AN ALGORITHM FOR FINDING ALL PREPROJECTIVE COMPONENTS OF THE AUSLANDER-REITEN QUIVER

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Abstract. The Auslander-Reiten quiver of a finite-dimensional associative algebra A encodes information about the finite dimensional representations of A and their homomorphisms. A component of the Auslander-Reiten quiver is called preprojective if it does not admit oriented cycles and each of its modules can be shifted into a projective module using the Auslander-Reiten translation. Preprojective components play an important role in the present research on algebras of finite and tame representation type. We present an algorithm which detects all preprojective components of a given algebra.

1. Introduction

Representation theory deals with the category of representations of an algebra A over a commutative ring k. We consider the case that k is an algebraically closed field and A is associative and finite-dimensional as k-space. Moreover, we are only interested in finite-dimensional representations which we consider as finite-dimensional right A-modules. Together with the A-homomorphisms these modules form the category mod$-A$.

The key which allows to study mod$-A$ with combinatorial algorithms is the observation that A and to a large extend also mod$-A$ can be studied using quivers. Remember, that a quiver is nothing but a directed graph. Namely, it was observed by Gabriel (see [Ga1], [Ga2]) that any basic finite-dimensional algebra over an algebraically closed field k is of the shape $k[Q]/I$ where Q is a finite quiver and I is an admissible ideal of the path algebra $k[Q]$. Since any finite-dimensional algebra is Morita equivalent to a basic algebra and Morita equivalent algebras have equivalent module categories, it is no loss of generality to study only the representation theory of basic algebras.

The quiver describing the module category mod$-A$ is usually called the *Auslander-Reiten quiver* of A. Its set of vertices consists of a minimal complete set of representatives of the isomorphism classes of indecomposable modules and the number of arrows from X to Y is given by the k-dimension of the space $\operatorname{rad}_A(X,Y)/\operatorname{rad}_A^2(X,Y)$. Note, that the Auslander-Reiten quiver is even a translation quiver meaning that it carries an additional structure, namely the *Auslander-Reiten translation* τ_A.

In general, it is hard to calculate components of the Auslander-Reiten quiver. One exception are the so-called *preprojective components* which are components without oriented cycles where every vertex can be shifted into a projective module by some power of τ_A.

Preprojective component play a decisive role in the study of algebras of finite representation type, the reason being that any simply connected algebra satisfies the separation criterion of Bautista-Larrión and therefore its Auslander-Reiten quiver has a preprojective component (see [Bo]). Nevertheless, there are many algebras that do not satisfy

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the separation criterion whereas their Auslander-Reiten quivers do have preprojective components.

We will see in section 3 that concerning preprojectve components it suffices to consider algebras $A = k[Q]/I$ where Q is directed i.e. there are no oriented cycles in Q. The aim of this paper is to present an algorithm which produces an initial subquiver $Q^\star$ of Q such that the algebra $A^\star = k[Q^\star]/I \cap k[Q^\star]$ has the following properties:

- The Auslander-Reiten quiver of $A^\star$ has preprojective components which contain all indecomposable projective modules over $A^\star$.
- A preprojective component of the Auslander-Reiten quiver of $A^\star$ is a preprojective component of the Auslander-Reiten quiver of A if and only if it does not contain any indecomposable radical summand of an indecomposable projective module associated with a direct neighbour of $Q^\star$ in Q.
- All preprojective components of the Auslander-Reiten quiver of A appear as above.

Since modules in preprojective components are characterised by their dimension vectors and by [KP] only a finite initial piece of a preprojective component can contain a given module, this is a finite algorithm which allows to check the existence of preprojective components and to construct all of them.

Let us outline the contents of the paper. In section 2 we provide the necessary definitions and preliminaries. For more background we refer to [ARS], [GR] and [Ri]. Section 3 is devoted to the study of completely preprojective algebras which are defined by the property that every indecomposable projective module appears in some preprojective component. In section 4 we present our algorithm and show that the first of the above properties holds. Finally, in section 5 we establish the uniqueness of $A^\star$ and prove the two remaining properties.

2. Definitions and preliminaries

2.1. Let A be a basic finite-dimensional k-algebra with a chosen decomposition $1 = \sum_{i=1}^n e_i$ of the unit element of A into orthogonal primitive idempotents. We form a k-linear category $\mathcal{A}$ using $\{1, \dots, n\}$ as set of objects and $e_y A e_x$ as space $\mathcal{A}(x, y)$ of morphisms from x to y. The composition is the multiplication in A. The category mod$-A$ of finite-dimensional right A-modules is equivalent to the category mod$-\mathcal{A}$ of k-linear contravariant functors from $\mathcal{A}$ to the category mod$-k$ of finite-dimensional k-spaces. We will denote the functors in this category as $\mathcal{A}$-modules. The equivalence sends an A-module M to the functor which associates the space Me_x to the object x and the right multiplication by a with each morphism $a \in e_y A e_x$.

Note that $\mathcal{A}$ is a *spectroid* in the sense of [GR] meaning that $\mathcal{A}$ is a k-linear category with finite-dimensional morphism spaces such that the endomorphism algebra of each object is local and any two objects are isomorphic iff they are equal. The assumption that the algebra A we started with is finite-dimensional is reflected by the fact that the spectroid $\mathcal{A}$ is finite i.e. has a finite set of objects. We will follow [GR] by considering finite spectroids $\mathcal{A}$ rather than finite-dimensional algebras A.

If we choose a minimal complete set of representatives of isomorphism classes of indecomposable modules in mod$-\mathcal{A}$, then the induced full subcategory ind$-\mathcal{A}$ will also be a spectroid but usually with an infinite set of objects.

Examples for objects in ind$-\mathcal{A}$ are the indecomposable projective modules $P_x = \mathcal{A}(-,x)$ and the indecomposable injective objects $I_x = \mathrm{D}\,\mathcal{A}(x,-)$ for all objects x of $\mathcal{A}$ where $\mathrm{D} = \mathrm{Hom}_k(-,k)$ is the usual duality.

2.2. If $\mathcal{S}$ is an arbitrary spectroid then its Jacobson radical $\mathrm{rad}\,\mathcal{S}$ consists of all non-isomorphisms in $\mathcal{S}$. The quiver of $Q_{\mathcal{S}}$ has the set of objects of $\mathcal{S}$ as set of vertices and the number of arrows from x to y is the dimension of the space $\mathrm{rad}\,\mathcal{S}(x,y)/\mathrm{rad}^2\,\mathcal{S}(x,y)$.

The k-linear *path category* kQ of an arbitrary quiver Q has the vertex set of Q as objects and the set of oriented paths from x to y in Q as basis of the morphism space $kQ(x,y)$. A k-linear functor $\pi : kQ_{\mathcal{S}} \to \mathcal{S}$ which is the identity on the objects and maps the set of arrows form x to y to a set of morphisms inducing a basis of the space $\mathrm{rad}\,\mathcal{S}(x,y)/\mathrm{rad}^2\,\mathcal{S}(x,y)$ is called a *presentation* of $\mathcal{S}$. If $\mathcal{A}$ is a finite spectroid, then any presentation $\pi : kQ_{\mathcal{A}} \to \mathcal{A}$ is full and therefore $\mathcal{A}$ can be identified with $kQ_{\mathcal{A}}/I$ where I is the kernel of π.

2.3. Due to results of Auslander and Reiten the quiver of the spectroid ind$-\mathcal{A}$ for a finite spectroid $\mathcal{A}$ carries the additional structure of a translation i.e. it is a translation quiver which is usually called the *Auslander-Reiten quiver* and denoted by $\Gamma_{\mathcal{A}}$. In order to introduce the translation let us first recall the notion of almost split morphisms and minimal morphisms in mod$-\mathcal{A}$. A morphism $g : Y \to Z$ in mod$-\mathcal{A}$ is said to be *right almost split* if it is not a retraction but each morphism $g' : Y' \to Z$ which is not a retraction factors as $g' = gh$. It is easy to see that the existence of such a g forces Z to be indecomposable. A morphism $g : Y \to Z$ is called *right minimal* if each endomorphism $h : Y \to Y$ satisfying $g = gh$ has to be an isomorphism. Left almost split and left minimal morphisms are defined in the dual fashion.

