod}_A(\mathbf{d})$ is integrally closed in its total quotient ring. It is well-known that if $\mathrm{mod}_A(\mathbf{d})$ is normal then it is nonsingular in codimension one, that is, the set of singular points in $\mathrm{mod}_A(\mathbf{d})$ is of codimension at least two (see [12, Chapter 11]). We shall need the following consequence of Serre's normality criterion.

Theorem 1.3. *Let A be a triangular algebra, $\mathbf{d} \in K_0(A)$ a positive vector, and assume that $\mathrm{mod}_A(\mathbf{d})$ is irreducible and a complete intersection. Then $\mathrm{mod}_A(\mathbf{d})$ is normal if and only if $\mathrm{mod}_A(\mathbf{d})$ is nonsingular in codimension one.*

Proof. See [12, Chapter 18]. □

The following characterization of maximal orbits in the module varieties of tame quasi-tilted algebras will be crucial in our investigations.

Proposition 1.4. *Let A be a tame quasi-tilted algebra, $\mathbf{d} \in K_0(A)$ a positive vector, and M a module in $\mathrm{mod}_A(\mathbf{d})$. Then $\mathcal{O}(M)$ is a maximal $G(\mathbf{d})$-orbit in $\mathrm{mod}_A(\mathbf{d})$ if and only if $\mathrm{Ext}^1_A(M', M'') = 0$ for any decomposition $M = M' \oplus M''$ of M.*

Proof. It is a direct consequence of [29, Theorem 3], [30, Corollary 3] and the well-known fact (see for example [6, (1.1)]) that any short exact sequence $0 \to U \to W \to V \to 0$ of modules gives a degeneration $U \oplus V \in \overline{\mathcal{O}(W)}$. We refer to [2, Proposition 3.6] for a detailed proof. □

The following important fact has been proved in [21, Proposition 2.1].

Proposition 1.5. *Let A be an algebra, $\mathbf{d} \in K_0(A)$ a positive vector and M, N modules in $\mathrm{mod}_A(\mathbf{d})$ with $\mathcal{O}(N) \subseteq \overline{\mathcal{O}(M)}$. Then for any A-module L the inequalities $\dim_K \mathrm{Hom}_A(N, L) \geq \dim_K \mathrm{Hom}_A(M, L)$ and $\dim_K \mathrm{Hom}_A(L, N) \geq \dim_K \mathrm{Hom}_A(L, M)$ hold.*

As a direct consequence of the above proposition and the formulae on the dimension of an orbit we get the following useful fact.

Corollary 1.6. *Let A be an algebra, $\mathbf{d} \in K_0(A)$ a positive vector, M, N modules in $\mathrm{mod}_A(\mathbf{d})$ with $\mathcal{O}(N) \subseteq \overline{\mathcal{O}(M)}$. Then for any A-module L we have $\dim \mathcal{O}(M \oplus L) - \dim \mathcal{O}(N \oplus L) \geq \dim \mathcal{O}(M) - \dim \mathcal{O}(N)$.*

2. Tubular families of modules

In this section we recall some facts on the module categories of tame concealed and tubular algebras, needed in the proof of our main result.

Assume that A is a tame concealed algebra. Recall from [23, (4.3)] that in this case the Auslander–Reiten quiver Γ_A of A has the form

$$\Gamma_A = \mathcal{P} \vee \mathcal{T} \vee \mathcal{Q},$$

where $\mathcal{P}$ is a preprojective component, $\mathcal{Q}$ is a preinjective component, and $\mathcal{T} = (\mathcal{T}^{(\lambda)})_{\lambda \in \mathbb{P}_1(K)}$ is a family of pairwise orthogonal sincere standard stable tubes separating $\mathcal{P}$ from $\mathcal{Q}$ (in the sense of [23, (3.1)]). Almost all tubes in $\mathcal{T}$ are homogeneous (of rank 1). The ordering of the above families from left to right indicates that there are nonzero maps only from a family to itself or to its right. The indecomposable projective modules lie in $\mathcal{P}$ while the indecomposable injective modules lie in $\mathcal{Q}$. Moreover, we know that $\operatorname{gl.dim} A \leq 2$ and

$$\begin{array}{lll} \operatorname{pd}_A X \leq 1, & \operatorname{id}_A X \leq 2, & \text{for } X \in \mathcal{P}, \\ \operatorname{pd}_A X \leq 1, & \operatorname{id}_A X \leq 1, & \text{for } X \in \mathcal{T}, \\ \operatorname{pd}_A X \leq 2, & \operatorname{id}_A X \leq 1, & \text{for } X \in \mathcal{Q}. \end{array}$$

Suppose now that A is tubular. Then by [23, (5.2)] Γ_A is of the form

$$\Gamma_A = \mathcal{P}_0 \vee \mathcal{T}_0 \vee \Big(\bigvee_{q \in \mathbb{Q}^+} \mathcal{T}_q \Big) \vee \mathcal{T}_\infty \vee \mathcal{Q}_\infty,$$

where $\mathcal{P}_0$ is a preprojective component, $\mathcal{Q}_\infty$ is a preinjective component, $\mathcal{T}_0 = (\mathcal{T}_0^{(\lambda)})_{\lambda \in \mathbb{P}_1(K)}$ is a family of ray tubes, $\mathcal{T}_\infty = (\mathcal{T}_\infty^{(\lambda)})_{\lambda \in \mathbb{P}_1(K)}$ is a family of coray tubes, and each $\mathcal{T}_q$, $q \in \mathbb{Q}^+$, where $\mathbb{Q}^+$ is the set of all positive rationals, is a family $(\mathcal{T}_q^{(\lambda)})_{\lambda \in \mathbb{P}_1(K)}$ of sincere stable tubes. Further, each $\mathcal{T}_q$, $q \in \mathbb{Q}^+ \cup \{0, \infty\}$, consists of pairwise orthogonal standard tubes, and almost all of them are homogeneous. The indecomposable projective A-modules lie in $\mathcal{P}_0 \vee \mathcal{T}_0$ while the indecomposable injective A-modules lie in $\mathcal{T}_\infty \vee \mathcal{Q}_\infty$. Each tubular family $\mathcal{T}_q$, $q \in \mathbb{Q}^+ \cup \{0, \infty\}$, separates

