Schumacher's Quantum Coding Revisited

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Abstract

We review the different models of noiseless coding of a source emitting quantum states: quantum encoding vs. arbitrary (unphysical) encoding, fidelity and trace norm difference vs. entanglement fidelity. We show the coding theorem in the most restrictive of these models, and prove the strong converse in the most powerful model.

Keywords

Quantum source, entanglement, fidelity, noiseless coding, strong converse.

I. Introduction

The subject of quantum information is very young, in fact one of its major roots is the paper [10] (which appeared only after its successor [8]). It is concerned with the following basic problem of information theory:

Consider a source emitting pure quantum states π on a fixed complex Hilbert space $\mathcal{H}$, say for definiteness from a finite set P, with probabilities $P(\pi)$. Suppose that we get the states $\pi_1, \ldots, \pi_n$ from n independent emissions, and we want to encode their joint state $\pi^n = \pi_1 \otimes \cdots \otimes \pi_n$ using as little resources as possible and such that the original state can be recovered from the encoding within a good approximation. The resource is to be basically the dimension of the Hilbert space of the encodings. As Schumacher [10] observed this gives an operational definition of the von Neumann entropy of the ensemble.

In the following section we will precisely define the model(s), and discuss their relation, in section IV we will prove the noiseless coding theorem for the weakest model: quantum encoding together with entanglement fidelity. In section V we will first review the different proofs for the (weak) converse which appeared so far, and prove the strong converse in the most powerful model: arbitrary encoding together with (ordinary) fidelity.

II. The different models

Fix the complex Hilbert space $\mathcal{H}$, $d = \dim \mathcal{H} < \infty$. $\mathfrak{L}(\mathcal{H})$ denotes the algebra of (bounded) linear operators of $\mathcal{H}$, $\mathfrak{L}(\mathcal{H})_*$ (understood as the predual under the trace pairing).

A *(discrete memoryless) quantum source* (q–DMS) is a pair (P, P) with a finite set $\mathsf{P} \subset \mathfrak{L}(\mathcal{H})_*$ of states on $\mathfrak{L}(\mathcal{H})$ and a p.d. P on P. The average state of the source is $P\mathsf{P} = \sum_{\pi \in \mathsf{P}} P(\pi)\pi$.

An *n–block code* for the q–DMS (P, P) is a pair $(\varepsilon_*, \delta_*)$ where $\varepsilon_* : \mathsf{P}^n \to \mathfrak{L}(\mathcal{H})_*$ maps P^n into the states on $\mathfrak{L}(\mathcal{K})$ (with some Hilbert space $\mathcal{K}$), and $\delta_* : \mathfrak{L}(\mathcal{K})_* \to \mathfrak{L}(\mathcal{H})_*^{\otimes n}$ is trace preserving and completely positive (i.e. it is a physical state transformation, see [3]).

We say that $(\varepsilon_*, \delta_*)$ is *quantum encoding* if ε_* is the restriction to P^n of a trace preserving and completely positive map $\varepsilon_* : \mathfrak{L}(\mathcal{H})_*^{\otimes n} \to \mathfrak{L}(\mathcal{K})$. If there is no condition on ε_* we say that $(\varepsilon_*, \delta_*)$ is *arbitrary encoding.*

For an n–block code $(\varepsilon_*, \delta_*)$ define

1. the *(average) fidelity*

$$\bar{F} = \bar{F}(\varepsilon_*, \delta_*) = \sum_{\pi^n \in \mathsf{P}^n} P^n(\pi^n) \cdot \mathrm{Tr}\,((\delta_* \varepsilon_* \pi^n)\pi^n)$$

2. the *(average) distortion*

$$\bar{D} = \bar{D}(\varepsilon_*, \delta_*) = \sum_{\pi^n \in \mathsf{P}^n} P^n(\pi^n) \cdot \frac{1}{2} \|\delta_* \varepsilon_* \pi^n - \pi^n\|_1$$

3. the *entanglement fidelity* (see [11])

$$F_e = F_e(\varepsilon_*, \delta_*) = \mathrm{Tr}\,\left(((\delta_* \varepsilon_* \otimes \mathrm{id})\Psi_{P\mathsf{P}}^{\otimes n})\Psi_{P\mathsf{P}}^{\otimes n}\right)$$

where $\Psi_{P\mathsf{P}}$ is a purification of $P\mathsf{P}$, i.e. it is a pure state on an extended system (by tensor product with some space $\mathcal{H}_0$), and $P\mathsf{P} = \mathrm{Tr}_{\mathcal{H}_0} \Psi_{P\mathsf{P}}$ (cf. [11] where it is proved that F_e does not depend on the purification chosen). Note that this makes sense only if $(\varepsilon_*, \delta_*)$ is quantum encoding.

