STABLE HOMOTOPY OF ABELIAN Γ-GROUPS

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0. Introduction

Let Γ be the small category of finite pointed sets. We will assume that objects of Γ are sets $[n]$, where

$$[n] := \{0, 1, ..., n\}$$

with basepoint 0, $n \geq 0$. An *abelian* Γ-*group* or a left Γ-module is the same as a functor $F : \Gamma \to Ab$ with property $F([0]) = 0$. We let Γ-*mod* denote the category of such objects. It is well-known that any such object defines a spectrum (see [7], [1]). We let $\pi_*^{st}(F)$ denote the homotopy of the corresponding spectrum. Let Ω be the small category of all nonempty finite sets and surjections. We will assume that the objects of Ω are the sets $\underline{n} := \{1, ..., n\}$, $n \geq 1$. In [5] we proved that there is an equivalence of categories

$$cr : \Gamma - mod \to Ab^{\Omega}.$$

Based on this fact we construct a spectral sequence, whose abutment is the stable homotopy of abelian Γ-groups and which gives a nice relationship between the homology of the symmetric groups and the stable homotopy of Γ-groups. Our spectral sequence is in the same spirit as the spectral sequence of Lück [3] (see also [8]), and uses in an essential way the fact that the category of Ω is an so called EI-category (see [3]).

For a functor $F : \Gamma \to Ab$ we let F_n denote the value of $cr(F)$ on $\underline{n}$. A functor F is called of degree $\leq d$ if $F_n = 0$ for all $n > d$.

1. Lück-Slominska spectral sequence for stable homotopy

1.1. Generalites on Ω-modules. Let Ω-*mod* be the category of all covariant functors from Ω to *Ab*. Similarly *mod*-Ω denotes the category of all contravariant functors from Ω to the category of abelian groups. The objects of Ω-*mod* (resp. *mod*-Ω) are called left (resp. right) Ω-modules. The categories Ω-*mod* and *mod*-Ω are abelian categories with sufficiently many projective and injective objects. For any n one defines

$$\Omega^n := \mathbf{Z}[\mathrm{Hom}_\Omega(\underline{n}, -)] \ \text{ and } \ \Omega_n := \mathbf{Z}[\mathrm{Hom}_\Omega(-, \underline{n})].$$

Here $\mathbf{Z}[S]$ denotes the free abelian group generated by a set S. It is a consequence of the Yoneda lemma that Ω^n (resp Ω_n), $n \geq 1$, are projective generators of the category Ω-*mod* (resp. *mod*-Ω). Any surjective map $\underline{m} \to \underline{n}$ yields morphisms of Ω-modules $\Omega_m \to \Omega_n$ and $\Omega^n \to \Omega^m$. Similarly the functors

$$\Gamma^n := \mathbf{Z}[\mathrm{Hom}_\Gamma([n], -)] \ \text{ and } \ \Gamma_n := \mathbf{Z}[\mathrm{Hom}_\Gamma(-, [n])]$$

are projective generators in the category of Γ-modules. It is well known that there is a bifunctor $- \otimes_\Omega - : mod\text{-}\Omega \times \Omega\text{-}mod \to Ab$ which is right exact with respect to each variable and preserves sums. Moreover, for any left Ω-module F one has an isomorphism

$$F_n \cong \Omega^n \otimes_\Omega F. \tag{1.1.1}$$

The derived functors of $- \otimes_\Omega -$ with respect to each variable are isomorphic and we will denote the common value by $\mathrm{Tor}^\Omega(-,-)$. The following lemma was proved in [6].

1.2. Lemma. *Let $T : \Gamma \to Ab$ be a covariant functor. Then there exists a natural isomorphism:*

$$\pi_*^{st}(T) \cong Tor_*^\Omega(t, cr(T)),$$

where $t : \Omega^{op} \to Ab$ is a functor, which assigns 0 to $\underline{k}$, if $k > 1$ and $\mathbf{Z}$ *to $\underline{1}$.*