$$\mathcal{P}_q = \mathcal{P}_0 \vee \Big(\bigvee_{p < q} \mathcal{T}_q \Big) \qquad \text{from} \qquad \mathcal{Q}_q = \Big(\bigvee_{p > q} \mathcal{T}_q \Big) \vee Q_\infty,$$

and again the ordering of families from left to right indicates that there are nonzero maps only from these families to itself or to the families to its right. In particular, we know that $\operatorname{gl.dim} A \leq 2$ and

$$\begin{array}{lll} \operatorname{pd}_A X \leq 1, & \operatorname{id}_A X \leq 2, & \text{for } X \in \mathcal{P}_0 \vee \mathcal{T}_0, \\ \operatorname{pd}_A X \leq 1, & \operatorname{id}_A X \leq 1, & \text{for } X \in \mathcal{T}_q,\ q \in \mathbb{Q}^+, \\ \operatorname{pd}_A X \leq 2, & \operatorname{id}_A X \leq 1, & \text{for } X \in \mathcal{T}_\infty \vee \mathcal{Q}_\infty. \end{array}$$

From now till the end of the section we assume that A is tame concealed or tubular. For A tubular we fix $q \in \mathbb{Q}^+$ and write $\mathcal{P}$, $\mathcal{T} = (\mathcal{T}^{(\lambda)})$, $\mathcal{Q}$, instead of $\mathcal{P}_q$, $\mathcal{T}_q$, $\mathcal{Q}_q$, respectively.

Fix $\lambda \in \mathbb{P}_1(K)$ and denote by r the rank of the tube $\mathcal{T}^{(\lambda)}$. Then for any indecomposable A-module X in $\mathcal{T}^{(\lambda)}$ there exists a unique sectional path $X_1 \to X_2 \to \cdots \to X_m = X$ with X_1 lying on the mouth of $\mathcal{T}^{(\lambda)}$, and then m is called the regular length of X, denoted by $\mathrm{rl}(X)$. We will call the modules X_i/X_{i-1}, $i = 1, \dots, m$, where $X_0 = 0$, the regular composition factors of X. In particular X_1 is called the regular socle of X and X/X_{m-1} is called the regular top of X.

Denote by $\mathbf{h}$ the dimension-vector of an indecomposable A-module lying on the mouth of a homogeneous tube of $\mathcal{T}$. It is known that $\mathbf{h}$ does not depend on the choice of a homogeneous tube. We have the following characterization of the families $\mathcal{P}$, $\mathcal{T}$, $\mathcal{Q}$.

Proposition 2.1. *Let X be an indecomposable A-module. Then X belongs to $\mathcal{P}$ (respectively, $\mathcal{T}$, $\mathcal{Q}$) if and only if $\langle \mathbf{h}, \mathbf{dim}\, X\rangle_A < 0$ (respectively, $\langle \mathbf{h}, \mathbf{dim}\, X\rangle_A = 0$, $\langle \mathbf{h}, \mathbf{dim}\, X\rangle_A > 0$).*

Proof. See [23, (4.3), (5.2)]. □

Let $\lambda_1, \dots, \lambda_t$ be the indices of all nonhomogeneous tubes in $\mathcal{T}$ and $r_1, \dots, r_t$ denote the ranks of the tubes $\mathcal{T}^{(\lambda_1)}, \dots, \mathcal{T}^{(\lambda_t)}$, respectively. Fix k in $\{1, \dots, t\}$ and define $I^{(\lambda_k)}$ to be the set of all integers from 0 to $r_k - 1$. In this set we introduce addition modulo r_k. Let $X_0^{(\lambda_k)}, \dots, X_{r_k-1}^{(\lambda_k)}$ be a numbering of indecomposable modules lying on the mouth of $\mathcal{T}^{(\lambda_k)}$ such that $\tau_A^{-1} X_i^{(\lambda_k)} = X_{i+1}^{(\lambda_k)}$, $i = 0, \dots, r_k - 1$. We define $\mathbf{e}_i^{(\lambda_k)} = \mathbf{dim}\, X_i^{(\lambda_k)}$ for $i = 0, \dots, r_k - 1$. We have

$$\mathbf{e}_0^{(\lambda_k)} + \cdots + \mathbf{e}_{r_k-1}^{(\lambda_k)} = \mathbf{h}$$

for each $k = 1, \dots, t$.

Let X be an indecomposable module from the family $\mathcal{T}$. If X belongs to a homogeneous tube then $\mathbf{dim}\, X = p\mathbf{h}$, where $p = \mathrm{rl}(X)$. If there exists $k \in \{1, \dots, t\}$ such that $X \in \mathcal{T}^{(\lambda_k)}$ then

$$\mathbf{dim}\, X = p\mathbf{h} + \mathbf{e}_i^{(\lambda_k)} + \cdots + \mathbf{e}_{i+s-1}^{(\lambda_k)},$$

where p and s are, respectively, the quotient and the remainder obtained from the division of $\mathrm{rl}(X)$ by r_k, and $X_i^{(\lambda_k)}$ is the regular socle of X. In general, if $M \in \mathrm{add}\, \mathcal{T}$ and $\mathbf{d} = \mathbf{dim}\, M$ then we may write $\mathbf{d}$ in a unique way in the form

$$\mathbf{d} = p\mathbf{h} + \sum_{k=1}^{t} \sum_{i=0}^{r_k-1} p_i^{(\lambda_k)} \mathbf{e}_i^{(\lambda_k)}$$

with $p \geq 0$, $p_i^{(\lambda_k)} \geq 0$, $i = 0, \dots, r_k - 1$, $k = 1, \dots, t$, and at least one $p_i^{(\lambda_k)}$, $i = 0, \dots, r_k - 1$, equals 0, for each $k = 1, \dots, t$. The above presentation of $\mathbf{d}$ is called the canonical decomposition of $\mathbf{d}$ (see [22]). Moreover, we shall abbreviate it by $\mathbf{d} = p\mathbf{h} + \mathbf{d}'$, where $\mathbf{d}' = \sum_{k=1}^{t} \sum_{i=0}^{r_k-1} p_i^{(\lambda_k)} \mathbf{e}_i^{(\lambda_k)}$.

