A combinatorial description of blocks in $\mathcal{O}(\mathcal{P}, \Lambda)$ associated with $sl(2)$-induction

V. Futorny, S. König and V. Mazorchuk

Abstract

We study the category $\mathcal{O}(\mathcal{P}, \Lambda)$, where Λ is an admissible category of dense weight $sl(2)$-modules. We give a combinatorial description of projectively stratified algebras, arising from $\mathcal{O}(\mathcal{P}, \Lambda)$ and prove a double centralizer property. Moreover, we determine the characters of tilting modules in $\mathcal{O}(\mathcal{P}, \Lambda)$ and prove that the finite-dimensional algebra associated with the principal block of our $\mathcal{O}(\mathcal{P}, \Lambda)$ is its own Ringel dual.

1 Introduction

Together with its definition in [BGG], two basic facts of the category $\mathcal{O}$, associated with a simple complex finite-dimensional Lie algebra $\mathfrak{G}$, were established. The first states that $\mathcal{O}$ decomposes into blocks, each of which is a module category over a finite-dimensional algebra (these algebras belong to the class of quasi-hereditary algebras, defined later in [CPS]). The second one is the celebrated BGG reciprocity between simple, projective and Verma modules in $\mathcal{O}$. Another crucial result about $\mathcal{O}$ was obtained much later by Soergel in [S2]. There a combinatorial description of the finite-dimensional algebras which correspond to blocks of $\mathcal{O}$ was given. In the case of the principal block of $\mathcal{O}$, the combinatorial datum of this description is the so-called coinvariant algebra together with the big projective module.

In [FKM] we proposed a natural generalization $\mathcal{O}(\mathcal{P}, \Lambda)$ of $\mathcal{O}$, which corresponds to an admissible category Λ of (infinite-dimensional in general) modules over a parabolic subalgebra $\mathcal{P}$ of $\mathfrak{G}$. In fact, we have shown that under some natural conditions, the obtained categories decompose into blocks, each of which is a module category over a projectively stratified algebra. We also found an analogue of BGG reciprocity.

The aim of this paper is to obtain an analogue of Soergel's combinatorial description for $\mathcal{O}(\mathcal{P}, \Lambda)$ in the case, when the semisimple part of the Levi factor of $\mathcal{P}$ is isomorphic to $sl(2, \mathbb{C})$ (see [FKM, Section 10]). In particular, we show (Theorem 3 and Corollary 7 in Section 6), that the combinatorial datum is again the coinvariant algebra and the big projective module in the principal block of $\mathcal{O}(\mathcal{P}, \Lambda)$.

Theorem A. *Let $P(L)$ be the big projective module in the principal block of the category $\mathcal{O}(\mathcal{P}, \Lambda(V(\lambda, \gamma)))$. Then* End $_{\mathcal{O}(\mathcal{P},\Lambda)}(P(L))$ *is the coinvariant algebra.*

Moreover, Theorem 4 in Section 7 provides us with a double centralizer property.

Theorem B. *Let B denote the (projectively stratified finite-dimensional) algebra associated with the principal block $\mathcal{O}(\mathcal{P},\Lambda)_{triv}$. Then B is isomorphic to the endomorphism algebra of the big projective module, viewed as a module over its endomorphism ring.*

We also construct tilting modules in $\mathcal{O}(\mathcal{P},\Lambda)$ and determine their characters. In Section 10 (Theorem 7) we establish Ringel self–duality.

Theorem C. *The projectively stratified algebra of the principal block $\mathcal{O}(\mathcal{P},\Lambda)_{triv}$ is its own Ringel dual.*

The paper is organized as follows. In Section 2 we introduce our main objects. In Section 3 we use Mathieu's localization technique to reduce the study of an arbitrary category $\mathcal{O}(\mathcal{P},\Lambda)$ to a special case $\Lambda = \Lambda(V(\mathfrak{l},\gamma))$. In Section 4 we define a functor $E : \mathcal{O}(\mathcal{P},\Lambda(V(\mathfrak{l},\gamma))) \to \mathcal{O}$ and establish its basic properties. In fact, we show that it transfers generalized Verma modules to Verma modules and projective objects to projective objects. In Section 5 we prove that $\mathcal{O}(\mathcal{P},\Lambda)$ is equivalent to a full subcategory of $\mathcal{O}$. In Section 6 we determine the endomorphism algebra of the big projective module in the principal block of $\mathcal{O}(\mathcal{P},\Lambda)$. In fact, we prove that this is the coinvariant algebra. In Section 7 we establish the double centralizer property for $\mathcal{O}(\mathcal{P},\Lambda)$. In Section 8 we define and investigate a subclass of tilting modules in $\mathcal{O}$ which we call strong tilting modules. Further, in Section 9 we define a notion of tilting module in $\mathcal{O}(\mathcal{P},\Lambda)$ and prove their existence and uniqueness. In fact, we show, that E transfers a tilting module in $\mathcal{O}(\mathcal{P},\Lambda)$ into a strong tilting module in $\mathcal{O}$ and this map is bijective. Finally, in Section 10 we determine the multiplicities of generalized Verma modules occuring in a standard filtration of a tilting module in $\mathcal{O}(\mathcal{P},\Lambda)$ and prove that the projectively stratified algebra associated with the principal block of $\mathcal{O}(\mathcal{P},\Lambda)$ is isomorphic to its Ringel dual.

2 Main objects

Let $\mathfrak{A}$ denote the Lie algebra $sl(2,C)$ with a fixed root basis $e = X_\alpha$, $f = X_{-\alpha}$, $h = H_\alpha$, where α is a root of $\mathfrak{A}$. For $\gamma \in \mathbb{C}$ and $\lambda \in \mathbb{C}/2\mathbb{Z}$ let $V(\lambda,\gamma)$ denote the unique weight $\mathfrak{A}$-module (see [FM1]), satisfying the following conditions:

1. λ is the support of $V(\lambda,\gamma)$ and all weight spaces of $V(\lambda,\gamma)$ are one-dimensional,
2. γ is the unique eigenvalue of the Casimir operator $C = (h+1)^2 + 4fe$ on $V(\lambda,\gamma)$,
3. f acts bijectively on $V(\lambda,\gamma)$.

