APPROXIMATION OF HUNT PROCESSES BY MULTIVARIATE POISSON PROCESSES

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Abstract

We prove that arbitrary Hunt processes on a general state space can be approximated by multivariate Poisson processes starting from each point of the state space. The key point is that no additional regularity assumption on the state space and on the underlying transition semigroup is used.

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1. Introduction

In [MRZ98], it was proved that any Hunt process associated with a Dirichlet form on a general state space can be approximated by Markov chains (more precisely, by multivariate Poisson processes, see (2.1) below) whose construction is based on the Yoshida approximation of the generator and tightness arguments. Among other things, the above mentioned approximation scheme gave a new proof for the existence of Hunt processes for a large class of Dirichlet forms (i.e. strictly quasi-regular ones). But, as mentioned in the introduction of [MRZ98], the price for the new existence theorem is that one gets only the approximation of the path space measures P_x for quasi-every x in the state space.

This paper is a continuation of the discussion initiated in [MRZ98]. In fact, we shall prove that *any* Hunt process (associated to a Dirichlet form or not) can be approximated by multivariate Poisson processes and this approximation works for all P_x, i.e. *each* point x of the state space (see Theorem 2.1 below). This generalizes the approximations result in [MRE98] considerably. However, we do not recover the existence result of [MRE98] mentioned above.

We should mention that under some additional regularity assumptions on the state space (e.g. if it is a locally compact separable metric space) and on the underlying transition semigroup (e.g. if it is a Feller semigroup), the Yoshida approximation by Markov chains to the underlying process is well-known (cf. the beautiful exposition in [EK86]). However, because of the involved tightness argument, it is difficult (at least to our knowledge) to relax the regularity assumptions. The strategy we use in this paper is to change the topology of the state space, and then to implement the tightness argument in the new topological space. Note that although we change the topology during the proof, the limit process is always a Hunt process w.r.t. the original topology, as it was pointed out in [MRZ98, Theorem 4.4]. In this paper we shall eventually prove moreover that also the weak convergence takes place w.r.t. the original topology (see Section 5 below) which completes the results in [MRZ98].

2. Statement of the Main Result

Throughout this paper we assume that E is a metrizable Lusin space (cf. [DMe75. III. 16]). Let d be a compatible metric on E. We adjoin an extra point Δ (the cemetery) to E and write E_Δ for $E \cup \{\Delta\}$. If E is locally compact, then Δ can be considered either as an isolated point of E_Δ, or as a point "at infinity" of E_Δ with the topology of the one point compactification. We select one of the above two topologies and fix it. If E is not locally compact then we simply consider Δ as an isolated point of E_Δ. Let $(\Omega, \mathcal{F}, (Z_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ be a Hunt process with state space E (see [MR92, IV. 1.13] for the definition of Hunt processes on non-locally compact spaces). The transition semigroup and resolvent of kernels related to $(Z_t)_{t\geq 0}$ are defined in a standard way, i.e.

$$\begin{aligned} P_t f(x) :=& P_t(x,f) := E_x[f(Z_t)], \qquad \forall\, x \in E_\Delta, \quad t\geq 0, \quad f\in B_b(E_\Delta). \\ R_\alpha f(x) :=& R_\alpha(x,f) := E_x\Big[\int_0^\infty e^{-\alpha t} f(Z_t)\,dt\Big] \\ =& \int_0^\infty e^{-\alpha t} P_t f(x)\,dt, \qquad \forall\, \alpha>0, \quad x\in E_\Delta, \quad f\in B_b(E_\Delta). \end{aligned}$$

Here and henceforth $B_b(E_\Delta)$ denotes all the bounded Borel functions on E_Δ. For fixed $n \in \mathbb{N}$, let $\{Y^n(k),\ k=0,1,\cdots,\}$ be a Markov chain in E_Δ with some initial distribution ν and transition function nR_n, and let $(\Pi_t^n)_{t\geq 0}$ be a Poisson process with parameter n, i.e.