In our considerations we will also need notions of generic decomposition and generic module. Recall, that in general if $\mathbf{d}$ is a dimension-vector then a decompositon $\mathbf{d} = \mathbf{d}_1 + \cdots + \mathbf{d}_m$ is called generic if the set of all modules of the form $X_1 \oplus \cdots \oplus X_m$, with X_i indecomposable of dimension-vector $\mathbf{d}_i$, $i = 1, \dots, m$, contains a dense and open subset of $\operatorname{mod}_A(\mathbf{d})$ (see [15], [19]). A module M in $\operatorname{mod}_A(\mathbf{d})$ whose $G(\mathbf{d})$-orbit $\mathcal{O}(M)$ is dense in $\operatorname{mod}_A(\mathbf{d})$ is said to be generic.

Let $\mathcal{T}^{(\lambda)}$ be a nonhomogeneous tube of rank r. We assign to $\mathcal{T}^{(\lambda)}$ a quiver $\Delta^{(\lambda)}$. The vertices of $\Delta^{(\lambda)}$ are given by the elements of the set $I^{(\lambda)}$, that is, by the numbers $0, \dots, r-1$, and we have an arrow from $i-1$ to i for $i = 1, \dots, r$ (remember that in the set $I^{(\lambda)}$ we have addition modulo r).

We need an explicit description of the generic decomposition of the dimension-vector $\mathbf{d}$ of the form

$$\mathbf{d} = \mathbf{d}' = \sum_{k=1}^{t} \sum_{i=0}^{r_k-1} p_i^{(\lambda_k)} \mathbf{e}_i^{(\lambda_k)}.$$

It will be done by induction on $\max\{p_i^{(\lambda_k)} \mid i = 0, \dots, r_k - 1,\ k = 1, \dots, t\}$. For each $k = 1, \dots, t$, let $Q_1^{(k)}, \dots, Q_{s_k}^{(k)}$ be the components of the full subquiver of $\Delta^{(\lambda_k)}$ whose set of vertices is $\{i \in I^{(\lambda_k)} \mid p_i^{(\lambda_k)} > 0\}$. Denote by $M_j^{(k)}$ the set of vertices of $Q_j^{(k)}$ and put $\mathbf{d}_j^{(k)} = \sum_{i \in M_j^{(k)}} \mathbf{e}_i^{(k)}$, $j = 1, \dots, s_k$, $k = 1, \dots, t$. Put also $q_i^{(\lambda_k)} = \max\{p_i^{(\lambda_k)} - 1, 0\}$, $i = 0, \dots, r_k - 1$, $k = 1, \dots, t$, and

$$\mathbf{x} = \sum_{k=1}^{t} \sum_{i=0}^{r_k-1} q_i^{(\lambda_k)} \mathbf{e}_i^{(\lambda_k)}.$$

If $\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_l$ is the generic decomposition of $\mathbf{x}$ then

$$\mathbf{d} = \sum_{k=1}^{t} \sum_{j=1}^{s_k} \mathbf{d}_j^{(k)} + \mathbf{x}_1 + \cdots + \mathbf{x}_l$$

will be the required generic decomposition of $\mathbf{d}$. The vectors $\mathbf{d}_j^{(k)}$, $j = 1, \dots, s_k$, $k = 1, \dots, t$, will be called the large summands of $\mathbf{d}$. Observe, that if $\mathbf{d} = \mathbf{d}_1 + \cdots + \mathbf{d}_m$ is the generic decomposition of $\mathbf{d} = \mathbf{d}'$ then there are unique (up to isomorphism) indecomposable modules X_j of dimension-vectors $\mathbf{d}_j$, $j = 1, \dots, m$. The module $R = \bigoplus_{j=1}^{m} X_j$ will be called the generic module for $\mathbf{d}$. Direct summands of R corresponding to the large summands of $\mathbf{d}$ will also be called large. Observe that $\operatorname{Ext}_A^1(R, R) = 0$, $\operatorname{pd}_A R = \operatorname{id}_A R = 1$, and hence $q_A(\operatorname{\mathbf{dim}} R) = \dim_K \operatorname{End}_A(R)$.

3. Maximal orbits

Assume that A is tame concealed or tubular and use the notation introduced in Section 2. In particular, we have the trisection $\Gamma_A = \mathcal{P} \vee \mathcal{T} \vee \mathcal{Q}$, where $\mathcal{T}$ is our distinguish $\mathbb{P}_1(K)$-family of stable tubes. We fix also the dimension-vector $\mathbf{d}$ of a module M from $\operatorname{add}\mathcal{T}$ with the canonical decomposition $\mathbf{d} = p\mathbf{h} + \mathbf{d}'$. We shall describe all maximal $G(\mathbf{d})$-orbits in $\operatorname{mod}_A(\mathbf{d})$ and prove that $\operatorname{mod}_A(\mathbf{d})$ is irreducible and a complete intersection of dimension $a(\mathbf{d})$.

