Gauss decomposition with prescribed semisimple part: short proof

Vladimir Chernousov *

Fakultät Mathematik, Universität Bielefeld,

Postfach 10 01 31, 33501 Bielefeld, Germany

e-mail: chernous@mathematik.uni-bielefeld.de

Erich W. Ellers †

Department of Mathematics, University of Toronto,

100 St.George Street, Toronto, Ontario M5S 3G3, Canada

e-mail: ellers@math.toronto.edu

Nikolai Gordeev ‡

Department of Mathematics, Russian State Pedagogical University,

Moijka 48, 191 186 St. Petersburg, Russia

e-mail: algebra@ivt.rgpu.spb.ru

Abstract

We give a uniform short proof of the fact that the intersection of every non-central conjugacy class in a Chevalley group and a big Gauss cell is non-empty and that this intersection contains elements with any prescribed semisimple part. The interest in this property stems at least in part from its relation to Ore's and Thompson's conjectures in the theory of finite groups.

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1 Introduction

Let H be a finite non-cyclic simple group. O. Ore and J. Thompson conjectured the following:

Conjecture 1 (O. Ore). Every element $h \in H$ is a commutator.

Conjecture 2 (J. Thompson). There exists a conjugacy class $C \subset H$ such that $C \cdot C = H$.

We refer to [EG2], [EG3], [O], [AH] for the results, historic remarks, the literature and generalizations of these conjectures. Note only that Conjecture 2 implies clearly Conjecture 1 and it was proved for alternating groups by Hsü [H] in 1965 and for sporadic groups by Neubüser, Pahlings and Cleuvers [NPC] in 1984. Recently Thompson's conjecture was also proved for simple groups of Lie type over fields with more than 8 elements by the second and third authors [EG2].

The main ingredient (and the most difficult part) of the proof of Conjecture 2 for groups of Lie type in [EG2] is the so-called Gauss decomposition with a prescribed semisimple part (see [EG1]). To formulate it, recall first that any such group G is a group with a BN-pair. Denote by B^- the opposite subgroup. It follows then from the Bruhat decomposition that $G = B^-NB$. We will call this decomposition the Gauss decomposition and the cell B^-B the big Gauss cell. The main result of [EG1] states that the intersection of an arbitrary non-central conjugacy class $C \subset G$ and the big Gauss cell always contains an element with any prescribed semisimple part $h \in H$, where $H = B \cap N$. For generalizations of this statement see [EG4], [MP].

The original proof of the Gauss decomposition with a prescribed semisimple part is too long and consists of case-by-case considerations. The purpose of the present work is to give a more or less uniform and short proof of it for all Chevalley groups over arbitrary fields and for all twisted forms of Chevalley groups over finite fields.

In the first part of the paper we carry out a reduction to the case of groups of rank at most 2. The main idea of the reduction is due to the first author and it appeared during our joint discussions of the results of [EG1] and [EG2].

In the second part we deal with groups of rank 1 and 2. This is the most lengthy part of our argument. In § 5 and § 6 we present two different proofs for groups of rank 2. The first one is uniform, but it works only for groups over fields with more than 4 elements. For the sake of expository completeness we include the second proof which is based on case-by-case considerations and allows to include all groups over fields with 3 or 4 elements.

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2 Notation and statement of the theorem

Throughout the paper we use the following notation.

K — an arbitrary field and $\overline{K}$ — a fixed algebraic closure;

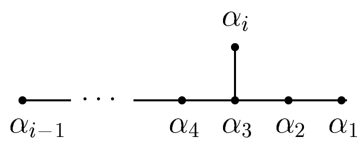
G — a simple simply connected algebraic group defined and split over K;

T — a maximal K-split torus in G;

$N = N_G(T)$ — the normalizer of T in G;

$\Sigma = \Sigma(T, G)$ — the root system of G relative to T;

$\Pi = \{\alpha_1, \dots, \alpha_n\}$ — a basis of Σ. The numeration of roots in Π corresponds to [B], tables I – X except groups of types E_i, $i = 6, 7, 8$; for type E_i we number simple roots as follows:



$\Sigma^{\pm}$ — sets of positive and negative roots;

W — the Weyl group;

B (resp. B^-) – the Borel subgroup corresponding to Σ^+ (resp. Σ^-);

U (resp. U^-) – the unipotent radical of B (resp. B^-).

If K is a finite field, one defines a Frobenius map

$$F : G(\overline{K}) \longrightarrow G(\overline{K})$$

(see [C2]). In this paper we will only consider the cases when F is a product of a graph automorphism ρ and a Frobenius field automorphism θ. The subgroup $G(K)^F \leq G(K)$ of fixed points is a finite twisted Chevalley group which has its own root system Σ_ρ and its own system of simple roots Π_ρ corresponding respectively to the $\langle\rho\rangle$-orbits in Σ and Π. Recall that the sets $\Sigma^{\pm}$ and the Weyl group W are ρ-stable and all subgroups of G defined above are also F-stable. The pair $(B(K)^F, N(K)^F)$ is a BN-pair for the group $G(K)^F$ (see [C1]).

Further we will deal only with K-points of the groups G, T, N, $B^{\pm}$, $U^{\pm}$ and their subgroups of F-fixed elements. So for simplicity instead of

$$G(K),\ N(K),\ B^{\pm}(K),\ U^{\pm}(K),\ G(K)^F,\ N(K)^F,\ B^{\pm}(K)^F,\ U^{\pm}(K)^F$$

we will write

$$G,\ N,\ B^{\pm},\ U^{\pm},\ G^F,\ N^F,\ (B^{\pm})^F,\ (U^{\pm})^F.$$

When there is no danger of confusion we will often omit the symbol F in G^F, N^F, $(B^{\pm})^F$, $(U^{\pm})^F$. Thus, throughout the paper G will denote either K-points of a Chevalley group or its twisted form. By abuse of language, we will also identify elements of the group N with their images in the Weyl group W. Thus, w_α denotes either the reflection in W corresponding to a root $\alpha \in \Sigma$ or any its inverse image in the group N. We denote the longest element in W by w_0 .

Furthermore, for the groups $T(K)$, $T(K)^F$ we will use the symbol H. So in our notation the big Gauss cell is $U^- HU$. Throughout the paper the element h in the decomposition $g = v\,h\,u$, where $v \in U^-$, $h \in H$, $u \in U$, will be called the semisimple part of g.

When we deal with twisted Chevalley groups over finite fields we will often omit index ρ in the notation Σ_ρ, Π_ρ and for such groups our numeration of roots coincides with that of [C1].