$$P\,(\Pi_t^n = k) = e^{-nt}\,\frac{(nt)^k}{k!}.$$

Assume that $(\Pi_t^n)_{t\geq 0}$ is independent of $\{Y^n(k),\ k=0,1,\cdots,\}$ and define

$$Z_t^n = Y^n(\Pi_t^n), \qquad \forall\, t\geq 0. \tag{2.1}$$

Then $(Z^n_t)_{t\ge0}$ is a strong Markov process in E_Δ. Following [J74] we call $(Z^n_t)_{t\ge0}$ multivariate Poisson process with intensity n and state space E_Δ. Define

$$P^n_t f := e^{-nt}\sum_{k=0}^{\infty}\frac{(nt)^k}{k!}(nR_n)^k f, \qquad \forall f\in B_b(E_\Delta). \tag{2.2}$$

It is well-known that $(P^n_t)_{t\ge0}$ is the transition semigroup of $(Z^n_t)_{t\ge0}$, i.e., for all $f\in B_b(E_\Delta)$, $t,s\ge0$, we have

$$E[f(Z^n_{t+s})\mid\sigma(Z^n_{s'},\ s'\le s)] = (P^n_t f)(Z^n_s).$$

(see [EK86, IV, 2]). Note that $(P^n_t)_{t\ge0}$ is a strongly continuous contraction semigroup on the Banach space $(B_b(E_\Delta),\ \|\cdot\|_{E_\Delta})(\|f\|_{E_\Delta} := \sup_{x\in E_\Delta}|f(x)|)$, and the corresponding generator is given by

$$L^n u(x) = n(nR_n u(x)-u(x)) = n\int_{E_\Delta}(u(y)-u(x))\,nR_n(x,dy) \tag{2.3}$$

for all $u\in B_b(E_\Delta)$.

Let $D_{E_\Delta}[0,\infty)$ be the space of all cadlag functions from $[0,\infty)$ to E_Δ, equipped with the Skorohod topology related to d (see [EK86, III]). Let P_x (resp. P^n_x) be the law of $(Z_t)_{t\ge0}$ (resp. $(Z^n_t)_{t\ge0}$) on $D_{E_\Delta}[0,\infty)$ with initial distribution δ_x for $x\in E_\Delta$.

Now we can state our main result.

Theorem 2.1. (cf. Theorem 5.3 below) For any $x\in E_\Delta$, P^n_x weakly converges to P_x on $D_{E_\Delta}[0,\infty)$ (with respect to the Skorohod topology related to d), when $n\to\infty$.

3. Construction of a Compact Metric Space $(\overline{E},\rho)$

Without loss of generality we may assume that $\Omega=\Omega_{E_\Delta} := D_{E_\Delta}[0,\infty)$ and $(Z_t)_{t\ge0}=(X_t)_{t\ge0}$ be the coordinate process on Ω_{E_Δ}. As usual we set $X_\infty := \Delta$. Let $(\mathcal{F}^n_t)_{t\ge0}$, $(\mathcal{F}_t)_{t\ge0}$ be the natural right continuous filtration of $(X_t)_{t\ge0}$ completed w.r.t. $(P^n_x)_{x\in E_\Delta}$, $(P_x)_{x\in E_\Delta}$ respectively. By a routine argument following [Ge75], one can check that $M_n := (\Omega_{E_\Delta},(X_t)_{t\ge0},\ (\mathcal{F}^n_t)_{t\ge0},(P^n_x)_{x\in E_\Delta})$

and $M := (\Omega_{E_\Delta},\ (X_t)_{t\geq 0},\ (\mathcal{F}_t)_{t\geq 0},\ (P_x)_{x\in E_\Delta})$ are Hunt processes. For a Borel subset $B \in \mathcal{B}(E_\Delta)$, we define

$$\sigma_B := \inf\{t \geq 0 \mid X_t \in B\},$$
$$\tau_B := \inf\{t \geq 0 \mid X_t \in B \text{ or } X_{t-} \in B\}.$$

(Here and henceforth we make the convention that $X_{0-} = X_0$). Note that if B is open, then σ_B coincides with τ_B.

From now on we fix $x_0 \in E_\Delta$. Following LeJan [L83], we define

$$\mathrm{Cap}_{x_0}(B) := E_{x_0}[e^{-\tau_B}], \qquad B \in \mathcal{B}(E_\Delta).$$

It has been shown in [L83] that Cap_{x_0} is a Choquet capacity on E_Δ. Moreover, for any $f \in \mathcal{B}_b(E_\Delta)$, $R_\alpha f$ is Cap_{x_0}-quasi-continuous, i.e., there exists an increasing sequence of closed set $(F_k)_{k\in\mathbb{N}}$ such that $\mathrm{Cap}_{x_0}(E_\Delta \backslash F_k) \downarrow 0$ and $R_\alpha f$ is continuous on F_k for each $k \in N$.