Lemma 3.1. *If $\mathcal{O}(N)$ is maximal in $\operatorname{mod}_A(\mathbf{d})$ then $N \in \operatorname{add}\mathcal{T}$.*

Proof. Assume N has an indecomposable direct summand $P \in \mathcal{P}$, and put $\mathbf{p} = \mathbf{dim}\,P$. Since M has no indecomposable direct summands from $\mathcal{P}$ it follows that $\operatorname{Hom}_A(M, P) = 0$, and consequently

$$\langle \mathbf{d}, \mathbf{p}\rangle_A = \langle \mathbf{dim}\,M, \mathbf{dim}\,P\rangle_A = -\dim_K \operatorname{Ext}^1_A(M, P) \le 0,$$

because $\operatorname{pd}_A M \le 1$. On the other hand, $N = P \oplus L$ for some A-module L and $\operatorname{Ext}^1_A(L, P) = 0$ by the maximality of $\mathcal{O}(N)$ and Proposition 1.4. Moreover, we have $\dim_K \operatorname{Ext}^1_A(P, P) \le \dim_K \operatorname{End}_A(P)$ [23, (4.3), (5.2)]. Then we obtain

$$\begin{aligned} 0 &\ge \langle \mathbf{d}, \mathbf{p}\rangle_A = \langle \mathbf{dim}\,N, \mathbf{dim}\,P\rangle_A \\ &= \dim_K \operatorname{Hom}_A(N, P) - \dim_K \operatorname{Ext}^1_A(P, P) + \dim_K \operatorname{Ext}^2_A(N, P) \\ &\ge \dim_K \operatorname{End}_A(P) - \dim_K \operatorname{Ext}^1_A(P, P) \ge 0. \end{aligned}$$

Hence $\langle \mathbf{d}, \mathbf{p}\rangle_A = 0$. In particular, we have $\dim_K \operatorname{End}_A(P) = \dim_K \operatorname{Ext}^1_A(P, P)$ and $\operatorname{Ext}^1_A(M, P) = 0$. The first equality implies that $q_A(\mathbf{p}) = 0$, and hence $\dim_K \operatorname{Hom}_A(P, \tau_A M) = \langle \mathbf{p}, \mathbf{dim}\,\tau_A M\rangle_A > 0$. Since $\operatorname{pd}_A M \le 1$ we obtain $\operatorname{Ext}^1_A(M, P) = D\operatorname{Hom}_A(P, \tau_A M) \ne 0$, a contradiction. Similarly we prove that N has no indecomposable direct summands from $\mathcal{Q}$. □

Now we are ready to describe the maximal orbits in $\operatorname{mod}_A(\mathbf{d})$. Before the formulation of the result we introduce a notation. If $N \in \operatorname{add}\mathcal{T}$, then for each $\lambda \in \mathbb{P}_1(K)$, we denote by $N^{(\lambda)}$ the direct sum of all indecomposable direct summands of N belonging to $\mathcal{T}^{(\lambda)}$. For a generic module R, a large summand of R which is a direct summand of $R^{(\lambda)}$ for some $\lambda \in \mathbb{P}_1(K)$ will be also called a large summand of $R^{(\lambda)}$.

Proposition 3.2. *Let R be a generic module for $\mathbf{d}'$. The $G(\mathbf{d})$-orbit of a module $N \in \operatorname{mod}_A(\mathbf{d})$ is maximal if and only if $N \in \operatorname{add}\mathcal{T}$ and its decompositon $N = \bigoplus_{\lambda\in\mathbb{P}_1(K)} N^{(\lambda)}$ satisfies the following condition. For each $\lambda \in \mathbb{P}_1(K)$, we have either $N^{(\lambda)}$ is obtained from $R^{(\lambda)}$ by replacing a large summand of $R^{(\lambda)}$ by an indecomposable summand, or $N^{(\lambda)} = L^{(\lambda)} \oplus R^{(\lambda)}$, where $L^{(\lambda)}$ is an indecomposable module whose neither regular socle nor regular top appears as a regular composition factor of an indecomposable summand of $R^{(\lambda)}$.*

Proof. The proof of the fact that each maximal orbit is of the required form does not differ from the proof of [22, Theorem 3.5]. The remaining part follows from the characterization of maximal orbits given in Proposition 1.4. □

Notice that for each λ we have that $\mathbf{dim}\, N^{(\lambda)} - \mathbf{dim}\, R^{(\lambda)}$ is a positive multiplicity of $\mathbf{h}$. Observe also that $\mathcal{O}(R)$ is a unique maximal $G(\mathbf{d}')$-orbit in $\operatorname{mod}_A(\mathbf{d}')$, thus it really justifies that R is called generic.

Using the above proposition and Proposition 1.1 we get the following information about the dimensions of maximal orbits.

Corollary 3.3. *Each maximal orbit in* $\operatorname{mod}_A(\mathbf{d})$ *has dimension* $a(\mathbf{d}) - p$.

As an immediate consequence of the previous proposition and Corollary 1.2 we also have the following corollary.

Corollary 3.4. *The variety* $\operatorname{mod}_A(\mathbf{d})$ *is a complete interesection of dimension* $a(\mathbf{d})$. *Moreover, the union of all maximal orbits in* $\operatorname{mod}_A(\mathbf{d})$ *is the open sheet of* $\operatorname{mod}_A(\mathbf{d})$ *and consists of nonsingular points.*

Now we may prove the following proposition.

Proposition 3.5. *The variety* $\operatorname{mod}_A(\mathbf{d})$ *is irreducible.*