Clearly, $V(\lambda,\gamma)$ is an indecomposable $\mathfrak{A}$-module, generated by any $V(\lambda,\gamma)_{\mu+k\alpha}$, $\mu \in \lambda$ for $k \in \mathbb{N}$ big enough.

Call a weight $\mathfrak{A}$-module V with finite-dimensional weight spaces admissible, provided f acts bijectively on V. By definition, any $V(\lambda,\gamma)$ is admissible.

Let $\tilde{\Lambda} = \tilde{\Lambda}(V(\lambda,\gamma))$ denote the category of $\mathfrak{A}$-modules, defined as follows: the objects of $\tilde{\Lambda}$ are all admissible submodules and all admissible quotients of all modules having the form $V(\lambda,\gamma)\otimes F$, where F is a finite-dimensional $\mathfrak{A}$-module; the homomorphisms of $\tilde{\Lambda}$ are those homomorphisms of $\mathfrak{A}$-modules, whose kernel is an admissible module. Clearly, $\tilde{\Lambda}$ is an abelian category (i.e. that it is closed under operations of taking admissible submodules and quotients), moreover, $\tilde{\Lambda}$ is closed under taking finite direct sums.

Remark 1. *In the case, when $V(\lambda,\gamma)$ is simple (this means $\gamma \neq (l+1)^2$ for all $l \in \lambda$), any submodule in $V(\lambda,\gamma)\otimes F$, where F is finite-dimensional, is admissible (see, for example, [CF]). The objects of $\tilde{\Lambda}$ are all the quotients and submodules of modules $V(\lambda,\gamma)\otimes F$.*

In the case, when $V(\lambda,\gamma)$ is not simple, $\tilde{\Lambda}$ still is a full subcategory of the category of $\mathfrak{A}$-modules. It is easy to see, that $\tilde{\Lambda}$ inherits an abelian structure from the category of all $\mathfrak{A}$-modules. In fact, let M_1 and M_2 be two weight modules with finite dimensional weight spaces and let $\varphi : M_1 \to M_2$ be a morphism. Then f acts injectively on $\ker(\varphi) \subset M_1$. Using the bijective action of f on M_2 (which has finite dimensional weight spaces) we also get that f acts surjectively on $\ker(\varphi)$. In a similar way, one can check that f acts bijectively on $\mathrm{coker}(\varphi)$.

Now let $\mathfrak{G}$ be a complex simple finite-dimensional Lie algebra and P be a parabolic subalgebra of $\mathfrak{G}$ such that $P = (\mathfrak{A}\oplus\mathfrak{H}_{\mathfrak{A}})\oplus\mathfrak{N}$, $\mathfrak{A}\oplus\mathfrak{H}_{\mathfrak{A}}$ reductive, $\mathfrak{N}$ nilpotent, $\mathfrak{H}_{\mathfrak{A}}$ abelian and $\mathfrak{A}$ as above. The category $\tilde{\Lambda}$ can be extended in a unique way to a category $\Lambda = \Lambda(V(\lambda,\gamma))$ of $\tilde{\mathfrak{A}} = \mathfrak{A}\oplus\mathfrak{H}_{\mathfrak{A}}$-modules, which satisfies the following conditions:

1. any $M \in \Lambda$ belongs to $\tilde{\Lambda}$, when viewed as an $\mathfrak{A}$-module,
2. any $M \in \Lambda$ is $\mathfrak{H}_{\mathfrak{A}}$-diagonalizable,
3. for any $M \in \Lambda$ and any $\mathfrak{H}_{\mathfrak{A}}$-diagonalizable finite-dimensional $\tilde{\mathfrak{A}}$-module F the module $M\otimes F$ decomposes into a direct sum of indecomposable modules from Λ,
4. the homomorphisms in Λ are those homomorphisms of $\tilde{\mathfrak{A}}$-modules, whose restriction to $\mathfrak{A}$ belongs to $\tilde{\Lambda}$.

Following [FKM, Section 3], we define $\mathcal{O}(\mathcal{P},\Lambda)$ to be the category of $\mathfrak{G}$-modules, whose objects are finitely generated and $\mathfrak{N}$-finite $\mathfrak{G}$-modules, which decompose into a direct sum of modules from Λ, when wieved as $\tilde{\mathfrak{A}}$-modules and whose homomorphisms are those homomorphisms of $\mathfrak{G}$-modules, whose kernel decomposes into a direct sum of modules from Λ, when viewed as an $\tilde{\mathfrak{A}}$-module. By [FKM, Section 4], $\mathcal{O}(\mathcal{P},\Lambda)$ has a block decomposition (with finitely many simples in each block) and by [FKM, Section 10], this decomposition can be choosen such that each block is equivalent to the module category over a projectively stratified finite-dimensional algebra. Moreover, if γ is not a square of an integer, this algebra is quasi-hereditary and in all cases there is an analogue of the BGG-reciprocity. From Remark 1 it follows, that the category $\mathcal{O}(\mathcal{P},\Lambda)$ is a full subcategory of the category of all $\mathfrak{G}$-modules and it inherits the abelean structure from the last category. Further, $\mathcal{O}(\mathcal{P},\Lambda)$ is closed under taking finite direct sums and under tensoring with finite-dimensional $\mathfrak{G}$-modules.

3 Mathieu's localization and the first equivalence

The aim of this section is to prove the following result:

Theorem 1. *The categories $\mathcal{O}(\mathcal{P},\Lambda(V(\lambda_1,\gamma)))$ and $\mathcal{O}(\mathcal{P},\Lambda(V(\lambda_2,\gamma)))$ are (blockwise) equivalent (i.e. they are independent on λ).*

To prove this we will use Mathieu's localization, proposed in [M] as a tool for classifying simple dense modules with finite-dimensional weight spaces. We refer the reader to [M] for all technical details.