Remark. In [L83] it was assumed that E is a locally compact metrizable space. But the argument in [L83] goes through for a metrizable Lusin space E. See Section 5 in Chapter IV of [MR92].

Let Q_+, Q_+^* denote the non-negative, respectively strictly positive rational numbers. Let J_0 be a countable family of nonnegative bounded continuous functions on E_Δ which separates the points of E_Δ and contains the constant function 1. Let J_1 be the smallest family of excessive functions containing $\{R_\alpha u \mid u \in J_0,\ \alpha \in Q_+^*\}$ and having the properties that if $u, v \in J_1,\ \alpha, c_1, c_2 \in Q_+^*$, then $R_\alpha u,\ c_1 u + c_2 v,\ u \wedge v$ are all in J_1. Clearly, J_1 separates the points of E_Δ and contains the constant function 1.

Lemma 3.1. There exists an increasing sequence of compact subsets $\{E_k\}_{k\in\mathbb{N}}$ of E_Δ such that

(i) $E_{x_0}[e^{-\sigma_{E_\Delta \backslash F_k}}] \to 0$ as $k \to \infty$;

(ii) $u|_{E_k}$ is continuous for each $u \in J_1$ and $k \in \mathbb{N}$.

Proof. By [L83, Lemma 2] (see also the argument of [MR92, IV. Section 5]) each $u \in J_1$ is Cap_{x_0}-quasi-continuous. By [FOT94, Lemma 7.1.1], J_1 is a countable family. Therefore, it follows from e.g. [MR92, III. 3.3] that there exists a common increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of E_Δ such that $\mathrm{Cap}_{x_0}(E_\Delta\backslash F_k) \to 0$ and $u|_{E_k}$ is continuous for all $u \in J_1$ and $k \in \mathbb{N}$. On the other hand, by [MR92, Theorem IV 1.15] there exists an increasing sequence of compact subsets $(F'_k)_{k\in\mathbb{N}}$ of E_Δ such that $P_{x_0}[\lim_{k\to\infty} \sigma_{E_\Delta\backslash F'_k} < \infty] = 0$. Let $E_k = F_k \cap F'_k$. Then $(E_k)_{k\in\mathbb{N}}$ satisfies assertions (i) and (ii). ∎

Note that by Lemma 3.1 (i) we may assume that $x_0 \in E_k$ for all $k \in \mathbb{N}$. Since J_1 is countable. We may write $J_1 = \{u_j \mid j \in \mathbb{N}\}$. Set $Y := \bigcup_{k\geq 1} E_k$, where E_k is as specified in the above lemma. We define for $x, y \in Y$

$$\rho(x,y) = \sum_{j=1}^{\infty} \frac{1}{2^j}(|u_j(x) - u_j(y)| \wedge 1). \tag{3.1}$$

Let $\overline{E}$ be the completion of Y with respect to ρ. Then $\overline{E}$ is a compact metric space. Let $J_1 - J_1$ denote all the functions which are expressed by a difference of two functions in J_1. For $u \in J_1 - J_1$, we denote by $\overline{u}$ the ρ-continuous extension of $u|_Y$ to $\overline{E}$. Let J be the collection of all such $\overline{u}$. Then J is a linear lattice and contains 1. Hence J is ρ-uniformly dense in $C_b(\overline{E})$ (:=all the bounded ρ-continuous functions on $\overline{E}$).

Remark. The metric ρ induces a topology on E_Δ which is called the Ray-topology and has been intensively discussed in the literature (see [Ge75, S88]).

Note that the ρ-topology and the d-topology coincide on E_k for each k, hence by Lemma 3.1 $(Z_t)_{t\geq 0}$ with initial distribution δ_{x_0} can be regarded as a cadlag process with state space $\overline{E}$. Let $(Z^n_t)_{t\geq 0}$ be defined by (2.1). By making use of the following lemma, one can show that for each $n \in \mathbb{N}$, $(Z^n_t)_{t\geq 0}$ with initial distribution δ_{x_0} can also be regarded as a cadlag process with state space $\overline{E}$.