Proof. We have proved in [2, Theorem 2] (in a rather complicated way) that $\operatorname{mod}_A(p\mathbf{h})$ is irreducible. Here we present a simpler proof of this fact. It follows from [23, (5.2)] and [10, Section 1] that, for each $i = 1, \dots, p$, there exists a regular map $\rho_i : K \to \operatorname{mod}_A(i\mathbf{h})$ whose image intersects all but finitely many orbits of indecomposable modules in $\operatorname{mod}_A(i\mathbf{h})$. Take a module M with $\mathbf{dim}\, M \leq p\mathbf{h}$ and of the form $M = \bigoplus_{j=1}^{t} H_j$, where H_j are indecomposable modules such that $\mathbf{dim}\, H_j = l_j\mathbf{h}$ for some positive integres l_j and $\mathcal{O}(H_j) \cap \rho_{l_j}(K) = \emptyset$. We also allow M to be 0. Choose positive integers $m_1, \dots, m_s$ such that $\sum_{j=1}^{t} l_j + \sum_{k=1}^{s} m_k = p$ and consider the set $\mathcal{X}(M, m_1, \dots, m_s)$ defined as the closure of the image of the morphism $G(\mathbf{d}) \times K^s \to \operatorname{mod}_A(p\mathbf{h})$ given by $(g, \lambda_1, \dots, \lambda_s) \mapsto g(M \oplus \rho_{m_1}(\lambda_1) \oplus \cdots \oplus \rho_{m_s}(\lambda_s))$. Clearly, $\mathcal{X}(M, m_1, \dots, m_s)$ is irreducible. On the other hand, the maximal dimension of orbits in $\operatorname{mod}_A(p\mathbf{h})$ is not bigger than $a(p\mathbf{h}) - p$, hence the dimension of $\mathcal{X}(M, m_1, \dots, m_s)$ is not bigger than $a(p\mathbf{h}) - p + m$, where $m = m_1 + \cdots + m_s$. Observe that the sum $\bigcup \mathcal{X}(M, m_1, \dots, m_s)$ is finite and contains all maximal orbits in $\operatorname{mod}_A(p\mathbf{h})$, and hence is equal $\operatorname{mod}_A(p\mathbf{h})$. Since $\dim \mathcal{X}(0, 1, \dots, 1)$ equals $a(p\mathbf{h})$ and the dimensions of the remaining sets $\mathcal{X}(M, m_1, \dots, m_s)$ are less than $a(p\mathbf{h})$ we infer, using the fact that each irreducible component of $\operatorname{mod}_A(p\mathbf{h})$ has dimension $a(p\mathbf{h})$, that $\mathcal{X}(0, 1, \dots, 1) = \operatorname{mod}_A(p\mathbf{h})$, and hence is the unique irreducible component of $\operatorname{mod}_A(p\mathbf{h})$.

Let R denote a generic module for $\mathbf{d}'$ and $R_0 = R$, $R_1, \ldots, R_k$ be all pairwise nonisomorphic modules obtained from R by replacing large summands of R by indecomposable modules in such a way that if a summand X is replaced by a module Y then $\mathbf{dim}\, Y - \mathbf{dim}\, X$ is a positive multiplicity of $\mathbf{h}$. There exist nonnegative integers $n_0, \ldots, n_k$ such that $\mathbf{dim}\, R_i - \mathbf{dim}\, R_0 = n_i\mathbf{h}$ for any $i = 0, \ldots, k$. For $i = 0, \ldots, k$, we define $\mathcal{Y}_i$ to be the set of all modules of the form $Q \oplus H$, where $Q \in \overline{\mathcal{O}(R_i)}$ and $\mathbf{dim}\, H = (p - n_i)\mathbf{h}$. It follows from the above considerations that the sets $\mathcal{Y}_0, \ldots, \mathcal{Y}_k$ are irreducible. We also have the inequalities $\dim \mathcal{Y}_i \le a(\mathbf{d}) - n_i$, $i = 0, \ldots, k$. On the other hand, using Proposition 3.2 and Corollary 1.2 we know that $\mathrm{mod}_A(\mathbf{d}) = \mathcal{Y}_0 \cup \mathcal{Y}_1 \cup \cdots \cup \mathcal{Y}_s$ and each irreducible component of $\mathrm{mod}_A(\mathbf{d})$ has dimension $a(\mathbf{d})$. Hence we get $\mathrm{mod}_A(\mathbf{d}) = \mathcal{Y}_0$, thus $\mathrm{mod}_A(\mathbf{d})$ is irreducible. □

4. Normality

Let $\mathbf{d}$ be the dimension-vector of a module M from $\mathrm{add}\, \mathcal{T}$ and $\mathbf{d} = p\mathbf{h} + \mathbf{d}'$ its canonical decomposition. In considerations concerning normality of $\mathrm{mod}_A(\mathbf{d})$ we will need some notation. Recall that an indecomposable module is called homogeneous if it lies in a homogeneous tube in the Auslander–Reiten quiver. Indecomposable modules which are not homogeneous are said to be nonhomogeneous. In general every module being a direct sum of homogeneous (respectively, nonhomogeneous) indecomposable modules will be called homogeneous (respectively, nonhomogeneous).

Let I be the set indexing the families of tubes in Γ_A. If A is tame concealed then $I = \{*\}$, while for A tubular we have $I = \mathbb{Q}^+ \cup \{0, \infty\}$. With each tubular family $\mathcal{T}_i$, $i \in I$, we may relate the dimension-vector $\mathbf{h}_i$ of modules lying on the mouth of homogeneous tubes in $\mathcal{T}_i$ (in particular $\mathbf{h}_* = \mathbf{h}$). Take a nonhomogeneous module N and a sequence $(m_i)_{i\in I}$ of nonnegative integers with all but a finite number of m_i's equal 0. We define the subset $\mathcal{W}(N, (m_i))$ of $\mathrm{mod}_A(\mathbf{dim}\, N + \sum_{i\in I} m_i\mathbf{h}_i)$ as the set of all modules of the form $E \oplus \bigoplus_{i\in I} H_i$, where $E \simeq N$ and H_i is a homogeneous module from $\mathrm{add}\, \mathcal{T}_i$ with $\mathbf{dim}\, H_i = m_i\mathbf{h}_i$, $i \in I$. Observe that $\mathrm{mod}_A(\mathbf{d})$ is a finite sum of sets $\mathcal{W}(N, (m_i))$ satisfying the condition $\mathbf{dim}\, N + \sum_{i\in I} m_i\mathbf{h}_i = \mathbf{d}$. Thus, in order to prove that $\mathrm{mod}_A(\mathbf{d})$ is normal, it is enough to show, by Theorem 1.3, that if a set $\mathcal{W}(N, (m_i))$ contains a module Z with $\mathrm{Ext}^2_A(Z, Z) \neq 0$ then $\dim \mathcal{W}(N, (m_i)) \le a(\mathbf{d}) - 2$. Notice that $\mathrm{pd}_A H \le 1$ and $\mathrm{id}_A H \le 1$ for each homogeneous module H, hence $\mathcal{W}(N, (m_i))$ contains a module Z with $\mathrm{Ext}^2_A(Z, Z) \neq 0$ if and only if $\mathrm{Ext}^2_A(N, N) \neq 0$. Moreover, we have the inequality

$$\dim \mathcal{W}(N, (m_i)) \le \max_{Z\in\mathcal{W}(N,(m_i))} \dim \mathcal{O}(Z) + \sum_{i\in I} m_i.$$

We start with the following lemma.