Denote by U_f the localization of $U(\mathfrak{G})$ with respect to the powers of $f = X_{-\alpha}$ and let θ_x, $x \in \mathbb{C}$ be the unique polynomial family of automorphisms of U_f, such that $\theta_x(u) = f^x u f^{-x}$ for all $u \in U_f$ and all $x \in \mathbb{Z}$.

Proof. We can assume that $\lambda_1 \neq \lambda_2$. Since $\mathbb{C}$ is one-dimensional over itself, there exists $x \in \mathbb{C}$ such that $\lambda_1 = \lambda_2 + x\alpha$. Moreover, $x \notin \mathbb{Z}$ according to our assumption.

By definition of $\mathcal{O}(\mathcal{P},\Lambda)$, f acts bijectively on any module $V \in \mathcal{O}(\mathcal{P},\Lambda)$. Thus any V can be trivially extended to an U_f-module.

Now suppose that M is an U_f-module and $0 \neq v \in M$ such that $H_\alpha v = av$ for some $a \in \mathbb{C}$. Then for any integer y we have $\theta_y(H_\alpha)v = f^y H_\alpha f^{-y} v = (a+2y)v$. Since the family θ_y is polynomial (in y) by definition, we have, that $\theta_y(H_\alpha)v = (a+2y)v$ for any $y \in \mathbb{C}$. From this it follows immediately, that the twist by θ_{-x} (resp. θ_x) is a well-defined functor from $\mathcal{O}(\mathcal{P},\Lambda(V(\lambda_1,\gamma)))$ to $\mathcal{O}(\mathcal{P},\Lambda(V(\lambda_2,\gamma)))$ (resp. from $\mathcal{O}(P,\Lambda(V(\lambda_2,\gamma)))$ to $\mathcal{O}(\mathcal{P},\Lambda(V(\lambda_1,\gamma))))$. Since the composition of θ_x and θ_{-x} is an identity, we easily conclude that these fuctors are mutually inverse. The block version follows immediately. This completes the proof. □

4 From $\mathcal{O}(\mathcal{P},\Lambda)$ to $\mathcal{O}$

According to Theorem 1, the properties of the category $\mathcal{O}(\mathcal{P},\Lambda(V(\lambda,\gamma)))$ do not depend on λ. Now we recall that for a fixed γ there exists at least one $V(\lambda,\gamma)$, which is not simple. More precisely, if γ is a square of an integer, such $V(\lambda,\gamma)$ is unique and if γ is not a square of an integer, there are precisely two non-isomorphic non-simple modules $V(\lambda',\gamma)$ and $V(\lambda'',\gamma)$. Let $V(\mathfrak{l},\gamma)$ be a non-simple module. The aim of this section is to define and investigate a functor from $\mathcal{O}(\mathcal{P},\Lambda(V(\mathfrak{l},\gamma)))$ to $\mathcal{O}$.

For $M \in \mathcal{O}(\mathcal{P},\Lambda(V(\mathfrak{l},\gamma)))$ denote by $E(M)$ the space of locally e-finite elements of M. Since e is locally ad-nilpotent, $E(M)$ is a $\mathfrak{G}$-submodule of M. On morphisms, E is restriction of a homomorphism $\varphi : M \to N$ to $E(\varphi) : E(M) \to E(N)$. We note, that from [FKM, Section 10, Section 4] it follows that any object M in $\mathcal{O}(\mathcal{P},\Lambda(V(\mathfrak{l},\gamma)))$ has finite length (as a $\mathfrak{G}$-module). Hence $E(M)$ also has a finite length. Thus we obtain, that E is a well-defined functor from $\mathcal{O}(\mathcal{P},\Lambda(V(\mathfrak{l},\gamma)))$ to $\mathcal{O}$. Our main goal in this and some of the next sections is to study the properties of this functor E. We note that an analogous functor was used in [M, Lemma A1].

Lemma 1. *$E(M) = 0$ if and only if $M = 0$.*

Proof. We have to prove the "only if" part. Since $E(M)$ is defined as the locally e-finite part of M and e is locally ad-nilpotent, $E(M)$ is an $\mathfrak{A}$-module (moreover, it is a $\mathfrak{G}$-module). By definition of $\mathcal{O}(\mathcal{P}, \Lambda(V(\mathfrak{l}, \gamma)))$, M decomposes into a direct sum of modules from $\tilde{\Lambda}$, thus it is sufficient to prove our statement for indecomposable modules in Λ.

Suppose, that γ is not a square of an integer. Then any indecomposable in Λ has the form $V(\hat{\lambda}, \hat{\gamma})$ for some $\hat{\lambda} \in \mathbb{C}/2\mathbb{Z}$ and $\hat{\gamma} \in \mathbb{C}$ and is not simple (see [FKM, Section 10] or [FM2, Example 2] or directly apply [K, Theorem 5.1]) by our assumption on Λ. Since $V(\hat{\lambda}, \hat{\gamma})$ is not simple and f acts bijectively on it, it should have a non-zero highest weight submodule. Hence its subspace of locally e-finite elements is non-zero.

Now suppose that γ is a square of an integer. Then, by [FKM, Section 10], any indecomposable module in Λ is either some non-simple $V(\hat{\lambda}, \hat{\gamma})$ or is a self-extension of some $V(\hat{\lambda}, \hat{\gamma})$, moreover, this self-extension in Λ exists if and only if $V(\hat{\lambda}, \hat{\gamma})$ itself has length 3. For $V(\hat{\lambda}, \hat{\gamma})$ everything is clear (analogous to the previous case). To complete the proof, consider a self-extension $V \in \Lambda$ of some $V(\hat{\lambda}, \hat{\gamma})$ of length 3. By definition of Λ, there is a finite-dimensional $\tilde{\mathfrak{A}}$-module F such that V is a direct summand of $F \otimes V(\hat{\lambda}, \hat{\gamma})$. Since $V(\hat{\lambda}, \hat{\gamma})$ has a non-zero highest weight submodule and tensoring with a finite-dimensional module is an exact functor, we conclude, that the space of locally e-finite vectors in V is non-zero. $\square$