The importance of right minimal and right almost split morphisms (and dually the left minimal and left almost split morphisms) comes from the following fact. If Z and U are objects in ind$-\mathcal{A}$ and $g : Y \to Z$ is a right minimal and right almost split morphism, then the number of arrows form $U \to Z$ in $\Gamma_{\mathcal{A}}$ equals the multiplicity of U as direct summand of Y.

For each object x of $\mathcal{A}$ the inclusion ι_x of the submodule $\mathrm{rad}\,P_x = (\mathrm{rad}\,\mathcal{A})P_x$ into P_x is right minimal and right almost split. Dually the projection from I_x onto the factor of I_x by the socle of I_x is left minimal and left almost split. If Z in ind$-\mathcal{A}$ is not projective or X in ind$-\mathcal{A}$ is not injective, then the fundamental result of Auslander and Reiten (see [AR]) says that there is a short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in mod$-\mathcal{A}$ (called *Auslander-Reiten sequence*) where f is left minimal and left almost split and g is right minimal and right almost split. Moreover, the modules X and Z uniquely determine each other. Therefore, by putting $X = \tau_{\mathcal{A}} Z$ and $Z = \tau_{\mathcal{A}}^{-} X$ we obtain mutually inverse bijections between the non-projective objects of ind$-\mathcal{A}$ and the non-injective objects of ind$-\mathcal{A}$. The map $\tau_{\mathcal{A}}$ is called the *Auslander-Reiten translation.* It follows that for Z and U in ind$-\mathcal{A}$ with Z non-projective the number of arrows in $\Gamma_{\mathcal{A}}$ from U to Z equals the number of arrows from $\tau_{\mathcal{A}} Z$ to U. This is the defining property of a general translation quiver.

In order to avoid case by case inspections for the indecomposable projective modules we put $\tau_{\mathcal{A}} P_x = 0$ and for the indecomposable injective modules I_x we put $\tau_{\mathcal{A}}^{-} I_x = 0$.

2.4. The components of $\Gamma_{\mathcal{A}}$ are usually hard to calculate. The situation is much easier for preprojective components. A (connected) component Θ of $\Gamma_{\mathcal{A}}$ is said to be *preprojective* if it does not contain oriented cycles and for every vertex Z of Θ there is a non-negative integer n such that $\tau_{\mathcal{A}}^n Z$ is projective.

For a module M in mod$-\mathcal{A}$ the vector $\mathbf{dim}\, M = (\dim_k M(x))_{x\in\mathcal{A}} \in \mathbb{Z}^n$ is called the *dimension vector* of M. Since for every exact sequence $0 \to X \longrightarrow Y \longrightarrow Z \to 0$ the equation $\mathbf{dim}\, X + \mathbf{dim}\, Z = \mathbf{dim}\, Y$ holds the dimension vector of the end term Z of an Auslander-Reiten sequence can be easily calculated once the dimension vectors of the start term X and the indecomposable summands of the middle term Y are known.

By writing $\mathcal{A}$ as $kQ_{\mathcal{A}}/I$ the modules P_x are practically given. With some effort one can also decompose the radicals $\operatorname{rad} P_x$ into indecomposable summands. Because by [Ha] an indecomposable module in a preprojective component is uniquely determined by its dimension vectors, preprojective components can be calculated easily by induction using the dimension vectors of the indecomposable projective modules and their indecomposable radical summands. The crucial point is that one has to know which indecomposable projective modules actually appear in preprojective components. This is the question we will answer.

2.5. We need some device which allows to induce Auslander-Reiten sequences from subspectroids. Let $\mathcal{B}$ be a full subspectroid of the finite spectroid $\mathcal{A}$, then the canonical restriction functor $\operatorname{res}_{\mathcal{B}} : \text{mod}-\mathcal{A} \to \text{mod}-\mathcal{B}$ has a right adjoint R and a left adjoint L which both are functors mod$-\mathcal{B} \to$ mod$-\mathcal{A}$. For an object x of $\mathcal{A}$ and a module U in mod$-\mathcal{B}$ we have $RU(x) = \operatorname{Hom}_{\mathcal{B}}(\operatorname{res}_{\mathcal{B}} P_x, U)$ and $LU(x) = U \otimes_{\mathcal{B}} \operatorname{res}_{\mathcal{B}} \mathcal{A}(x,-) \cong \operatorname{D}\operatorname{Hom}_{\mathcal{B}}(U, \operatorname{res}_{\mathcal{B}} I_x))$. Consequently, the functor R is left exact and maps the indecomposable injective module I_x over $\mathcal{B}$ to the corresponding module I_x over $\mathcal{A}$. Similarly, L is right exact and maps P_x over $\mathcal{B}$ to P_x over $\mathcal{A}$. In [Sc] it is shown that for Z in ind$-\mathcal{B}$ the module LZ in ind$-\mathcal{A}$ satisfies $R\tau_{\mathcal{B}}Z = \tau_{\mathcal{A}}LZ$. Dually for X in ind$-\mathcal{B}$ the module RX is in ind$-\mathcal{A}$ and $L\tau_{\mathcal{B}}^- X = \tau_{\mathcal{A}}^- RX$.

For a spectroid $\mathcal{S}$ a full subspectroid $\mathcal{T}$ is called *initial* provided that for any non-zero map $f : x \to y$ in $\mathcal{S}$ if y lies in $\mathcal{T}$, then also x lies in $\mathcal{T}$. In an analogous way a full subquiver Q' of a quiver Q is said to be initial if for any arrow $x \to y$ if y lies in Q', then also x lies in Q'.

Let $\mathcal{B}$ be a full subspectroid of a finite spectroid $\mathcal{A}$ with $\mathcal{A} = kQ_{\mathcal{A}}/I$. Then $\mathcal{B}$ is initial iff the full subquiver Q' of $Q_{\mathcal{A}}$ associated with the objects of $\mathcal{B}$ is initial. In this case $Q_{\mathcal{B}} = Q'$ and $\mathcal{B} = kQ'/I \cap kQ'$.

For an initial subspectroid $\mathcal{B}$ of $\mathcal{A}$ every module N over $\mathcal{B}$ can be considered as module over $\mathcal{B}$ be putting $N(x) = 0$ for all x which belong to $\mathcal{A}$ but not to $\mathcal{B}$. Moreover, this inclusion mod$-\mathcal{B} \to$ mod$-\mathcal{A}$ coincides with the functor L. Therefore the formulas from [Sc] degenerate to $R\tau_{\mathcal{B}}Z = \tau_{\mathcal{A}}Z$ and $\tau_{\mathcal{B}}^- X = \tau_{\mathcal{A}}^- RX$. In contrast to LN the module RN is usually not isomorphic to N. Nevertheless, because $RN(x) = N(x)$ for all x in $\mathcal{B}$, there is a canonical inclusion of N into RN which allows to consider N as submodule of RN.. We will frequently use that an Auslander-Reiten sequence $0 \to X \longrightarrow Y \longrightarrow Z \to 0$ in mod$-\mathcal{A}$ where X and Z belong to mod$-\mathcal{B}$ is actually also an Auslander-Reiten sequence in mod$-\mathcal{B}$.

2.6. For a preprojective component Θ of $\Gamma_{\mathcal{A}}$ we denote by $\mathcal{S}(\Theta)$ the full subspectroid of ind$-\mathcal{A}$ associated with the vertices of Θ. The defining property of right minimal and right almost split maps shows that any presentation $k\Theta \to \mathcal{S}(\Theta)$ is full. With the same argument one obtains that $\mathcal{S}(\Theta)$ is an initial subspectroid of ind$-\mathcal{A}$.

3. Completely preprojective algebras

3.1. A finite-dimensional algebra resp. a finite spectroid $\mathcal{A}$ is called *completely preprojective* if every indecomposable projective module lies in a preprojective component. Note, that $\mathcal{A}$ (i.e. $Q_{\mathcal{A}}$) is not assumed to be connected. A preprojective component Θ of $\Gamma_{\mathcal{A}}$ is said to be *complete* if all indecomposable projective modules P_x belong to Θ.