1.3. The chain complex $\mathcal{K}_*(U)$. Let U be a right Ω-module. One defines the chain complex $\mathcal{K}_*(U)$ in the category of right Ω-modules as follows. Take an object $\underline{x}$ of the category Ω. Then $\mathcal{K}_k(U)(\underline{x})$ is defined as the direct sum over all sequences $n_0 > n_1 > \cdots > n_k$ of

$$U_{n_k}\otimes_{\Sigma_{n_k}}\mathbf{Z}[\Omega(\underline{n}_{k-1},\underline{n}_k)]\otimes_{\Sigma_{n_{k-1}}}\cdots\otimes_{\Sigma_{n_1}}\mathbf{Z}[\Omega(\underline{n}_0,\underline{n}_1)]\otimes_{\Sigma_{n_0}}\mathbf{Z}[\Omega(\underline{x},\underline{n}_0)].$$

Here $\Omega(\underline{n},\underline{m})$ denotes the set of morphisms from the category $\underline{n}$ to $\underline{m}$ in the category Ω. A typical element of this module is denoted by

$$(a; x \xrightarrow{f_0} n_0 \xrightarrow{f_1} n_1 \to \cdots \xrightarrow{f_k} n_k)$$

or by $(a; f_k \mid \cdots \mid f_1 \mid f_0)$ with $a \in U_{n_k}$. Then the boundary of $\mathcal{K}_k(U)(\underline{x})$ is given by

$$d(a; f_k \mid \cdots \mid f_1 \mid f_0) =$$

$$= \sum_{i=0}^{k-1}(-1)^i(a; f_k \mid \cdots \mid f_{i+1}f_i \mid \cdots \mid f_0)+$$

$$+(-1)^k(f_k^*(a); f_{k-1} \mid \cdots \mid f_0).$$

1.4. Lemma. *The homology of $\mathcal{K}_*(U)$ is zero in positive dimensions and is U in dimension zero.*

Proof. One puts $\mathcal{K}_{-1}(U) = U$ and one defines the augmentation $\mathcal{K}_*(U) \to \mathcal{K}_{-1}(U)$ by $d(a; x \xrightarrow{f_0} n_0) = f_0^*(a),$. Here $x \in \Omega$ and $a \in U_{n_0}$. Now one defines a map $h_k : \mathcal{K}_k(U) \to \mathcal{K}_{k+1}(U)$, $k \geq -1$ by

$$h_{-1}(a) = (a; x \xrightarrow{id} x), \quad a \in T(x)$$

$$h_k(a; f_k \mid \cdots \mid f_1 \mid f_0) = (a; f_k \mid \cdots \mid f_1 \mid f_0 \mid id), \text{ if } x > n_0$$

and

$$h_k(a; f_k \mid \cdots \mid f_1 \mid f_0) = 0 \text{ if } x = n_0.$$

Then easy calculations shows that $dh + hd = id$.

1.5. The complex $\mathcal{K}_*(t)$. Now we take $U = t$. Since $\Omega(-,\underline{1}) \cong \underline{\mathbf{Z}}$, where $\underline{\mathbf{Z}}$ is the constant functor with values $\mathbf{Z}$, we see that the complex $\mathcal{K}_*(t)$ in low dimensions looks as follows:

$$\underline{\mathbf{Z}} \leftarrow \bigoplus_{n>1} \mathbf{Z} \otimes_{\Sigma_n} \Omega_n \leftarrow \bigoplus_{n>m>1} \mathbf{Z} \otimes_{\Sigma_m} \mathbf{Z}[\Omega(\underline{n},\underline{m})] \otimes_{\Sigma_n} \Omega_n \leftarrow \cdots \quad .$$

thanks to Lemma 1.4 we know that this complex is a (nonprojective) resolution of t. In 1.9 we shall consider the hyperhomology spectral sequence associate to this resolution. To do this, we need Lemma 1.6 and Lemma 1.8.

1.6. Lemma. *Let M be a Σ_n-representation and let U_M be a right Ω-module given by*

$$U_M = M \otimes_{\Sigma_n} \Omega_n.$$

Then for any left Ω-module T one has an isomorphism

$$Tor_*^{\Omega}(U_M, T) \cong Tor_*^{\Sigma_n}(M, T_n).$$

Proof. One observes that for $M = \mathbf{Z}[\Sigma_n]$ one has $U_M \cong \Omega_n$. Therefore U_M is projective in this case and the lemma is true for $M = \mathbf{Z}[\Sigma_n]$. Since $\mathbf{Z}[\Omega(x, n)]$ is a free Σ_n-module for any x, one sees that the value of the functor Ω_n lie in the category of free Σ_n-modules. Therefore the functor $M \mapsto U_M$ is exact from the category of Σ_n-modules to the category of right Ω-modules. Now one can use the axiomatic characterisation of Tor functors to finish the proof.