If α is a root, we denote the root subgroup corresponding to α by U_α. If G is a Chevalley group, $U_\alpha = \langle x_\alpha(u) \mid u \in K\rangle$ is a one parameter subgroup. In the twisted cases U_α is a either one, two or three parameter subgroup. For any fixed order of roots every element $u \in U$ (resp. U^-) can be written uniquely as a product $u = \prod_{\alpha\in\Sigma^+} u_\alpha$, where $u_\alpha \in U_\alpha$, (resp. $U_{-\alpha}$). We will call the element u_α the α-component of u.

We also use the Steinberg notation $h_\alpha(t)$, $t \in K^\star$, for the corresponding semisimple element in $H \cap \langle U_{-\alpha}, U_\alpha\rangle$. Note that in the case of a twisted Chevalley group G one has

$$h_\alpha(t) = h_\beta(t)h_{\rho(\beta)}(t^\theta)\cdots h_{\rho^k(\beta)}(t^{\theta^k}),$$

where $\{\beta,\, \rho(\beta), \dots, \rho^k(\beta)\}$ is the $\langle\rho\rangle$-orbit of $\beta \in \Sigma$ which corresponds to the root $\alpha \in \Sigma_\rho$.

We are now in a position to formulate the main result of the paper. Let G be either a Chevalley group over an arbitrary field or a twisted Chevalley group over a finite field. Consider a group

$$\Gamma = \langle\, s,\, G\,\rangle$$

such that G is a normal subgroup in Γ and the conjugation $g \to sgs^{-1}$ is a diagonal automorphism of G. Note that we do not exclude the case when s is central in Γ. Recall also that it follows from the definition of diagonal automorphisms (see [St], §§ 10, 11) that the subgroups $U_{\pm\alpha}$, $B^{\pm}$, H, N are stable under s.

Theorem 2.1 *Let $f = s\,g \in \Gamma$ be a non-central element. Then for every $h \in H$ there exists an element $x \in G$ such that*

$$x\,f\,x^{-1} = s\,v\,h\,u$$

for some $v \in U^-$, $u \in U$.

3 Reduction to groups of rank at most 2

For any subset $\Pi' \subset \Pi$ denote by Σ' a root subsystem generated by Π'. Let also $G' = \langle\, U_\alpha \mid \alpha \in \Sigma' \,\rangle$.

Lemma 3.1 *Let*

$$V = \prod_{\beta \in \Sigma^+ \setminus (\Sigma')^+} U_\beta$$

where the product is taken in an arbitrary order. Then for any $x \in G'$ *we have* $x\, V\, x^{-1} = V$.

Proof. This follows immediately from the Chevalley commutator formula (see [St], [C2]). □

The fact that any non-central conjugacy class in G intersects the big Gauss cell is well known. We indicate a short proof of it for the convenience of the reader.

Proposition 3.2 *Let* $f = s\, g \in \Gamma$ *be a non-central element. Then* f *is conjugate in* Γ *to an element* f' *of the form* $f' = v\, s\, t\, u$, *where* $v \in U^-$, $t \in H$, $u \in U$.

Proof. It suffices to prove that $G = B\, B^- B$. For groups of rank 1 the last statement is trivial. Let G be a group of an arbitrary rank and let $g \in G$. By the Gauss decomposition, $G = B^-\, W\, B$. We may then assume that g is of the form $g = v\, w$, where $v \in B^-$, $w \in W$. Suppose now that $w = w_\beta\, w'$, where $\beta \in \Pi$ and the length $l(w')$ of w' is less than that of w. Let $v = v'\, v_{-\beta}$, where

$$v' = \prod_{\alpha \in \Sigma^+ \setminus \beta} v_{-\alpha}$$

and $v_{-\alpha} \in U_{-\alpha}$, $v_{-\beta} \in U_{-\beta}$. The group $\langle\, U_\beta,\, U_{-\beta} \,\rangle$ has rank 1 and so we can put $v_{-\beta}\, w_\beta = v_1\, v_2\, v_3$ for some $v_1,\, v_3 \in U_\beta$, $v_2 \in H\, U_{-\beta}$. Lemma 3.1 shows that $v_1^{-1}\, v'\, v_1 \in U^-$. Since $l(w') < l(w)$, we also have $(w')^{-1} U_\beta\, w' \in U$ (see [St], Appendix, Lemma 19), hence $(w')^{-1} v_3\, w' \in U$. It follows that g is of the form

$$g = v_1 \left[\left(v_1^{-1}\, v'\, v_1 \right) v_2 \right]\; w' \left[(w')^{-1} v_3\, w' \right]$$

and an inductive argument completes the proof. □

Proposition 3.3 *Let* $f = s\, g \in \Gamma$ *be a non-central element. Then* f *is conjugate in* Γ *to an element* f' *of the form* $f' = v\, s\, t\, u$, *where* $v \in U^-$, $t \in H$, $u \in U$ *and at least for one simple root* $\alpha \in \Pi$ *an* α*-component* $v_{-\alpha}$ *or* u_α *is non-trivial.*

Proof. Applying Proposition 3.2 we may assume that $f = v\,s\,t\,u$. Suppose all simple α-components of the elements v and u are trivial. Since f is non-central, we may additionally assume that a γ-component $u_\gamma \neq 1$ for some root $\gamma \in \Sigma^+$ (if $u = v = 1$, then $u_\gamma\, f\, u_\gamma^{-1}$ is an appropriate element for some $\gamma \in \Sigma^+$). Pick an element $w \in W$ such that $w(\gamma) = \alpha \in \Pi$ and let $w = w_{\beta_n} \cdots w_{\beta_1}$ be a reduced decomposition of w. Consider the following sequences:

$$
\begin{aligned}
v_0 &= v, & v_1 &= w_{\beta_1} v w_{\beta_1}^{-1}, & \ldots,\quad & v_n = w_{\beta_n} v_{n-1} w_{\beta_n}^{-1},\\
st_0 &= st, & st_1 &= w_{\beta_1} st w_{\beta_1}^{-1}, & \ldots,\quad & st_n = w_{\beta_n} st_{n-1} w_{\beta_n}^{-1},\\
u_0 &= u, & u_1 &= w_{\beta_1} u w_{\beta_1}^{-1}, & \ldots,\quad & u_n = w_{\beta_n} u_{n-1} w_{\beta_n}^{-1}.
\end{aligned}
$$

If $u_{i-1} \in U$ (resp. $v_{i-1} \in U^-$) and the β_i-component of u_{i-1} (resp. v_{i-1}) is trivial, then by Lemma 3.1 we have $u_i \in U$ (resp. $v_i \in U^-$). It follows immediately that among the elements

$$f_0 = f,\ f_1 = w_{\beta_1} f_0 w_{\beta_1}^{-1}, \ldots, f_n = w_{\beta_n} f_{n-1} w_{\beta_n}^{-1}$$

there is at least one with the prescribed property. $\square$

We are now ready to reduce the case of an arbitrary group G in the statement of our main theorem to the case of groups of rank at most 2. Assume that $r = \operatorname{rank} G > 2$ and that for all groups of rank ≤ 2 the statement holds. Let $G_1,\ G_2 \leq G$ be the subgroups of G generated by $\langle\, U_{\pm\alpha_1}, \ldots, U_{\pm\alpha_{r-1}} \,\rangle$ and $\langle\, U_{\pm\alpha_2}, \ldots, U_{\pm\alpha_r} \,\rangle$ respectively.

