Spherical Coxeter Groups and Hyperelliptic 3-Manifolds *

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Abstract

The groups generated by reflections in faces of Coxeter polyhedra in three-dimensional Thurston's geometries are studied. It is shown that for a wide class of Coxeter groups there are finite index subgroups which uniformize hyperelliptic 3-manifolds.

Let X^3 be one of three-dimensional geometries $\mathbb{H}^3$, $\mathbb{E}^3$, $\mathbf{S}^3$, $\mathbb{H}^2 \times \mathbb{E}^1$ or $\mathbf{S}^2 \times \mathbb{E}^1$ [6]. In the paper we consider *3-manifolds* M^3 which are quotient-spaces $M^3 = X^3/\Gamma$, where Γ is a discrete group of isometries acting on X^3 without fixed points. A 3-manifold M^3 is said to be *hyperelliptic*, if there exists an isometric involution τ such that the quotient-space $M^3/\langle\tau\rangle$ is homeomorphic to the three-domensional sphere $\mathcal{S}^3$. It was shown in [5] that in each of three-dimensional geometries there exist hyperelliptic manifolds. The method of constructing of hyperelliptic manifolds in [5] was based on the study of right-angled polyhedra with hamiltonian cycles. In the present paper we consider a wider class of Coxeter polyhedra and demonstrate their connection with hyperelliptic manifolds.

Let P be a Coxeter polyhedron (i.e. a polyhedron with dihedral angles of the form π/n, where $n \geq 2$ is integer) in one of the above three-dimensional geometries. Its *skeleton* P^1 is the weighted graph whose vertices and edges coincide with vertices and edges of the polyhedron P. For each edge $e \in P^1$ its weight is equal to n if the dihedral angle corresponding to this edge in P is equal to π/n.

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Recall [3] that a graph $H = (V(H), E(H))$ is said to be a *spanning subgraph* of a graph $G = (V(G), E(G))$ if sets of vertices of these graphs coincide: $V(H) = V(G)$, and the set of edges of H is a subset of the set of edges of G: $E(H) \subseteq E(G)$.

Two weighted graphs G and H with integer positive weights are said to be *homeomorphic* if these graphs can be obtained from some weighted graph by the subdivision of its edges with the weight 2.

Recall some facts from the theory of Coxeter groups [1]. Let $p \subset \mathbf{S}^3$ be a Coxeter polyhedron, and $\delta(p)$ be the group generated by reflections in it's faces. Remark, that all groups $\delta(p)$ are finite. Schemes of Coxeter groups $\delta(p)$ and skeletons p^1 corresponding to three-dimensional spherical Coxeter polyhedra p are listed in the following table.

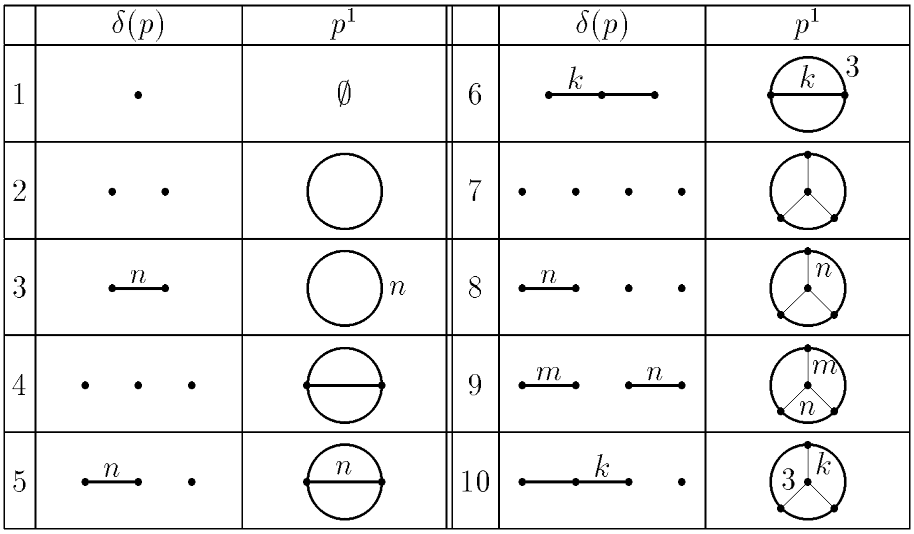

Table 1.