Lemma 4.1. *If* $\mathcal{O}(N)$ *is an orbit of codimension 1 in* $\mathrm{mod}_A(\mathbf{d})$ *then we have* $\mathrm{Ext}_A^2(N,N)=0$.

Proof. If $\mathcal{O}(N)$ is a maximal orbit then $\mathrm{Ext}_A^2(N,N)=0$ by Proposition 3.2. Hence we may assume that $\mathcal{O}(N)$ is not maximal. Then from Proposition 3.5 there is exactly one maximal orbit in $\mathrm{mod}_A(\mathbf{d})$. It is only possible for $p=0$, and the unique maximal orbit is of the form $\mathcal{O}(R)$, where R is a generic module for $\mathbf{d}$. Since the orbit $\mathcal{O}(N)$ is not maximal, applying Proposition 1.4, we deduce that there exist two different indecomposable direct summands N_1 and N_2 of N such that $\mathrm{Ext}_A^1(N_2,N_1)\neq 0$. Thus we have a nonsplittable exact sequence

$$(*)\qquad 0\longrightarrow N_1\longrightarrow L\longrightarrow N_2\longrightarrow 0.$$

If we write N in the form $N=N_0\oplus N_1\oplus N_2$ we get a nonsplittable exact sequence

$$0\longrightarrow N_0\oplus N_1\longrightarrow N_0\oplus L\longrightarrow N_2\longrightarrow 0,$$

and it follows that $N\in\overline{\mathcal{O}(N_0\oplus L)}\setminus\mathcal{O}(N_0\oplus L)$. Since $\mathcal{O}(R)$ is a unique maximal orbit in $\mathrm{mod}_A(\mathbf{d})$ we get $\mathcal{O}(N_0\oplus L)=\mathcal{O}(R)$, and hence $R=R_0\oplus R_1$, where $R_0\simeq N_0$ and $R_1\simeq L$. In particular we get $\mathrm{pd}_A N_0\leq 1$ and $\mathrm{id}_A N_0\leq 1$. So we have to prove that $\mathrm{Ext}_A^2(N_1\oplus N_2,N_1\oplus N_2)=0$. Notice that, by Corollary 1.6, the codimension of $\mathcal{O}(N_1\oplus N_2)$ in $\mathrm{mod}_A(\mathbf{dim}\,L)$ is equal 1.

Since $\mathrm{Hom}_A(N_1,L)\neq 0$ and $L\in\mathrm{add}\,\mathcal{T}$ we have $N_1\in\mathcal{P}\vee\mathcal{T}$. Similarly, we have $N_2\in\mathcal{T}\vee\mathcal{Q}$. If $N_1\in\mathcal{T}$ or $N_2\in\mathcal{T}$, the claim is obvious. Thus we may assume that $N_1\in\mathcal{P}$ and $N_2\in\mathcal{Q}$. We know that $\mathrm{pd}_A N_1\leq 1$ and $\mathrm{id}_A N_2\leq 1$, so we only have to show that $\mathrm{Ext}_A^2(N_2,N_1)=0$. We shall prove first that $\mathrm{Ext}_A^1(N_1,N_1)=0$. Notice that $N_2\in\mathcal{Q}$ implies $\langle\mathbf{dim}\,N_2,\mathbf{dim}\,L\rangle_A=-\dim_K\mathrm{Ext}_A^1(N_2,L)\leq 0$, and hence

$$\begin{aligned}\dim_K\mathrm{Hom}_A(N_1,L)&=\langle\mathbf{dim}\,N_1,\mathbf{dim}\,L\rangle_A\\&=q_A(\mathbf{dim}\,L)-\langle\mathbf{dim}\,N_2,\mathbf{dim}\,L\rangle_A\\&\geq q_A(\mathbf{dim}\,L)=\dim_K\mathrm{End}_A(L).\end{aligned}$$

Applying $\mathrm{Hom}_A(N_1,-)$ to the short exact sequence $(*)$ we obtain an exact sequence

$$\begin{array}{ccccccc}0&\longrightarrow&\mathrm{Hom}_A(N_1,N_1)&\longrightarrow&\mathrm{Hom}_A(N_1,L)&\xrightarrow{\varphi}&\mathrm{Hom}_A(N_1,N_2)\\&&&\longrightarrow&\mathrm{Ext}_A^1(N_1,N_1)&\longrightarrow&\mathrm{Ext}_A^1(N_1,L).\end{array}$$

On the other hand,

$$\begin{gathered}\dim_K\mathrm{Hom}_A(N_1,N_1)+\dim_K\mathrm{Hom}_A(N_1,N_2)=\dim_K\mathrm{Hom}_A(N_1,N_1\oplus N_2)\\\leq\dim_K\mathrm{End}_A(N_1\oplus N_2)-1=\dim_K\mathrm{End}_A(L)\leq\dim_K\mathrm{Hom}_A(N_1,L),\end{gathered}$$

because the codimension of $\mathcal{O}(N_1\oplus N_2)$ in $\mathrm{mod}_A(\mathbf{dim}\,L)$ equals 1. This shows that φ is an epimorphism. Since $\mathrm{Ext}_A^1(N_1,L)=D\overline{\mathrm{Hom}}_A(L,\tau_AN_1)=0$, we get $\mathrm{Ext}_A^1(N_1,N_1)=0$.