Proposition 3.3 shows that any $f \in \Gamma$ up to conjugation is of the form

$$f = v\,s\,t\,u, \tag{1}$$

where an α-component of u or v is non-trivial for some $\alpha \in \Pi$. Assume first that $\alpha \in \Sigma_2^+ = \langle\, \alpha_2, \ldots, \alpha_r \,\rangle^+$. Let us write our prescribed semisimple element h and the semisimple part t of f in the form

$$
\begin{aligned}
t &= h_{\alpha_1}(t_1')\, h_{\alpha_2}(t_2') \cdots h_{\alpha_r}(t_r'),\ t_i' \in K^\star,\\
h &= h_{\alpha_1}(t_1)\, h_{\alpha_2}(t_2) \cdots h_{\alpha_r}(t_r),\ t_i \in K^\star,
\end{aligned} \tag{2}
$$

(see [St], [C1]). Substituting (2) into (1) we may rewrite f in the form

$$f = v'\ [v_1\,(s\, h_{\alpha_1}(t_1')\, h_{\alpha_2}(t_2') \cdots h_{\alpha_r}(t_r'))\, u_1]\ u' \tag{3}$$

where $v_1 \in G_2 \cap U^-$, $u_1 \in G_2 \cap U$ and

$$v' \in \prod_{\beta \in \Sigma^+ \setminus \Sigma_2^+} U_{-\beta},\quad u' \in \prod_{\beta \in \Sigma^+ \setminus \Sigma_2^+} U_\beta.$$

Denote $\Gamma_2 = \langle\, s h_{\alpha_1}(t'_1),\, G_2 \,\rangle$ and let $f_1 = v_1\,(s\,h_{\alpha_1}(t'_1)\,h_{\alpha_2}(t'_2)\cdots h_{\alpha_r}(t'_r))\,u_1$. By our construction, the α-component of v_1 or u_1 is non-trivial, implying f_1 is non-central in Γ_2. Note also that for any $x \in G_2$ Lemma 3.1 shows that

$$x\,v'\,x^{-1} \in \prod_{\beta\in\Sigma^+\setminus\Sigma_2^+} U_{-\beta}, \quad x\,u'\,x^{-1} \in \prod_{\beta\in\Sigma^+\setminus\Sigma_2^+} U_{\beta}.$$

Therefore, applying induction we have that for any parameter $\widetilde{t}_2 \in K^\star$ there exists $x \in \Gamma_2$ such that the element $x\,f\,x^{-1}$ lies in the big Gauss cell and its semisimple part is of the form

$$h' = s\,h_{\alpha_1}(t'_1)\,h_{\alpha_2}(\widetilde{t}_2)\,h_{\alpha_3}(t_3)\cdots h_{\alpha_r}(t_r). \tag{4}$$

To modify parameters t'_1, $\widetilde{t}_2$ we need to pick a parameter $\widetilde{t}_2$ in such a way that the element h' is non-central in $\Gamma_1 = \langle\, s\,h_{\alpha_r}(t_r),\, G_1 \,\rangle$.

Lemma 3.4 *There exists $\widetilde{t}_2 \in K^\star$ such that the element (4) is non-central in Γ_1.*

Proof. We may assume that $|K| > 2$, since otherwise by Proposition 3.2 the statement of Theorem 2.1 is trivial. Take any parameter $\widetilde{t}_2$ and suppose that h' is central in Γ_1. We claim then that for any $z \in K^\star$, $z \neq 1$, the element

$$h'\,h_{\alpha_2}(z) = s\,h_{\alpha_1}(t'_1)\,h_{\alpha_2}(\widetilde{t}_2 z)\,h_{\alpha_3}(t_3)\cdots h_{\alpha_r}(t_r)$$

is non-central in Γ_1. Indeed, it suffices to observe that according to our agreement on numeration of the roots in Π the subsystem generated by α_1, α_2 has type A_2, hence $h_{\alpha_2}(z)$ is non-central in G_1. □

Thus, without loss of generality we may assume that our element f is of the form $f = v\,h'\,u$, where $v \in U^-$, $u \in U$, h' is given by (4) and h' is non-central in Γ_1. Arguing as above, we can rewrite f in the form

$$f = v'\,\left[v_1\,(s\,h_{\alpha_1}(t'_1)\,h_{\alpha_2}(\widetilde{t}_2)\,h_{\alpha_3}(t_3)\cdots h_{\alpha_r}(t_r))\,u_1\right]\,u'$$

where v_1, $u_1 \in G_1$ and

$$v' \in \prod_{\beta\in\Sigma^+\setminus\Sigma_1^+} U_{-\beta}, \quad u' \in \prod_{\beta\in\Sigma^+\setminus\Sigma_1^+} U_{\beta}.$$

Here Σ_1 is a subsystem generated by roots $\alpha_1, \ldots, \alpha_{r-1}$. It remains to note that the element in the square bracket is non-central in Γ_1 and hence the same argument as above shows that we can modify parameters t'_1, $\widetilde{t}_2$ as we want.

Now let $\alpha \in \{\alpha_1, \ldots, \alpha_{r-1}\}$ and let f be again of the form (1) and its semisimple part t is given by (2). This case is slightly different from the previous one for here α_{r-1}, α_r can generate a subsystem of type B_2 or $A_1 \times A_1$. However as a preliminary step we can first modify the parameter t_1' in the decomposition (2) so that the element $s\,t$ will be non-central in Γ_2. This fact then will imply that the element in the square bracket in (3) is non-central in Γ_2 and the rest of the proof works in exactly the same fashion as above.

4 Groups of rank 1

For Chevalley groups of rank 1, i. e. for $SL_2(K)$, and for twisted groups of the form $SL_2(K^\theta)$ the statement of Theorem 2.1 can be easily verified by direct computations. So we need to consider only twisted forms G^F of a Chevalley group G of type either A_2, C_2, or G_2. In this section we will work with both groups G, G^F simultaneously. So for the group G (resp. G^F) we will use the standard notation $B^\pm$, H, $U^\pm$ (resp. $(B^\pm)^F$, H^F, $(U^\pm)^F$).

Let $\Pi_\rho = \{\alpha\}$. We begin with the following proposition which (as we see below) eventually will be crucial for the proof.