Theorem 1:

$$\bar{D}^2 \leq 1 - \bar{F} \leq \bar{D} \text{ and } 1 - \bar{F} \leq 1 - F_e$$

Proof: The first double inequality is in [13] (even for lossy processes), the other is in [11]. ■

The n–block code $(\varepsilon_*, \delta_*)$ is called an $(n, \lambda)_{\bar{F}}$*–code* if $1 - \bar{F}(\varepsilon_*, \delta_*) \leq \lambda$. Similarly for F_e. The n–block code $(\varepsilon_*, \delta_*)$ is called an $(n, \lambda)_{\bar{D}}$*–code* if $\bar{D}(\varepsilon_*, \delta_*) \leq \lambda$.

The *rate* of an n–block code $(\varepsilon_*, \delta_*)$ is defined as $R(\varepsilon_*, \delta_*) = \frac{1}{n} \log \dim \mathcal{K}$.

From the previous theorem it is clear that the most restrictive model is where we have to find an $(n, \lambda)_{F_e}$–code with quantum encoding, whereas the most powerful model is where we have to find an $(n, \lambda)_{\bar{F}}$–code with arbitrary encoding (equivalently we may use $\bar{D}$). Now we define for a q–DMS (P, P)

1. the λ–(quantum,F_e)*–rate* as

$$R_{q,F_e}(\lambda) = \limsup_{n\to\infty} \min\{R(\varepsilon_*, \delta_*) : \ (\varepsilon_*, \delta_*) \text{ an } \\ (n, \lambda)_{F_e}\text{–code with qu. encoding}\}$$

2. the λ–(quantum,$\bar{F}$)*–rate* as

$$R_{q,\bar{F}}(\lambda) = \limsup_{n\to\infty} \min\{R(\varepsilon_*, \delta_*) : \ (\varepsilon_*, \delta_*) \text{ an } \\ (n, \lambda)_{\bar{F}}\text{–code with qu. encoding}\}$$

3. the λ–(arbitrary,$\bar{F}$)–*rate* as

$$R_{a,\bar{F}}(\lambda) = \limsup_{n\to\infty} \min\{R(\varepsilon_*, \delta_*) : \ (\varepsilon_*, \delta_*) \text{ an } (n,\lambda)_{\bar{F}}\text{–code with arb. encoding}\}$$

Despite our lot of definitions the situation turns out to be quite simple:

Theorem 2: For all $\lambda \in (0,1)$ the three λ–rates of the q–DMS (P, P) are equal to the von Neumann entropy of the ensemble (P, P):

$$R_{q,F_e}(\lambda) = R_{q,\bar{F}}(\lambda) = R_{a,\bar{F}}(\lambda) = H(P\mathsf{P})$$

where $H(\rho) = -\mathrm{Tr}\,(\rho \log \rho)$ is the von Neumann entropy, see [9].

Proof: Between the first two members of the chain we have "$\geq$" by theorem 1, between the second and the third "$\geq$" is obvious. $R_{q,F_e}(\lambda) \leq H(P\mathsf{P})$ follows from the noiseless coding theorem IV.5. Finally $R_{a,\bar{F}}(\lambda) \geq H(P\mathsf{P})$ follows from the strong converse theorem V.8 ■

On the history of this theorem see the remarks in the sections IV and V.

III. Shadows

In this section we introduce some combinatorial concepts useful in the later discussion.

Construct *variance–typical projectors* $\Pi^n_{V,\rho,\alpha}$ using typical (or *generated*) sequences in the sense of Wolfowitz [15]: for a diagonalization $\rho = \sum_j q_j \pi_j$ let $s_j = \sqrt{q_j(1-q_j)}$ and

$$\mathcal{T}^n_{V,\rho,\alpha} = \{(j_1, \dots, j_n) : \forall j \ |N(j|j^n) - nq_j| \leq s_j \alpha \sqrt{n}\}$$

with $N(j|j^n) = |\{i : j = j_i\}|$, and define

$$\Pi^n_{V,\rho,\alpha} = \sum_{(j_1,\dots,j_n)\in\mathcal{T}^n_{V,\rho,\alpha}} \pi_{j_1} \otimes \cdots \otimes \pi_{j_n}$$

For a state ρ define $\mu(\rho)$ as the minimal nonzero eigenvalue of $\sqrt{\rho(\mathbb{1}-\rho)}$, and $N(\rho) = \dim \mathrm{supp}\, \sqrt{\rho(\mathbb{1}-\rho)}$. Finally define $K = 2^{\frac{\log e}{e}}$.