Here labels m, n, and k on edges are integers $m, n \geq 3$ and $k = 3, 4, 5$. To simplify notations in figures, we omit the labels 2 on edges of weighted graphs.

For a given spherical Coxeter polyhedron p with the skeleton $p^1 \neq \emptyset$ we shall consider Coxeter polyhedra P in spaces X^3, such that their skeletons P^1 contain spanning subgraphs homeomorphic to the skeleton p^1. In each case there is a wide class of such polyhedra. In particular, if P^1 contains a spanning subgraph homeomorphic to $\bigcirc$ (see case 2 of table 1), then P is a right-angled hamiltonian polyhedron.

The following theorem shows that there exist finite index subgroups of Coxeter groups which are fundamental groups of hyperelliptic manifolds.

Theorem. *Let p be a Coxeter polyhedron in $\mathbf{S}^3$ with the skeleton p^1, and P be a Coxeter polyhedron in X^3 with the skeleton P^1, where $X^3 = \mathbb{H}^3$, $\mathbb{E}^3$, $\mathbf{S}^3$, $\mathbf{S}^2 \times \mathbb{E}^1$ or $\mathbb{H}^2 \times \mathbb{E}^1$. Denote by $\delta(p)$ and $\Delta(P)$ corresponding Coxeter groups. Assume, that P^1 contains a proper spanning subgraph T which is homeomorphic to p^1, and weights of all edges from $P^1 \setminus T$ equal 2. Then there exists a torsion-free subgroup $G < \Delta(P)$ of index $|\Delta(P) : G| = 2|\delta(p)|$ such that the manifold $M^3 = X^3/G$ is hyperelliptic.*

Proof. Here we shall use terminology and basic facts from the theory of three-dimensional orbifolds [7, ch.13].

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be orbifolds with the universal covering space $\mathcal{O}$ and orbifold fundamental groups Γ_1 and Γ_2 respectively. Thus, the following canonical covering maps hold: $\pi_1 : \mathcal{O} \longrightarrow \mathcal{O}_1 = \mathcal{O}/\Gamma_1$ and $\pi_2 : \mathcal{O} \longrightarrow \mathcal{O}_2 = \mathcal{O}/\Gamma_2$. Assume that $\Gamma_1 \subset \Gamma_2$ and denote by $\pi : \mathcal{O}_1 \longrightarrow \mathcal{O}_2$ the orbifold covering induced by this group inclusion. The *branching index* of the covering π in a point $x \in \mathcal{O}_1$ is the number $|(\Gamma_2)_{\tilde{x}} : (\Gamma_1)_{\tilde{x}}|$, where $\tilde{x} = \pi_1^{-1}(x)$ and $(\Gamma_i)_{\tilde{x}}$ denotes the stabilizer of the point $\tilde{x}$ in the group Γ_i. In particular, if the covering π is unbranched in the point $x \in \mathcal{O}_1$, then we get $|(\Gamma_2)_{\tilde{x}} : (\Gamma_1)_{\tilde{x}}| = 1$ and $(\Gamma_1)_{\tilde{x}} = (\Gamma_2)_{\tilde{x}}$.

Let X and Y be metric spaces. For the covering $\pi : X \longrightarrow Y$ we consider the group $\mathrm{Cov}_\pi(X, Y) = \{h \in \mathrm{Homeo}(X) : \pi \circ h = \pi\}$. The covering π is said to be a *regular G–covering branched over the weighted graph $T \subset Y$* if the following three conditions are satisfied:

(i) $\mathrm{Cov}_\pi(X, Y) \cong G$;

(ii) the map $\overline{\pi} : X \setminus \pi^{-1}(T) \longrightarrow Y \setminus T$ induced by the map π, is an unbranched covering;

(iii) the quotient space $X/\mathrm{Cov}_\pi(X, Y)$ is an orbifold with underlying space Y and singular set T.

Two coverings $\pi : X \longrightarrow Y$ and $\pi' : X' \longrightarrow Y$ are said to be *equivalent* if there exists a homeomorphism $h : X \longrightarrow X'$ such that $\pi = \pi' \circ h$.