Lemma 3.2 (cf. [MRZ98, Lemma 3.8 and 3.9]) For $n \geq 2$, $U \subset E_\Delta$ open, and $x \in E_\Delta$, we have

$$E^n_x[e^{-2\sigma_U}] \leq E_x[e^{-\sigma_U}]. \tag{3.2}$$

Hence for each $n \in \mathbb{N}$

$$E^n_{x_0}[e^{-\sigma_{E_\Delta \setminus E_k}}] \to 0, \quad \text{as} \quad k \to \infty. \tag{3.3}$$

Proof. For any $x \in E_\Delta$ we denote $E_x[e^{-\sigma_U}]$ by $e(x)$. Since $(n-1)R_n e(x) \le e(x)$ for each $n \ge 2$, it follows that $((n-1)R_n)^k e(x) \le e(x)$ for all $k \in \mathbb{N}$. Hence

$$\begin{aligned} P^n_t e(x) =& e^{-nt} \sum_{k=0}^{\infty} \frac{(nt)^k}{k!} (nR_n)^k e(x) \\ =& e^{-nt} \sum_{k=0}^{\infty} \frac{(nt)^k}{k!} \Big(\frac{n}{n-1}\Big)^k ((n-1)R_n)^k e(x) \\ \le& e^{-nt} \sum_{k=0}^{\infty} \frac{\big(\frac{n^2}{n-1}\, t\big)^k}{k!} e(x) = e^{(1+\frac{1}{n-1})t} e(x). \end{aligned}$$

This gives

$$e^{-2t} P^n_t e(x) \le e(x), \qquad x \in E_\Delta.$$

But $\lim_{t\to 0} e^{-2t} P^n_t f(x) = f(x)$ holds for all $x \in E_\Delta$ and $f \in \mathcal{B}_b(E_\Delta)$. Hence $e(x)$ is 2-excessive related to (P^n_t) and consequently, we have

$$E^n_x[e^{-2\sigma_U} e(X_{\sigma_U})] \le e(x). \tag{3.4}$$

Since obviously for any $x \in E_\Delta$,

$$X_{\sigma_U} \in U \qquad P^n_x - \text{a.s. on } \{\sigma_U < \infty\},$$

we have that for any $x \in E_\Delta$,

$$e^{-2\sigma_U} \le e^{-2\sigma_U} e(X_{\sigma_U}), \qquad P^n_x - \text{a.s.}.$$

Therefore, (3.2) follows from (3.4). (3.3) is a direct consequence of Lemma 3.1 (i) and (3.2). ∎

Let $\widetilde{P}_{x_0}$ (resp. $\widetilde{P}^n_{x_0}$) be the law of (Z_t) (resp. (Z^n_t)) on $D_{\overline{E}}[0, \infty)$ which is equipped with the Skorohod topology related to ρ. The corresponding expectation will be denoted by $\widetilde{E}_{x_0}[\,\cdot\,]$ (resp. $\widetilde{E}^n_{x_0}[\,\cdot\,]$). The coordinate process on $D_{\overline{E}}[0, \infty)$ will be denoted again by $(X_t)_{t\ge 0}$.

Lemma 3.3. Let $f \in C_b(\overline{E})$. Then for any $T>0$ and $\varepsilon>0$ there exists a function $g \in J_1 - J_1$ such that

$$\widetilde{E}_{x_0}\Big[\sup_{t\le T} |\overline{R_1 g} - f|\,(X_t)\Big] \le \varepsilon \tag{3.5}$$

and

$$\widetilde{E}^n_{x_0}\Big[\sup_{t\le T} |\overline{R_1 g} - f|\,(X_t)\Big] \le \varepsilon \qquad \text{for all} \qquad n \in \mathbb{N}. \tag{3.6}$$

Proof. Since J is uniformly dense in $C(\overline{E})$. We can take a function $u \in J_1 - J_1$ such that $\|\overline{u} - f\|_{\overline{E}} \le \frac{\varepsilon}{3}$. By Lemma 3.1 we can find $k_0 \in \mathbb{N}$ such that

$$\widetilde{E}_{x_0}[e^{-\sigma_{\overline{E}\setminus E_{k_0}}}] = E_{x_0}[e^{-\sigma_{E_\Delta\setminus E_{k_0}}}] \le \frac{\varepsilon}{6} e^{-2(T+1)}(\|\overline{u}\|_{\overline{E}}+1)^{-1}.$$

