Circular flow numbers of regular multigraphs

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Abstract

The circular flow number $F_c(G)$ of a graph $G = (V, E)$ is the minimum $r \in \mathbb{Q}$ such that G admits a flow ϕ with $1 \leq \phi(e) \leq r - 1$, for each $e \in E$.

We determine the circular flow number of some regular multigraphs. In particular, we characterize the bipartite $(2t+1)$-regular graphs ($t \geq 1$). Our results imply that there are gaps for possible circular flow numbers for $(2t+1)$-regular graphs, e.g. there is no cubic graph G with $3 < F_c(G) < 4$.

We further show that there are snarks with circular flow number arbitrary close to 4, answering a question of X. Zhu.

1 Introduction and basic definitions

In this paper we consider multigraphs $M = (V, E)$, with vertex set V and edge set E. Each edge is incident to precisely two different vertices, that is, we do not allow loops.

An *orientation* D of M is an assignment of a direction to each edge, and for $v \in V$, $D^+(v)$ ($D^-(v)$) is the set of edges whose head (tail) is incident to v.

Let $k \geq 2$ be a positive integer and $\phi : E \longrightarrow \{0, 1, \ldots, k-1\}$ be a function. If for all $v \in V$,

$$\sum_{e \in D^{+}(v)} \phi(e) = \sum_{e \in D^{-}(v)} \phi(e), \tag{1}$$

then ϕ is a *k-flow* on M. If, in addition, $\phi(e) \neq 0$, for all $e \in E$, then ϕ is a *nowhere-zero* k-flow on M. We also say that M has a (nowhere-zero) k-flow.

Tutte introduced the theory of integer flows on graphs and we refer the reader to [8, 11] or [15] for a more extensive introduction.

Tutte [13] showed that the graph property of having a k-flow is independent of the orientation of its edges, and hence determining for which value k a multigraph has a nowhere-zero k-flow is a problem about multigraphs, not directed multigraphs. Further if a graphs has a nowhere-zero k-flow than it has one for every $k' \geq k$. So the problem is to determine the smallest possible number for which a graph has such a flow. Tutte conjectured that bridgeless graphs have a nowhere-zero 5-flow (5-Flow-Conjecture), and that they have a nowhere-zero 3-flow if they do not have edge 3-cuts (3-Flow-Conjecture).

Seymour [10] proved that every bridgeless graph has a nowhere-zero 6-flow and Jaeger [6] proved that bridgeless graphs without edge 3-cut have a nowhere-zero 4-flow. Both conjectures are equivalent to its restrictions on some regular graphs, namely, it suffices to prove the 3-Flow Conjecture for 5-regular graphs and to prove the 5-Flow Conjecture for 3-regular graphs.

In [12] the author gave some sufficient conditions under which 3-regular graphs have a nowhere-zero 5-flow. It is also shown that a minimal counter-example to the 5-Flow Conjecture has at least 44 vertices.

Another approach to these conjectures might be to allow rational flow values. To be precise, let $M = (V, E)$ be a multigraph and D an orientation of its edges. Let ϕ be a function from E into the positive rational numbers and $r \in \mathbb{Q}^{+}$. If $1 \leq \phi(e) \leq r - 1$ for each $e \in E$ and (1) is satisfied for every vertex of M, then ϕ is called a *nowhere-zero r-flow* on M. If we allow $0 \leq \phi(e) < 1$ then ϕ is a *pseudo* r-flow on M. The minimum number r such that a multigraph M has a nowhere-zero r-flow is the *circular flow number* $F_c(M)$ of M.

This approach is closely related to the theory of orientations of graphs, see [8] or Chapter 9 in [15].

Let (D_1, ϕ_1) and (D_2, ϕ_2) be flows on $M = (V, E)$. The *sum* of (D_2, ϕ_2) and

(D_1, ϕ_1) is the flow (D, ϕ) with orientation

$$D := D_1|_{E \setminus \{e | \phi_2(e) > \phi_1(e)\}} \cup D_2|_{E \setminus \{e | \phi_1(e) \geq \phi_2(e)\}},$$

and

$$\phi(e) := \begin{cases} \phi_1(e) + \phi_2(e) & \text{if } e \text{ received the same direction under } D_1 \text{ and } D_2 \\ |\phi_1(e) - \phi_2(e)| & \text{otherwise} \end{cases}$$

for $e \in E$.

