Some Observations on Inverses of Band Matrices and Low Rank Perturbations of Triangular Matrices

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1 Introduction

One of the many fields, in which P. Rozsa's contributions play important roles, is that of band matrices and their inverses. See e.g. [8], [9], [10],[6]. This field is not only of a certain mathematical beauty, but is also influenced by applications. For example E. Asplund, whose paper [1] is often cited as the first one to treat inverses of band matrices, mentions that his paper was initiated by the paper [2] of S. Asplund, dealing with the numerical solution of boundary value problems.

The same could be said of these notes, which grew out of intensive discussions concerning certain models of metabolisms, see [4]. Here it was of importance to exhibit in some matrices square submatrices of full rank, or equivalently, describe all singular submatrices, which have to be avoided for that purpose. The matrices involved turned out to be inverses of Hessenberg matrices or slightly perturbed Hessenberg matrices.

In this connection we will treat here the following problem:

Problem 1 Characterize all nonsingular n-by-n matrices B, whose inverses $A = B^{-1}$ are rank one perturbations of lower triangular matrices, i.e.,

$$A = L + uv^T,$$

where L is lower triangular and $u, v \in K^n$.

We see easily that B can be of very different forms. If L is nonsingular, then by the Sherman-Morrison formula, see, e.g., [5, p. 51], we get

$$B = (L + uv^T)^{-1} = L^{-1} - sL^{-1}uv^TL^{-1}, \qquad s = (1 + v^TL^{-1}u)^{-1}.$$

Hence B is of the same form as A. On the other hand, if L is strictly lower triangular, then B is an unreduced lower Hessenberg matrix, containing many zero entries.

It is not too difficult to discuss also the more general problem of perturbations of rank p:

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Problem 2 Characterize all nonsingular n-by-n matrices B, whose inverses $A = B^{-1}$ are rank p perturbations of lower triangular matrices, i.e.,

$$A = L + UV^T,$$

where L is lower triangular and U, V are n-by-p matrices.

Solutions to these two problems are given in section 4. They are intimately connected to the description of inverses of band matrices. In section 3 we give two new characterizations, which we feel to be slightly simpler than those in the original paper [1] by E. Asplund. They are given in terms of a few rank conditions.

Section 2 contains introductory material.

2 Notations and Definitions

All matrices involved are over the field K, where $K = R, C$. A square n-by-n matrix $H = (h_{ij})$ is a lower (upper) Hessenberg matrix if the relation $h_{ij} = 0$ for $j > i+1 (i > j+1)$ holds.

It is *unreduced* if in addition $h_{j,j+1} \neq 0, j = 1, \ldots, n-1$ $(h_{j+1,j} \neq 0, j = 1, \ldots, n-1)$.

It is a *band matrix of width* p if $h_{ij} = 0$ for $j > i + p$, and it is a *strict band matrix of width* p, if in addition $h_{i,j} \neq 0, \quad 1 \leq i, j \leq n, j - i = p$. Observe here that formally also negative bandwidths are possible. But for a nonsingular matrix the bandwidth is necessarily nonnegative.

The following result for the inverse of a block matrix is well-known, see e.g., [7].

If

$$G = \begin{pmatrix} G_1 & G_2 \\ G_3 & G_4 \end{pmatrix} \tag{2.1}$$

is nonsingular, where G_2 and G_3 are square, and G_2 is nonsingular, then the Schurcomplement

$$S = G_3 - G_4 G_2^{-1} G_1 \tag{2.2}$$

is nonsingular and

$$G^{-1} = \begin{pmatrix} -S^{-1} G_4 G_2^{-1} & S^{-1} \\ G_2^{-1} + G_2^{-1} G_1 S^{-1} G_4 G_2^{-1} & -G_2^{-1} G_1 S^{-1} \end{pmatrix} \tag{2.3}$$

Observe that we can also write it in the form

$$G^{-1} = \begin{pmatrix} 0 & 0 \\ G_2^{-1} & 0 \end{pmatrix} + UV^T \tag{2.4}$$

where

$$U = \begin{pmatrix} S^{-1} \\ -G_2^{-1} G_1 S^{-1} \end{pmatrix}$$

and

$$V^T = (G_4 G_2^{-1} \qquad I).$$

For an n-by-n matrix A we denote by $A(r,s)$ the r-by-s submatrix of A formed by the first r rows and last s columns of A. We have the following rank relation:

$$\mathrm{rank}(A) \leq \mathrm{rank}(A(r,s)) + (n-r) + (n-s) = 2n - (r+s) + \mathrm{rank}(A(r,s)), \tag{2.5}$$

as A is formed from $A(r,s)$ by adding $n-r$ rows and $n-s$ columns.