Proposition 4.1 *For any $h \in H^F$ there exists $y \in U^F$ such that*

$$y\, w_\alpha \in (U^-)^F\, h\, U^F$$

Proof. Let ω_i be a fundamental weight of the group G and let

$$\lambda_i : G \longrightarrow GL(V)$$

be the corresponding rational irreducible representation. Consider a basis $e = \{e_1, \ldots, e_m\}$ of V consisting of the weight-vectors and such that the vectors e_1, e_m correspond to the weights ω_i and $w_\alpha(\omega_i)$ respectively. The map λ_i provides us then with the matrix representation

$$\widetilde{\lambda}_i : G \longrightarrow GL_m$$

By our construction, the matrix $\widetilde{\lambda}_i\,(w_\alpha)$ clearly is of the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & d_1 \\ 0 & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & * & \cdots & * & * \\ d_m & & \cdots & & * \end{pmatrix} \tag{5}$$

where $d_1 \cdot d_m \neq 0$.

Let

$$h = h_{\alpha_1}(t^\theta)\, h_{\alpha_2}(t) \in G^F \leq G,$$

where $\{\alpha_1,\, \alpha_2\} = \Pi$ is the basis of the root system Σ (here we assume that α_1 is the short root). Clearly, we have $\alpha_2 = \rho(\alpha_1)$, so Π is the $\langle\, \rho\,\rangle$-orbit corresponding to the root $\alpha \in \Pi_\rho$. Since $\langle\, \omega_i, \alpha_j\,\rangle = \delta_{ij}$ (see [St]), we get

$$\widetilde{\lambda}_i(h)(e_1) = t^\nu e_1, \tag{6}$$

where $\nu = 1$ or θ, implying that for any elements $v \in (U^-)^F$, $u \in U^F$ the $(1,1)$-entry of the matrix $\widetilde{\lambda}_i(v\,h\,u)$ is equal to t^ν. It means that for any element $g \in (U^-)^F\, H\, U^F$ from the big Gauss cell of the group G^F the $(1,1)$-entry of the corresponding matrix $\widetilde{\lambda}_i(g)$ defines the semisimple part h of g uniquely.

If $g \in G^F$ is not in the big cell, then g is of the form $g = w_\alpha\, h'\, u'$ for some $h' \in H^F$, $u' \in U^F$ and one can easily verify that the $(1,1)$-entry of the matrix $\widetilde{\lambda}_i(g)$ equals 0.

Put $a = t^\nu$. From what has been said above and from (5), (6) it follows easily that to complete the proof of the proposition it suffices to find $y \in U^F$ such that the $(1,m)$-entry of $\widetilde{\lambda}_i(y)$ is a/d_m.

Case $\Sigma = A_2$: Then $G^F = SU_3(K)$. Take the first fundamental weight ω_1. Then $\widetilde{\lambda}_1$ is the natural representation, hence every matrix $\widetilde{\lambda}_1(y)$, $y \in U^F$, has the form

$$\begin{pmatrix} 1 & l & -q \\ 0 & 1 & l^\theta \\ 0 & 0 & 1 \end{pmatrix} \tag{7}$$

where $l, q \in K$ and $l\,l^\theta + q + q^\theta = 0$ (see [St], [C2]). Since the norm map $K \to K^\theta$ is surjective, there exists $y \in U^F$ with the property $-q = a/d_3$.

Case $\Sigma = C_2$: Here $G = Sp_4(K)$. Take again the first fundamental weight ω_1. Then $\widetilde{\lambda}_1$ is the natural representation, and it has four weights ε_1, ε_2, $-\varepsilon_2$, $-\varepsilon_1$. If $y \in U^F$, then

$$y = x_{2\varepsilon_2}(l)\, x_{\varepsilon_1-\varepsilon_2}(l^\theta) x_{2\varepsilon_1}(q)\, x_{\varepsilon_1+\varepsilon_2}(l^{1+\theta} + q^\theta),$$

where $l,\, q \in K$ (see [St], §11, Lemma 63). Put $l = 0$. Then the $(1,4)$-entry of $\widetilde{\lambda}_1(y)$ is q. Therefore, we can put $q = a/d_4$.

Case $\Sigma = G_2$: Pick the second fundamental weight ω_2. Then $\widetilde{\lambda}_2$ is the adjoint representation, hence the vector space V coincides with the Lie algebra $\mathfrak{g}$ of G. Denote the element of the Chevalley basis of $\mathfrak{g}$ corresponding to the root $\gamma \in \Sigma$ by e_γ.

As above we will also use the following notation: $\Pi = \{\alpha_1,\, \alpha_2\}$; $\Pi_\rho = \{\alpha\}$. The element $w_\alpha \in W^\rho \leq W$ clearly coincides with -1 . By the construction,

$e_{3\alpha_1+2\alpha_2}$ is the highest weight vector and $w_\alpha(e_{3\alpha_1+2\alpha_2}) = \pm\, e_{-3\alpha_1-2\alpha_2}$ is the lowest weight vector.

Consider the following two elements in G^F:

$$y(l) = x_{2\alpha_1+\alpha_2}(l^\theta)x_{3\alpha_1+2\alpha_2}(l),$$
$$y_0 = x_{\alpha_1}(1)x_{\alpha_2}(1)x_{\alpha_1+\alpha_2}(1)x_{2\alpha_1+\alpha_2}(1).$$

According to [C1], p. 247, we have

$$\begin{aligned} y(l)(e_{-3\alpha_1-2\alpha_2}) &= -l^2 e_{3\alpha_1+2\alpha_2} + A_1, \\ y_0(e_{-3\alpha_1-2\alpha_2}) &= e_{3\alpha_1+2\alpha_2} + A_2, \end{aligned} \tag{8}$$

where A_1, A_2 are linear combinations of vectors from the Chevalley basis different from $e_{3\alpha_1+2\alpha_2}$. Furthermore, if $h_\alpha(l) = h_{\alpha_1}(l^\theta)\, h_{\alpha_2}(l) \in G^F$, then we have

$$\begin{aligned} h_\alpha(l)(e_{-3\alpha_1-2\alpha_2}) &= l^{-1}\, e_{-3\alpha_1-2\alpha_2}, \\ h_\alpha(l)(e_{3\alpha_1+2\alpha_2}) &= l\, e_{3\alpha_1+2\alpha_2}. \end{aligned} \tag{9}$$

Now (8), (9) show that the $(1, 14)$-entries of the matrices $\widetilde{\lambda}_2(y(l))$ and $\widetilde{\lambda}_2(h_\alpha(l)\, y_0\, h_\alpha(l^{-1}))$ are $-l^2$ and l^2 respectively. So it remains to observe that $|K| = 3^{2b+1}$ (see [St], [C1]), therefore $|K^*| = 2\,d$, where d is odd, implying every element $x \in K^*$ can be written in the form $\pm l^2$. The proof of the proposition is complete. $\square$

Using Proposition 4.1 it is easy to complete the proof of Theorem 2.1 for twisted groups of rank 1. Indeed, without loss of generality we may assume that $g \notin B^F$. Then $f = s\, g \in s\, B^F\, w_\alpha\, B^F$. After conjugating (if necessary) we may even assume that $f \in w_\alpha\, B^F\, s$. Let $f = w_\alpha\, b\, s$, where $b \in B^F$. Then for an element y from Proposition 4.1 the semisimple part of the element

$$y\, f\, y^{-1} = (y\, w_\alpha)\, b\, (s\, y^{-1}\, s^{-1})\, s$$

is the product of the semisimple part of $y\, w_\alpha$ and that of b. It implies that we can get every semisimple part in the Gauss decomposition of $y\, f\, y^{-1}$.