Corollary 1. *Under the notations of Lemma 1 the following holds: For any $\mu \in \operatorname{supp} M$ and any $k \in \mathbb{N}$ big enough $M_{\mu-k\alpha}$ belongs to the locally e-finite part of V.*

Proof. Follows directly from the proof of Lemma 1. $\square$

We recall, that any module in $\mathcal{O}(\mathcal{P}, \Lambda)$ is a weight $\mathfrak{G}$-module with finite-dimensional weight spaces ([CF]). Recall, that any $M \in \mathcal{O}(\mathcal{P}, \Lambda)$ has finite length. From this and Corollary 1 we deduce the following:

Corollary 2. *Let $M \in \mathcal{O}(\mathcal{P}, \Lambda(V(\mathfrak{l}, \gamma)))$. Then for any $\mu \in \operatorname{supp} M$ and for any $k \in \mathbb{N}$ big enough we have $\dim M_{\mu-k\alpha} = \dim E(M)_{\mu-k\alpha}$.*

Lemma 2. *Let $L \in \mathcal{O}(\mathcal{P}, \Lambda(V(\mathfrak{l}, \gamma)))$ be a simple object. Then $E(L)$ is of length 1 or 2 (as a $\mathfrak{G}$-module).*

Proof. Recall, that $V(\mathfrak{l}, \gamma)$ is not simple. First we note, that according to [FKM, Proposition 2] and [FKM, Remark 2], any simple $L \in \mathcal{O}(\mathcal{P}, \Lambda(V(\mathfrak{l}, \gamma)))$ is a quotient of a generalized Verma module $M_{\mathcal{P}}(W)$, where $W \in \Lambda$ is a simple object (see, for example, [FKM, Section 2]).

Further, if L is a simple object in $\mathcal{O}(\mathcal{P}, \Lambda(V(\mathfrak{l}, \gamma)))$ (but it is not a simple $\mathfrak{G}$-module), we have that L has no non-trivial $\mathfrak{G}$-submodules, on which f act bijectively. Since f acts bijectively on L itself, we conclude that any $\mathfrak{G}$-submodule of L is a quotient of $M_{\mathcal{P}}(\tilde{W})$, where $\tilde{W}$ is a submodule in W. Now it is easy to see that the length of L equals the length of W and $E(L)$ is generated by a locally e-finite part of W. Since W is a simple object in Λ, we obtain that its length is bounded by 3 and the length of its locally e-finite part is bounded by 2. By Lemma 1 we know that $E(M)$ is non-trivial, which completes the proof. $\square$

For a simple object $W \in \Lambda$ we will denote by $L_P(W)$ the unique simple (as an object in $\mathcal{O}(\mathcal{P}, \Lambda)$) quotient of $M_P(W)$. We note, that $L_P(W)$ is a simple $\mathfrak{G}$-module if and only if W is a simple $\mathfrak{A}$-module. The next statement one more times generalizes the ideas used in the proof of Lemma 1:

Lemma 3. *Let $M \in \mathcal{O}(\mathcal{P}, \Lambda(V(\mathfrak{l}, \gamma)))$ and F be a finite-dimensional $\mathfrak{G}$-module. Then $E(M \otimes F) \simeq E(M) \otimes F$ (as $\mathfrak{G}$-modules).*

Proof. The inclusion $E(M) \otimes F \to M \otimes F$ factors through $E(M \otimes F)$. Exactness of $\_ \otimes F$ for a finite-dimensional F implies the assertion. $\square$

Lemma 4. *Let W be a simple object in $\Lambda(V(\mathfrak{l}, \gamma))$ and $M_{\mathcal{P}}(W) \in \mathcal{O}(\mathcal{P}, \Lambda(V(\mathfrak{l}, \gamma)))$. Then $E(M_{\mathcal{P}}(W))$ is a Verma module in $\mathcal{O}$.*

Proof. Recall that W is not simple and isomorphic to some $V(\hat{\lambda}, \hat{\gamma})$. We have that $E(M_{\mathcal{P}}(W)) \simeq M_{\mathcal{P}}(\hat{W})$, where $\hat{W}$ is the locally e-finite part of W. Since $\hat{W}$ is a Verma module over $\mathfrak{A}$ we obtain that $M_{\mathcal{P}}(\hat{W})$ is a Verma module over $\mathfrak{G}$. $\square$

The following statement describes the key property of E.

Proposition 1. *The functor E sends projective objects from $\mathcal{O}(\mathcal{P}, \Lambda(V(\mathfrak{l}, \gamma)))$ to projective modules in $\mathcal{O}$.*

Proof. First we consider the case $\mathfrak{G} = \mathfrak{A}$. According to [FKM, Section 10], an indecomposable projective object in Λ is either some $V(\hat{\lambda}, \hat{\gamma})$ or its selfextension of length two, which appears as a direct summand in $F \otimes V(\mathfrak{l}, \gamma)$ for some finite-dimensional F. If $V(\hat{\lambda}, \hat{\gamma})$ itself is projective, then $\hat{\gamma}$ is not the square of an integer and hence the corresponding e-finite part is the unique simple (=projective =Verma) module in the indecomposable block of category $\mathcal{O}$. This means that in this case the statement is true. If γ is the square of an integer, we can assume that $\gamma = 0$. Thus $V(\mathfrak{l}, \gamma)$ is projective in Λ (see [FKM, Section 10] again) and its e-finite part is projective in $\mathcal{O}$. In this case to obtain the statement we only need to recall that the functor $F \otimes \_$ is exact.