The following well-known result is the special case $p=1$ of Theorem 3.2. It is given here separately, because we are occasionally refering to it.

Theorem 2.6 *For a given nonsingular n-by-n matrix A the following are equivalent.*

(a) A^{-1} is an unreduced lower Hessenberg matrix.

(b) $A = L + uv^T$, where L is a strictly lower triangular matrix and $u, v \in K^n$

Let us denote by $\mathcal{G}(n)$ the set of all matrices described in the previous theorem, i.e. all nonsingular n-by-n matrices of the form $L + uv^T$, L strictly lower triangular and $u, v \in K^n$. In particular, $\mathcal{G}(1)$ are the nonzero 1-by-1 matrices.

The matrix class $\mathcal{T}(n)$ is defined in the following way. A matrix A belongs to this class, if A is a block lower triangular matrix with diagonal blocks A_{ii} of order n_i, and $A_{ii} \in \mathcal{G}(n_i)$. $\mathcal{T}(n)$ consists of all inverses of nonsingular lower Hessenberg matrices, not necessarily unreduced. In fact, $A \in \mathcal{T}(n)$ is the inverse of an unreduced Hessenberg matrix iff $n_1 = n$ and the inverse of a triangular matrix iff $n_i = 1, i = 1, \ldots, n$. We call an n-by-n matrix B a *generalized (lower) Hessenberg matrix*, if $n = 1$ or $B(n-1, n-1) \in \mathcal{T}(n-1)$. We remark that for triangular $B(n-1, n-1)$, B is an unreduced Hessenberg matrix. The other extreme case, namely that in $B(n-1, n-1)$ there is only one block of dimension $n-1$, yields a full matrix B of the form $B = L + uv^T$, where L is a nonsingular lower triangular matrix and $u, v \in K^n$.

3 Inverses of Band Matrices

Our first observation concerns a well-known result, often attributed to E. Asplund. We complement the usual characterization by two more, in our opinion simpler equivalent conditions.

Theorem 3.1 *Let A be a nonsingular n-by-n matrix and $p \le n-2$. The following are equivalent.*

1. *$A^{-1} = (b_{ij})$ is a band matrix of width p, i.e. $b_{ij} = 0$ for $j > i + p$.*
2. *Let C be a submatrix of A such that $a_{ij} \in C$ implies $j + p > i$. Then the rank of C is less or equal to p.*
3. *$\operatorname{rank}(A(r, n+p-r)) \le p$ for all $p+1 \le r \le n-1$.*
4. *$\operatorname{rank}(A(r, n+p-r)) = p$ for all $p+1 \le r \le n-1$.*

Proof The equivalence of (1) and (2) is Asplunds result. Continuing, we see at once that (4) $\to$ (3). The implication (2) $\to$ (3) is trivial from the observation, that each $A(r, n+p-r)$ is one of the submatrices described in (2). On the other hand, all submatrices in (2) are contained in some $A(r, n+p-r)$, where $p \le r \le n$. While for $r = p$ and $r = n$ the rank of $A(r, n+p-r)$ is trivially $\le p$, (even equal to p by the nonsingularity of A), this is true for the other cases by (3). It remains to show that (3) implies (4). First by (2.5) $n = \operatorname{rank}(A) \le \operatorname{rank}(A(r, n+p-r) + 2n - (n+p)$, which implies $\operatorname{rank}(A(r, n+p-r) \ge p$. Using (3), the relation (4) follows.

□

We would like to mention [3, Theorem 3.1], where still another characterization in terms of the vanishing of certain p-minors is given.

We will also complement a second result of Asplund, which gives a characterization of the slightly stronger condition of strict bandedness of A^{-1}.