By (3.2) for all $n \in \mathbb{N}$ we then have

$$\widetilde{E}^n_{x_0}[e^{-2\sigma_{\overline{E}\setminus E_{k_0}}}] = E^n_{x_0}[e^{-2\sigma_{E_\Delta\setminus E_{k_0}}}] \le \frac{\varepsilon}{6} e^{-2(T+1)}(\|\overline{u}\|_{\overline{E}}+1)^{-1}.$$

The above two inequalities imply that

$$\begin{aligned}
\widetilde{P}_{x_0}[\sigma_{\overline{E}\setminus E_{k_0}} \le T+1] &\le \frac{1}{6}\,\frac{\varepsilon}{\|\overline{u}\|_{\overline{E}}+1}\,e^{-(T+1)} \le \frac{1}{6}\,\frac{\varepsilon}{\|\overline{u}\|_{\overline{E}}+1},\\
\widetilde{P}^n_{x_0}[\sigma_{\overline{E}\setminus E_{k_0}} \le T+1] &\le \frac{1}{6}\,\frac{\varepsilon}{\|\overline{u}\|_{\overline{E}}+1}.
\end{aligned}$$

Since u is the difference of two excessive functions and E_{k_0} is compact, there exists $\alpha \in Q^*_+$ such that $\|\alpha R_\alpha u - u\|_{E_{k_0}} \le \frac{\varepsilon}{3}$. Then

$$\widetilde{E}_{x_0}\Big[\sup_{t\le T} |\overline{\alpha R_\alpha u} - f|\,(X_t)\Big] \le \frac{\varepsilon}{3} + \widetilde{E}_{x_0}\Big[\sup_{t\le T} |\overline{\alpha R_\alpha u} - \overline{u}|\,(X_t)\Big] \le \varepsilon.$$

Similarly, for $n \in \mathbb{N}$,

$$\widetilde{E}^n_{x_0}\Big[\sup_{t\le T} |\overline{\alpha R_\alpha u} - f|\,(X_t)\Big] \le \varepsilon.$$

The proof is completed by employing the resolvent equation to rewrite $\alpha R_\alpha u$ as $R_1 g$ for some $g \in J_1 - J_1$. ■

4. Convergence of the Finite Dimensional Distributions

Lemma 4.1 Let $f = R_1 u$ for some $u \in \mathcal{B}_b(E_\Delta)$. Then for any $T > 0$,

$$\sup_{t\le T}\sup_{x\in E_\Delta} |P_t^n f(x) - P_t f(x)| \longrightarrow 0, \qquad n\to\infty. \tag{4.1}$$

In particular, if $u \in J_1 - J_1$, then $(t,x) \to P_t f(x)$ is uniformly continuous on $[0,T]\times Y$ where $Y := \bigcup_{k\ge 1} E_k$ is equipped with ρ-topology.

Proof. We assume first that $f = R_\alpha R_1 u$ for some $\alpha \in Q_+^*$ and $u \in \mathcal{B}_b(E_\Delta)$. In this case we have (cf. the proof of [MRZ98, Theorem 4.3])

$$\sup_{t\le T}\sup_{x\in E_\Delta} |P_t^{n_1} f(x) - P_t^{n_2} f(x)| \le T\Big(\frac{1}{n_1}+\frac{1}{n_2}\Big)\,\|w\|_{E_\Delta} \tag{4.2}$$

with $w := R_1(\alpha R_\alpha u - u) - (\alpha R_\alpha u - u) \in \mathcal{B}_b(E_\Delta)$.

Indeed by (2.2) we have for $n_1, n_2 \in \mathbb{N}$

$$\begin{aligned} P_t^{n_1} f - P_t^{n_2} f &= \int_0^t \frac{d}{ds}(P_s^{n_1} P_{t-s}^{n_2} f)\, ds \\ &= \int_0^t P_s^{n_1} P_{t-s}^{n_2} (L^{n_1} - L^{n_2}) f\, ds, \end{aligned}$$

consequently

$$\sup_{t\le T}\sup_{x\in E_\Delta} |P_t^{n_1} f(x) - P_t^{n_2} f(x)| \le T \sup_{x\in E_\Delta} |L^{n_1} f(x) - L^{n_2} f(x)|. \tag{4.3}$$

Using the explicit formula (2.3) one can check that

$$(L^{n_1} - L^{n_2}) f = R_{n_1} w - R_{n_2} w$$

with w specified as above. Consequently (4.2) follows from (4.3).