1.7. Partitions. In the next section we need the well known relationship between partitions and surjections: The set of orbits of the set $\Omega(\underline{n}, \underline{m})$ under the action of the group $\Sigma_n \times \Sigma_m$ is the same as the set of partitions of n of length m. Here a *partition* or a 1-*partition* is a sequence of natural numbers $\lambda = (\lambda_1, \cdots, \lambda_k)$, such that $k \geq 1$, $1 \leq \lambda_1 \leq \cdots \leq \lambda_k$. For a partition λ we put $l(\lambda) = k$ and $s(\lambda) = \lambda_1 + \cdots + \lambda_k$. If $s(\lambda) = m$, then we say that λ is a partition of m. For a partition λ we put $\Sigma_\lambda = \Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_k} \subset \Sigma_{s(\Sigma_\lambda)}$. We let $\Pi(m)$ be the set of all partitions of m. Moreover for $m > k$ we let $\Pi(m, k)$ be the set of partitions of m of length k. If $\lambda = (\lambda_1, \cdots, \lambda_k)$, is such a partition, we let $\underline{\lambda} : \underline{m} \to \underline{k}$ be the map which takes the first λ_1 element to 1, the next λ_2 element to 2 and so on.

For $n \geq 2$ an *n-partition* λ is the same as a partition $(\lambda_1, \cdots, \lambda_k)$ together with $(n-1)$-partitions λ^i of λ_i for each $i = 1, \cdots, k$. For an n-partition λ we put

$$s(\lambda) = \lambda_1 + \cdots + \lambda_k$$

$$l(\lambda) = (l(\lambda^1) + \cdots + l(\lambda^k), k).$$

and

$$\Sigma_\lambda = \Sigma_{\lambda^1} \times \cdots \times \Sigma_{\lambda^k} \subset \Sigma_{s(\lambda)}.$$

We call $s(\lambda)$ and $l(\lambda)$ the *sum* and *length* of λ. So by definition $s(\lambda) \in \mathbf{N}$, while $l(\lambda) \in \mathbf{N}^n$. For $n > n_1 > \cdots > n_k$ we let $\Pi(n, n_1, n_2, \cdots n_k)$ be the set of k-partitions λ such that $s(\lambda) = n$ and $l(\lambda) = (n_1, n_2, \cdots n_k)$. Any such partition defines a chain of maps

$$\underline{n} \to \underline{n}_1 \to \cdots \to \underline{n}_k$$

by iterating the above construction. For example

$$3 + 5 = (1 + 2) + (1 + 2 + 2)$$

is a 2-partition, whose sum is 8 and length is $(5, 2)$. Moreover

$$\Sigma_\lambda = \Sigma_1 \times \Sigma_2 \times \Sigma_1 \times \Sigma_2 \times \Sigma_2 \subset \Sigma_8,$$

while the corresponding chain of maps is

$$\underline{8} \xrightarrow{f} \underline{5} \xrightarrow{g} \underline{2},$$

where $f(1) = 1, f(2) = f(3) = 2, f(4) = 3, f(5) = f(6) = 4, f(7) = f(8) = 4$ and $g(1) = g(2) = 1, g(3) = g(4) = g(5) = 2$.

1.8. Lemma. *Let $n > n_1 > \cdots > n_k$. Then the Σ_n-module*

$$\mathbf{Z} \otimes_{\Sigma_{n_k}} \mathbf{Z}[\Omega(\underline{n}_{k-1}, \underline{n}_k)] \otimes_{\Sigma_{n_{k-1}}} \cdots \otimes_{\Sigma_{n_1}} \mathbf{Z}[\Omega(\underline{n}, \underline{n}_1)]$$

is isomorphic to

$$\bigoplus \mathbf{Z}[\Sigma_n / \Sigma_\lambda]$$

where the sum is taken over all $\lambda \in \Pi(n, n_1, \cdots, n_k)$.