The following theorem relates rational valued flows and integer flows on graphs.

Theorem 1.1 *Let M be a multigraph and k a natural number. The following statements are equivalent:*

1. *there are two integers $1 \leq p < q$ such that $1 < \frac{q}{p} + 1 \leq k$ and M has a nowhere-zero $(\frac{q}{p} + 1)$-flow.*
2. *there are two integers $1 \leq p < q$ such that $1 < \frac{q}{p} + 1 \leq k$ and M has a nowhere-zero $(\frac{q}{p} + 1)$-flow, and for each $e \in E(M)$ there is an integer n such that $\phi(e) = \frac{n}{p}$.*
3. *M has an integer nowhere-zero k-flow.*

Proof. The statement that 2) or 3) implies 1) is trivial.

We prove that statement 1) implies 3). Let (D, ϕ) be a nowhere-zero $(\frac{q}{p} + 1)$-flow on M and $F \subseteq E(M)$ be the set of edges e where $\phi(e)$ is not an integer. If $|F| = 0$ we are done.

We proceed by induction on $|F|$. Each edge e with non-integer flow value is contained in a circuit C_e of such edges. Choose $e \in F$ such that $\phi(e) - 1$ or $k - 1 - \phi(e) = \min\{\phi(e) - 1, k - 1 - \phi(e) | e \in F\}$, say $\phi(e) - 1 = q$ is minimum. Let (D_e, q) be the pseudo flow on the directed circuit C_e with $q(e) = q$ $(\forall e \in C_e)$, and where e has the opposite direction as in D. The sum of (D, ϕ) and (D_e, q) is a nowhere-zero r-flow (D', ϕ') on M, and $|F'| \leq |F| - 1$. By induction hypothesis (D', ϕ') can be converted into an integer nowhere-zero k-flow on M. If $k - 1 - \phi(e)$ is the minimum of $\phi(e) - 1$ and $k - 1 - \phi(e)$ we proceed analogously.

That 1) implies 2) can be proved similarly, when we define F to be the set of edges where $\phi(e)$ is not of the form $\frac{n}{p}$ for a positive integer n. $\square$

Jaeger proposed the following conjecture.

Conjecture 1.2 (Jaeger [7]) *For all integers $p \geq 1$, every $4p$-edge-connected graph has a nowhere-zero $(2 + \frac{1}{p})$-flow.*

It is easy to see that it suffices to prove the 3-Flow-Conjecture for edge 4-connected 5-regular graphs. Therefore, for $p = 1$ Conjecture 1.2 is the 3-Flow-Conjecture, and in [7] it is shown that Conjecture 1.2 for $p = 2$ implies Tutte's 5-Flow Conjecture.

2 The Circular Flow Number of some Graphs

We will need the concept of balanced valuations on multigraphs which is due to Bondy [1] and Jaeger [5]. A *balanced valuation* on a multigraph G is a function w from the vertex set V of G into the real numbers such that for all $X \subseteq V$: $|\sum_{v \in X} w(v)| \leq |\partial_G(X)|$, where $\partial_G(X)$ is the set of edges with precisely one end in X in (multi-) graph G.

Using Theorem 1.1.2 the theory of flows can be translated into the theory of balanced valuations by the following theorem of Jaeger.

Theorem 2.1 ([5]) *A multigraph G has a nowhere-zero $(\frac{q}{p}+1)$-flow $(0 < p < q)$ if and only if there is a balanced valuation w of G such that for all $v \in V(G)$ there is an integer k_v such that $k_v \equiv d_G(v) \bmod 2$ and $w(v) = k_v \frac{q+p}{q-p}$.*

The complete graph on n vertices is denoted by K_n. If n is odd then K_n is eulerian and hence its circular flow number is 2. Jensen (see [2]) showed that if n is even and $n > 4$ then K_n has a nowhere-zero 3-flow. This is the best possible value for integer flows for these graphs. We will prove a refinement of Jensen's result.