5 Groups of rank 2 over big fields

Throughout this section, G denotes either a Chevalley group over an arbitrary field or a twisted Chevalley group over a finite field of rank 2. Let $\Pi = \{\alpha, \beta\}$. If $\gamma \in \Sigma$, let also $G_\gamma = \langle U_{\pm\gamma} \rangle$, $H_\gamma = H \cap G_\gamma$. Recall that H_γ consists of elements of the form $h_\gamma(t)$.

Denote the group of diagonal automorphisms of a root subgroup U_γ by D_γ. For any two roots $\gamma,\,\delta\in\Sigma$ we have the canonical map $\mu_{\gamma\delta}:H_\gamma\longrightarrow D_\delta$, where

$$\mu_{\gamma\delta}(h):u\longrightarrow h\,u\,h^{-1}$$

Proposition 5.1 *Assume*
i) $\mu_{\alpha\beta}\neq 1,\ \mu_{\beta\alpha}\neq 1$;
ii) *for any pair* $d_1\in D_\alpha,\ d_2\in D_\beta$ *there exists* $h_\alpha\in H_\alpha$ *such that* $\mu_{\alpha\alpha}(h_\alpha)\neq d_1,\ \mu_{\alpha\beta}(h_\alpha)\neq d_2$.

Then Theorem 2.1 holds for G.

The proof is analogous to the corresponding argument in §3. Namely, let f=s g. By Proposition 3.3, one can assume that

$$f=v\,s\,t\,u, \tag{10}$$

where $v\in U^-$, $t\in H$, $u\in U$ and a γ-component of v or u is non-trivial for $\gamma=\alpha$ or β. Let $C=\{x\,f\,x^{-1}\mid x\in G\}$ and let $C'\subset C$ consist of all elements which have the decomposition of the form (10).

Lemma 5.2 *Let f be as above and let $\gamma\in\Pi$. Assume that $s\,t$ is non-central in $\langle\, s\,t,\,G_\gamma\,\rangle$ or the γ-component of v or u is non-trivial. Then for an arbitrary parameter $z\in K^\star$ the set C' contains an element with the semisimple part $t\,h_\gamma(z)$.*

Proof. Rewrite (10) in the form

$$f=v'\,[v_{-\gamma}\,s\,t\,u_\gamma]\,u',$$

where

$$v'\in\prod_{\delta\in\Sigma^+\setminus\{\gamma\}}U_{-\delta},\qquad u'\in\prod_{\delta\in\Sigma^+\setminus\{\gamma\}}U_\delta.$$

The conditions of the lemma guarantee that the element in the square bracket is non-central in $\langle\, s\,t,\,G_\gamma\,\rangle$. Since the sets

$$\prod_{\delta\in\Sigma^+\setminus\{\gamma\}}U_{-\delta},\qquad\prod_{\delta\in\Sigma^+\setminus\{\gamma\}}U_\delta$$

are invariant under conjugation by elements from G_γ, the statement follows from the fact that the Gauss decomposition has been already proved for groups of rank 1. □

Let us come back to the proof of the proposition. We claim first that up to conjugation one can assume that the α-component of v or u in the decomposition (10) is non-trivial. Assume the contrary. Then by Proposition 3.3

the β-component of u or v is non-trivial. Applying Lemma 5.2 we get that C' contains an element f' with the semisimple part of the form $t\,h_\beta(z)$. By the condition i) we can pick the parameter z such that $s\,t\,h_\beta(z)$ does not commute with some $u_\alpha \in U_\alpha$. It follows then that the element $u_\alpha\, f'\, u_\alpha^{-1}$ is as required.

Thus without loss of generality we may assume that our element f is of the form (10) and the α-component of v or u is non-trivial. Let

$$t = h_\alpha(t'_\alpha)\, h_\beta(t'_\beta), \qquad h = h_\alpha(t_\alpha)\, h_\beta(t_\beta).$$

By ii), there is a parameter t_0 such that the elements $s\,h_\alpha(t_0)\,h_\beta(t'_\beta)$ and $s\,h_\alpha(t_0)\,h_\beta(t_\beta)$ are non-central in $\langle\, s\,h_\alpha(t_0),\, G_\beta \,\rangle$ and $\langle\, s\,h_\beta(t_\beta),\, G_\alpha \,\rangle$ respectively. It follows then that applying Lemma 5.2 three times we can first pass from the element f to an element with the semisimple part $s\,h_\alpha(t_0)\,h_\beta(t'_\beta)$, then to that of $s\,h_\alpha(t_0)\,h_\beta(t_\beta)$ and finally to $s\,h_\alpha(t_\alpha)\,h_\beta(t_\beta)$. The proposition is proved. $\square$

Corollary 5.3 *Theorem 2.1 holds for all groups of rank* 2 *except possibly the following cases:*

$A_2(q)$, $B_2(q)$, ${}^2A_3(q^2)$, ${}^2A_4(q^2)$, $\qquad q = 3;$

$G_2(q)$, ${}^3D_4(q^3)$, $\qquad q = 3, 4;$

Proof. If the basic field K consists of two elements, there is nothing to prove since Proposition 3.3 shows that up to conjugation any element $f \in G$ lies in the big Gauss cell. In the remaining cases (except the above mentioned) it is a routine matter to verify the conditions i), ii) in Proposition 5.1. Let us verify these conditions, for example, for type ${}^2F_4(q^2)$, where $q^2 \neq 2$.