If $\mathcal{A}$ is completely preprojective, then any cycle in $Q_{\mathcal{A}}$ would give rise to a cycle in a preprojective component of $\Gamma_{\mathcal{A}}$. Hence the quiver $Q_{\mathcal{A}}$ of a completely preprojective finite spectroid $\mathcal{A}$ has to be directed i.e. does not admit oriented cycles. Similarly one sees that $Q_{\mathcal{A}}$ is connected if and only if there is precisely one preprojective component which is obviously complete.

Proposition. *Let $\mathcal{A}$ be a finite spectroid and Θ a preprojective component of the Auslander-Reiten quiver $\Gamma_{\mathcal{A}}$. If we denote by $\mathcal{B}$ the full subcategory of $\mathcal{A}$ associated with all objects x such that $Z(x) \neq 0$ for some Z in Θ, then the following assertions hold:*

(1) $\mathcal{B} = \{x \in \mathcal{A} \mid P_x \in \Theta\}$.
(2) *$\mathcal{B}$ is an initial subspectroid of $\mathcal{A}$.*
(3) *$\tau_{\mathcal{B}}^- X = \tau_{\mathcal{A}}^- X$ and $\tau_{\mathcal{B}} X = \tau_{\mathcal{A}} X$ for each object X in Θ.*
(4) *Θ is a complete preprojective component of $\Gamma_{\mathcal{B}}$.*
(5) *For each y in $\mathcal{A}$ and not in $\mathcal{B}$ none of the indecomposable direct summands of* $\operatorname{res}_{\mathcal{B}} P_y$ *lies in Θ.*

Proof. (1) If P_x is in Θ, then $P_x(x) = \mathcal{A}(x,x) \neq 0$ and therefore x is in $\mathcal{B}$. Conversely, if $Z(x) \neq 0$ for Z in Θ, then by the Yoneda lemma $\operatorname{Hom}_{\mathcal{A}}(P_x, Z) \neq 0$ holds. Since $\mathcal{S}(\Theta)$ is initial, also P_x belongs to Θ.

(2) Suppose $\mathcal{A}(x,y) \neq 0$ and P_y in Θ. Again by the Yoneda lemma $\operatorname{Hom}_{\mathcal{A}}(P_x, P_y) \neq 0$ and therefore P_x is in Θ.

(3) For Z in Θ we first consider the case that Z is not projective and we dispose of an Auslander-Reiten sequence $0 \to X \longrightarrow Y \longrightarrow Z \to 0$ in mod$-\mathcal{A}$. On the other hand Z is also a $\mathcal{B}$-module and there is an Auslander-Reiten sequence $0 \to X' \longrightarrow Y' \longrightarrow Z \to 0$ in mod$-\mathcal{B}$. Because for each indecomposable summand U of Y there is a non-zero morphism from U to Z and from X to U also X and Y are $\mathcal{B}$-modules and therefore $\tau_{\mathcal{A}} Z = X = X' = \tau_{\mathcal{B}} Z$. For Z projective we get $R\tau_{\mathcal{B}} X = \tau_{\mathcal{A}} Z = 0$ and therefore also $\tau_{\mathcal{B}} X = 0$.

If X is a non-injective module in Θ then the same arguments as above yield $\tau_{\mathcal{B}}^- X = \tau_{\mathcal{A}}^- X$. In case X is injective, then $\tau_{\mathcal{A}}^- X = 0$. But the module RX is injective as well and thus $\tau_{\mathcal{B}}^- X = \tau_{\mathcal{A}}^- RX = 0$.

(4) follows from (1) and (3) immediately.

(5) Let y be in object of $\mathcal{A}$ but not in $\mathcal{B}$. We assume that there is an indecomposable direct summand of $\operatorname{res}_{\mathcal{B}} P_y$ which lies in Θ and obtain $0 \neq \operatorname{Hom}_{\mathcal{B}}(\operatorname{res}_{\mathcal{B}} P_y, U) = RU(y)$. Hence the canonical inclusion of U into the indecomposable module RU is not an isomorphism and thus U cannot be an injective $\mathcal{A}$-module. By (3) the non-zero module $X = \tau_{\mathcal{A}}^- U$ lies in Θ. The equation $U = \tau_{\mathcal{A}} X = R\tau_{\mathcal{B}} X = RU$ gives a contradiction. □

3.2. The proposition above shows that all preprojective components can always be found as complete preprojective components for some initial subspectroid. After a preparatory lemma we will use this result to show that we only need to consider finite spectroids whose quivers are directed.

Lemma. *Let $\mathcal{B}$ be an initial subspectroid of a finite spectroid $\mathcal{A}$. Suppose that Θ is a preprojective component of $\Gamma_{\mathcal{B}}$. If for all y in $\mathcal{A}$ but not in $\mathcal{B}$ the indecomposable direct summands U of $\operatorname{res}_{\mathcal{B}} P_y$ do not belong to Θ, then Θ is a preprojective component of $\Gamma_{\mathcal{A}}$.*

Proof. Since $\mathcal{S}(\Theta)$ is initial in ind$-\mathcal{B}$ the assumption on the summands U of the restrictions $\operatorname{res}_{\mathcal{B}} P_y$ shows that $RX = X$ for all X in Θ and hence $\tau_{\mathcal{A}}X = \tau_{\mathcal{B}}X$, $\tau_{\mathcal{A}}^{-}X = \tau_{\mathcal{B}}^{-}X$. This implies that the Auslander-Reiten sequences constituting Θ remain Auslander-Reiten sequences over $\mathcal{A}$. Thus it remains to show that Θ is closed under neighbours in $\Gamma_{\mathcal{A}}$. For this reason we prove that for an arrow $X \to Y$ in $\Gamma_{\mathcal{A}}$ the start term X lies in Θ iff the end term Y lies in Θ. We only establish one direction because the other is similar. Suppose that X belongs to Θ and consider a left minimal and left almost split map $X \to M$. It suffices to show that M is a $\mathcal{B}$-module. In case X is not injective, then as seen above $\tau^{-}X$ and consequently M are $\mathcal{B}$-modules. For X injective we now that $M = X/\operatorname{soc} X$ is a $\mathcal{B}$-module as well. □

For an arbitrary finite spectroid $\mathcal{A}$ we denote by $\mathcal{A}^{init}$ the full subspectroid given by all objects x of $\mathcal{A}$ such that no predecessor of x in $Q_{\mathcal{A}}$ lies on an oriented cycle of $Q_{\mathcal{A}}$. Clearly $\mathcal{A}^{init}$ is an initial subspectroid of $\mathcal{A}$ whose quiver is directed.

Theorem. *Let $\mathcal{A}$ be a finite spectroid.*

(a) *If $\Gamma_{\mathcal{A}}$ has a preprojective component Θ, then all modules in Θ are $\mathcal{A}^{init}$-modules and Θ is a preprojective component of $\Gamma_{\mathcal{A}^{init}}$ which does not contain any indecomposable summand of a restriction $\operatorname{res}_{\mathcal{A}^{init}} P_y$ for all y in $\mathcal{A}$ but not in $\mathcal{A}^{init}$.*

[b) *If Θ is a preprojective component of $\Gamma_{\mathcal{A}^{init}}$ which does not contain any indecomposable summand of a restriction $\operatorname{res}_{\mathcal{A}^{init}} P_y$ for all y in $\mathcal{A}$ but not in $\mathcal{A}^{init}$, then Θ is a preprojective component of $\Gamma_{\mathcal{A}}$.*

Proof. The first part follows from the proposition because every initial completely preprojective subspectroid of $\mathcal{A}$ has to be contained in $\mathcal{A}^{init}$. The second part follows from the lemma. □

Thus for looking for preprojective components one only has to consider the initial subspectroid $\mathcal{A}^{init}$ whose quiver is directed. For computational purposes it is worth noting that for checking if the indecomposable summands of the restrictions of $\operatorname{res}_{\mathcal{A}^{init}} P_y$ appear in preprojective components of $\Gamma_{\mathcal{A}^{init}}$ one only has to know the dimension vectors of these modules.