Taking into account the right continuity of $t \to P_t^n f(x)$, we conclude from (4.2) that for each $x \in E_\Delta$, there exists a right continuous function, say $P_t^* f(x),\ t \in [0,\infty)$, such that for any $T > 0$

$$\sup_{t\le T}\sup_{x\in E_\Delta} |P_t^n f(x) - P_t^* f(x)| \longrightarrow 0, \qquad n\to\infty. \tag{4.4}$$

On the other hand, if we define for $\beta > 0,\ n \in \mathbb{N}$

$$R_\beta^n f(x) = E_x^n\Big[\int_0^\infty e^{-\beta t} f(X_t)\, dt\Big], \qquad \forall\, x \in E_\Delta.$$

Then one can check that (cf. [MRZ98, Lemma 4.1])

$$R_\beta^n f = \Big(\frac{n}{\beta+n}\Big)^2 R_{\frac{\beta n}{\beta+n}} f + \frac{1}{\beta+n} f.$$

Letting $n \to \infty$ we see that for all $\beta > 0$

$$\int_0^\infty e^{-\beta t} P_t^n f(x)\, dt = R_\beta^n f(x) \to R_\beta f(x) = \int_0^\infty e^{-\beta t} P_t f(x)\, dt.$$

By the uniqueness of the Laplace transform, we conclude that $P_t^* f(x) = P_t f(x)$ for all $t \in [0, \infty)$. Hence (4.1) is verified for $f = R_\alpha R_1 u$, which extends to the general situation immediately since by the resolvent equation $\lim\limits_{\alpha\to\infty} \|(\alpha - 1)R_\alpha R_1 f - R_1 f\|_{E_\Delta} = 0$ for all $f \in \mathcal{B}_b(E_\Delta)$. The last assertion of the lemma follows from (4.1) and the fact that $(t, x) \to P_t^n f(x)$ is uniformly continuous on $[0, T] \times Y$ for each $n \in \mathbb{N}$ and $f \in J_1 - J_1$. ∎

Proposition 4.2. For any $m \in \mathbb{N}$, $f \in C_b(\overline{E}^m)$, $0 = t_1 < t_2 < \cdots < t_m$,

$$\widetilde{E}_{x_0}^n[f(X_{t_1}, X_{t_2}, \cdots, X_{t_m})] \to \widetilde{E}_{x_0}[f(X_{t_1}, X_{t_2}, \cdots, X_{t_m})], \qquad n \to \infty. \tag{4.5}$$

Proof. By virtue of Stone-Weirestrass theorem, we need only to show that for any $m \in \mathbb{N}$, $f_1, f_2, \cdots, f_m \in C_b(\overline{E})$, $0 = t_1 < t_2 < \cdots < t_m$,

$$\widetilde{E}_{x_0}^n[f_1(X_{t_1}) f_2(X_{t_2}) \cdots f_m(X_{t_m})] \to \widetilde{E}_{x_0}[f_1(X_{t_1}) f_2(X_{t_2}) \cdots X_{t_m})], \qquad n \to \infty. \tag{4.6}$$

Note that (4.6) holds for $m = 1$ by (4.1). Suppose (4.6) holds for $m = k$, we are going to show that then it holds also for $m = k + 1$. To this end let $f_1, f_2, \cdots, f_k \in C_b(\overline{E})$ and $f_{k+1} = \overline{R_1 u}$ for some $u \in J_1 - J_1$. Let φ be the ρ-continuous extension of $(P_{t_{k+1}-t_k} f_{k+1})|_Y$. Then $f_k \varphi \in C_b(\overline{E})$. By the induction assumption and the Markov property of $(Z_t)_{t\geq 0}$, we get

$$\widetilde{E}_{x_0}^n[f_1(X_{t_1}) \cdots (f_k \varphi)(X_{t_k})] \to \widetilde{E}_{x_0}[f_1(X_{t_1}) \cdots f_k(X_{t_k}) f_{k+1}(X_{t_{k+1}})]. \tag{4.7}$$