Let $\Pi = \{\alpha_1,\, \alpha_2,\, \alpha_3,\, \alpha_4\}$ be the basis of the root system of type F_4. Then there are two $\langle\,\rho\,\rangle$-orbits in Π: $\{\alpha_1,\, \alpha_4\}$ and $\{\alpha_2,\, \alpha_3\}$ (see [St], [C2]). Let α (resp. β) be the root in Π_ρ corresponding to $\{\alpha_2,\, \alpha_3\}$ (resp. $\{\alpha_1,\, \alpha_4\}$). Consider the following elements in G^F:

$$h_\alpha(t) = h_{\alpha_2}(t)\, h_{\alpha_3}(t^\theta), \qquad h_\beta(t) = h_{\alpha_1}(t)\, h_{\alpha_4}(t^\theta).$$

Clearly, we have

$$h_\alpha(t)\, x_{\alpha_1}(u)\, h_\alpha(t)^{-1} = x_{\alpha_1}(t^{-1}\, u)$$

$$h_\beta(t)\, x_{\alpha_2}(u)\, h_\beta(t)^{-1} = x_{\alpha_2}(t^{-1}\, u),$$

implying $\mu_{\alpha\beta} \neq 1$, $\mu_{\beta\alpha} \neq 1$. Furthermore, by [St] (see the proof of Theorem 36 in § 11), we have

$$h_\alpha(t)\, x_\alpha(u, v)\, h_\alpha(t)^{-1} = x_\alpha(t^{2-2\theta}u,\, t^{2\theta}v).$$

This means that the map

$$(\,\mu_{\alpha\alpha}, \mu_{\alpha\beta}\,) \,:\, K^* \longrightarrow D_\alpha \times D_\beta, \quad t \longrightarrow (\,\mu_{\alpha\alpha}(h_\alpha(t)),\, \mu_{\alpha\beta}(h_\alpha(t))\,)$$

is injective, hence ii) follows, since by assumption $|K^\star| > 2$. $\square$

6 Groups of rank 2 : crystallographic cases

We keep the notation of the previous section. In particular, G denotes either a Chevalley group over an arbitrary field or a twisted Chevalley group over a finite field of rank 2. Let $\Pi = \{\alpha_1, \alpha_2\}$. We begin with two observations which allow to restrict ourselves with consideration of elements of a very specific form.

Lemma 6.1 *Let $\alpha \in \Pi$. Then for every non-central element $f = s\,g$ there exists a root $\gamma \in \Sigma^+$, $\gamma \neq \alpha$, such that up to conjugation f is of the form*

$$f = w_\gamma\, s\, t\, g_\alpha\, u,$$

where $t \in H$, $g_\alpha \in G_\alpha$, $u \in U$ and the α-component of u is trivial.

Proof. Let $W_\alpha = \{1, w_\alpha\}$ and let $P_\alpha = B\, W_\alpha\, B$ be the corresponding parabolic subgroup. Recall that the decomposition of the Weyl group into the double cosets

$$W = \cup_{w_i} W_\alpha\, w_i\, W_\alpha$$

gives the decomposition

$$G = \cup_{w_i} P_\alpha\, w_i\, P_\alpha. \tag{11}$$

Since f is non-central we may assume that $g \notin P_\alpha$. Then the decomposition (11) shows that up to conjugation f can be written in the form $f = w_i\, s\, t\, p_\alpha$, where $p_\alpha \in P_\alpha$, $t \in H$ and $w_i \neq 1$.

The result will follow if we show that representatives of the double cosets can be chosen among reflections. To this end, recall that G is a group of rank 2, hence every element from the Weyl group W is either a reflection or a rotation. If w_i is a rotation, then the element $w_i\, w_\alpha$ is clearly a reflection and it is a representative of the same coset $W_\alpha\, w_i\, W_\alpha$, as required. □

Lemma 6.2 *Let $f = w_\gamma\, s\, t\, g_\alpha\, u$ be as in Lemma 6.1. Then for every parameter $z \in K^\star$ there exists $y \in G$ such that*

$$y\, f\, y^{-1} = v_{-\gamma}\, h_\gamma(z)\, s\, t\, g_\alpha\, u',$$

where $v_{-\gamma} \in U_{-\gamma}$, $v_{-\gamma} \neq 1$, $u' \in U$ and the α-component of u' is trivial.

Proof. By Proposition 4.1, there is $y \in U_\gamma$ such that $y\, w_\gamma = v_{-\gamma}\, h_\gamma(z)\, u_\gamma$, where $v_{-\gamma} \in U_{-\gamma}$, $u_\gamma \in U_\gamma$. Clearly, $v_{-\gamma} \neq 1$, since otherwise we would have $w_\gamma \in U_\gamma$. Then after conjugation by y we have

$$y\, f\, y^{-1} = v_{-\gamma}\, h_\gamma(z)\, u_\gamma\, s\, t\, g_\alpha\, (u\, y^{-1})$$

and it remains to note that

$$[u_\gamma, s\, t\, g_\alpha] \in \prod_{\delta \in \Sigma^+ \setminus \alpha} U_\delta$$

if α is a simple root and $\gamma \neq \alpha$. $\square$

We now proceed with the proof of Theorem 2.1 for groups of rank 2.

Case $\mathbf{A_2}$: In the above notation put $\alpha = \alpha_2$. Let our prescribed semisimple element h has parameters t_1, t_2, i.e. $h = h_{\alpha_1}(t_1)\, h_{\alpha_2}(t_2)$. Clearly, $W = W_{\alpha_2} \cup W_{\alpha_2}\, w_{\alpha_1}\, W_{\alpha_2}$. Then Lemma 6.2 shows that up to conjugation f is of the form

$$f = v_{-\alpha_1}\, [s\, h_{\alpha_1}(t_1)\, g_{\alpha_2}]\, u,$$

where $u \in U$, $v_{-\alpha_1} \in U_{-\alpha_1}$ is non-trivial and $g_{\alpha_2} \in G_{\alpha_2}$.

We claim that one can additionally assume that $f' = s\, h_{\alpha_1}(t_1)\, g_{\alpha_2}$ is non-central in $\Gamma_2 = \langle\, s\, h_{\alpha_1}(t_1),\, G_{\alpha_2}\, \rangle$. Indeed, if f' is central in Γ_2, then for any non-trivial element $u_{\alpha_1+\alpha_2} \in U_{\alpha_1+\alpha_2}$ the element $u_{\alpha_1+\alpha_2}\, f\, u_{\alpha_1+\alpha_2}^{-1}$ is as required, since

$$1 \neq [u_{\alpha_1+\alpha_2}, v_{-\alpha_1}] = u_{\alpha_2} \in U_{\alpha_2}. \tag{12}$$

Since for groups of rank 1 our theorem has been already proved, we conclude that after conjugation by a suitable element $\gamma \in \Gamma_2$ the element f' is of the form

$$f' = v_{-\alpha_2}\, s\, h_{\alpha_1}(t_1)\, h_{\alpha_2}(t_2)\, u_{\alpha_2}.$$

Applying Lemma 3.1 we get finally that after conjugation by γ the element f has the same semisimple part $h = h_{\alpha_1}(t_1)\, h_{\alpha_2}(t_2)$.