Proof. It is enough to observe that $\Pi(n, n_1, \cdots, n_k)$ is the space of orbits under the Σ_n action on

$$* \times_{\Sigma_{n_k}} \Omega(\underline{n}_{k-1}, \underline{n}_k) \times_{\Sigma_{n_{k-1}}} \cdots \times_{\Sigma_{n_1}} \Omega(\underline{n}, \underline{n}_1).$$

The way how to choose an representative of an element of $\Pi(n, n_1, \cdots, n_k)$ was described in 1.7.

1.9. Corollary. *Let T be an abelian Γ-group. Then there exists a spectral sequence*

$$E^1_{pq} = \bigoplus H_q(\Sigma_\lambda, T_{s(\lambda)}) \Longrightarrow \pi^{st}_{p+q}(T),$$

where the sum is taken over all $n_1 > \cdots > n_p > 1$ and $\lambda \in \Pi(n_1, \cdots, n_p)$. In particular

$$E^1_{00} = T_1, \quad E^1_{0q} = 0, \ q > 0,$$

$$E^1_{1q} = \bigoplus_{n>1} H_q(\Sigma_n, T_n)$$

and

$$E^1_{2q} = \bigoplus_{n>m>1} \bigoplus_{\lambda \in \Pi(n,m)} H_q(\Sigma_\lambda, T_n).$$

Proof. It is a consequence of the hyperhomology spectral sequence associated to the complex 1.5, together with Lemma 1.6, Lemma 1.8 and Shapiro lemma in group homology theory.

1.10. Consequences. The rest of this section is a collection of some straightforward consequences of Corollary 1.9.

a) Let T be an abelian Γ-group of degree $\leq d$. By Proposition 2.5 we get, that $E^1_{pq} = 0$ as soon as $p \geq d$. Therefore the groups $\pi^{st}_i T$ are torsion groups for $i \geq d$. Moreover, if $l > d$ is a prime and $i \geq d$, then $\pi^{st}_i T$ has no l-torsion part.

b) Let T be an abelian Γ-group. Then $\pi^{st}_0 T$ is the cokernel of the map $(T_2)_{\Sigma_2} \to T_1$, while $\pi^{st}_1 T \cong ker\ (d_1)/(Imd_2)$. Here

$$d_1 : \bigoplus_{n>1} (T_n)_{\Sigma_n} \to T_1$$

is induced by the unique map $\underline{n} \to \underline{1}$, while the restriction of

$$d_2 : \bigoplus_{n>m>1} \bigoplus_{\lambda \in \Pi(n,m)} (T_n)_{\Sigma_\lambda} \to \bigoplus_{n>1} (T_n)_{\Sigma_n}$$

on the summand corresponding to λ has only two nontrivial components, the first is the canonical projection $(T_n)_{\Sigma_\lambda} \to (T_n)_{\Sigma_n}$ induced by the inclusion $\Sigma_\lambda \subset \Sigma_n$ and the second component is (-1) times the composition

$$(T_n)_{\Sigma_\lambda} \to T_m \to (T_m)_{\Sigma_n}.$$

Here the first morphism is induced by $\underline{\lambda} : \underline{n} \to \underline{m}$.

c) Assume $deg\ T \leq 3$. Then the E^1-term of this spectral sequence is as follows

$$\begin{array}{ccccccc}
 & & \dots & & & & \\
0 & & H_2(\Sigma_2, T_2) \oplus H_2(\Sigma_3, T_3) & \leftarrow & H_2(\Sigma_2, T_3) & & 0 \\
0 & & H_1(\Sigma_2, T_2) \oplus H_1(\Sigma_3, T_3) & \leftarrow & H_1(\Sigma_2, T_3) & & 0 \\
T_1 & \leftarrow & H_0(\Sigma_2, T_2) \oplus H_0(\Sigma_3, T_3) & \leftarrow & H_0(\Sigma_2, T_3) & & 0
\end{array}$$