Then one has (cf. [15])

Lemma 3: For every state ρ and $n > 0$

$$\mathrm{Tr}\,(\rho^{\otimes n} \Pi^n_{V,\rho,\alpha}) \geq 1 - 2N(\rho) e^{-2\mu(\rho)^2\alpha^2}$$

and with $\Pi^n = \Pi^n_{V,\rho,\alpha}$

$$\Pi^n \exp\left(-nH(\rho) - Kd\alpha\sqrt{n}\right) \leq \Pi^n \rho^{\otimes n} \Pi^n \leq \leq \Pi^n \exp\left(-nH(\rho) + Kd\alpha\sqrt{n}\right)$$

$$\operatorname{Tr}\Pi^n_{V,\rho,\alpha} \leq \exp\left(nH(\rho) + Kd\alpha\sqrt{n}\right)$$

Every η–*shadow* B of $\rho^{\otimes n}$ (this means $0 \leq B \leq \mathbb{1}$ and $\operatorname{Tr}(\rho^{\otimes n}B) \geq \eta$) satifies

$$\operatorname{Tr} B \geq \left(\eta - 2N(\rho)e^{-2\mu(\rho)^2\alpha^2}\right)\exp\left(nH(\rho) - Kd\alpha\sqrt{n}\right)$$

Proof: The first estimate is just the well known Hoeffding inequality [4]. The second estimate is the key: to prove it let $\pi^n = \pi_{j_1} \otimes \cdots \otimes \pi_{j_n}$ one of the eigenprojections of $\rho^{\otimes n}$ which contributes to $\Pi^n_{V,\rho,\alpha}$. Then

$$\operatorname{Tr}(\rho^{\otimes n}\pi^n) = q_{j_1}\cdots q_{j_n} = \prod_j q_j^{N(j|j^n)}$$

Taking logs and using the defining relation for the $N(j|j^n)$ we find

$$\begin{aligned}
\left|\sum_j -N(j|j^n)\log q_j - nH(\rho)\right| &\leq \sum_j -\log q_j |N(j|j^n) - nq_j| \\
&\leq \sum_j -\alpha\sqrt{n}\sqrt{q_j}\log q_j \\
&= 2\alpha\sqrt{n}\sum_j -\sqrt{q_j}\log\sqrt{q_j} \\
&\leq 2d\frac{\log e}{e}\alpha\sqrt{n}
\end{aligned}$$

Now the bound on $\operatorname{Tr}\Pi^n_{V,\rho,\alpha}$ is immediate since $\operatorname{Tr}\Pi^n\rho^{\otimes n}\Pi^n \leq 1$. Finally let B an η–shadow of $\rho^{\otimes n}$. Then

$$\begin{aligned}
\operatorname{Tr}\left(\Pi^n\rho^{\otimes n}\Pi^n B\right) &= \operatorname{Tr}(\rho^{\otimes n}B) - \\
&\qquad - \operatorname{Tr}\left((\rho^{\otimes n} - \Pi^n\rho^{\otimes n}\Pi^n)B\right) \\
&\geq \eta - 2N(\rho)e^{-2\mu(\rho)^2\alpha^2}
\end{aligned}$$

From the upper bound on $\Pi^n\rho^{\otimes n}\Pi^n$ the last inequality of the lemma follows. ∎

IV. Noiseless coding

Let $\alpha > 0$. The Schumacher *scheme with constant* α for the q–DMS (P, P) is the following family of n–block codes $(\varepsilon_*, \delta_*)$ with quantum encoding: define $\Pi^n = \Pi^n_{V,P\mathsf{P},\alpha}$ and the Hilbert space $\mathcal{K} = \operatorname{supp}\Pi^n$, and

$$\begin{aligned}
\varepsilon_* : \mathfrak{L}(\mathcal{H})_*^{\otimes n} &\longrightarrow \mathfrak{L}(\mathcal{K})_* \\
\sigma &\longmapsto \Pi^n\sigma\Pi^n + \frac{1 - \operatorname{Tr}(\sigma\Pi^n)}{\dim\mathcal{K}}\mathbb{1} \\
\delta_* : \mathfrak{L}(\mathcal{K})_* &\longrightarrow \mathfrak{L}(\mathcal{H})_*^{\otimes n} \\
\sigma &\longmapsto \sigma
\end{aligned}$$

Remark 4: Essentially the Schumacher scheme was first defined in [10], with a slightly different definition of Π^n. The great contribution of [10] was to notice the possibility and importance of having a *typical subspace*, and the following theorem is just a variation of the original argument. Subsequently there appeared minor modifications and refinements (see [8] and [7]), but all rely on one or another notion of typical subspace of $\mathcal{H}^{\otimes n}$.