Theorem 3.2 *For a nonsingular n-by-n matrix A and $p \leq n-2$ the following are equivalent:*

1. *$A^{-1} = (b_{ij})$ is a strict band matrix of width p, i.e. $b_{ij} = 0$ for $j > i+p$ and $b_{ij} \neq 0$ for $j = i+p$.*
2. *$A = L + UV^T$, where U, V are n-by-p matrices of rank p and L is a band matrix of width $-p$.*
3. *$\operatorname{rank}(A(r, n+p-r)) = p$ for $p+1 \leq r \leq n-1$, and $\operatorname{rank}(A(p,p)) = p$.*

Proof The equivalence of (1) and (2) is due to Asplund and uses essentially the relations (2.3) for the inverse of block matrices. To show that (1) and (2) imply (3), we write $U^T = (U_1^T, U_2^T), V^T = (V_1^T, V_2^T)$, where U_1 and V_2 are p-by-p. Then due to (2) $A(p,p) = U_1 V_2^T$. Both factors are nonsingular, because otherwise the first p rows or last p columns of A would be linearly dependent. Thus we have $\operatorname{rank}(A(p,p) = p$. The first statement of (3) follows from (1) and Theorem 3.1.

For the reverse implication (3) $\rightarrow$ (1) we use (2.3), applied to $G = A$, where the nonsingular upper right submatrix G_2 is given by $A(p,p)$. By Theorem 3.1 the upper right block of A^{-1} is lower triangular, by (2.3) it is S^{-1}, hence nonsingular. So A^{-1} is a strict band matrix of width p.

□

Using the class of matrices $\mathcal{T}(r)$ we can describe the inverses of band matrices in a more detailed form. Let us assume that $B = A^{-1}$ is a band matrix of width p. If $C = B(n-p+1, n-p+1)$ is nonsingular, C is a nonsingular Hessenberg matrix and hence its inverse is a matrix in $\mathcal{T}(n-p+1)$. Hence A is by (2.4) applied to $G = A^{-1}$ and the nonsingular upper right corner $G_2 = C$ of the form

$$A = \begin{pmatrix} 0 & 0 \\ C^{-1} & 0 \end{pmatrix} + UV^T \tag{3.3}$$

where U, V are n-by-$(p-1)$ and of full rank $p-1$. This allows to exhibit areas where the ranks of submatrices are at most $p-1$, namely where the matrix $C^{-1} \in \mathcal{T}(n-p+1)$ has zero entries, and those areas where the ranks are bounded by p.

4 Low Rank perturbations of Triangular Matrices

We discuss in the beginning of this section matrices the inverses of which are rank one perturbations of lower triangular matrices. In the introduction we had already formulated this as Problem 1. We can prove

Theorem 4.1 *For a nonsingular n-by-n matrix A the following are equivalent:*

1. *A^{-1} is a generalized lower Hessenbergmatrix.*
2. *There exists a lower triangular matrix L and vectors $u, v \in K^n$ with*

$$A = L + uv^T$$

and $u_1 v_n \neq 0$.

Proof The case $n = 1$ is obviously true, so let now $n > 1$. We assume first that A^{-1} is a generalized Hessenberg matrix, i.e.

$$A^{-1} = \begin{pmatrix} a & T \\ c & b^T \end{pmatrix} \tag{4.2}$$

where $T \in \mathcal{T}(n-1)$. T^{-1} is a lower Hessenberg matrix, which is in general not unreduced. But then by (2.3) with S of dimension 1-by-1 we see that A has the required form (2), where the entry S^{-1} in position $(1,n)$ is nonzero. For the inverse statement we use that $L + uv^T$ can be written in the form of (2.4):

$$L + uv^T = \begin{pmatrix} o & o \\ \tilde{L} & o \end{pmatrix} + uv^T, \tag{4.3}$$

where $\tilde{L}$ is block triangular with diagonal blocks which are unreduced Hessenberg matrices. Here we remark that in $L + uv^T$ we can always assume (by eventually changing the first component of v and the last component of u) that the elements of L in position $(1,1)$ and (n,n) vanish. This uses the fact that $(L + uv^T)_{1,n} = u_1 v_n \neq 0$. It is easy to see that $\tilde{L}$ is nonsingular. We get from (2.1), that the inverse $(L + uv^T)^{-1}$ is a generalized Hessenberg matrix.