3.3. The next lemma will allow to recognise complete preprojective components from finite initial pieces. We recall that in a quiver Q a vertex x is said to be a *predecessor* of a vertex y it there is a path $x = x_0 \to x_1 \to \cdots \to x_n = y$ in Q. Moreover, X is called *direct predecessor* of y if there is an arrow $x \to y$ in Q. The notions of *successor* and *direct successor* are defined in the dual way.

Note, that in a preprojective component Θ of $\Gamma_{\mathcal{A}}$ each module X has only finitely many predecessors.

Lemma. *Suppose, that $\mathcal{A}$ is a finite spectroid with directed quiver $Q_{\mathcal{A}}$ and Θ' is full directed subquiver of $\Gamma_{\mathcal{A}}$.*

If Θ' is closed under predecessors in $\Gamma_{\mathcal{A}}$, contains all indecomposable projective $\mathcal{A}$-modules, and each vertex in Θ' has only finitely many predecessors, then the full subquiver Θ of $\Gamma_{\mathcal{A}}$ supported by all non-zero objects of the shape $\tau_{\mathcal{A}}^{-n}X$ with X in Θ' and an integer

$n \geq 0$ is a union of preprojective components of $\Gamma_{\mathcal{A}}$ which contains all indecomposable projectives.

In particular, if Θ' is connected, then Θ is a complete preprojective component.

Proof. Since clearly all indecomposable projectives belong to Θ, the following remains to show.

(1) Θ is a union of components of $\Gamma_{\mathcal{A}}$.
(2) Θ is a directed quiver.
(3) For each vertex Z of Θ there exists a non-negative integer n and x in $\mathcal{A}$ such that $\tau^n_{\mathcal{A}} Z = P_x$.

(1) Let $X \to Y$ be an arrow in $\Gamma_{\mathcal{A}}$. We show that X belongs to Θ if and only Y belongs to Θ. If Y is in Θ, then there is a minimal non-negative integer n with $\tau^n_{\mathcal{A}} Y$ in Θ'. In case there exists a non-negative integer $m < n$ such that $\tau^m_{\mathcal{A}} X$ is projective, then X is in Θ by definition. Otherwise we obtain an arrow $\tau^n_{\mathcal{A}} X \to \tau^n_{\mathcal{A}} Y$. Because Θ' is closed under predecessors, $\tau^n_{\mathcal{A}} X$ has to lie in Θ' and consequently X in Θ.

If X is in Θ, then we assume that Y does not belong to Θ which implies that Y is not projective and we obtain an Auslander-Reiten sequence $0 \to U \longrightarrow V \longrightarrow Y \to 0$. By the definition of Θ the module $U = \tau_{\mathcal{A}} Y$ cannot belong to Θ as well. On the other hand X is a direct summand of V which, as seen above, shows that its predecessor U is in Θ. Thus we found a contradiction.

(2) We assume that there is cycle $X = Z_0 \to Z_1 \to \cdots \to Z_t = X$ in Θ. There is a non-negative integer n such that all $\tau^n_{\mathcal{A}} Z_j$ are non-zero and at least one of these modules lies in Θ'. Because Θ' is closed under predecessors, then the whole cycle lies in Θ' which is a contradiction.

(3) By definition for X in Θ there exists n with $\tau^n_{\mathcal{A}} X$ in Θ'. Because each point of the directed quiver Θ' has only finitely many predecessors, we find another m such that $\tau^{m+n}_{\mathcal{A}} X$ is projective. □

3.4. Loosely speaking, our strategy is, to build up a completely preprojective finite spectroid $\mathcal{A}$ inductively by adding vertices starting from the sources. To formulate the induction step we define that $\mathcal{A}$ is a *one-point extension* of the subspectroid $\mathcal{B}$ by the object x iff x is a sink of $Q_{\mathcal{A}}$ and the only object of $\mathcal{B}$ which does not belong to $\mathcal{A}$. This implies that $\mathcal{B}$ is initial and $\operatorname{res}_{\mathcal{B}} P_x = \operatorname{rad} P_x$.

We will need the notion of directing modules. If $\mathcal{A}$ is a finite spectroid and X, Y are objects in ind$-\mathcal{A}$, then a *path of length* n from X to Y in ind$-\mathcal{A}$ is a sequence $X = X_0 \to X_1 \to \cdots \to X_n = Y$ of non-zero radical morphisms $X_{i-1} \to X_i$ in ind$-\mathcal{A}$. The indecomposable module X is called *directing* if there does not exist a path of length $n > 0$ which starts and ends in X. A possibly decomposable module $Z = \oplus_{i=1}^m Z_i$ with Z_i indecomposable is said to be *directing* of there do not exist summands Z_s, Z_t and a non-projective Y in ind$-\mathcal{A}$ such that there is a path from Z_s to $\tau_{\mathcal{A}} Y$ and a path from Y to Z_t in ind$-\mathcal{A}$. It was shown in [HR] that both concepts of directing modules coincide for indecomposables. Moreover, it was proved that for a one-point extension $\mathcal{A}$ of $\mathcal{B}$ by x the projective module P_x is directing in ind$-\mathcal{A}$ iff the radical $\operatorname{rad} P_x$ is directing in mod$-\mathcal{B}$.

Note, that it is easy to check whether a module whose indecomposable summands are all preprojective is directing, because in this case all the involved paths in ind$-\mathcal{A}$ come from paths in preprojective components. In particular, each indecomposable module in a preprojective component is directing itself.

Recall, furthermore, that a path $Z_0 \to Z_1 \to \cdots \to Z_m$ in the Auslander-Reiten quiver $\Gamma_{\mathcal{A}}$ is said to be *sectional* provided $Z_{i-2} \neq \tau_{\mathcal{A}} Z_i$ for all $i = 2, \dots, n$.

Theorem. *Let the finite spectroid $\mathcal{A}$ be a one-point extension of $\mathcal{B}$ by x. We decompose* $\operatorname{rad} P_x = \oplus_{i=1}^n U_i$ *into indecomposable summands. Suppose that $\mathcal{B}$ is completely preprojective and* $\operatorname{rad} P_x$ *has the following properties:*

(a) *All summands U_i lie in preprojective components.*
(b) *The module* $\operatorname{rad} P_x$ *is a directing $\mathcal{B}$-module.*
(c) *For each object y of $\mathcal{B}$ and summand U_j any path $U_j = Z_0 \to Z_1 \to \cdots \to Z_m = P_y$ in $\Gamma_{\mathcal{B}}$ has to be sectional.*

Then $\mathcal{A}$ is completely preprojective and in the preprojective components of $\Gamma_{\mathcal{A}}$ there is no proper projective successor of P_x.

Proof. We define Θ'' as the full subquiver of the union of the preprojective components of $\Gamma_{\mathcal{B}}$ given by all modules Z such that any path from some U_j to Z is sectional. In order to construct the quiver Θ', we add the vertex P_x and as many arrows from each U_j contained in Θ'' to P_x as this module appears as summand in $\operatorname{rad} P_x$. We wish to apply the preceeding lemma and have to check the required properties.

By construction Θ' is directed and each vertex has only finitely many predecessors. Since $\mathcal{B}$ is completely preprojective, condition (c) shows that all modules P_y for y in $\mathcal{A}$ belong to Θ'. Thus it remains to prove that Θ' is a full and predecessor closed subquiver of $\Gamma_{\mathcal{A}}$. Both assertions can be derived from the following claim: If Z is a module in Θ' and $g : N \to Z$ is right minimal and right almost split in mod$-\mathcal{A}$, then all indecomposable direct summands of N belong to Θ'.

If Z is not projective, then Z is a $\mathcal{B}$-module in Θ'' and there is an Auslander-Reiten sequence $0 \to X \longrightarrow N \xrightarrow{g} Z \to 0$ in mod$-\mathcal{A}$ but also an Auslander-Reiten sequence $0 \to X' \longrightarrow N' \xrightarrow{g} Z \to 0$ in mod$-\mathcal{B}$. Assuming $X \neq X'$ from $X = \tau_{\mathcal{A}} Z = R\tau_{\mathcal{B}} Z = RX'$ we obtain $RX'(x) = X(x) \neq 0$. Thus $\operatorname{Hom}_{\mathcal{B}}(U_j, X') \neq 0$ for some U_j and hence there is a path from U_j to X' yielding a non-sectional path from U_j to Z. This is a contradiction to Z in Θ''. Therefore all summands of N belong to Θ''.