Case $\mathbf{B_2}$ does not differ from the previous one. Indeed, we put $\alpha = \alpha_1$. Then there are three double cosets:

$$P_{\alpha_1},\ P_{\alpha_1}\, w_{\alpha_2}\, P_{\alpha_1},\ P_{\alpha_1}\, w_{\alpha_1+2\alpha_2}\, P_{\alpha_1}.$$

In view of Lemma 6.2 it follows that for any parameter $z \in K^\star$ up to conjugation we have

$$f = v_{-\gamma}\, [\, s\, h_\gamma(z)\, g_{\alpha_1}\,]\, u,$$

where $v_{-\gamma} \in U_{-\gamma}$, $v_{-\gamma} \neq 1$ and $\gamma = w_{\alpha_2}$ or $w_{\alpha_1+2\alpha_2}$. Clearly, in both cases $H = H_\gamma \times H_{\alpha_1}$. Then putting $h = h_\gamma(t_\gamma)\, h_{\alpha_1}(t_1)$ we get first that up to conjugation f is of the form

$$f = v_{-\gamma}\, [s\, h_\gamma(t_\gamma)\, g_{\alpha_1}]\, u.$$

The rest of the proof works exactly as in the previous case if instead of commutator relation (12) one applies the relations

$$[u_{\alpha_1+2\alpha_2}, v_{-\alpha_2}] = u_{\alpha_1+\alpha_2}\, u_{\alpha_1},$$
$$[u_{\alpha_2}, v_{-\alpha_1-2\alpha_2}] = v_{-\alpha_1-\alpha_2}\, v_{-\alpha_1}.$$

Note that here $u_{\alpha_1+\alpha_2}$, u_{α_1}, and $v_{-\alpha_1-\alpha_2}$, $v_{-\alpha_1}$ are non-trivial elements (see [St], Lemma 33).

Case $\mathbf{G_2}$: Denote $\alpha = \alpha_2$. Then there are four double cosets:

$$P_{\alpha_2},\ P_{\alpha_2}\, w_{\alpha_1}\, P_{\alpha_2},\ P_{\alpha_2}\, w_{3\alpha_1+\alpha_2}\, P_{\alpha_2},\ P_{\alpha_2}\, w_{\alpha_1}\, w_{3\alpha_1+2\alpha_2}\, P_{\alpha_2}.$$

Obviously we have decompositions

$$H = H_{\alpha_1} \times H_{\alpha_2}, \quad H = H_{3\alpha_1+\alpha_2} \times H_{\alpha_2},$$

for $3\alpha_1+\alpha_2$ and α_2 are both long roots. Repeating word by word the previous proofs and using the commutator relations

$$\begin{aligned} [u_{3\alpha_1+\alpha_2}, v_{-\alpha_1}] &= u_{\alpha_2}\, u_{2\alpha_1+\alpha_2}\, u_{\alpha_1+\alpha_2} u_{3\alpha_1+2\alpha_2} \\ [u_{3\alpha_1+2\alpha_2}, v_{-3\alpha_1-\alpha_2}] &= u_{\alpha_2} \end{aligned} \tag{13}$$

(note that here $u_{\alpha_2} \neq 1$, see [C2]) we get easily that up to conjugation any element

$$f \notin P_{\alpha_2}\, w_{\alpha_1}\, w_{3\alpha_1+2\alpha_2}\, P_{\alpha_2}$$

lies in the big Gauss cell and has the prescribed semisimple part h.

The case $f \in P_{\alpha_2}\, w_{\alpha_1}\, w_{3\alpha_1+2\alpha_2}\, P_{\alpha_2}$ also does not require much additional work. Indeed, the roots $\delta = 3\alpha_1 + 2\alpha_2$ and $\gamma = \alpha_1$ are orthogonal, implying $g_\gamma\, g_\delta = g_\delta\, g_\gamma$ for every $g_\gamma \in G_\gamma$, $g_\delta \in G_\delta$. Arguing then just as above we have the following analog of Lemma 6.2.

Lemma 6.3 *Let $f = w_\gamma\, w_\delta\, s\, t\, g_\alpha\, u$, where γ and δ are orthogonal and $\alpha \in \Pi$ is a simple root. Assume that $H = \langle H_\gamma, H_\alpha \rangle$. Then for every parameter $z \in K^*$ there exists $y \in G$ such that*

$$y\, f\, y^{-1} = v_{-\gamma}\, v_{-\delta}\, h_\gamma(z)\, s\, t\, g_\alpha\, u',$$

where $v_{-\gamma} \in U_{-\gamma}$, $v_{-\delta} \in U_{-\delta}$, $u' \in U$, $t \in H$ and $v_{-\gamma}$, $v_{-\delta}$ are non-trivial.

Let $h = h_{\alpha_1}(t_1)\, h_{\alpha_2}(t_2)$. By Lemma 6.3, we may put

$$f = v_{-3\alpha_1-2\alpha_2}\, v_{-\alpha_1}\, [\, s\, h_{\alpha_1}(t_1)\, g_{\alpha_2}\,]\, u'.$$

After conjugation by a suitable element we may assume additionally that the element $f' = [s\, h_{\alpha_1}(t_1)\, g_{\alpha_2}]$ is non-central in $\Gamma_2 = \langle\, s\, h_{\alpha_1}(t_1), G_{\alpha_2} \,\rangle$. Indeed, the relations (13) and

$$[u_{3\alpha_1+\alpha_2}, v_{-3\alpha_1-2\alpha_2}] = v_{-\alpha_2}$$

shows that for an arbitrary non-trivial $u_{3\alpha_1+\alpha_2} \in U_{3\alpha_1+\alpha_2}$ we get that

$$u_{3\alpha_1+\alpha_2}\, f\, (u_{3\alpha_1+\alpha_2})^{-1} = v'\, [\, s\, h_{\alpha_1}(t_1)\, g'_{\alpha_2}\,]\, u'$$

where

$$v' \in \prod_{\nu \in \Sigma^+ \setminus \alpha_2} U_{-\nu}, \;\; u' \in \prod_{\nu \in \Sigma^+ \setminus \alpha_2} U_{\nu}.$$

and the $(-\alpha_2)$-component of $s\, h_{\alpha_1}(t_1)\, g'_{\alpha_2}$ is non-central. The rest of the proof coincides with the final part of the proof in previous cases.

Cases ^{2}A$_3$, ^{2}A$_4$: The root system has type $B_2 = C_2$. Let $\alpha = \alpha_1$ be the short root. We have then three double cosets

$$P_{\alpha_1},\;\; P_{\alpha_1}\, w_{\alpha_2}\, P_{\alpha_1},\;\; P_{\alpha_1}\, w_{\alpha_1+\alpha_2}\, P_{\alpha_1}.$$

One can check that all the arguments from the previous cases can be carried over to our situation provided we have the following statement.