□

For completeness the general case of nonsingular matrices $L + uv^T$, where not necessarily $u_1 v_n \neq 0$, has still to be discussed. It can be easily reduced to the situation of Theorem 4.1.

Theorem 4.4 *For a given nonsingular n-by-n matrix A the following are equivalent.*

1. *There exists a lower triangular matrix L and vectors $u, v \in K^n$ with*

$$A = L + uv^T$$

2. *There exist numbers r, s with $1 \leq r \leq s \leq n$ such that $B = A^{-1}$ is a block lower triangular matrix with diagonal blocks B_1, B_2, B_3 satisfying*

   - *B_1 and B_3 are nonsingular lower triangular of order $r-1$ and $n-s$ (possibly void)*
   - *B_2 is nonsingular generalized Hessenberg matrix of order $s-r+1$.*

Proof By Theorem 4.1 the implication (2) $\to$ (1) is obvious.

Assume now that A is of the form (1). Let k be the upper bandwidth of A, i.e the largest integer such that there is r satisfying $a_{r,r+k} \neq 0$. Observe that the case $k = n-1$ is treated in Theorem 4.1. Clearly $k \geq 0$. The case $k = 0$ implies that A^{-1} is lower triangular, so that (2) holds.

If $k > 0$ then $u_r \neq 0$ and $v_s \neq 0, s = r + k$. In addition due to the maximality of k we have $u_i = 0$ for $i < r$ and $v_j = 0$ for $j > s$. Hence A is block triangular with (at most) three diagonal blocks $A_i, i = 1,2,3$ with dimensions as in (2). The middle block A_2 satisfies the assumptions of Theorem 4.1, thus we have that its inverse B_2 is a generalized Hessenberg matrix. Hence A^{-1} is of the form in (2).

□

We turn now to Problem 2 of the introduction and give a generalization of the previous Theorem 4.1 to the case of rank p perturbations of lower triangular matrices. Here we have

Theorem 4.5 *Let A be an n-by-n nonsingular matrix over K, and $B = A^{-1}$. Let p be an integer, $1 \le p \le n$. The following are equivalent.*

1. $\text{rank}(B(r, n-r)) = p$ *for all* $p+1 \le r \le n-p-1$ *and* $B(n-p, n-p)$ *nonsingular.*
2. $A = L + UV^T$, *where L is lower triangular, U, V are n-by-p and $A(p,p)$ is nonsingular.*

We remark that for $2p > n-2$ the first statement of (1) is void. (1) is in this case just the statement: $B(n-p, n-p)$ nonsingular.

Proof (1) $\to$ (2): We apply Theorem 3.1 to $B(n-p, n-p)$. By (1) condition (4) is satisfied and hence $(B(n-p, n-p))^{-1}$ is a band matrix of width p. For $2p \ge n-2$ this is trivially true. We apply now (2.3) to $G = B$, and see that the first term in (2.3) is lower triangular. This shows (2).

(2) $\to$ (1): If $2p > n-2$, we have only to show that $B(n-p, n-p)$ is nonsingular. This is an immediate consequence of (2.3) applied to $G = A$ where $G_2 = A(p,p)$.

Now consider the case $2p \le n-2$. Then from the nonsingularity of $A(p,p)$ we get that the first p rows of U and the last p columns of V have rank p. Then we can, by eventually changing the last p rows of U, assume that the last p columns of L are zero. Similarly we can force the first p rows of L to be zero. Hence we have

$$A = \begin{pmatrix} 0 & 0 \\ \tilde{L} & 0 \end{pmatrix} + UV^T, \tag{4.6}$$

where $\tilde{L}$ is an $(n-p)$-by-$(n-p)$ bandmatrix of width p. It also follows that $\tilde{L}$ is nonsingular. We get from (2.1) and (2.4) that $\tilde{L}^{-1} = B(n-p, n-p)$. By Theorem 3.1 applied to $B(n-p, n-p)$ we see that the submatrices of $B(n-p, n-p)$ formed from their first r rows and their last $(n-p)+(p-r)$ columns have rank p, $1 \le r \le n-p$. Now we use that these submatrices are just $B(r, n-r)$ to prove (2).

□

Finally let us remark that there seems not to be a *simple* counterpart of Theorem 4.4 for $p > 1$.

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