In case $Z = P_y$ for y in $\mathcal{B}$ clearly $N = \operatorname{rad} P_y$ lies in mod$-\mathcal{B}$ and the same arguments as above apply. For $Z = P_x$ by (a) and (b) all indecomposable summands U_j of $\operatorname{rad} P_z$ lie in Θ''.

In particular, we have seen that Θ'' is a full and predecessor closed subquiver of $\Gamma_{\mathcal{S}}$ which finally implies that P_x cannot have a proper projective successor. □

3.5. If $\mathcal{A}$ is a completely preprojective finite spectroid, then x in $\mathcal{A}$ is called *strong sink* if P_x does not have a proper projective successor in the Auslander-Reiten quiver of $\mathcal{A}$. Every strong sink is obviously a sink of $Q_{\mathcal{A}}$ but the converse does not hold in general. We saw that the extension object x considered in the preceeding theorem is actually a strong sink. This observation allows to formulate a converse of Theorem 3.4.

Proposition. *Let $\mathcal{A}$ be a completely preprojective finite spectroid and x a strong sink of $\mathcal{A}$. If we denote by $\mathcal{B}$ the initial subspectroid supported by all objects different from x, then $\mathcal{B}$ is completely preprojective and* $\operatorname{rad} P_x = \oplus_{i=1}^n U_i$ *satisfies the properties (a), (b), (c) from the theorem above.*

Proof. In order to show that $\mathcal{B}$ is completely preprojective, we want to use again the lemma from the previous section and consider the full subquiver Θ' of $\Gamma_{\mathcal{A}}$ supported by all $Z \neq P_x$ such that there does not exist a non-sectional path from some U_j to Z. All Z in Θ' are $\mathcal{B}$-modules because otherwise $0 \neq Z(x) \cong \mathrm{Hom}_{\mathcal{A}}(P_x, Z)$ and therefore there is a path from P_x to Z inducing an non-sectional path from some U_j to Z. It follows that Θ is a directed and predecessor closed full subquiver of $\Gamma_{\mathcal{B}}$ such that each vertex has only finitely many predecessors. It remains to prove that Θ' contains all indecomposable projective $\mathcal{B}$-modules.

Let us assume the existence of some P_y with P_y not in Θ'. This means that there is a non-sectional path $U_j = Z_0 \to Z_1 \to \cdots \to Z_m = P_y$. All Z_i have to be $\mathcal{B}$-modules because otherwise $\mathrm{Hom}_{\mathcal{A}}(P_x, Z_i) \neq 0$ and the Yoneda lemma would yield that x is not a strong sink. Let us chose t as the smallest index such that $Z_t = \tau_{\mathcal{A}} Z_{t+2}$. Without loss of generality we may assume that there is no non-sectional path from U_j to Z_t. By [HR, Proposition 2] we obtain $\mathrm{Hom}_{\mathcal{B}}(U_j, Z_t) \neq 0$ implying $RZ_t(x) \neq 0$ and hence $RZ_t \neq Z_t$. On the other hand we know $Z_t = \tau_{\mathcal{A}} Z_{t+2} \cong R\tau_{\mathcal{B}} Z_{t+2} = RZ_t$.

Since the preprojective module P_x is a directing $\mathcal{A}$-module, $\operatorname{rad} P_x$ is a directing $\mathcal{B}$-module by [HR] which implies (a) and (b). Finally, we have seen above that all indecomposable projective $\mathcal{B}$-modules belong to Θ' yielding (c). □

4. The algorithm

4.1. In view of the reduction to $\mathcal{A}^{init}$ introduced in the last section from now on we will only consider spectroids $\mathcal{A}$ whose quivers are directed. We know that any preprojective component comes from a completely preprojective initial subspectroid which we could find by inductive one-point extension using the theorems at the end of the preceeding section. The problem is that for each consecutive one-point extension we need a strong sink and it is not clear how to arrange this. Before we formulate the algorithm solving this problem we will illustrate it at an easy example.

Let the spectroid $\mathcal{A}$ be given by the quiver from Figure 1 equipped with the relation $\gamma\beta$. If we make inductive one-point extensions along the sequence $(1, 2, 3, 4)$, then Theorem 3.4 always applies and therefore $\mathcal{A}$ is completely preprojective. If we use the sequence $(1, 3, 4)$, then we obtain a completely preprojective initial subspectroid $\mathcal{B}$ but now the simple projective radical P_1 of P_2 has a non-sectional path to P_4 in the finite preprojective Auslander-Reiten quiver $\Gamma_{\mathcal{B}}$ which we display in Figure 2. As usual the Auslander-Reiten translation is indicated by dotted arrows.

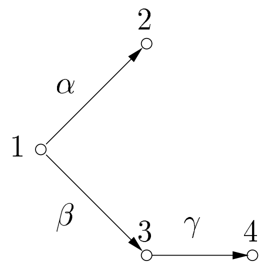

Figure 1

4.2. We inductively define an ascending sequence $\mathcal{A}_0, \mathcal{A}_1, \ldots$ of initial subspectroids of $\mathcal{A}$ starting with the empty spectroid as $\mathcal{A}_0$ which we agree to be completely preprojective. For $n > 0$ we distinguish two cases.

Case 1: $\mathcal{A}_{n-1}$ is not completely preprojective. If this happens, we put $\mathcal{A}_n := \mathcal{A}_{n-1}$.

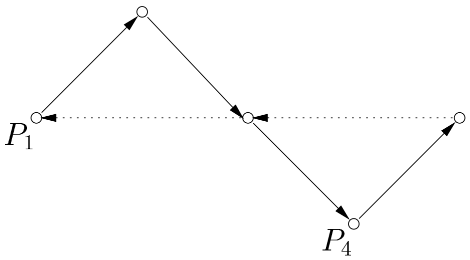

FIGURE 2

Case 2: $\mathcal{A}_{n-1}$ is completely preprojective. We define a quiver Q_n with three types of vertices. Firstly, there are all x in $\mathcal{A}$ but not in $\mathcal{A}_{n-1}$ but $\operatorname{rad} P_x$ is a $\mathcal{A}_{n-1}$-module. Secondly, there are all x in $\mathcal{A}$ but not in $\mathcal{A}_{n-1}$ such that there is an indecomposable summand U of $\operatorname{rad} P_x$ which belongs to a preprojective component of $\Gamma_{\mathcal{A}_{n-1}}$. Thirdly, there is a vertex ∞ different from all objects of $\mathcal{A}$. Note, that there may be vertices which are of the first and second type simultaneously.

This set of vertices is equipped with four types of arrows. Firstly, for all x, y in $\mathcal{A}$ we have an arrow $\sigma_{x,y} : x \to y$ provided that y is a proper successor of x in $Q_{\mathcal{A}}$. Secondly, for all x, y in $\mathcal{A}$ there is an arrow $\sigma'_{x,y} : x \to y$ if there is a preprojective component of $\Gamma_{\mathcal{A}_{n-1}}$ containing an indecomposable direct summand U of $\operatorname{rad} P_x$ and an indecomposable direct summand V of $\operatorname{rad} P_y$ such that there is a non-sectional path from U to V. Thirdly, for all x in $\mathcal{A}$ such that $\operatorname{rad} P_x$ is an $\mathcal{A}_{n-1}$-module there is an arrow $\sigma_{\infty,x} : \infty \to x$ provided that there is an indecomposable summand of $\operatorname{rad} P_x$ which does not lie in a preprojective component of $\Gamma_{\mathcal{A}_{n-1}}$. Fourthly, there is a loop $\sigma_{\infty,\infty} : \infty \to \infty$.

We continue distinguishing again two subcases.

Subcase 2.1: Q_n does not have sources. We put $\mathcal{A}_n := \mathcal{A}_{n-1}$.