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be orbifolds with underlying spaces X and Y respectively, and let T be a weighted graph in Y. An orbifold covering $\pi : \mathcal{O}_1 \longrightarrow \mathcal{O}_2$

is said to be *regular G–covering branched over T* if the associated covering $\pi_X : X \longrightarrow Y$ of underlying spaces does satisfy the above conditions (i), (ii), and (iii).

Denote by $\delta^+(p)$ and $\Delta^+(P)$ subgroups of groups $\delta(p)$ and $\Delta(P)$ of index two consisting of all orientation-preserving isometries. Then $\Omega^3 = X^3/\Delta^+(P)$ is the orbifold whose underlying space the three-dimensional space $\mathcal{S}^3$ and whose singular set is the weighted graph P^1. Analogously, $\omega^3 = \mathbf{S}^3/\delta^+(p)$ is the orbifold whose underlying space is $\mathcal{S}^3$ and whose singular set is the weighted graph p^1.

Lemma 1. *Let p be a spherical Coxeter polyhedron with the skeleton $p^1 \subset \mathcal{S}^3$. Then, up to equivalence, there exists an unique regular $\delta^+(p)$–covering $\pi : \Sigma^3 \longrightarrow \mathcal{S}^3$ branched over p^1. Moreover, Σ^3 is homeomorphic to $\mathcal{S}^3$.*

Proof. Remark, that the universal covering

$$\pi_p \ : \ \mathcal{S}^3 \ \longrightarrow \ \mathcal{S}^3 = \mathbf{S}^3/\delta^+(p),$$

branched over the skeleton $p^1 \subset \mathcal{S}^3$, does satisfy to conditions of the lemma. Assume that there exists another regular $\delta^+(p)$–covering $\pi \ : \ \Sigma^3 \ \longrightarrow \ \mathcal{S}^3$ branched over p^1. Let us demonstrate that the coverings π_p and π are equivalent. Indeed, these coverings induce unbranched $\delta^+(p)$–coverings $\overline{\pi_p}$ and $\overline{\pi}$ such that

$$\overline{\pi_p} \ : \ \mathcal{S}^3 \setminus \pi_p^{-1}(p^1) \ \longrightarrow \ \mathcal{S}^3 \setminus p^1$$

and

$$\overline{\pi} \ : \ \Sigma^3 \setminus \pi^{-1}(p^1) \ \longrightarrow \ \mathcal{S}^3 \setminus p^1.$$

Remark, that these coverings are determined by appropriate epimorphisms ϕ_p and ϕ of the fundamental group $\Gamma = \pi_1(\mathcal{S}^3 \setminus p^1)$ onto the group $\delta^+(p)$. Thus, $\pi_1(\mathcal{S}^3 \setminus \pi_p^{-1}(p^1)) = \mathrm{Ker}\ \phi_p$ and $\pi_1(\Sigma^3 \setminus \pi^{-1}(p^1)) = \mathrm{Ker}\ \phi$. Since coverings π_p and π are branched over p^1, the epimorphisms ϕ_p and ϕ have the following property: if γ_e is a single loop around the edge e of the graph p^1 with the weight n_e, then orders of elements $\phi_p(\gamma_e)$ and $\phi(\gamma_e)$ in the group $\delta^+(p)$ are equal to n_e.

For the set $A = \{\gamma_e^{n_e} \mid e \in p^1\}$ we denote by $\mathrm{Ncl}(A)$ its normal closure in the group Γ. Consider the canonical homomorphism

$$\varphi \ : \ \Gamma \ \longrightarrow \ \Gamma/\mathrm{Ncl}(A).$$

In virtue of above properties of the epimorphisms ϕ_p and ϕ, there exist epimorphisms

$$\theta_p \ : \ \Gamma/\mathrm{Ncl}(A) \ \longrightarrow \ \delta^+(p)$$

and

$$\theta \ : \ \Gamma/\mathrm{Ncl}(A) \ \longrightarrow \ \delta^+(p)$$

such that $\phi_p = \theta_p \circ \varphi$ and $\phi = \theta \circ \varphi$.