Theorem 2.2 *For $k \geq 1$ holds $F_c(K_{2k+2}) = 2 + \frac{2}{k}$.*

Proof. Let $V = \{v_1, \ldots, v_{2k+2}\}$ be the set of vertices of K_{2k+2}. Define $w(v_i) = (-1)^i(k+1)$. We show that w is a balanced valuation.

Let $X \subset V(K_{2k+2})$ and $|X| = l$. Then $K_{2k+2}[X] = K_l$ and $|\partial(X)| = l(2k+2-l)$. Since $|\partial(X)| = |\partial(V \setminus X)|$ and $\sum_{v \in X} w(v) = -\sum_{v \in V \setminus X} w(v)$ we may assume that $l \leq k+1$. Then holds $\sum_{v \in X} w(v) \leq l(k+1) \leq l(2k+2-l) = |\partial(X)|$. Since equality holds when $l = k+1$ the statement is proved. □

Tutte [13] proved that a cubic multigraph is bipartite if and only if it has nowhere-zero 3-flow. The following theorem generalizes this result.

Theorem 2.3 *Let $k \geq 1$, for a $(2k+1)$-regular multigraph G the following statements are equivalent:*

1. *G is bipartite.*
2. *$F_c(G) = 2 + \frac{1}{k}$.*
3. *G admits a $(2 + \frac{r}{t})$-flow, and $\frac{1}{k} \leq \frac{r}{t} < \frac{2}{2k-1}$.*

Proof. $(1 \Rightarrow 2)$ Let $V(G) = A \cup B$ where A and B are independent sets. Since G is regular it follows that $|A| = |B|$. By König's Theorem, G is edge colorable with $2k+1$ colors. Let c be such a coloring with colors $c_1, c_2, \ldots, c_{2k+1}$, and let $e \in C_1$ if $c(e) = c_i$ and $i \leq k$ and $e \in C_2$ otherwise. Direct all edges in C_1 from A to B, and all edges in C_2 from B to A. Define $\phi(e) = 1 + \frac{1}{k}$ if $e \in C_1$, and $\phi(e) = 1$ if $e \in C_2$. Then ϕ is a nowhere-zero $(2 + \frac{1}{k})$-flow in G.

By Theorem 2.1 there is a balanced valuation w on G with $|w(v)| = 2k+1$, for all $v \in V(G)$. This is best possible, and hence $F_c(G) = 2 + \frac{1}{k}$.

The implication $(2 \Rightarrow 3)$ is trivial, hence we have to show

$(3 \Rightarrow 1)$ From Theorem 2.1 it follows that G has a balanced valuation w with $w(v) = k_v \frac{2t+r}{r}$ and $k_v \equiv 1 \bmod 2$. Since $2k < \frac{2t+r}{r}$, it follows that $2k+1 \geq |k_v \frac{2t+r}{r}| > |2kk_v|$, and consequently $|k_v| = 1$, i.e. $w(v) \in \{\pm \frac{2t+r}{r}\}$.

Let V^+ be the set of vertices where $w(v) > 0$. Assume that there is an edge uv in $G[V^+]$. Then $4k \geq w(u) + w(v) = 2 + 4\frac{t}{r} > 2 + 4(k - \frac{1}{2}) = 4k$, a contradiction. Hence V^+ is an independent set and $V \setminus V^+$ as well. □

Further characterizations of $(2k+1)$-regular bipartite graphs can be deduced from the restriction of Theorem 9.2.3 in [15] to $(2k+1)$-regular graphs.

For $k = 1$, Theorem 2.3 says that there is no cubic graph G with $3 < F_c(G) < 4$. On the other side it follows from results of Zhu [16] and Goddyn, Tarsi, Zhang [3], that for any rational number $2 \leq r \leq 4$ there is a planar graph G with $F_c(G) = r$.

Theorem 2.3 immediately raises the question whether there are $(2k+1)$-regular graphs G with $F_c(G) = 2 + \frac{2}{2k-1}$. We will show that such graphs exist. For this we will need the following well known theorem.

Theorem 2.4 *A cubic graph has a nowhere-zero 4-flow if and only if it is edge 3-colorable.*

Let G be a cubic graph, and $\mathcal{F}$ a 2-factor of G. Let $G^k_{\mathcal{F}}$ ($k \geq 1$) be the $(2k+1)$-regular multigraph, obtained from G by replacing each edge of $\mathcal{F}$ by k parallel edges.