Subcase 2.2: Q_n has sources. We choose a source x and define $\mathcal{A}_n$ to be the full subspectroid of $\mathcal{A}$ supported by all objects z in $\mathcal{A}_{n-1}$ and x.

Finally, we define $\mathcal{A}^\star$ to be the union of all the subcategories $\mathcal{A}_n$ and state some easy observations.

Remark. (a) *The sequence $\mathcal{A}_0, \mathcal{A}_1, \ldots$ is an ascending sequence of full subspectroids.*
(b) *There is a non-negative integer $n^\star$ such that $\mathcal{A}^\star = \mathcal{A}_{n^\star}$.*
The sequence $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_{n^\star}$ is strictly ascending.
(c) *Each source of $Q_{\mathcal{A}}$ lies in $\mathcal{A}^\star$.*

4.3. Before analysing the subspectroids $\mathcal{A}_n$, we want to establish a technical lemma. To formulate it, we suppose that $\mathcal{A}$ is a finite spectroid with directed quiver and $\mathcal{B}$ is an initial subspectroid. By $\mathcal{C}$ we denote the full subspectroid of $\mathcal{A}$ given by all objects x not belonging to $\mathcal{B}$. The subspectroid $\mathcal{C}$ is *final* in the obvious sense. The $\mathcal{C}$-$\mathcal{B}$-bimodule B is obtained by restricting the $\mathcal{A}$-$\mathcal{A}$-bimodule $\mathcal{A}(-,-)$ to $\mathcal{B}$ in the first and to $\mathcal{C}$ in the second argument. We recall that modules over $\mathcal{A}$ can be interpreted as triples $U = (U_\omega, \gamma_U, U_\alpha)$ where U_ω in mod$-\mathcal{C}$, U_α in mod$-\mathcal{B}$ and γ_U in $\operatorname{Hom}_{\mathcal{C}}(U_\omega, \operatorname{Hom}_{\mathcal{B}}(B, U_\alpha))$. More precisely, an equivalence from mod$-\mathcal{A}$ to the category of these triples is given by sending an $\mathcal{A}$-module U to its restriction U_ω to $\mathcal{C}$, its restriction U_α to $\mathcal{B}$ and the morphism γ_U which maps for x in $\mathcal{C}$ an element u in $U(x)$ to $\mathcal{A}(y, x) \to U_\alpha(x), a \mapsto U(f)(u)$. Note, that $\operatorname{Hom}_{\mathcal{B}}(B, U_\alpha)$ is considered as object of mod$-\mathcal{C}$ using the $\mathcal{C}$-module structure of B in the usual way.

We need the following easy observation. If $U = (U_\omega, \gamma_U, U_\alpha)$ is an indecomposable module in $\mathrm{mod}-\mathcal{A}$ with $U_\omega \neq 0$, then for each indecomposable summand U' of U_α there is an object x of $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{B}}(\mathrm{res}_{\mathcal{B}} P_x, U') \neq 0$. The reason is, that otherwise the triple $(0, 0, U')$ would be a non-trivial direct summand of U.

Lemma. *If x in $\mathcal{C}$ and U is an indecomposable direct summand of $\mathrm{rad}\, P_x$ satisfying $U \neq \mathrm{res}_{\mathcal{B}} U$, then for each indecomposable direct summand U' of $\mathrm{res}_{\mathcal{B}} U$ there exists an object y of $\mathcal{C}$ which is a proper predecessor of x in $Q_{\mathcal{A}}$, such that there is an indecomposable summand V of $\mathrm{rad}\, P_y$ satisfying $\mathrm{Hom}_{\mathcal{B}}(\mathrm{res}_{\mathcal{B}} V, U') \neq 0$.*

In particular, for each x in $\mathcal{C}$ and each indecomposable summand U' of $\mathrm{res}_{\mathcal{B}} P_x$ there is an object y of $\mathcal{C}$ which is a predecessor of x in $Q_{\mathcal{A}}$, such that there is an indecomposable direct summand V' of $\mathrm{rad}\, P_y$ which is a $\mathcal{B}$-module and a predecessor of U' in $\mathrm{ind}-\mathcal{B}$.

Proof. We define $\mathcal{A}'$ as the full subspectroid of $\mathcal{A}$ supported by all proper predecessors of x and $\mathcal{B}'$ as the full subspectroid of $\mathcal{A}'$ given by all objects from $\mathcal{B}$. We observe that $\mathcal{A}'$ and $\mathcal{B}'$ are initial subspectroids of $\mathcal{A}$. Moreover, $\mathrm{rad}\, P_x$ and therefore also U are actually $\mathcal{A}'$-modules. Applying the remarks above to the indecomposable summand U' of $\mathrm{res}_{\mathcal{B}'} U = U_\alpha$ we obtain an object y in $\mathcal{A}'$ but not in $\mathcal{B}'$ satisfying $\mathrm{Hom}_{\mathcal{B}'}(\mathrm{res}_{\mathcal{B}'} P_y, U') \neq 0$. Since $\mathrm{res}_{\mathcal{B}'} P_y = \mathrm{res}_{\mathcal{B}} P_y$ for all y in $\mathcal{A}'$, we also get $\mathrm{Hom}_{\mathcal{B}}(\mathrm{res}_{\mathcal{B}} P_y, U') \neq 0$. Because y does not belong to $\mathcal{B}$, we know $\mathrm{res}_{\mathcal{B}} P_y = \mathrm{res}_{\mathcal{B}} \mathrm{rad}\, P_y$ which shows the existence of an indecomposable summand V of $\mathrm{rad}\, P_y$ with $\mathrm{Hom}_{\mathcal{B}}(\mathrm{res}_{\mathcal{B}} V, U') \neq 0$.

In order to prove the additional assertion, we define $l(x)$ to be the number of proper predecessors of x in $Q_{\mathcal{A}}$ which do not belong to $\mathcal{B}$. If $l(x) = 0$, then $\mathrm{rad}\, P_x = \mathrm{res}_{\mathcal{B}} P_x$ and the claim is clear using $y = x$. Now we suppose $l(x) > 0$ and consider an indecomposable direct summand U' of $\mathrm{res}_{\mathcal{B}} P_y$. We find an indecomposable direct summand U of $\mathrm{rad}\, P_x$ such that U' is a summand of $\mathrm{res}_{\mathcal{B}} U$.

If $\mathrm{res}_{\mathcal{B}} U = U$, then $U = U'$ and we choose $y = x$ again. Otherwise we apply the lemma to obtain a proper predecessor y of x in $Q_{\mathcal{A}}$ lying in $\mathcal{C}$, such that there is an indecomposable summand V of $\mathrm{rad}\, P_y$ satisfying $\mathrm{Hom}_{\mathcal{B}}(\mathrm{res}_{\mathcal{B}} V, U') \neq 0$. Hence there is also an indecomposable summand V' of $\mathrm{res}_{\mathcal{B}} V$ with $\mathrm{Hom}_{\mathcal{B}}(V', U') \neq 0$. The module V' is an indecomposable direct summand of $\mathrm{res}_{\mathcal{B}} P_y$ where $l(y) < l(x)$. By induction there is a predecessor y' of y in $Q_{\mathcal{A}}$ not belonging to $\mathcal{B}$ such that there is an indecomposable summand V'' of $\mathrm{rad}\, P_{y'}$ which is a predecessor of V' in $\mathrm{ind}-\mathcal{B}$. $\square$

4.4. After this preparation we are ready to study the subspectroids $\mathcal{A}_n$.

Lemma. *For every integer $n \geq 0$ the following assertions hold:*

(1) *$\mathcal{A}_n$ is an initial subspectroid of $\mathcal{A}$.*
(2) *$\mathcal{A}_n$ is completely preprojective.*
(3) *If $n > 0$ and $\mathcal{A}_n$ is a one-point extension of $\mathcal{A}_{n-1}$ by x, then the full subquiver of $\Gamma_{\mathcal{A}_n}$ supported by the predecessors of P_x is a predecessor closed full subquiver of $\Gamma_{\mathcal{A}}$ such that $\tau_{\mathcal{A}} X = \tau_{\mathcal{A}_n} X$ for all its vertices X.*

Proof. We apply induction on n and observe that for $n = 0$ nothing is to prove. For $n > 0$ we know by induction that (1), (2), (3) hold for $\mathcal{A}_{n-1}$. Thus we are in the case 2 of the algorithm. For the subcase 2.1 that $\mathcal{A}_n = \mathcal{A}_{n-1}$ we are done. Thus there is a source x of Q_n that the the objects of $\mathcal{A}_n$ are obtained from the objects of $\mathcal{A}_{n-1}$ by adding x. We decompose $P_x = \oplus_{i=1}^{n} U_i$ into indecomposable direct summands U_i.