By [4], the group $\Gamma/\mathrm{Ncl}(A)$ is the group of the orbifold whose underlying space is $\mathcal{S}^3$, and whose singular set is the weighted graph p^1. Hence, this orbifold is ω^3. Therefore, $\Gamma/\mathrm{Ncl}(A) \cong \delta^+(p)$. Moreover, θ_p and θ are epimorphisms of the group $\delta^+(p)$ onto itself. Since the group of spherical isometries $\delta^+(p)$ is finite, θ_p and θ are isomorphic. Thus, the epimorphism ϕ_p differs from the epimorphism ϕ on some automorphism α of the group $\delta^+(p)$, that is $\phi_p = \alpha \circ \phi$. Whence, the kernels $\mathrm{Ker}\phi_p$ and $\mathrm{Ker}\phi$ coincide and coverings $\overline{\pi_p}$ and $\overline{\pi}$ are equivalent.

By Fox theorem [2], each of the coverings $\overline{\pi_p}$ and $\overline{\pi}$ admits the unique, up to equivalence, extension to the regular $\delta^+(p)$–covering branched over p^1. Therefore, the coverings π_p and π are equivalent. The proof of the lemma is finished.

$\square$

Following [5], we call an orientable orbifold $\mathcal{O}$ *hyperelliptic sphere* if its underlying space is the three-dimensional sphere $\mathcal{S}^3$ and its singular set is a link (in particular, a knot) in $\mathcal{S}^3$ with singularity index 2 on each component. It was shown in [5] that for each hyperelliptic sphere $\mathcal{O}$ there exists a manifold M^3 with an involution τ such that $M^3/\langle\tau\rangle = \mathcal{O}$.

Consider a proper spanning subgraph T of the graph P^1 which is homeomorphic to the graph p^1.

Lemma 2. *Let $\pi : \Sigma^3 \longrightarrow \Omega^3$ be an orbifold regular $\delta^+(p)$–covering branched over T. Then Σ^3 is a hyperelliptic sphere.*

Proof. Denote $\Gamma_0 = \Delta^+(P)$, then $\Omega^3 = X^3/\Gamma_0$ and $\Sigma^3 = X^3/\Gamma$, where $\Gamma \lhd \Gamma_0$ and $\Gamma_0/\Gamma \cong \delta^+(p)$. Consider canonical coverings $\sigma : X^3 \to \Sigma^3 = X^3/\Gamma$ and $\omega : X^3 \to \Omega^3 = X^3/\Gamma_0$. Let us fix an arbitrary point $z \in \Sigma^3$ and its preimage $\tilde{z} \in \sigma^{-1}(z)$. Denote $z' = \omega(\tilde{z})$.

In a small enough ball $V_{\tilde{z}} = B(\tilde{z}, \epsilon)$ the orbifold covering $\pi : \Sigma^3 \longrightarrow \Omega^3$ can be presented in the form $\pi : V_{\tilde{z}}/\Gamma_{\tilde{z}} \longrightarrow V_{\tilde{z}}/(\Gamma_0)_{\tilde{z}}$, where $\Gamma_{\tilde{z}}$ and $(\Gamma_0)_{\tilde{z}}$ are stabilizers of the point $\tilde{z}$ in groups Γ and Γ_0, respectively. In particular, the branching index of the covering π in the point z is equal to $|(\Gamma_0)_{\tilde{z}} : \Gamma_{\tilde{z}}|$.

For the point $z' \in \Omega^3$ one of the following cases holds:

(i) $z' \notin P^1$;

(ii) z' is an interior point of an edge $e \in P^1 \setminus T$;

(iii) z' is an interior point of an edge of the graph T;

(iv) z' is a 2-valent vertex of the graph T;

(v) z' is a 3-valent vertex of the graph T.