Finally, applying $\mathrm{Hom}_A(-, N_1)$ to $(*)$ we get an exact sequence

$$0 = \mathrm{Ext}^1_A(N_1, N_1) \longrightarrow \mathrm{Ext}^2_A(N_2, N_1) \longrightarrow \mathrm{Ext}^2_A(L, N_1) = 0,$$

because $\mathrm{pd}_A L \leq 1$. Therefore $\mathrm{Ext}^2_A(N_2, N_1) = 0$, and this finishes the proof. □

The next step in the proof is the following.

Lemma 4.2. *Let N be a nonhomogeneous module with $\mathrm{Ext}^2_A(N, N) \neq 0$ and $(m_i)_{i\in I}$ be a sequence of nonnegative integers with $\mathbf{dim}\, N + \sum_{i\in I} m_i \mathbf{h}_i = \mathbf{d}$ and $m_j > 0$ for some $j \in I$. Then $\dim \mathcal{W}(N, (m_i)) \leq a(\mathbf{d}) - 2$.*

Proof. Let Z be a module from $\mathcal{W}(N, (m_i))$. Then we may write Z in the form $Z \simeq N \oplus H \oplus L$, where H is a direct sum of all indecomposable homogeneous direct summands of Z from $\mathcal{T}_j$. Since $\mathrm{Ext}^2_A(N, N) \neq 0$ we may also write $N = N_0 \oplus N_1 \oplus N_2$, where N_1 and N_2 are indecomposable modules with $\mathrm{Ext}^2_A(N_2, N_1) \neq 0$. In particular, we know that $\mathrm{id}_A N_1 = 2$ and $\mathrm{pd}_A N_2 = 2$.

Assume first that $j = *$ (if A is concealed) and $j \in \mathbb{Q}^+$ (if A is tubular). The condition $\mathrm{pd}_A N_2 = 2$ implies that $N_2 \in \mathcal{Q}_j$ (we write $\mathcal{Q}_*$ for $\mathcal{Q}$), so $\langle \mathbf{dim}\, N_2, \mathbf{h}_j \rangle_A < 0$ by Proposition 2.1. It follows that $\mathrm{Ext}^1_A(N_2, H) \neq 0$. Let $\mathcal{X}_Z$ be the set of all modules of the form $N_0 \oplus N_1 \oplus L \oplus V$, where V is the middle term of a nonsplittable short exact sequence with the left end H and the right end N_2. It is clear that $\mathcal{X}_Z \cap \mathcal{W}(N, (m_i)) = \emptyset$, as the multiplicity of N_2 in each module belonging to $\mathcal{X}_Z$ is smaller than for modules belonging to $\mathcal{W}(N, (m_i))$. We have also $\mathcal{X}_Z \cap \mathcal{U} = \emptyset$, where $\mathcal{U}$ denotes the open sheet of $\mathrm{mod}_A(\mathbf{d})$, because $\mathrm{id}_A N_1 = 2$. Hence, if we put $\mathcal{X} = \bigcup_{Z\in\mathcal{W}(N,(m_i))} \mathcal{X}_Z$ then we get $\dim \overline{\mathcal{X}} \leq a(\mathbf{d}) - 1$. Observe that $\mathcal{W}(N, (m_i)) \subseteq \overline{\mathcal{X}}$. Consequently, in order to prove that $\dim \mathcal{W}(N, (m_i)) \leq a(\mathbf{d}) - 2$, it is enough to show that for each irreducible component $\mathcal{Z}$ of $\overline{\mathcal{X}}$ there exists an open subset $\mathcal{Z}_0$ of $\mathcal{Z}$ with $\mathcal{W}(N, (m_i)) \cap \mathcal{Z} \subseteq \mathcal{Z} \setminus \mathcal{Z}_0$.

Let $\mathcal{Z}$ be an irreducible component of $\overline{\mathcal{X}}$. Let $\mathcal{Z}_1$ be the set of all points of $\mathcal{Z}$ which do not belong to the remaining irreducible components of $\overline{\mathcal{X}}$. Observe that $\mathcal{Z}_1$ is an open subset of $\mathcal{Z}$. Hence, if we define $\mathcal{Z}_0$ as the intersection of $\mathcal{Z}_1$ with the set of all orbits of maximal dimension in $\mathcal{Z}$, we also get an open nonempty subset of $\mathcal{Z}$. In addtion, $\mathcal{Z}_0$ does not intersect any irreducible component of $\overline{\mathcal{X}}$ different from $\mathcal{Z}$. Hence, it follows that each orbit in $\mathcal{Z}_0$ is a maximal orbit in $\overline{\mathcal{X}}$, thus $\mathcal{Z}_0 \cap \mathcal{W}(N, (m_i)) = \emptyset$.

Now we only have to consider the case of A tubular and $j \in \{0, \infty\}$. Without loss of generality, we may assume that $j = 0$. We deal again with a module $Z \in \mathcal{W}(N, (m_i))$ and keep its decomposition $Z \simeq N_0 \oplus N_1 \oplus N_2 \oplus H \oplus L$. By [23, (5.2)] we know that $\langle \mathbf{d}, \mathbf{h}_0 \rangle_A < 0$. Since $\mathrm{id}_A N_1 = 2$ we have $N_1 \in \mathcal{P}_0 \vee \mathcal{T}_0$, thus $\langle \mathbf{dim}\, N_1, \mathbf{h}_0 \rangle_A \geq 0$. As a consequence we obtain $\langle \mathbf{dim}\, N_0 + \mathbf{dim}\, N_2 + \mathbf{dim}\, L, \mathbf{h}_0 \rangle_A < 0$, and hence $\mathrm{Ext}^1_A(N_0 \oplus N_2 \oplus L, H) \neq 0$. So we can define $\mathcal{X}_Z$ as the set of all modules from $\mathrm{mod}_A(\mathbf{d})$ of the form $N_1 \oplus V$, where V is the middle term of a nonsplittable short exact sequence with the left end H and the right end $N_0 \oplus N_2 \oplus L$. Observe that such a module V has no homogeneous direct summands from $\operatorname{add} \mathcal{T}_0$ of dimension-vector $m_0\mathbf{h}_0$. Thus it follows again that $\mathcal{X}_Z \cap \mathcal{W}(N, (m_i)) = \emptyset$. Obviously $\mathcal{X}_Z \cap \mathcal{U} = \emptyset$. Therefore, as above, we get $\dim \mathcal{W}(N, (m_i)) \leq a(\mathbf{d}) - 2$. □

Now we are ready to prove the following proposition.