Theorem 5: The Schumacher scheme has rate

$$R(\varepsilon_*, \delta_*) \leq H(P\mathsf{P}) + \frac{Kd\alpha}{\sqrt{n}}$$

and entanglement fidelity

$$F_e(\varepsilon_*, \delta_*) \geq 1 - 4N(P\mathsf{P})e^{-2\mu(P\mathsf{P})^2\alpha^2}$$

Proof: The rate estimate is immediate from the shadow lemma III.3. For the fidelity consider a purification of $P\mathsf{P} = \sum_j q_j |\varphi_j\rangle\langle\varphi_j|$, e.g. the projector of $|\psi\rangle = \sum_j \sqrt{q_j}|\varphi_j\rangle\otimes|\varphi_j\rangle$ on $\mathfrak{L}(\mathcal{H}^{\otimes 2})$. With that the fidelity estimate follows easily from the shadow lemma III.3. ■

By slightly changing the definition of the subspace used we arrive at the JHHH–*scheme* [7]: just take for Π^n the projector

$$\Pi^n_{H(\cdot)\leq R} = \text{l. c. supp}_{H(\nu)\leq R} \Pi^n_{V,\nu,0}$$

(the least common support) with some rate $R \geq 0$. Then in [7] it is proved that this gives universally good compression of all sources (P, P) with $H(P\mathsf{P}) < R$. For one thing (see [7])

$$\operatorname{Tr} \Pi^n_{H(\cdot)\leq R} \leq (n+1)^{d^2+d} \exp(nR)$$

and for the fidelity one has

Theorem 6: Let $(\varepsilon_*, \delta_*)$ the JHHH–scheme with rate R and block length n. Then for every q–DMS (P, P)

$$\bar{F}(\varepsilon_*, \delta_*) \geq 1 - 2(n+1)^d \exp\left(-n \cdot \min_{H(\nu)\geq R} D(\nu\|P\mathsf{P})\right)$$

Proof: First note that by direct calculation for ν codiagonal with a state ρ we have

$$\Pi^n_{V,\nu,0}\rho^{\otimes n}\Pi^n_{V,\nu,0} = \Pi^n_{V,\nu,0} \exp(-nD(\nu\|\rho) - nH(\nu))$$

(see also [14]). Fix a diagonalization $P\mathsf{P} = \sum_j q_j \pi_j$ and observe

$$\Pi^n_{H(\cdot)\leq R} \geq \sum_{\nu\in\mathbb{C}[\pi_1,\ldots,\pi_d], H(\nu)\leq R} \Pi^n_{V,\nu,0}$$

Using the simple fact that $\Pi_{V,\nu,0} \neq 0$ only if $\nu \in \frac{1}{n}\mathbb{N}[\pi_j|j]$ and $\operatorname{Tr}\Pi^n_{V,\nu,0} \leq \exp(nH(\nu))$ we find as in the previous theorem

$$\begin{aligned} 1-\bar{F}(\varepsilon_*,\delta_*) &\leq 2 \sum_{\nu\in\frac{1}{n}\mathbb{N}[\pi_j|j],H(\nu)>R} \exp(-nD(\nu\|P\mathsf{P}) \\ &\leq 2(n+1)^d \exp\left(-n\cdot\min_{H(\nu)\geq R} D(\nu\|P\mathsf{P})\right) \end{aligned}$$

where the last estimate is by *type counting*: there are at most $(n+1)^d$ different ν diagonal in the basis $\{\pi_j|j\}$ and $\Pi^n_{V,\nu,0} \neq 0$. ■

V. Strong converse

The first proofs in [10], [8] for the optimality of the Schumacher scheme where valid only under the additional assumption that δ_* is of the form $\delta_*(\sigma) = U\sigma U^*$ for a unitary embedding U of $\mathcal{K}$ into $\mathcal{H}^{\otimes n}$. Also they achieved the bound $H(P\mathsf{P})$ only in the limit of $\lambda \to 0$ (so they proved a *weak converse*). The proof of [2] removed the restriction on δ_*, but still yields only a weak converse. Also it works with some surprising and difficult fidelity estimates, involving even mixed state fidelity, see [6] (We may note that they seem to be related to our inequalities of theorem II.1). We should also mention the work [1] where a weak converse was proved for quantum encodings and using entanglement fidelity. The criticism of the authors on [2] however is unjustified: by the discussion in section II their result is weaker than that of [2].