Now consider the general case. Recall the construction of projective modules in $\mathcal{O}$ and $\mathcal{O}(\mathcal{P}, \Lambda)$ ([BGG, Section 4], [FKM, Section 4]). In $\mathcal{O}(\mathcal{P}, \Lambda)$ any projective occures as a direct summund in the pojection on the corresponding block of a module having the form

$$P(V, k) = U(\mathfrak{G}) \bigotimes_{U(\mathcal{P})} \left((U(\mathfrak{N})/(U(\mathfrak{N})\mathfrak{N}^k)) \otimes V \right),$$

where k is a big enough positive integer and V is a projective in Λ. Clearly, E commutes with the induction from $\mathcal{P}$ to $\mathfrak{G}$. Applying now Lemma 3 we have

$$E(P(V, k)) = U(\mathfrak{G}) \bigotimes_{U(\mathcal{P})} \left((U(\mathfrak{N})/(U(\mathfrak{N})\mathfrak{N}^k)) \otimes E(V) \right).$$

Now $E(V)$ is projective in the corresponding $sl(2)$ category $\mathcal{O}$. From this and the construction of $P(V,k)$ it follows that if the projection of $P(V,k)$ on the block of $\mathcal{O}(\mathcal{P},\Lambda)$ is projective, then the corresponding projection of $E(P(V,k))$ on the block of $\mathcal{O}$ is also projective. To complete the proof, we just have to mention that E commutes with projections on blocks. $\square$

5 Further properties of E: the second equivalence

The aim of this section is to study E in more detail. In fact we will prove that E is a full functor and that it induces an equivalence between $\mathcal{O}(\mathcal{P},\Lambda)$ (Λ as in Section 4) and a full subcategory of $\mathcal{O}$. Till the end of the section we assume that $\Lambda = \Lambda(V(\mathfrak{l},\gamma))$, where $V(\mathfrak{l},\gamma)$ is a non-simple module as in Section 4.

Lemma 5. *Let M and N be in $\mathcal{O}(\mathcal{P},\Lambda)$ and $\varphi : M \to N$ be a non-zero homomorphism (of $\mathfrak{G}$-modules). Then the $\varphi(E(M)) \neq 0$.*

Proof. Let $m \in M$ be such that $\varphi(m) \neq 0$. Recall that f acts injectively (even bijectively) on M and N, and by Corollary 2 there exists a positive integer k such that $f^k \cdot m \in E(M)$. Now $\varphi(f^k \cdot m) = f^k \cdot \varphi(m) \neq 0$. $\square$

Corollary 3. *E produces an equivalence of $\mathcal{O}(\mathcal{P},\Lambda)$ with a (not necessary full) subcategory of $\mathcal{O}$.*

So far we have no evidence, why E should be a full functor. The rest of this section will be devoted to establishing this fact. According to the proof of Lemma 2, for any simple object $L \in \mathcal{O}(\mathcal{P},\Lambda)$ there exists a unique simple subquotient $\hat{L}$ of $E(L)$ on which f acts injectively. Since $E(L)$ belongs to $\mathcal{O}$, it is characterized by its highest weight, which we will denote by $\hat{E}(L)$. So, using the usual notation for the simple quotient of a Verma module ([D]), we can write that $\hat{L} \simeq L(\hat{E}(L))$. As usual, for a $\mathfrak{G}$-weight λ by $P(\lambda)$ we will denote the projective cover of $L(\lambda)$ (in $\mathcal{O}$). For a simple $L \in \mathcal{O}(\mathcal{P},\Lambda)$ denote by $P(L)$ its projective cover in $\mathcal{O}(\mathcal{P},\Lambda)$.

Lemma 6. *For any $M \in \mathcal{O}(\mathcal{P},\Lambda)$ and any simple L in $\mathcal{O}(\mathcal{P},\Lambda)$ holds $(M : L) = (E(M) : L(\hat{E}(L)))$.*

Proof. We can restrict our attention to a block of $\mathcal{O}(\mathcal{P},\Lambda)$, which is a module category over a projectively stratified algebra ([FKM, Theorem 3]). Now the statement follows from Corollary 2 by induction with respect to the poset indexing simple modules in our block. $\square$

From Lemma 6 one can easily deduce an analogue of the Kazhdan-Lusztig (Conjecture=) Theorem for generalized Verma modules $M_{\mathcal{P}}(W)$, where W is a simple object in Λ (this was proved first time in [KM]). In fact, we just reduce the problem to the known result for category $\mathcal{O}$ (see [BB, BK, S2]).

Corollary 4. *Let W be a simple object in Λ and L be a simple object in $\mathcal{O}(\mathcal{P},\Lambda)$. Then $(M_{\mathcal{P}}(W) : L) = (M(\hat{E}(L_{\mathcal{P}}(W))) : L(\hat{E}(L)))$.*

Proof. First we note that $E(M_{\mathcal{P}}(W))$ is a Verma module and $M(\hat{E}(L_{\mathcal{P}}(W))) \subset E(M_{\mathcal{P}}(W))$. Moreover, $E(M_{\mathcal{P}}(W))/M(\hat{E}(L_{\mathcal{P}}(W)))$ is a direct sum of finite-dimensional $\mathfrak{A}$-modules, hence $(E(M_{\mathcal{P}}(W)) : L(\hat{E}(L))) = (M(\hat{E}(L_{\mathcal{P}}(W))) : L(\hat{E}(L)))$. The rest follows from Lemma 6. $\square$

Lemma 7. *$E(M)$ is indecomposable if and only if M is indecomposable.*

Proof. Clearly, it is enough to prove that $E(M)$ is indecomposable as soon as M is indecomposable. Suppose that $E(M) = N_1 \oplus N_2$. Since f acts injectively on $E(M)$ it acts injectively on both N_i, $i = 1, 2$. Recall, that N_i, $i = 1, 2$ are $\mathfrak{G}$-submodules in M. Let M_i, $i = 1, 2$ denote the set of all $v \in M$ such that $f^k(v) \in N_i$ for some k big enough. From Corollary 1 it follows that $M = M_1 \oplus M_2$ as a vector space. Since f is a locally ad-nilpotent element in $\mathfrak{G}$ it follows that both M_i, $i = 1, 2$ are $\mathfrak{G}$-submodules. This contradicts the assumtion, that M is indecomposable. $\square$