On (1): For each object z of $\mathcal{A}$ which does not belong to $\mathcal{A}_{n-1}$ such that $\mathrm{rad}\, P_z$ is not in $\mathrm{mod}-\mathcal{A}_{n-1}$ there exists a proper predecessor y in $Q_{\mathcal{A}}$ not belonging to $\mathcal{A}_{n-1}$. Therefore there is an arrow $\sigma_{y,z}$ in Q_n showing that z is not a source. Consequently, $\operatorname{rad} P_x$ is a $\mathcal{A}_{n-1}$-module and all proper predecessors of x in $Q_{\mathcal{A}}$ are in $\mathcal{A}_{n-1}$ yielding that $\mathcal{A}_n$ is initial as well.

On (2): We want to apply Theorem 3.4 and have to check the conditions (a), (b), (c).

The absence of an arrow $\infty \to x$ shows that all summands U_i lie in preprojective components which is (a). The failure of (b) would mean that there is a non-sectional path from a summand U_i to a summand U_j yielding a loop in x. For (c) we assume the existence of a non-sectional path from some U_j to P_z where z is in $\mathcal{A}_{n-1}$. Then there is also an indecomposable summand V of $\operatorname{rad} P_z$ such that there is a non-sectional path from U_j to V. By the inductive construction of $\mathcal{A}_{n-1}$ there has to be $t < n$ such that $\mathcal{A}_t$ is the one-point extension of $\mathcal{A}_{t-1}$ by z. By induction we know that property (c) of the lemma holds for $\mathcal{A}_t$. Hence, there is a non-sectional path from U_j to V in the preprojective components of the Auslander-Reiten quiver of $\mathcal{A}_{t-1}$. In particular, U_j is an $\mathcal{A}_{t-1}$-module and we obtain an arrow $x \to z$ in Q_t. This is a contradiction to z being chosen as source of Q_t.

On (3): It suffices to show that each right minimal and right almost split morphism $g : Y \to X$ in $\operatorname{mod}-\mathcal{A}_n$ where X occurs as predecessor of P_x in $\Gamma_{\mathcal{A}_n}$ is actually right minimal and right almost split in $\operatorname{mod}-\mathcal{A}$. For X projective this is obvious. For X not projective it is enough to prove that $\tau_{\mathcal{A}}X = \tau_{\mathcal{A}_n}X$. If we assume that this equality does not hold, then the formula $\tau_{\mathcal{A}}X = R\tau_{\mathcal{A}_n}X$ shows the existence of an object y in $\mathcal{A}$ but not in $\mathcal{A}_n$ satisfying $\operatorname{Hom}_{\mathcal{A}_n}(\operatorname{res}_{\mathcal{A}_n} P_y, \tau_{\mathcal{A}_n}X) \neq 0$. We obtain the existence of an indecomposable direct summand U of $\operatorname{rad} P_y$ and an indecomposable direct summand U' of the restriction $\operatorname{res}_{\mathcal{A}_n} U$ such that there is an non-sectional path from U' to some U_i. By Lemma 4.3 there is another object z not belonging to $\mathcal{A}_n$ which has an indecomposable summand V' of $\operatorname{rad} P_z$ being an $\mathcal{A}_n$-module and a predecessor of U' in $\operatorname{ind}-\mathcal{A}_n$. Altogether we found a non-sectional path from V' to U_i in the preprojective component of $\mathcal{A}_{n-1}$. Thus z is a predecessor of x in Q_n which contradicts the fact that x is a source. $\square$

Remark. *Part (b) of the lemma shows that in the algorithm case 1 never occurs and part (c) implies that the predecessors of P_x in $\operatorname{ind}-\mathcal{A}$ are precisely the predecessors of P_x in the preprojective components of $\Gamma_{\mathcal{A}_n}$.*

Since $\mathcal{A}^\star = \mathcal{A}_{n^\star}$ the lemma immediately yields our first main result.

Theorem. *Suppose $\mathcal{A}$ is a finite spectroid with directed quiver. Then the following assertions hold:*

(1) *$\mathcal{A}^\star$ is an initial subspectroid of $\mathcal{A}$.*
(2) *$\mathcal{A}^\star$ is completely preprojective.*
(3) *For all x in $\mathcal{A}^\star$ the predecessors of P_x in the preprojective components of $\Gamma_{\mathcal{A}^\star}$ are precisely the predecessors of P_x in $\operatorname{ind}-\mathcal{A}$.*

5. Properties of $\mathcal{A}^\star$

5.1. We continue to study finite spectroids $\mathcal{A}$ whose quivers are directed. In the previous section we introduced an algorithm producing an initial subspectroid $\mathcal{A}^*$ which is completely preprojective. The construction of $\mathcal{A}$ depends on the choices of sources in the quivers Q_n. Consequently, it seems possible that for different choices of sources the algorithm yields different subspectroids $\mathcal{A}^\star$. Our first aim in this section is to show that this is not the case.

We define X in ind$-\mathcal{A}$ to be *predecessor bounded* if there is a bound for the length n of paths $X_0 \to X_1 \to \cdots \to X_n = X$ in ind$-\mathcal{A}$ ending in X. Otherwise X is called *predecessor unbounded.* Clearly, X is predecessor bounded if it lies in a preprojective component.

The following Proposition follows immediately from the work of Liu (see [Li]).

Proposition. *Let $\mathcal{A}$ be a finite spectroid. If Ω is a component of $\Gamma_{\mathcal{A}}$ not containing any projective vertex, than each module X in Ω is predecessor unbounded.*

Proof. The component Ω is semiregular in the notation of [Li]. If Ω does not contain an oriented cycle, then it is even a left stable component with this property and by [Li, Theorem 3.3] for each vertex X of Ω all the modules $\tau_{\mathcal{A}}^m X$ with $m \geq 0$ are pairwise different. In particular, X is predecessor unbounded.

In the case that Ω contains an oriented cycle, by [Li, Theorem 3.6] it is a stable or coextended tube as introduced in [Ri]. It is easy to see that each object X of Ω is predecessor unbounded. □

Let us return to our algorithm.

Corollary. *If $\sigma_{\infty,x}$ is an arrow in Q_n with $x \neq \infty$, then P_x is predecessor unbounded in* ind$-\mathcal{A}$.

Proof. The existence of the arrow $\sigma_{\infty,x}$ means that there is a radical summand U_i of P_x which does not belong to any preprojective component of $\mathcal{A}_{n-1}$. On the other hand these preprojective components contain all indecomposable projective $\mathcal{A}_{n-1}$-modules. Therefore the component containing U_i satisfies the assumptions of Proposition 5.1. Thus U_i is predecessor unbounded in ind$-\mathcal{A}_{n-1}$. Since P_x is a successor of U_i in ind$-\mathcal{A}$ and $\mathcal{A}_{n-1}$ is an initial subspectroid of $\mathcal{A}$, we obtain that P_x is predecessor unbounded in ind$-\mathcal{A}$. □

5.2. In an analogous fashion as for ind$-\mathcal{A}$ we say that a vertex x of a quiver Q is *predecessor bounded* provided there is a bound for the length n of paths $x_0 \to x_1 \to \cdots \to x_n = x$ in Q ending in x.

Lemma. *If $x \neq \infty$ is a predecessor unbounded vertex of Q_n, then P_x is predecessor unbounded in* ind$-\mathcal{A}$.

Proof. We start out by proving the following claim: If $\alpha : x \to y$ is an arrow in Q_n with $x, y \neq \infty$, then P_x is a predecessor of P_y in ind$-\mathcal{A}$.