Lemma 2.5 *Let G be a cubic graph and $k \geq 1$. If $F_c(G) = 4$ then there is a 2-factor $\mathcal{F}$ such that $F_c(G^k_{\mathcal{F}}) = 2 + \frac{2}{2k-1}$.*

Proof. From Theorem 2.4 it follows that G is edge 3-colorable and hence it has an even 2-factor $\mathcal{F}$. Let $C_1, \ldots, C_n$ be the even circuits of $\mathcal{F}$, and color the vertices of each circuit black and white, alternately. Let B and W be the set of black and white vertices, respectively. Then $|B| = |W|$, and $G[B]$ and $G[W]$ consist of components with at most two vertices.

Define $w(v) = 2k$, if $v \in B$ and $w(v) = -2k$, if $v \in W$. We show that w is a balanced valuation on $G^k_{\mathcal{F}}$, and hence $G^k_{\mathcal{F}}$ admits a $(2 + \frac{2}{2k-1})$-flow by Theorem 2.1.

Let $X \subseteq V$, and $B' = X \cap B$, $W' = X \cap W$. We may assume that $|B'| - |W'| = t \geq 0$, and hence $\sum_{v \in X} w(v) = 2kt$.

If $t = 0$ then certainly $|\sum_{v \in X} w(v)| \leq |\partial(X)|$.

If $t > 0$ then the only possibility to create a contradicting imbalance between $|B'|$ and $|W'|$ is that X contains induced paths $p_1, \ldots, p_s$ and $q_1, \ldots, q_r$ of even length having both end vertices in B' and W', respectively, and $t = s - r$. But any of these paths contributes at least $2k$ edges to $\partial(X)$ and hence $|\partial(X)| \geq 2kt = |\sum_{v \in X} w(v)|$. Thus $F_c(G^k_{\mathcal{F}}) \leq 2 + \frac{2}{2k-1}$.

Since G is not bipartite there are two vertices u, v of B (or W) which are connected by a single edge. Hence $w(u) + w(v) = 4k = |\partial(\{u, v\})|$ and therefore $F_c(G^k_{\mathcal{F}}) = 2 + \frac{2}{2k-1}$. $\square$

Let v be a vertex of degree d in a multigraph M, $d \geq 2$, and $v_1, \ldots, v_d$ be the not necessarily distinct neighbors of v.

Let $u_1, \ldots, u_d$ be the vertices of degree $d-1$ in the complete bipartite graph $K_{d,d-1}$. Multigraph M' is a *Meredith-extension* of M (applied on v, with $d_M(v) = d$), if it is obtained from $M - v$ and $K_{d,d-1}$ by adding edges $v_i u_i$, for $i = 1, \ldots, d$. Figure 1 illustrates this extension for the case $d = 5$.

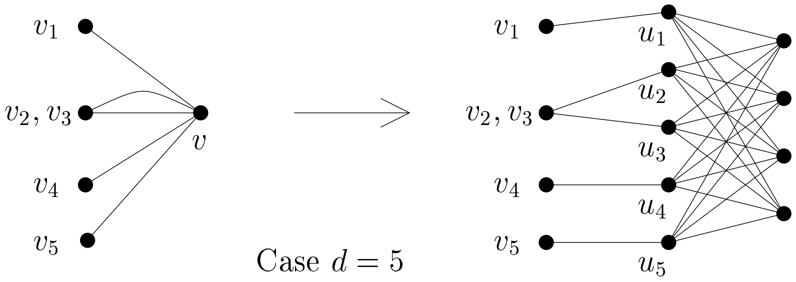


Figure 1 *Meredith-extension*

From a result of [9] it follows that a $(2k+1)$-regular multigraph M is class 2 if and only if its Meredith-extension is class 2. For flows we show

Theorem 2.6 *If M' is a Meredith-extension of a multigraph M, then*

$$F_c(M) = F_c(M').$$

Proof. Let $M = (V, E)$ be a multigraph and $u \in V$ with $d_M(v) = d \geq 2$. Let $M' = (V', E')$ be the Meredith-extension of M, applied on $u \in V$, and let A, B be the set of vertices of $K_{d,d-1}$ with degree d, $d-1$, respectively. Then $|A| = d - 1$ and $|B| = d$, and V' is the disjoint union of $V \setminus \{u\}$, A and B.