Proposition 2. *$E(P(L)) \simeq P(\hat{E}(L))$.*

Proof. By Lemma 7, $E(P(L))$ is indecomposable, since so is $P(L)$. Hence we only need to compute the unique simple quotient N of $E(P(L))$. Let $L \simeq L_P(W)$ for some simple $W \in \Lambda$ and $P(W)$ be the projective cover of W in Λ. Clearly, N is isomorphic to the unique simple quotient of $E(M_{\mathcal{P}}(P(W)))$ and hence to the unique simple quotient of $M_{\mathcal{P}}(E(P(W)))$. Let W' be the top of $E(P(W))$. From the proof of Proposition 1 it follows that $M_P(W') \simeq M(\hat{E}(L))$ and thus $N \simeq L(\hat{E}(L))$. $\square$

Corollary 5. *Let $M \in \mathcal{O}(\mathcal{P},\Lambda)$ and Q be a projective object in $\mathcal{O}(\mathcal{P},\Lambda)$. Then*

$$\dim \mathrm{Hom}_{\mathcal{O}(\mathcal{P},\Lambda)}(Q, M) = \dim \mathrm{Hom}_{\mathcal{O}}(E(Q), E(M)).$$

Proof. Clearly, we can assume, that $Q \simeq P(L)$ for some simple $L \in \mathcal{O}(\mathcal{P},\Lambda)$. Then we have $\dim \mathrm{Hom}_{\mathcal{O}(\mathcal{P},\Lambda)}(P(L), M) = (M : L)$ and $\dim \mathrm{Hom}_{\mathcal{O}}(E(P(L)), E(M)) = (E(M) : L(\hat{E}(L)))$ by Proposition 2. Now the statement follows from Lemma 6. $\square$

Corollary 5 and Lemma 5 suggest that E should be a full functor. Now we are able to prove this.

Theorem 2. *For any $M, N \in \mathcal{O}(\mathcal{P},\Lambda)$ holds*

$$\dim \mathrm{Hom}_{\mathcal{O}(\mathcal{P},\Lambda)}(M, N) = \dim \mathrm{Hom}_{\mathcal{O}}(E(M), E(N)).$$

Proof. First we note that by Corollary 5 the statement is true, when M is a projective module in $\mathcal{O}(\mathcal{P},\Lambda)$. Clearly, we can assume that M is indecomposable. Let $P(M)$ be a projective cover of M. By virtue of Lemma 5 it is enough to prove that for any $\varphi : E(M) \to E(N)$ there is ψ in $\mathcal{O}(\mathcal{P},\Lambda)$ such that $\varphi = E(\psi)$. We have that $E(P(M))$ is a projective cover of $E(M)$. Let $a : E(P(M)) \to E(M)$ be a canonical epimorphism. By Lemma 5 and Corollary 5, there exists an epimorphism $x : P(M) \to M$ and a homomorphism $y : P(M) \to N$ such that $a = E(x)$ and $\varphi \circ a = E(y)$. For $m \in M$ set $\psi(m) = y \circ x^{-1}(m)$. We have to show that this is a well-defined map. But $\ker a \subset \ker \varphi \circ a$, hence $\ker x \subset \ker y$ since f acts bijectively on $P(M)$ and E acts on homomorphisms by restriction. This means that ψ is well-defined. Since both x and y are $\mathfrak{G}$-morphisms we deduce that ψ is also a $\mathfrak{G}$-morphism. Clearly, $E(\psi) = \varphi$, since E is just a restriction. This completes the proof of our theorem. $\square$

Corollary 6. *E is a full functor. In particular, $\mathcal{O}(\mathcal{P}, \Lambda)$ is equivalent to a full subcategory of $\mathcal{O}$. Moreover, the image of a block of $\mathcal{O}(\mathcal{P}, \Lambda)$ is contained in a block of $\mathcal{O}$.*

6 Analogue for $\mathcal{O}(\mathcal{P}, \Lambda)$ of Soergel's Endomorphism Theorem

Recall ([S2]), that the principal block of $\mathcal{O}$ is the block $\mathcal{O}_{triv}$, containing the trivial (one-dimensional) $\mathfrak{G}$-module. Let $L(\mu_i)$, $i = 1, 2, \dots, n$ be a complete list of simple modules in $\mathcal{O}_{triv}$. Note, that $A = \{\mu_i\}$ coinsides with the orbit of 0 under the dot action of the Weyl group of $\mathfrak{G}$ (see, for example [D] or [S2]). Following [S2] we will call the projective module $P(w_0 \cdot 0)$, where w_0 is the longest element in the Weyl group, the big projective module. Let γ be the square of an integer and denote by $\mathcal{O}(\mathcal{P}, \Lambda)_{triv}$ the direct summand of $\mathcal{O}(\mathcal{P}, \Lambda(V(\mathfrak{l}, \gamma)))$ which has a non-trivial image in $\mathcal{O}_{triv}$ under E. Since E acts blockwise and $\mathcal{O}_{triv}$ is indecomposable, such indecomposable $\mathcal{O}(\mathcal{P}, \Lambda)_{triv}$ is unique. Let L be a simple object in $\mathcal{O}(\mathcal{P}, \Lambda)$ such that $\hat{E}(L) = w_0 \cdot 0$. It exists, since f acts injectively on $L(w_0 \cdot 0) = M(w_0 \cdot 0)$. Now the results of the previous section (in fact, Corollary 5 and Lemma 5) make it possible to describe the endomorphism algebra of the "big projective" module $P(L)$ in $\mathcal{O}(\mathcal{P}, \Lambda)_{triv}$. This is a direct analogue of the Soergel's result for $\mathcal{O}_{triv}$ (which we also use in the proof).