Then Horodecki [5] noticed that considering $\bar{D}$ instead of $\bar{F}$ drastically simplifies the proof. His argument is as follows:

Assume that we are given a code (ε_*,δ_*) with arbitrary encoding in the states on a k–dimensional Hilbert space and

$$\bar{D} = \frac{1}{2}\sum_{\pi^n\in\mathsf{P}^n} P^n(\pi^n)\|\pi^n - \delta_*\varepsilon_*\pi^n\|_1 \leq \lambda \leq \frac{1}{4}$$

So by Markov's inequality there is a subset $\mathcal{C} \subset \mathsf{P}^n$ with $P^n(\mathcal{C}) \geq 1-2\sqrt{\lambda}$ and

$$\forall\pi^n\in\mathcal{C}\ \|\pi^n - \delta_*\varepsilon_*\pi^n\|_1 \leq \sqrt{\lambda}$$

Now form the state $\sigma = \sum_{\pi^n\in\mathsf{P}^n} P^n(\pi^n)\varepsilon_*\pi^n$, then by Uhlmann's monotonocity of the quantum I–divergence [12]

$$\forall\pi^n\in\mathsf{P}^n\ D(\varepsilon_*\pi^n\|\sigma) \geq D(\delta_*\varepsilon_*\pi^n\|\delta_*\sigma)$$

Averaging we obtain

$$\sum_{\pi^n\in\mathsf{P}^n} P^n(\pi^n)D(\varepsilon_*\pi^n\|\sigma) \geq \sum_{\pi^n\in\mathsf{P}^n} P^n(\pi^n)D(\delta_*\varepsilon_*\pi^n\|\delta_*\sigma)$$

Now it is straightforward to calculate the l.h.s. of this to $H(\sigma) - \sum_{\pi^n \in \mathsf{P}^n} P^n(\pi^n) H(\varepsilon_* \pi^n)$, whereas the r.h.s evaluates similarly to $H(\delta_*\sigma) - \sum_{\pi^n \in \mathsf{P}^n} P^n(\pi^n) H(\delta_* \varepsilon_* \pi^n)$. Since $2\lambda, \sqrt{\lambda} \leq 1/2$ we can use a continuity property of H, cf. [9]: $\|\delta_*\sigma - (P\mathsf{P})^{\otimes n}\|_1 \leq 2\lambda$ implies

$$|H(\delta_*\sigma) - nH(P\mathsf{P})| \leq -2\lambda \log \frac{2\lambda}{d^n}$$

and (for $\pi^n \in \mathcal{C}$) $\|\delta_*\varepsilon_*\pi^n - \pi^n\|_1 \leq \sqrt{\lambda}$ implies

$$|H(\delta_*\varepsilon_*\pi^n) - H(\pi^n)| \leq -\sqrt{\lambda} \log \frac{\sqrt{\lambda}}{d^n}$$

Combining we get the chain of inequalities

$$\begin{aligned}
\log k \geq H(\sigma) &\geq H(\sigma) - \sum_{\pi^n \in \mathsf{P}^n} P^n(\pi^n) H(\varepsilon_*\pi^n) \\
&\geq H(\delta_*\sigma) - \sum_{\pi^n \in \mathsf{P}^n} P^n(\pi^n) H(\delta_*\varepsilon_*\pi^n) \\
&\geq nH(P\mathsf{P}) - \sum_{\pi^n \in \mathsf{P}^n} P^n(\pi^n) H(\pi^n) - \\
&\qquad - 2\sqrt{\lambda} \log d^n + 2\lambda \log \frac{2\lambda}{d^n} + \sqrt{\lambda} \log \frac{\sqrt{\lambda}}{d^n} \\
&= nH(P\mathsf{P}) - n(2\lambda + 3\sqrt{\lambda}) \log d + \\
&\qquad + 2\lambda \log 2\lambda + \sqrt{\lambda} \log \sqrt{\lambda}
\end{aligned}$$