Since M has an $F_c(M)$-flow there is, by Theorem 2.1, a corresponding balanced valuation w on V such that for each $v \in V$, $w(v) = k_v r$ $(1 < r \leq k_v r \leq 2k+1)$ and $k_v \equiv d_M(v) \bmod 2$. We may assume that $k_u \geq 1$, i.e. $w(u) = k_u r$. Note that $k_u r \leq d$. Define w' on V' as follows:

$$w'(v) = \begin{cases} w(v) & : \quad v \in V \setminus \{u\} \\ k_u r & : \quad v \in B \\ -k_u r & : \quad v \in A \end{cases}$$

Per definition we have $w'(v) = k_v r$ for each $v \in V'$ and $k_v \equiv d_{M'}(v) \bmod 2$. To show that w' is a balanced valuation on M' we have to show that $|\sum_{v \in X} w'(v)| \leq |\partial_{M'}(X)|$, for all $X \subseteq V'$.

Let $X \subseteq V'$. We may assume that $M'[X]$ is connected. Let $X' = (V \setminus \{u\}) \cap X$, $A' = A \cap X$ and $B' = B \cap X$. Then X is the disjoint union of these three sets, where each of them could possibly be empty. It holds

$$|\partial_{K_{d,d-1}}(A' \cup B')| \quad = \quad d(|A'| + |B'|) - |B'| - 2|A'||B'|, \tag{2}$$

$$|\partial_{M'}(X')| = |\partial_M(X')|, \quad (3)$$

and

$$|\partial_{M'}(X)| = |\partial_{M'}(X')| - |B'| + |\partial_{K_{d,d-1}}(A' \cup B')|. \quad (4)$$

Further

$$\sum_{v \in X'} w'(v) = \sum_{v \in X'} w(v), \quad (5)$$

$$\sum_{v \in X} w'(v) = \sum_{v \in X'} w'(v) + \sum_{v \in A' \cup B'} w'(v). \quad (6)$$

and henceforth it follows from $|\sum_{v \in X'} w(v)| \leq |\partial_M(X')|$ and (2), (3), (4), (5), (6) that

$$|\partial_{M'}(X)| \geq |\sum_{v \in X'} w'(v)| + d(|A'| + |B'|) - 2|B'| - 2|A'||B'| \quad (7)$$

If $|B'| \geq |A'|$ then $\sum_{v \in A' \cup B'} w'(v) = k_u r(|B'| - |A'|) \geq 0$ and hence it follows from (7) that

$$\begin{aligned}
|\partial_{M'}(X)| &\geq |\sum_{v \in X'} w'(v)| + d(|B'| - |A'|) + 2d|A'| - 2|B'| - 2|A'||B'| \\
&\geq |\sum_{v \in X'} w'(v)| + |\sum_{v \in A' \cup B'} w'(v)| + 2d|A'| - 2|B'| - 2|A'||B'| \\
&\geq |\sum_{v \in X} w'(v)| + 2d(|A'| - |B'|) \\
&\geq |\sum_{v \in X} w'(v)|.
\end{aligned}$$

If $|B'| < |A'|$ then $|\sum_{v \in A' \cup B'} w'(v)| = k_u r(|A'| - |B'|) \geq 0$ and hence it follows from (7) that

$$
\begin{aligned}
|\partial_{M'}(X)| &\geq |\sum_{v\in X'} w'(v)| + d(|A'| - |B'|) + 2(d-1)|B'| - 2|A'||B'| \\
&\geq |\sum_{v\in X'} w'(v)| + |\sum_{v\in A'\cup B'} w'(v)| + 2(d-1)|B'| - 2|A'||B'| \\
&\geq |\sum_{v\in X} w'(v)|.
\end{aligned}
$$

Thus w' is a balanced valuation on M' with $w'(v) = k_v r$ ($k_v \equiv d_{M'}(v) \bmod 2$) and hence $F_c(M') = F_c(M)$. $\square$

As an immediate consequence of Lemma 2.5 and Theorem 2.6 we obtain

Corollary 2.7 *There are infinitely many $(2k+1)$-regular simple graphs with circular flow number equal to $2 + \frac{2}{2k-1}$.*

Zhu asked in [17] whether there are graphs with circular flow number strictly between 4 and 5. We answer this question in the affirmative.