Theorem 3. *Let $\Lambda = \Lambda(V(\mathfrak{l}, \gamma))$. Then* $\mathrm{End}_{\mathcal{O}(\mathcal{P},\Lambda)}(P(L)) \simeq \mathrm{End}_{\mathcal{O}}(P(w_0 \cdot 0))$. *In fact,* $\mathrm{End}_{\mathcal{O}(\mathcal{P},\Lambda)}(P(L))$ *is the coinvariant algebra (see [S2]).*

Proof. Follows from Corollary 5, Corollary 6, Lemma 5 and [S2, Endomorphismensatz 3]. $\square$

Consider $\mathcal{O}(\mathcal{P}, \Lambda(V(\lambda, \gamma)))$ for γ being the square of an integer and $\lambda \in \mathbb{C}/2\mathbb{Z}$. Let $x \in \mathbb{C}$ be such that θ_x moves $\mathcal{O}(\mathcal{P}, \Lambda(V(\mathfrak{l}, \gamma)))$ to $\mathcal{O}(P, \Lambda(V(\lambda, \gamma)))$. Call $\theta_x(P(L))$ the big projective module in the principal block $\theta_x(\mathcal{O}(\mathcal{P}, \Lambda(V(\mathfrak{l}, \gamma)))_{triv})$ of $\mathcal{O}(\mathcal{P}, \Lambda(V(\lambda, \gamma)))$.

Corollary 7. $\mathrm{End}_{\mathcal{O}(\mathcal{P},\Lambda(V(\lambda,\gamma)))}(\theta_x(P(L)))$ *is the coinvariant algebra.*

Proof. Follows from Theorem 3 and Theorem 1. $\square$

Remark 2. *It is easy to see, that, as in the category $\mathcal{O}$, the big projective module in $\mathcal{O}(\mathcal{P},\Lambda)$ can be characterized as the unique indecomposable projective in the principal block of $\mathcal{O}(\mathcal{P},\Lambda)$ such that any Verma (or, in notation of [FKM], generalized Verma) module from the principal block occurs as a subquotient in the standard filtration of this projective module.*

7 Analogue for $\mathcal{O}(\mathcal{P},\Lambda)$ of Soergel's Double Centralizer Theorem

The next result is analogous to Soergel's description of the algebra corresponding to the principal block of $\mathcal{O}$ (see [S2]). It is usually called the *double centralizer property.*

Theorem 4. *Let B denote the (projectively stratified finite-dimensional) algebra associated with $\mathcal{O}(\mathcal{P},\Lambda)_{triv}$. Then B is isomorphic to the endomorphism algebra of the big projective module, viewed as a module over its endomorphism ring.*

It is more convenient to prove this theorem in "abstract" notations which we are going to introduce now. Let A (resp. B) denote the algebra associated with the principal block of $\mathcal{O}$ (resp. $\mathcal{O}(\mathcal{P},\Lambda)$). We recall, that according to Section 5, B is a (matrix) subalgebra of A. Let e be the primitive idempotent of A such that Ae is the big projective module in $\mathcal{O}_{triv}$. Then Be is the big projective module in $\mathcal{O}(\mathcal{P},\Lambda)_{triv}$ and $C = eAe = eBe$ is the coinvarian algebra, which is the endomorphism algebra of Ae and Be. Let $T = \mathrm{Hom}_{\ A}(Ae, \_)$ denote the Soergel's functor ([S2]). Recall, that by Soergel's Theorem ([S2, Struktursatz 2]), for any $M \in \mathcal{O}_{triv}$ and any projective $Q \in \mathcal{O}_{triv}$ holds

$$\mathrm{Hom}_{\ A}(M,Q) \simeq \mathrm{Hom}_{\ C=eAe}(T(M),T(Q)).$$

Proof of Theorem 4. We start from $B = \mathrm{Hom}_{\ B}(B,B)$. Applying the results from Section 5, we have $\mathrm{Hom}_{\ B}(B,B) \simeq \mathrm{Hom}_{\ A}(E(B),E(B))$. Now applying the mentioned result by Soergel we obtain $\mathrm{Hom}_{\ A}(E(B),E(B)) \simeq \mathrm{Hom}_{\ eAe}(T(E(B)),T(E(B)))$. We know from Theorem 3, that $eAe = eBe$. Recall, that $E(Be) = Ae$, hence $T(E(B)) = \mathrm{Hom}_{\ A}(Ae,E(B)) = \mathrm{Hom}_{\ A}(E(Be),E(B)) \simeq \mathrm{Hom}_{\ B}(Be,B) = eB$. Finally,

$$\mathrm{Hom}_{\ eAe}(T(E(B)),T(E(B))) \simeq \mathrm{Hom}_{\ eBe}(eB,eB).$$

Now we note, that B is a matrix subalgebra of A and we can apply the duality on A to the last endomorphism ring, obtaining $\mathrm{Hom}_{\ eBe}(eB,eB) \simeq \mathrm{Hom}_{\ eBe}(Be,Be)$, which completes the proof. $\square$

Remark 3. *According to [FP] or [FKM, Section 12] there is a canonical duality on B which can also be applied directly in the proof of Theorem 4.*

8 Strong tilting modules in $\mathcal{O}$

The rest of the paper will be devoted to the construction and study of tilting modules in $\mathcal{O}(\mathcal{P},\Lambda)$. Since by [FKM, Theorem 3] any finite block of $\mathcal{O}(\mathcal{P},\Lambda)$ corresponds to a projectively stratified algebras, one can just use an abstract result from [AHLU] to state the existence and uniqueness of (characteristic) tilting module for each block of $\mathcal{O}(\mathcal{P},\Lambda)$. Their result generalizes Ringel's fundamental theorem [R] from quasi–hereditary to standardly stratified algebras.

However we are going to use a slightly more symmetric definition of tilting module, which is more natural in our case. This means that we will not be able to apply results from [AHLU] directly, in particular, we will have to prove the existence of tilting modules. Finally, we will determine the multiplicities of generalized Verma modules occuring in a standard filtration of an indecomposable tilting module, thus determining the character of this tilting module. This generalizes the recent result of Soergel ([S3]).