Thus we proved

Theorem 7 (Weak converse) For every q–DMS (P, P)

$$\liminf_{\lambda \to 0} R_{a,\bar{D}}(\lambda) \geq H(P\mathsf{P})$$

■

In fact much more is true:

Theorem 8 (Strong converse) Let (P, P) a q–DMS and $(\varepsilon_*, \delta_*)$ an $(n, \lambda)_{\bar{F}}$–code with arbitrary encoding, and $\alpha > 0$. Then

$$\dim \mathcal{K} \geq \left(1 - \lambda - 4\sqrt{N(P\mathsf{P})} e^{-\mu(P\mathsf{P})^2 \alpha^2}\right) \cdot \\ \cdot \exp\left(nH(P\mathsf{P}) - Kd\alpha\sqrt{n}\right)$$

Proof: Let $B = \delta_*(\mathbb{1}_\mathcal{K})$ and $\Pi^n = \Pi^n_{V,P\mathsf{P},\alpha}$. Since $\varepsilon_*\pi^n \leq \mathbb{1}_\mathcal{K}$ for every $\pi^n \in \mathsf{P}^n$ it is clear that $\delta_* \varepsilon_* \pi^n \leq B$. Thus

$$\begin{aligned}
\mathrm{Tr}\ (B \cdot \Pi^n \pi^n \Pi^n) &\geq \mathrm{Tr}\ ((\delta_* \varepsilon_* \pi^n) \Pi^n \pi^n \Pi^n) \\
&= \mathrm{Tr}\ ((\delta_* \varepsilon_* \pi^n) \pi^n) \\
&\quad - \mathrm{Tr}\ (\delta_* \varepsilon_* \pi^n (\pi^n - \Pi^n \pi^n \Pi^n)) \\
&\geq \mathrm{Tr}\ ((\delta_* \varepsilon_* \pi^n) \pi^n) - \|\pi^n - \Pi^n \pi^n \Pi^n\|_1 \\
&= \mathrm{Tr}\ ((\delta_* \varepsilon_* \pi^n) \pi^n) - \sqrt{8(1 - \mathrm{Tr}\, \pi^n \Pi^n)}
\end{aligned}$$

(the last estimate by theorem II.1). Averaging over $P^{\otimes n}$ we find, with the shadow lemma III.3 and concavity of the square root:

$$\begin{aligned}
\mathrm{Tr}\ \big(\Pi^n (P\mathsf{P})^{\otimes n} \Pi^n B\big) &\geq \bar{F} - \sqrt{8\,(1 - \mathrm{Tr}\,(P\mathsf{P})^{\otimes n} \Pi^n)} \\
&\geq 1 - \lambda - 4\sqrt{N(P\mathsf{P})} e^{-\mu(P\mathsf{P})^2 \alpha^2}
\end{aligned}$$

Since by the shadow lemma III.3

$$\Pi^n (P\mathsf{P})^{\otimes n} \Pi^n \leq \Pi^n \exp\big(-nH(P\mathsf{P}) + K d\alpha\sqrt{n}\big)$$

we conclude

$$\begin{aligned}
\mathrm{Tr}\, B \geq \mathrm{Tr}\ (B\Pi^n) &\geq \Big(1 - \lambda - 4\sqrt{N(P\mathsf{P})} e^{-\mu(P\mathsf{P})^2 \alpha^2}\Big) \cdot \\
&\quad \cdot \exp\big(nH(P\mathsf{P}) - K d\alpha\sqrt{n}\big)
\end{aligned}$$

and with $\dim \mathcal{K} = \mathrm{Tr}\, \mathbb{1}_{\mathcal{K}} = \mathrm{Tr}\, B$ the proof is complete. ■

Corollary 9: Let $E_n = o(n)$ and $\lambda_n \leq 1 - e^{-E_n}$. Then for every sequence $(\varepsilon_{n*}, \delta_{n*})$ of $(n, \lambda_n)_{\bar{F}}$–codes with arbitrary encoding for the q–DMS (P, P)

$$\liminf_{n \to \infty} R(\varepsilon_{n*}, \delta_{n*}) \geq H(P\mathsf{P})$$

■

VI. Conclusion

We reviewed and unified the present knowledge in quantum source coding, and added the strong converse to Schumacher's quantum coding. Its proof is remarkable in that it employs a positive operator which is not necessarily bounded by $\mathbb{1}$ (this is why we could not directly apply the shadow lemma). Even though it has consequently no interpretation as a physical measurement, it can be analyzed to give information about the coding scheme.

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