But if we denote the ρ-continuous extension of $(P_{t_{k+1}-t_k}^n f_{k+1})|_Y$ by φ_n, then by the Markov property of $(Z_t^n)_{t\geq 0}$ we have

$$\widetilde{E}_{x_0}^n[f_1(X_{t_1}) \cdots (f_k \varphi_n)(X_{t_k})] = \widetilde{E}_{x_0}^n[f_1(X_{t_1}) \cdots f_k(X_{t_k}) f_{k+1}(X_{t_{k+1}})]. \tag{4.8}$$

By the above lemma we have $\|\varphi^n - \varphi\|_{\overline{E}} \to 0$ as $n \to \infty$. Therefore, (4.8) implies

$$\begin{aligned}
&|\widetilde{E}_{x_0}^n[f_1(X_{t_1}) \cdots f_{k+1}(X_{t_{k+1}})] - \widetilde{E}_{x_0}^n[f_1(X_{t_1}) \cdots (f_k \varphi)(X_{t_k})]| \\
\leq &\|f_1\|_{\overline{E}} \|f_2\|_{\overline{E}} \cdots \|f_k\|_{\overline{E}} \|\varphi^n - \varphi\|_{\overline{E}} \to 0
\end{aligned}$$

as $n \to \infty$, which together with (4.7) inplies that

$$\widetilde{E}^n_{x_0}[f_1(X_{t_1})\cdots f_{k+1}(X_{t_{k+1}})] \to \widetilde{E}_{x_0}[f_1(Xt_1)\cdots f_{k+1}(X_{t_{k+1}})]$$

for $f_1, f_2, \cdots, f_k \in C_b(\overline{E})$ and $f_{k+1} = \overline{R_1 u}$ with $u \in J_1 - J_1$. The proof is completed by employing Lemma 3.3. ■

5. Weak Convergence of $\{P^n_{x_0}\}_{n\in\mathbb{N}}$

Let $D_R[0,\infty)$ be the space of all real-valued cadlag functions on $[0,\infty)$.

Lemma 5.1 For any $f \in C_b(\overline{E})$, the laws of $\{f \circ Z^n\}_{n\in\mathbb{N}}$ with initial distribution δ_{x_0} form a tight family on $D_R[0,\infty)$.

Proof. Let $f \in C_b(\overline{E})$. Then by Lemma 3.3 for every $\varepsilon > 0$ and $T > 0$ there exists $g \in J_1 - J_1$ such that if we set $\xi^n(t) = \overline{R_1 g}(Z^n_t)$, then

$$E^n_{x_0}\Big[\sup_{t\le T} \mid \xi^n(t) - f(Z^n_t)|\Big] \le \varepsilon.$$

By the strong Markov property $\xi^n(t)$ is expressed by

$$\xi^n(t) - \xi^n(0) = M^n_t + \int_0^t L^n(R_1 g)(Z^n_s)\, ds$$

where $(M^n_t)_{t\ge 0}$ is a $P^n_{x_0}$-martingale. By (2.3) we have $L^n(R_1 g) = nR_n(R_1 g - g)$. Therefore, by the contraction property of nR_n, we have

$$\sup_{n\in\mathbb{N}} E^n_{x_0}\Big[\int_0^T \sup_{t\le T} |L^n(R_1 g)(Z^n_t)|\, ds\Big] \le T\,\|R_1 g - g\|_{E_\Delta} < \infty.$$

Thus the lemma is proved by applying [EK86, III. Theorem 9.4].

Proposition 5.2. As $n \to \infty$, $\widetilde{P}^n_{x_0}$ converges weakly to $\widetilde{P}_{x_0}$ on $D_{\overline{E}}[0,\infty)$ with respect to the Skorohod topology related to ρ.

Proof. By virtue of the above lemma, we see that by [EK86, III. Theorem 9.1] $\{\widetilde{P}^n_{x_0}\}_{n\in\mathbb{N}}$ is relatively compact on $D_{\overline{E}}[0,\infty)$. But by Proposition 4.2 and [EK86, III. Theorem 7.8] any weak limit of a subsequence of $\{\widetilde{P}^n_{x_0}\}_{n\in\mathbb{N}}$ must have the same finite dimensional distributions as $\widetilde{P}_{x_0}$ and hence coincides with $\widetilde{P}_{x_0}$. ■

Theorem 5.3. As $n \to \infty$, $P^n_{x_0}$ converges weakly to P_{x_0} on $D_{E_\Delta}[0,\infty)$ with respect to the Skorohod topology related to d.