The following statement seems to be known, see [17], but I have not seen a published proof so far.

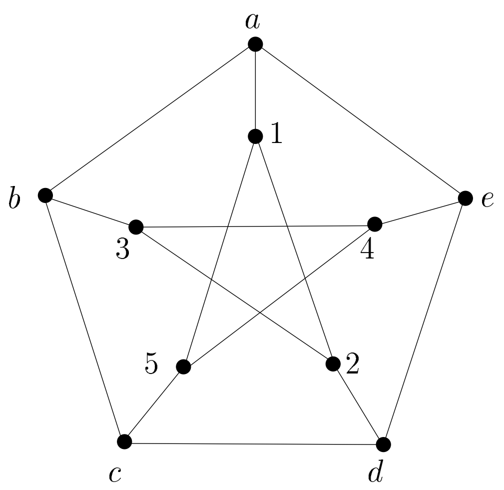


Figure 2 *Petersen graph P*

Proposition 2.8 *The Petersen graph has circular flow number 5.*

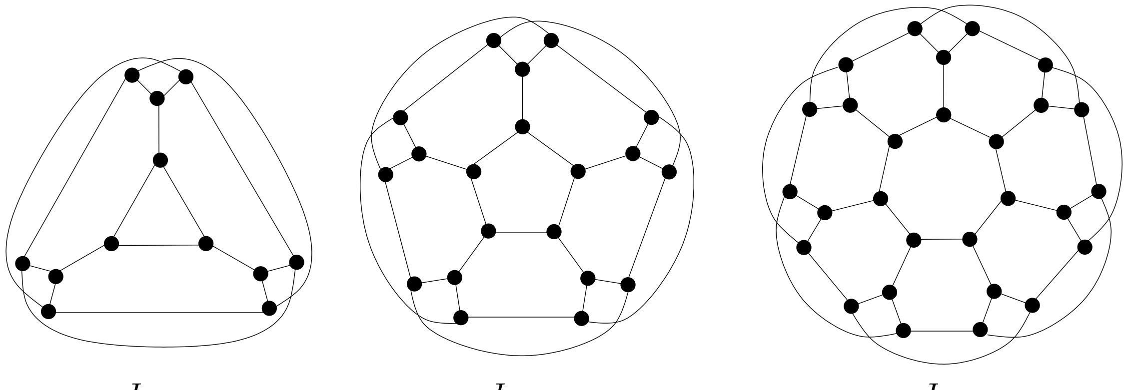
Proof. Let P be labeled as in Figure 2. Assume $F_c(P) < 5$. Then it follows from Theorem 2.1 that there is a balanced valuation w on P with $|w(v)| > \frac{5}{3}$, for all $v \in V(P)$. Hence, for the induced partition of $V(P)$ into two sets V_1 and V_2 ($|V_1| = |V_2|$) the induced connected components in $P[V_i]$, $i = 1, 2$, contain at most two vertices.

We may assume that $d, c \in V_1$. This implies $e, b, 2, 5 \in V_2$ and hence $a, 1 \in V_1$. Since $|V_1| = |V_2|$ one of 3 or 4 is in V_2, say 4. Then $5, 4$ and b are in V_2 and they induce a path of length 2, contradiction. Hence $F_c(P) = 5$. □

The *flower snark* J_{2k+1} is the snark with vertex set
$V(J_{2k+1}) = \{a_i, b_i, c_i, d_i | i = 0, 1, \ldots, 2k\}$ and edge set
$E(J_{2k+1}) = \{b_i a_i, b_i c_i, b_i d_i; a_i a_{i+1}; c_i d_{i+1}; d_i c_{i+1} | i = 0, 1, \ldots, 2k\}$. The graphs J_3, J_5 and J_7 are shown in Figure 3. This class of snarks was introduced by Isaacs [4].

Since these graphs are snarks it follows from Theorems 1.1 and 2.4 that $F_c(J_{2k+1}) > 4$, $k \geq 1$.

The Petersen graph can be obtained from J_3 by contracting the triangle to a single vertex. Thus it follows from Proposition 2.8 that $F_c(J_3) = 5$. However, all other flower snarks have smaller circular flow number as the following theorem shows.