Proposition 4.3. *The variety* $\operatorname{mod}_A(\mathbf{d})$ *is normal.*

Proof. It is enough to show that, if $\dim \mathcal{W}(N, (m_i)) \geq a(\mathbf{d}) - 1$ for some nonhomogeneous module N and a sequence $(m_i)_{i\in I}$ of nonnegative integers with $\mathbf{d} = \mathbf{dim}\, N + \sum_{i\in I} m_i\mathbf{h}_i$, then $\operatorname{Ext}^2_A(N, N) = 0$. If there exists $j \in I$ such that $m_j > 0$ then it follows from the above lemma. Thus we may assume that $m_i = 0$ for all $i \in I$. It is clear that if $\dim \mathcal{W}(N, (m_i)) = a(\mathbf{d})$ then $\mathbf{d} = \mathbf{d}'$ and N is a generic module for $\mathbf{d}$, and consequently $\operatorname{Ext}^2_A(N, N) = 0$. Hence we may assume that $\dim \mathcal{W}(N, (m_i)) = a(\mathbf{d}) - 1$. It means that $\mathcal{O}(N)$ is an orbit of codimension 1, and hence $\operatorname{Ext}^2_A(N, N) = 0$ by Lemma 4.1. □

References

[1] Bobiński, G. and Skowroński, A.: Geometry of directing modules over tame algebras, J. Algebra, in press (1999)

[2] Bobiński, G. and Skowroński, A.:, Geometry of modules over tame quasi-tilted algebras, Coll. Math. **79** (1999), 85–118.

[3] Bongartz, K.: Algebras and quadratic forms, J. London Math. Soc. **28**, 461–469 (1983)

[4] Bongartz, K.: A geometric version of the Morita equivalence, J. Algebra **139**, 159–171 (1991)

[5] Bongartz, K.: Minimal singularites of Dynkin quivers, Comment. Math. Helvetici **69**, 575–611 (1994)

[6] Bongartz, K.: On degenerations and extensions of finite dimensional modules, Advances Math. **121**, 245–287 (1996)

[7] Bongartz, K.: Some geometric aspects of representation theory, CMS Conf. Proc. **23**, 1–27 (1998)

[8] de Concini, C. and Strickland, E.: On the variety of complexes, Advances Math. **41**, 57–77 (1981)

[9] Crawley-Boewey, W.W.: On tame algebras and bocses, Proc. London. Math. Soc. **56**, 451–483 (1988)

[10] Dowbor, P. and Skowroński, A.: On the representation type of locally bounded categories, Tsukuba J. Math. **10**, 63–72 (1986)

[11] Drozd, Yu.A.: Tame and wild matrix problems, Lecture Notes in Math. **832**, Springer–Verlag, 1980, pp. 242–258

[12] Eisenbud, D.: *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics **150**, Springer–Verlag, 1996

[13] Gabriel, P.: Finite representation type is open, Lecture Notes in Math. **488**, Springer–Verlag, 1975, pp. 132–155

[14] Gabriel, P.: Auslander-Reiten sequences and representation-finite algebras, Lecture Notes in Math. **831**, Springer–Verlag, 1980, pp. 1–71

[15] Kac, V.G.: Infinite root systems, representations of graphs and invariant theory, Invent. Math. **56**, 57–92 (1980)
[16] Kraft, H.: *Geometrische Methoden in der Invariantentheorie*, Vieweg, 1984
[17] Kraft, H.: Geometric methods in representation theory, Lecture Notes in Math. **944**, Springer–Verlag, 1981, pp. 180–258
[18] Lenzing, H. and de la Peña, J.A.: Concealed-canonical algebras and separating tubular families, Proc. London Math. Soc., in press
[19] de la Peña, J.A.: On the dimension of the module-varieties of tame and wild algebras, Comm. Algebra **19**, 1795–1807 (1991)
[20] de la Peña, J.A. and Skowroński, A.: Geometric and homological characterizations of polynomial growth strongly simply connected algebras, Invent. Math. **126**, 287–296 (1996)
[21] Riedtmann, C.: Degenerations for representations of quivers with relations, Ann. Sci. École Norm. Sup. **19**, 275–301 (1986)
[22] Ringel, C.M.: The rational invariants of the tame quivers, Invent. Math. **58**, 217–239 (1980)
[23] Ringel, C.M.: *Tame algebras and integral quadratic forms*, Lecture Notes in Math. **1099**, Springer–Verlag, 1984
[24] Shafarevich, I.R.: *Basic algebraic geometry*, Graduate Texts in Math. **213**, Springer–Verlag, 1977
[25] Skowroński, A.: On omnipresent tubular families of modules, CMS Conf. Proc. **18**, 641–657 (1996)
[26] Skowroński, A.: Simply connected algebras of polynomial growth, Compositio Math. **109**, 99–133 (1997)
[27] Skowroński, A.: Tame quasi-tilted algebras, J. Algebra **203**, 470–490 (1998)
[28] Skowroński, A.: On the structure of periodic modules over tame algebras, Proc. Amer. Math. Soc., in press (1999)
[29] Skowroński, A. and Zwara, G.: Degenerations for indecomposable modules and tame algebras, Ann. Sci. École Norm. Sup. **31**, 153–180 (1998)
[30] Zwara, G.: Degenerations of finite dimensional modules are given by extensions, Compositio Math., in press.