Proof. Let $\varepsilon > 0$ be arbitrary. By Lemma 3.1 (i) and Lemma 3.2 we can inductively take a subsequence $\{k_l\}_{l\geq 1} \subset \mathbb{N}$ such that

$$\inf_n \{\widetilde{P}^n_{x_0}(B_l)\} \wedge \widetilde{P}_{x_0}(B_l) > 1 - \frac{\varepsilon}{2^l}$$

where

$$B_l := \Big\{ w \in D_{\overline{E}}[0,\infty) \mid w(t) \in E_{k_l} \text{ for all } t \in \Big[0, -\log\frac{\varepsilon}{2^l}\Big)\Big\}.$$

Set $B = \bigcap\limits_{l\geq 1} B_l$. Then B is a closed subset of $D_{\overline{E}}[0,\infty)$ (since each B_l is so by [Ek86, III. Proposition 5.2 and III. Lemma 5.1]). Since the ρ-topology and the d-topology coincide on each E_k, by [EK86, III. Proposition 6.5] one can check that B is also a closed subset of $D_{E_\Delta}[0,\infty)$ and the two topologics coincide on B. Moreover, $\widetilde{P}_{x_0}$ (resp. $\widetilde{P}^n_{x_0}$) coincides with P_{x_0} (resp. $P^n_{x_0}$) on B. Let $f \in C_b(D_{E_\Delta}[0,\infty))$. Then $f|_B$ is continuous on $B \subset D_{\overline{E}}[0,\infty)$ w.r.t. the trace topology of $D_{\overline{E}}[0,\infty)$. By Tietz's extension theorem f can be continuously extended to $D_{\overline{E}}[0,\infty)$, without enlarging the uniform norm $\|f\|_\infty := \sup\limits_{x\in D_{E_\Delta}[0,\infty)} |f(x)|$. Let $\overline{f}$ denote this extension. Thus

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \Big| \int_{D_{E_\Delta}[0,\infty)} f dP^n_{x_0} - \int_{D_{E_\Delta}[0,\infty)} dP_{x_0} \Big| \\
&\leq \overline{\lim_{n\to\infty}} \Big[\Big| \int_B f dP^n_{x_0} - \int_B f dP_{x_0} \Big| + \Big| \int_{D_{E_\Delta}[0,\infty)\setminus B} f dP^n_x \Big| + \Big| \int f_{D_{E_\Delta}[0,\infty)\setminus B} f dP_{x_0} \Big| \Big] \\
&\leq \overline{\lim_{n\to\infty}} \Big[\Big| \int_B f dP^n_{x_0} - \int_B f dP_{x_0} \Big| + 2\varepsilon \|f\|_\infty \\
&\leq \overline{\lim_{n\to\infty}} \Big[\Big| \int_{D_{\overline{E}}[0,\infty)} \widetilde{f} d\widetilde{P}^n_{x_0} - \int_{D_{\overline{E}}[0,\infty)} \widetilde{f} d\widetilde{P}_{x_0} \Big| + \Big| \int_{D_{\overline{E}}[0,\infty)\setminus B} \widetilde{f} d\widetilde{P}^n_{x_0} \Big| \\
&\quad + \Big| \int_{D_{\overline{E}}[0,\infty)\setminus B} \widetilde{f} d\widetilde{P}_{x_0} \Big| \Big] + 2\varepsilon \|f\|_\infty \\
&\leq \overline{\lim_{n\to\infty}} \Big| \int_{D_{\overline{E}}[0,\infty)} \widetilde{f} d\widetilde{P}^n_{x_0} - \int_{D_{\overline{E}}[0,\infty)} \widetilde{f} d\widetilde{P}_{x_0} \Big| + 4\varepsilon \|f\|_\infty = 4\varepsilon \|f\|_\infty.
\end{aligned}$$

Since ε was arbitrary, we conclude that

$$\lim_{n\to\infty} \int_{D_{\overline{E}}[0,\infty)} f dP^n_{x_0} = \int_{D_{E_\Delta}[0,\infty)} f dP_{x_0}.$$

Since $f \in C_b(D_{E_\Delta}[0,\infty))$ was arbitrary, this completes the proof. $\blacksquare$

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