Lemma 6.4 *H is generated by H_{α_1} and $H_{\alpha_1+\alpha_2}$.*

Proof. Examine first type $^2A_3$. We have $G \simeq SU_4(K)$ and the subgroups H_{α_1}, H_{α_2}, $H_{\alpha_1+\alpha_2}$ consist of all matrices of the form

$$\mathrm{diag}\,(\, t,\, t^{-1},\, t^{\theta},\, t^{-\theta}\,)$$
$$\mathrm{diag}\,(\, 1,\, s,\, s^{-1},\, 1\,)$$
$$\mathrm{diag}\,(\, l,\, l^{\theta},\, l^{-1},\, l^{-\theta}\,)$$

respectively, where $t,\, l \in K^*$ and $s \in K^{\theta}$ (see [St], [C2]). Letting $l = t^{-1}$ and multiplying the first and third matrices we get a matrix of the form

$$\mathrm{diag}\,(\, 1,\, t^{-1}\, t^{-\theta},\, t\, t^{\theta},\, 1\,).$$

This shows that $H_{\alpha_2} \subset \langle\, H_{\alpha_1+\alpha_2}, H_{\alpha_1}\,\rangle$, since the norm map $N_{K/K^\theta} :\, K \to K^\theta$ is surjective.

The case $^2A_4$ is entirely analogous to the previous one and so we omit details. □

Case $^3\mathbf{D_4}$: Denote a Chevalley group of type D_4 by G and let $G^F < G$ be the corresponding twisted subgroup of type $^3D_4$. Let

$$\Pi = \{\beta_1 = \varepsilon_1 - \varepsilon_2,\, \beta_2 = \varepsilon_2 - \varepsilon_3,\, \beta_3 = \varepsilon_3 - \varepsilon_4,\, \beta_4 = \varepsilon_3 + \varepsilon_4\},\ \Pi_\rho = \{\alpha_1,\, \alpha_2\}$$

be the bases of the root systems Σ and Σ_ρ, where α_1 and α_2 correspond to the ρ-orbits $\{\beta_1,\, \beta_3,\, \beta_4\}$ and $\{\beta_2\}$ respectively. Recall that the root system Σ_ρ has type G_2 and α_1 is a short root.

If we put $\alpha = \alpha_1$, we get four double cosets in G^F:

$$P_\alpha,\ P_\alpha\, w_{\alpha_2}\, P_\alpha,\ P_\alpha\, w_{\alpha_1+\alpha_2}\, P_\alpha,\ P_\alpha\, w_{\alpha_2}\, w_{2\alpha_1+\alpha_2}\, P_\alpha.$$

Lemma 6.5 *$H = \langle\, H_\gamma,\, H_\alpha\,\rangle$, where $\gamma = \alpha_2$ or $\alpha_1 + \alpha_2$.*

Proof. The case $\gamma = \alpha_2$ is trivial. Let $\gamma = \alpha_1 + \alpha_2$. Then we have

$$\begin{aligned}
H_{\alpha_1} &= \langle\, h_{\beta_1}(t)\, h_{\beta_3}(t^\theta)\, h_{\beta_4}(t^{\theta^2}) \mid t \in K^\star\,\rangle,\\
H_{\alpha_2} &= \langle\, h_{\beta_2}(s) \mid s \in (K^\theta)^\star\,\rangle,\\
H_{\alpha_1+\alpha_2} &= \langle\, h_{\varepsilon_1-\varepsilon_3}(l)\, h_{\varepsilon_2-\varepsilon_4}(l^\theta)\, h_{\varepsilon_2+\varepsilon_4}(l^{\theta^2}) \mid l \in K^\star\,\rangle.
\end{aligned}$$

Consider two elements

$$\begin{gathered}
h_1 = h_{\beta_1}(l^{-1})\, h_{\beta_3}(l^{-\theta})\, h_{\beta_4}(l^{-\theta^2}) \in H_{\alpha_1},\\
h_2 = h_{\varepsilon_1-\varepsilon_3}(l)\, h_{\varepsilon_2-\varepsilon_4}(l^\theta)\, h_{\varepsilon_2+\varepsilon_4}(l^{\theta^2}) \in H_{\alpha_1+\alpha_2}
\end{gathered}$$

Taking into account that $h_{\varepsilon_1-\varepsilon_3}(l) = h_{\beta_1}(l)\, h_{\beta_2}(l)$ and analogously $h_{\varepsilon_2-\varepsilon_4}(l) = h_{\beta_2}(l)\, h_{\beta_3}(l)$ and $h_{\varepsilon_2+\varepsilon_4}(l) = h_{\beta_2}(l)\, h_{\beta_4}(l)$ we easily get that

$$h_1 \cdot h_2 = h_{\beta_2}(l\, l^\theta\, l^{\theta^2}) \in H_{\alpha_2},$$

implying

$$H_{\alpha_2} \subset \langle H_{\alpha_1},\, H_{\alpha_1+\alpha_2}\,\rangle,$$

as required. $\square$

By Lemma 6.2, Lemma 6.3, we have that for any parameter $z \in K^\star$ any non-central $f \in G^F$ up to conjugation is of a form

$$v_{-\alpha_2}\,[\, s\, h_{\alpha_2}(z)\, g_{\alpha_1}\,]\, u,$$

or

$$v_{-\alpha_1-\alpha_2}\,[\, s\, h_{\alpha_1+\alpha_2}(z)\, g_{\alpha_1}\,]\, u,$$

or

$$v_{-\alpha_2}\, v_{-2\alpha_1-\alpha_2}\,[\, s\, h_{\alpha_2}(z)\, g_{\alpha_1}\,]\, u,$$

where $g_{\alpha_1} \in G_{\alpha_1}$, $u \in U$ and all $v_{-\alpha_2}$, $v_{-\alpha_1-\alpha_2}\, v_{-2\alpha_1-\alpha_2}$ are non-trivial. Therefore, arguing as above it remains to show that up to conjugation the element in the square bracket can been assumed to be non-central in $\Gamma_1 = \langle s\, h_\gamma(z),\, G_{\alpha_1} \rangle$, where $\gamma = \alpha_2$ or $\alpha_1 + \alpha_2$. But the last follows immediately from the following commutator formulas:

$$[\, v_{-\alpha_2},\, u_{\alpha_1+\alpha_2} \,] = u_{\alpha_1}\, u',$$

where $u_{\alpha_1} \neq 1$ and u' lies in the unipotent radical of the parabolic subgroup P_{α_1};

$$[\, v_{-\alpha_1-\alpha_2},\, u_{\alpha_2} \,] = v'\, v_{-\alpha_1},$$

where $v_{-\alpha_1} \neq 1$ and v' lies in the unipotent radical of the parabolic subgroup $w_0\, P_{\alpha_1}\, w_0^{-1}$;

$$[\, v_{-\alpha_2}\, v_{-2\alpha_1-\alpha_2},\, u_{3\alpha_1+\alpha_2} \,] = v'\, u_{\alpha_1},$$

where $u_{\alpha_1} \neq 1$ and v' lies in the unipotent radical of the parabolic subgroup $w_0\, P_{\alpha_1}\, w_0^{-1}$.

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