J_3 J_5 J_7

Figure 3

Theorem 2.9 *For all* $k \geq 1 : 4 < F_c(J_{2k+1}) \leq 4 + \frac{1}{k}$.

Proof. We define some directed circuits in J_{2k+1} and flows on it. The sum of these flows then gives a $(4 + \frac{1}{k})$-flow on J_{2k+1}.

For $1 \leq i \leq k$, let (D_i, ϕ_i) be the pseudo flow on the directed circuit $C_i = a_0$, b_0, c_0, $d_1, \ldots, d_l$, c_{l+1}, $d_{l+2}, \ldots, d_{2i-1}$, b_{2i-1}, a_{2i-1}, a_{2i}, $a_{2i+1}, \ldots, a_0$, ($l < i$, l odd), with $\phi_i(e) = \frac{1}{k}$ if $e \in C_i$ and $\phi_i(e) = 0$ otherwise.

For $k+1 \le i \le 2k$, let (D_i, ϕ_i) be the flow on the directed circuit $C_i = a_{2(i-k)-1}, a_{2(i-k)}, b_{2(i-k)}, c_{2(i-k)}, d_{2(i-k)-1}, b_{2(i-k)-1}, a_{2(i-k)-1}$, with $\phi_i(e) = 1$ if $e \in C_i$ and $\phi_i(e) = 0$ otherwise.

Let (D_{2k+1}, ϕ_{2k+1}) be the flow on the directed circuit $C_{2k+1} = c_0, d_{2k}, b_{2k}, c_{2k}, d_{2k-1}, \ldots, d_0, b_0, c_0$, with $\phi_{2k+1}(e) = 1$ if $e \in C_{2k+1}$ and $\phi_i(e) = 0$ otherwise.

Let (D_{2k+2}, ϕ_{2k+2}) be the flow on the directed circuit $C_{2k+2} = c_0, d_1, b_1, c_1, d_2, \ldots, d_{2k}, b_{2k}, c_{2k}, d_0, b_0, c_0$, with $\phi_{2k+2}(e) = 1$ if $e \in C_{2k+2}$ and $\phi_i(e) = 0$ otherwise.

Let (D_{2k+3}, ϕ_{2k+3}) be the flow on the directed circuit $C_{2k+3} = a_0, a_1 \ldots, a_{2k}, a_0$, with $\phi_{2k+3}(e) = 1$ if $e \in C_{2k+3}$ and $\phi_i(e) = 0$ otherwise.

The sum of $(D_1, \phi_1), (D_2, \phi_2), \ldots, (D_{2k+2}, \phi_{2k+2}), (D_{2k+3}, \phi_{2k+3})$ gives a $(4 + \frac{1}{k})$-flow on J_{2k+1}. $\square$

Corollary 2.10 *For each $\epsilon > 0$ there are infinitely many snarks G with*

$$4 < F_c(G) < 4 + \epsilon$$

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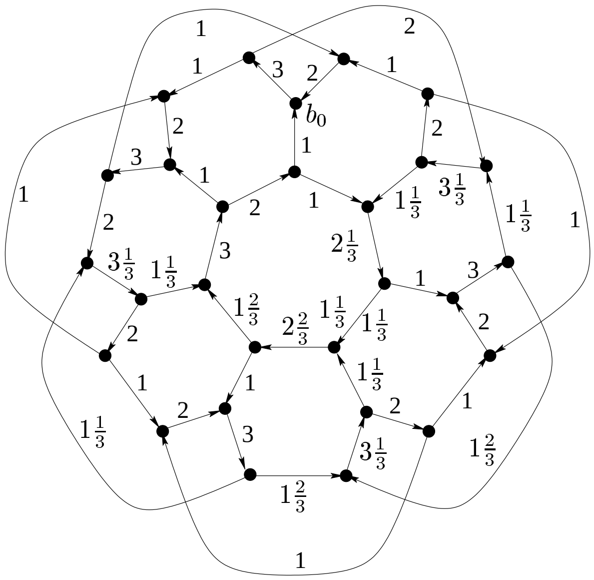

Figure 4 *A $(4 + \frac{1}{3})$-flow on J_7.*

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