The claim is obvious, if $\alpha = \sigma_{x,y}$ comes from a path in $Q_{\mathcal{A}}$. Otherwise $\alpha = \sigma'_{x,y}$ such that there is a non-sectional path $U = Z_0 \to Z_1 \to \cdots \to Z_m = V$ from an indecomposable direct summand U of $\operatorname{rad} P_x$ to an indecomposable direct summand V of $\operatorname{rad} P_y$ in a preprojective component of $\mathcal{A}_{n-1}$. We choose a minimal situation such that $Z_t = \tau_{\mathcal{A}_{n-1}} Z_{t+2}$ which implies that there is no non-sectional path from U to Z_t. By [HR, Proposition 2] we get $\operatorname{Hom}_{\mathcal{A}_{n-1}}(U, Z_t) \neq 0$ and thus $0 \neq \operatorname{Hom}_{\mathcal{A}_{n-1}}(\operatorname{res}_{\mathcal{A}_{n-1}} P_x, Z_t) = RZ_t(x) = \tau_{\mathcal{A}} Z_{t+2}(x)$. The Yoneda lemma yields a path from P_x to P_y in ind$-\mathcal{A}$.

Now we proceed with the proof of the lemma. That x is predecessor unbounded shows that for each integer $m \geq 0$ there is a path $x_0 \to x_1 \to \cdots \to x_m = x$ in Q_n. If for some m the vertex ∞ appears as x_t and without generality we assume that $x_l \neq \infty$ for all $l > t$, then by Corollary 5.1 and the claim P_x is predecessor unbounded in ind$-\mathcal{A}$. $\{x_1, \ldots, x_r\}$ In the case that all the paths can be formed without using ∞ because of the finiteness of $\mathcal{A}$ we find a path such that there exists $t > 0$ with $x_0 = x_t$. Using the claim this gives rise to a cycle of indecomposable projective modules starting and ending in P_{x_0}. Again by the claim, this makes P_x predecessor unbounded in ind$-\mathcal{A}$. $\square$

Theorem. *Let $\mathcal{A}$ be a finite spectroid with directed quiver. The set of objects of $\mathcal{A}^\star$ consists precisely of those objects of $\mathcal{A}$ such that P_x is predecessor bounded.*

Proof. Since $\mathcal{A}^\star$ is completely preprojective, P_x is obviously predecessor bounded in ind$-\mathcal{A}^\star$ for each x in $\mathcal{A}^\star$. Part (c) of Theorem 4.4 shows that this is also true in ind$-\mathcal{A}$. Conversely, if x is not in $\mathcal{A}^\star = \mathcal{A}_{n^\star}$, then by possibly passing to an appropriate predecessor in $Q_{\mathcal{A}}$ we may suppose that x belongs to $Q_{n^\star}$. The assumption that x is predecessor bounded implies that the full subquiver of $Q_{n^\star}$ given by the predecessors of x is directed and consequently has a source which is a source of $Q_{n^\star}$ as well, a contradiction. Thus x is a predecesoor unbounded vertex of $Q_{n^\star}$. Thus P_x is predecessor unbounded in ind$-\mathcal{A}$ by the Lemma. $\square$

Corollary. *$\mathcal{A}$ is completely preprojective if and only if $\mathcal{A} = \mathcal{A}^\star$.*

The characterisation of $\mathcal{A}^\star$ given in the theorem could be used to define $\mathcal{A}^\star$ abstractly. One is tempted to say that $\mathcal{A}^\star$ is something like a maximal initial completely preprojective subspectroid. We present an example which shows that this is literally incorrect and which illustrates the introduced notions. Let $\mathcal{A}$ be given by the quiver from Figure 3 subjected to the relations $\beta\alpha$ and $\beta'\alpha'$.

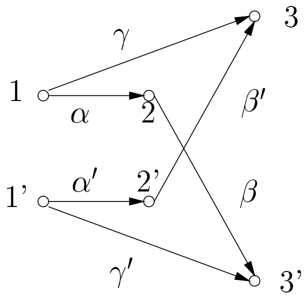

FIGURE 3

It turns out that $\mathcal{A}^\star = \mathcal{A}_4$ is the full subspectroid supported by $\{1, 1', 2, 2'\}$. The quiver Q_5 is shown in Figure 4. The one-point extension of $\mathcal{A}_\star$ by 3 is still completely preprojective and an initial subspectroid of $\mathcal{A}$.

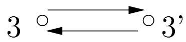

FIGURE 4

5.3. The following theorem summarizes our findings.

Theorem. *Let $\mathcal{A}$ be a finite spectroid with directed quiver.*

(a) *If Θ is a preprojective component of $\Gamma_{\mathcal{A}^\star}$ such that for each x in $\mathcal{A} \setminus \mathcal{A}^\star$ no indecomposable direct summand of* rad P_x *belongs to Θ, then Θ is a preprojective component of $\Gamma_{\mathcal{A}}$.*

(b) *If Θ is a preprojective component of $\Gamma_{\mathcal{A}}$, then Θ is a preprojective component of $\Gamma_{\mathcal{A}^\star}$ such that for each x in $\mathcal{A} \setminus \mathcal{A}^\star$ no indecomposable direct summand of* rad P_x *belongs to Θ.*

Proof. By Lemma 4.3 we know that Θ does not contain any indecomposable direct summand of $\operatorname{res}_{\mathcal{A}^\star} P_x$ for all x not in $\mathcal{A}^\star$. Thus (a) follows from Lemma 3.2.

In order to see (b) we observe that all projectives P_z in Θ are predecessor bounded. Hence the initial subspectroid induced by all z such that P_z belongs to Θ is contained in $\mathcal{A}^\star$. Consequently, Θ is a preprojective component of $\Gamma_{\mathcal{A}^\star}$. Assume there exists an indecomposable summand U of $\operatorname{rad} P_x$ with x not in $\mathcal{A}^\star$ such that U appears in Θ. Because $RU \neq U$, the module U cannot be an injective $\mathcal{A}^\star$-module and is of the shape $\tau_{\mathcal{A}^\star} V$ for some non-projective V in Θ. But this implies $\tau_{\mathcal{A}} V = R\tau_{\mathcal{A}^\star} V \neq \tau_{\mathcal{A}^\star} V$ which is a contradiction. □

5.4. A finite spectroid $\mathcal{A}$ of global dimension at most 1 is called *hereditary.* By [Ga1] it is of the form $\mathcal{A} = kQ$ where Q is a finite directed quiver. It is well-known (see e.g. [Ri]) that hereditary spectroids are completely preprojective. We want to use our setting to reprove this classical result.

An enumeration $\{x_1, \dots, x_r\}$ of the set of objects of a completely preprojective spectroid $\mathcal{A}$ is called a *construction* if $j \geq i$ provided that x_i is a predecessor of x_j in $Q_{\mathcal{A}}$. Equivalently, this means that $\mathcal{A}$ can be built up by one-point extensions along this enumeration. Our algorithm from section 4 produces special constructions which we call *strong constructions.* Example 4.1 shows that in general not all constructions are strong.

Theorem. *If $\mathcal{A}$ is a hereditary finite spectroid, then every construction of $\mathcal{A}$ is strong. In particular, $\mathcal{A}$ is completely preprojective.*

Proof. We want to show that given a construction $\{x_1, \dots, x_r\}$ for all $n = 0, \dots, r$ the subspectroid $\mathcal{A}_n$ may be chosen as the full subspectroid of $\mathcal{A}$ induced by $\{x_1, \dots, x_n\}$. For this purpose by induction it is sufficient to prove that x_n is always a source of Q_n. The heredity of $\mathcal{A}$ shows that all summands of $\operatorname{rad} P_{x_n}$ are projective and predecessors of projective modules are projective. Therefore arrows of type $\sigma_{\infty,x}$ and $\sigma'_{x,y}$ for $x, y \neq \infty$ do not exist. □

Remark. *Our algorithm also works for the much more general class of artin algebras (see* [ARS]*) if one is able to calculate the dimension vectors of the indecomposable direct summands of the radicals of the indecomposable projective modules.*

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