Here the homomorphisms $(T_3)_{\Sigma_2} \to (T_2)_{\Sigma_2} \oplus (T_3)_{\Sigma_3}$ is the same as the restriction of the homomorphism d_2 from b) on the summand corresponding to the partition $\lambda = (1, 2)$. From this description is clear that the first component factors through T_3 and therefore it yields the zero map in the higher homology terms. Thus the map

$$d_i : H_i(\Sigma_2, T_3) \to H_i(\Sigma_2, T_2) \oplus H_i(\Sigma_3, T_3),\ i \geq 1$$

is nothing but $x \mapsto (0, j_* x)$, where $j : \Sigma_2 \cong \Sigma_1 \times \Sigma_2 \subset \Sigma_3$ is the inclusion. Clearly the spectral sequence stops at the E^2 level and we get an exact sequence

$$0 \to Coker\ (d_{i+1}) \to \pi^{st}_{i+2}(T) \to Ker\ (d_i) \to 0,\ i \geq 0.$$

If $deg\ T \leq 2$, then the $\pi^{st}_i T$ are isomorphic to the cokernel and the kernel of the map $(T_2)_{\Sigma_2} \to T_1$ for $i = 0$ and $i = 1$, while $\pi^{st}_i T \cong H_{i-1}(\Sigma_2, T_2)$ for $i \geq 2$. This is a generalization of an observation made in [4].

d) Let l be a prime and let T be an abelian Γ-group of degree $\leq l$. By a) we know that $\pi^{st}_i(T)$ is a torsion group for $i \geq l$. We let $\pi^{st}(T, l)$ be the l-torsion part of $\pi^{st}(T)$. Similarly we let $H_i(\Sigma_n, T_n, l)$ be the l-torsion part of $H_i(\Sigma_n, T_n,)$. Then

$$\pi^{st}_i(T, l) \cong H_{i-1}(\Sigma_n, T_n, l),\ i \geq l.$$

Indeed, if a group $H_q(\Sigma_\lambda, T_{s(\lambda)})$ from Corollary 1.9, has an l-torsion part then Σ_l appears as a part in Σ_λ and this happens only when $l(\lambda) = 1$ and $s(\lambda) = l$.

2. On Dold-Puppe stable derived functors

2.1. Background. Let $T : Ab \to Ab$ be a functor. According to [DP] the stable derived functors of T are defined as follows. For an abelian group A one chooses a free simplicial resolution $P_* \to A$ in the category Ab. Then one defines

$$L_*^{st}T(A) := colim_i \pi_{*+i} T((S^i)_{ab} \otimes P_*).$$

It is well-known that the functors $L_*^{st}T$ are additive and for any finitely generated abelian group one has a short exact sequence

$$0 \to L_n^{st}T(\mathbf{Z}) \otimes A \to L_n^{st}T(A) \to L_{n-1}^{st}T(\mathbf{Z}) * A \to 0, \tag{2.1.1}$$

where as usual $*$ denotes the torsion product of abelian groups. Let $\tilde{T}$ be the abelian Γ-group given by $\tilde{T}(X) = T(X_{ab})$. Clearly $\pi_*^{st}\tilde{T} \cong L_*^{st}T(\mathbf{Z})$. We usually consider only such functors $T : Ab \to Ab$ which preserve direct colimits and are "right exact" in the sence that for any simplicial abelian group A_* one has an isomorphism $\pi_0 T(A_*) \cong T(\pi_0 A_*)$. For such functors (2.1.1) holds for any abelian group A. Moreover, in this case T is completly determined by the restriction of T on the subcategory ab of finitely generated free abelian groups and the degree of $\tilde{T}$ concides with degree of T as defined by Eilenberg and MacLane. Thus all results proved before can be applied to $L_*^{st}T(A)$. As we see in this section some results holds in a less restricted form. In order to do this we need the coinduction principle for polynomial functors. We will also use the following fact (see Korollar 5.20 of [2]):

$$L_*^{st}T = 0, \tag{2.1.2}$$

if $T(A) = B(A, A)$. Here $B : Ab \times Ab \to Ab$ is a bifunctor with the property $B(0, A) = B(A, 0) = 0$.