Let us study stabilizators $(\Gamma_0)_{\tilde{z}}$ and $\Gamma_{\tilde{z}}$ in each of the above cases. In the case (i) we have $\Gamma_{\tilde{z}} = (\Gamma_0)_{\tilde{z}} = \langle 1 \rangle$. In the case (ii), since all edges of the graph $P^1 \setminus T$ have weight 2, the dihedral angle at the edge e is equal to $\pi/2$. The covering $\pi : \Sigma^3 \to \Omega^3$ is unbranched over e, so $\Gamma_{\tilde{z}} = (\Gamma_0)_{\tilde{z}} \cong \mathbb{Z}_2$. In the case (iii) consider an edge $e \subset T$ having z' as an interior point. Let dihedral angle at the edge e be equal to π/n, $n \geq 2$. Then, $(\Gamma_0)_{\tilde{z}} \cong \mathbb{Z}_n$. Since π is the $\delta^+(p)$–covering branched over T, we get $|(\Gamma_0)_{\tilde{z}} : \Gamma_{\tilde{z}}| = n$. Therefore, $\Gamma_{\tilde{z}} \cong \langle 1 \rangle$. In the case (iv) the point z' is a 2-valent vertex of the graph T. There are three edges incidental to z' in P^1. Two of them belong to T. Since T is a spanning subgraph of the skeleton P^1 homeomorphic to p^1, we get that each of these three edges incidental to z' has singularity 2. Therefore $(\Gamma_0)_{\tilde{z}} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since the covering π is branched over the point $z' \in T$, we have $|(\Gamma_0)_{\tilde{z}} : \Gamma_{\tilde{z}}| = 2$, and so, $\Gamma_{\tilde{z}} \cong \mathbb{Z}_2$. In the case (v) the branching index $|(\Gamma_0)_{\tilde{z}} : \Gamma_{\tilde{z}}|$ of the covering π at the vertex z is equal to the order of the group $(\Gamma_0)_{\tilde{z}}$. Whence, $\Gamma_{\tilde{z}} = \langle 1 \rangle$.

Thus, in each of the above cases either $\Gamma_{\tilde{z}} = \langle 1 \rangle$ and z is non-singular point, or $\Gamma_{\tilde{z}} \cong \mathbb{Z}_2$ and z is singular point with singularity index 2. It means that singular set of the orbifold Σ^3 is locally homeomorphic to the open interval with singularity index 2. By lemma 1, the underlying space of the orbifold Σ^3 is homeomorphic to $\mathcal{S}^3$. Thus, the orbifold Σ^3 is hyperelliptic sphere, and proof of the lemma is finished.

□

Continue the proof of the theorem. Consider a regular $\delta^+(p)$–covering

$$\pi \ : \ \Sigma^3 = X^3/\Gamma \ \longrightarrow \ \Omega^3 = X^3/\Delta^+(P)$$

branched over T. In virtue of lemma 2, the orbifold Σ^3 is the hyperelliptic sphere uniformized by the group $\Gamma \triangleleft \Delta^+(P)$. As was shown in [5], if Σ^3 is the hyperelliptic sphere then there exists a manifold M^3 with an involution

τ such that $M^3/\langle\tau\rangle = \Sigma^3$. The covering $M^3 \longrightarrow \Sigma^3$ induces the groups inclusion $G \triangleleft \Gamma$, where $G = \pi_1(M^3)$ is the torsion-free subgroup of index 2 in the group Γ. It is easy to see that G is subgroup of $\Delta(P)$ of index $|\Delta(P) : G| = 2|\delta(p)|$. The theorem is proven.

□

Remark, that the particular case of the theorem for hamiltonian right-angled polyhedra (case 2 in the table above) was obtained in [5].

References

[1] Coxeter H. M. S. *Discrete groups generated by reflections,* Ann. Math. **35** (1934) P. 588–621.

[2] Fox R. H. *Covering spaces with singularities,* Algebraic Geometry and Topology (eds. R.H.Fox, D.C.Spencer, and A.W.Tucker), Princeton Univ. Press, Princeton, N. J. 1957. P. 243–257.

[3] Harary F. *Graph Theory.* Addison-Wesley Publishing Company, 1969.

[4] Haefliger A., Quach N.D. *Une presentation de groupe fundamental d'une orbifold,* Asterisque **116** (1984) P. 98–107.

[5] Mednykh A.D. *Three-dimensional hyperelliptic manifolds,* Ann. Global Anal. Geom. **8** (1990) P. 13–19.

[6] Scott P. *The geometries of 3-manifolds,* Bull. London Math. Soc. **15** (1986) P. 401–487.

[7] Thurston W. *The geometry and topology of 3-manifolds.* Lecture Notes, Princeton University 1980.

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