2.2. Coinduction Principle. Let $\widetilde{\mathcal{P}}_n$ be the category of functors from ab to Ab of degree $\leq n$. Moreover for any Σ_n-module M, one denotes by $M^\sharp$ the functor given by

$$M^\sharp(A) := H^0(\Sigma_n, M \otimes A^{\otimes n}).$$

2.3. Proposition. *Let $\mathcal{P}$ be a full subcategory of the category $\widetilde{\mathcal{P}}_n$, with following properties:*

i) *$M^\sharp \in \mathcal{P}$ for any right Σ_k-module M, $k \leq n$.*

ii) *If*

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0$$

is a short exact sequence in $\widetilde{\mathcal{P}}_n$ with $T'' \in \mathcal{P} \cap \widetilde{\mathcal{P}}_{n-1}$ and $T \in \mathcal{P}$ then $T' \in \mathcal{P}$.

iii) *If*

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0$$

is a short exact sequence in $\widetilde{\mathcal{P}}_n$ with $T' \in \mathcal{P} \cap \widetilde{\mathcal{P}}_{n-1}$ and $T'' \in \mathcal{P}$, then $T \in \mathcal{P}$.

Then $\mathcal{P} = \widetilde{\mathcal{P}}_n$.

Proof. Let $T \in \widetilde{\mathcal{P}}_n$ be a functor of degree n and define T_n to be the value of the last nontrivial cross-effect of T on $(\mathbf{Z} \mid \cdots \mid \mathbf{Z})$. Then the codiagonal $\mathbf{Z} \to \mathbf{Z} \bigoplus \cdots \bigoplus \mathbf{Z}$ yields a natural transformation $T \to T_n^\sharp$, whose kernel and cokernel have degree $< n$. Based on this fact one can easily deduce the proposition.

2.4. Lemma. *Let $T : Ab \to Ab$ be a functor of degree $\leq n$. We assume that T satisfies the conditions in 6.1. If $i \geq 1$, then $L_i^{st}T(\mathbf{Z})$ is a torsion group. Moreover if $l > n$ is a prime, then $L_i^{st}T(\mathbf{Z})$ has no l-torsion part.*

Proof. By the coinduction principle it is enough to consider the case, when $T = M^\sharp$. If one inverts $n!$, then for $M^\sharp$ one can apply (2.1.2).

2.5. The functors $M_i^\flat$. Let M be a Σ_n-module. We let $M_i^\flat$ be the functor $Ab \to Ab$ given by

$$A \mapsto Tor_i^{\Sigma_n}(M, A^{\otimes n}), \ i \geq 0.$$

Hence $M_0^\flat = M^\flat$. Obviously the value of $M_i^\flat$ on $\mathbf{Z}$ is $H_i(\Sigma_n, M)$. Easy calculations show that the value of the m-th cross-effect on $(\mathbf{Z} \mid \cdots \mid \mathbf{Z})$ is

$$\bigoplus_{k_1+\cdots+k_m=n} H_i(\Sigma_{k_1} \times \cdots \times \Sigma_{k_m}, M). \tag{2.5.1}$$

Here all $k_j > 0$. Hence for any $i > 0$, the n-th cross-effect vanishes and therefore the functors $M_i^\flat$ have degree $< n$.

Lemma 2.6. *If n is not a power of a prime l, then $\pi_*^{st} M_i^\flat$ has no l-torsion.*

Proof. For any $1 \le k \le n$ one considers the k-th cross-effect and the natural homomorphisms

$$T(A) \to T(A \mid \cdots \mid A) \to T(A),$$

where $T = M_i^\flat$. Easy calculations show that this composition is the same as the multiplication by $\binom{n}{k}$. Since π_*^{st} vanishes on

$$A \mapsto T(A \mid \cdots \mid A)$$

thanks to (2.1.2), we conclude that the multiplication by $\binom{n}{k}$ on $L_*^{st} M_i^\flat$ is zero. Now one can use the Hilfssatz 10.10 of [2] to finish the proof.

Proposition 2.7. *There exists a spectral sequence whose abutment is 0 and $E^2_{pq} = L_p^{st} M_q^\flat$. Moreover one has an isomorphism*

$$E^2_{0q} \cong Coker(\bigoplus_{k=1}^{k=n-1} H_q(\Sigma_k \times \Sigma_{n-k}, M) \to H_q(\Sigma_n, M)).$$

Proof. Let $P_* \to M$ be a free resolution of M in the category of Σ_n-modules. By (2.1.2) we know that $L_*^{st} M_*^\flat = 0$ if M is free over Σ_n. Because of this fact one sees that the hyperhomology spectral sequence for the chain complex $P_*^\flat$ in the category of functors Ab^{ab} gives the result. The calculation of E^2_{0q} can be deduced from 4.10 b).

Corollary 2.8. *One has isomorphisms*

$$L_0^{st} M^\flat = 0 = L_1^{st} M^\flat,$$

$$L_2^{st} M^\flat(\mathbf{Z}) \cong Coker(\bigoplus_{k=1}^{k=n-1} H_1(\Sigma_k \times \Sigma_{n-k}, M) \to H_1(\Sigma_n, M)).$$

We let $L_*^{st}T(\mathbf{Z})(l)$ and $H_*(\Sigma_n, M)(l)$ be the l-torsion part of $L_*^{st}T(\mathbf{Z})$ and $H_*(\Sigma_n, M)$ respectively.

Proposition 2.9. *Let l be a prime and $T : Ab \to Ab$ be a functor of degree $\leq l$. Then for any $n \geq 2$ there exist a natural isomorphism*

$$H_{n-1}(\Sigma_l, T_l)(l) \cong L_n^{st}T(\mathbf{Z})(l).$$

Proof. Let us remark that the statement is obvious if $degT < l$, because both groups in question are zero thanks to Proposition 2.4. Next we consider the case when $T = M^\flat$ where M is a representation of Σ_p. We consider the l-torsion part of the spectral sequence 2.7. We have $E^2_{pq} = 0$ if $p > 0$ and $q > 0$, because the degrees of the functors $M_q^\flat$ are $< l$ if $q > 0$ and one can use still Proposition 2.4. Since the spectral sequence converges and the abutment is zero, one gets that the only possible nontrivial differentials are isomorphisms:

$$E^2_{n0} = L_n^{st}M^\flat \to E^2_{0n-1}, \; n \geq 2.$$

Let us remark, that if G is a finite group with Sylow l-subgroup K and M is a G-module, then the natural map $H_i(K, M) \to H_i(G, M)$ is surjective on the l-torsion part. From this fact and the well-known structure of the Sylow subgroups in the symmetric groups follows that

$$E^2_{0n-1} \cong H_{n-1}(\Sigma_l, M)(l).$$

Hence the result is true for $T = M^\flat$. Now one can use the induction principle to finish the proof.

2.10. Induction Principle. Let $\mathcal{P}_n$ be the category of abelian Γ-groups of degree $\leq n$. For a based set X, one denotes by X_{ab} the free abelian group generated by X modulo the relation $* = 0$, where $*$ is the basepoint of X. For a Σ_n-module M, one defines the functor $M^\flat : \Gamma \to Ab$ by

$$M^\flat(X) = M \otimes_{\Sigma_n} (X_{ab})^{\otimes n}.$$

Then $M^\flat \in \mathcal{P}_n$. Let T be an object in $\mathcal{P}_n$. One defines the natural transformation from the last nontrivial cross-effect to our functor

$$\bar{P} : T(X \mid \cdots \mid X) \to T(X)$$

by the composition

$$T(X \mid \cdots \mid X) \subset T(X \vee \cdots \vee X) \stackrel{(1,\cdots,1)_*}{\longrightarrow} T(X)$$

One easily shows that

$$T(X \mid \cdots \mid X) \cong M \otimes_{\Sigma_n} (X_{ab})^{\otimes n},$$

where $M = T_n$. Moreover the transformation $\bar{P}$ factors trough $M^\flat$, and yields the natural transformation $M^\flat \to T$, whose kernel and cokernel have smaller degrees (see [6]). Based on this fact one obtains the following *induction principle*.

2.11. Proposition. *Let $\mathcal{P}$ be a full subcategory of the category $\mathcal{P}_n$. Consider the following properties of $\mathcal{P}$:*

i) *For any $k \le n$ and any right Σ_n-module M, the functor $M^\flat \in \mathcal{P}$.*

ii) *If*

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0$$

is a short exact sequence in $\mathcal{P}_n$ with $T'' \in \mathcal{P} \cap \mathcal{P}_{n-1}$ and $T' \in \mathcal{P}$ then $T \in \mathcal{P}$.

iii) *If*

$$0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0$$

is a short exact sequence in $\mathcal{P}_n$ with $T' \in \mathcal{P} \cap \mathcal{P}_{n-1}$ and $T \in \mathcal{P}$, then $T'' \in \mathcal{P}$.

If $\mathcal{P}$ satisfies properties i) - iii) *then $\mathcal{P} = \mathcal{P}_n$.*

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