First we recall the notion of tilting modules for $\mathcal{O}$ and study them from another point of view. The Chevalley anti–involution σ on $\mathfrak{G}$ give rises to a duality (i.e. an exact contravariant equivalence, preserving simple objects) on $\mathcal{O}$ (see, for example [J, Section 4.10]). For $M\in\mathcal{O}$ we will denote by M^* the corresponding dual module in $\mathcal{O}$. We note, that $L(\lambda)^*\simeq L(\lambda)$ for any $\lambda\in\mathfrak{H}^*$. Let $\mathcal{F}(\Delta)$ (resp. $\mathcal{F}(\bigtriangledown)$) denote the full subcategory of $\mathcal{O}$, which consists of all modules in $\mathcal{O}$ having a Verma flag, i.e. a filtration whose subquotients are Verma modules (resp. a dual Verma flag, i.e. a filtration, whose subquotients are $M(\lambda)^*$, $\lambda\in\mathfrak{H}^*$). A module $M\in\mathcal{O}$ is called tilting module, if it belongs to $\mathcal{F}(\Delta)\cap\mathcal{F}(\bigtriangledown)$. It is known, that indecomposable tilting modules are naturaly parametrized by simple modules, hence by $\lambda\in\mathfrak{H}^*$ ([R]). We will denote by $T(\lambda)$ the indecomposable tilting module which corresponds to $\lambda\in\mathfrak{H}^*$ (i.e. whose Verma flag starts with $M(\lambda)$).

Let α denote the simple root of $\mathfrak{G}$ which corresponds to the subalgebra $\mathfrak{A}$ and s_α denote the corresponding simple reflection on $\mathfrak{H}^*$. Suppose that $\lambda\in\mathfrak{H}^*$ is such that $s_\alpha(\lambda)=\lambda+k\alpha$ for some $k\in\mathbb{N}$. Consider the indecomposable projective module $P(\lambda)\in\mathcal{O}$, Clearly, there exists a Verma flag $P(\lambda)=P_0\supset P_1\supset P_2\supset\dots$ of $P(\lambda)$ such that $P_0/P_1\simeq M(\lambda)$ and $P_1/P_2\simeq M(s_\alpha(\lambda))$. Set $\hat{P}(\lambda)=P(\lambda)/P_2$. Then $\hat{P}(\lambda)$ has a Verma flag with $M(\lambda)$ on the top and $M(s_\alpha(\lambda))$ on the bottom. Define a class $K(\alpha)$ of modules in $\mathcal{O}$ as follows: if $\lambda\in\mathfrak{H}^*$ is such that $s_\alpha(\lambda)-\lambda\notin(\mathbb{Z}\alpha\setminus\{0\})$ then $K(\alpha)$ contains $M(\lambda)$; in the other case $K(\alpha)$ contains $\hat{P}(\lambda)$ if $s_\alpha(\lambda)-\lambda\in\mathbb{N}\alpha$ and $K(\alpha)$ contains $\hat{P}(s_\alpha(\lambda))$ if $\lambda-s_\alpha(\lambda)\in\mathbb{N}\alpha$.

Denote by $\mathcal{F}(\Delta)_\alpha$ (resp. $\mathcal{F}(\bigtriangledown)_\alpha$) the full subcategory of $\mathcal{O}$, containing all modules which admit a filtration with subquotients from $K(\alpha)$ (resp. with subquotients of the form M^*, $M\in K(\alpha)$). Since any module in $K(\alpha)$ has a Verma flag, we have $\mathcal{F}(\Delta)_\alpha\subset\mathcal{F}(\Delta)$, $\mathcal{F}(\bigtriangledown)_\alpha\subset\mathcal{F}(\bigtriangledown)$ and $\mathcal{O}(K(\alpha))=\mathcal{F}(\Delta)_\alpha\cap\mathcal{F}(\bigtriangledown)_\alpha\subset\mathcal{F}(\Delta)\cap\mathcal{F}(\bigtriangledown)$. Hence any module in $\mathcal{O}(K(\alpha))$ (if there is any) is a tilting module. So to determine $\mathcal{O}(K(\alpha))$ we have to find out which indecomposable tilting modules belong to it.

Lemma 8. *For any $M\in K(\alpha)$ (resp. M such that $M^*\in K(\alpha)$) and any finite-dimensional $\mathfrak{G}$-module F the module $F\otimes M$ belongs to $\mathcal{F}(\Delta)_\alpha$ (resp. $\mathcal{F}(\bigtriangledown)_\alpha$).*

Proof. Follows from the exactness of $F\otimes \_$ by standard arguments combined with the observation that we are considering objects which are induced from projective objects. □

Proposition 3. *$T(\lambda) \in \mathcal{O}(K(\alpha))$ if and only if either $s_\alpha(\lambda) - \lambda \notin \mathbb{Z}\alpha \setminus \{0\}$ or $\lambda - s_\alpha(\lambda) \in \mathbb{N}\alpha$.*

Proof. If $s_\alpha(\lambda) - \lambda \notin (\mathbb{Z}\alpha \setminus \{0\})$ then, according to the definition of $K(\alpha)$, any Verma module (resp. dual Verma module), occuring in the Verma flag (resp. dual Verma flag) of $T(\lambda)$ belongs to $K(\alpha)$ (resp. is of the form M^* for some $M \in K(\alpha)$). Hence $T(\lambda) \in \mathcal{O}(K(\alpha))$.

Recall that any increasing Verma flag of $T(\lambda)$ starts with $M(\lambda)$. From the definition of $K(\alpha)$ it follows that for any $\lambda \in \mathfrak{H}^*$ such that $s_\alpha(\lambda) - \lambda \in \mathbb{N}\alpha$ there are no modules in $K(\alpha)$ such that their increasing Verma flag starts with $M(\lambda)$. Since any filtration with quotients from $K(\alpha)$ can be extended to a Verma flag, we obtain that in the case $s_\alpha(\lambda) - \lambda \in \mathbb{N}\alpha$ the module $T(\lambda)$ cannot belong to $K(\alpha)$.

So we only have to prove that $T(\lambda) \in K(\alpha)$ in case $\lambda - s_\alpha(\lambda) \in \mathbb{N}\alpha$. This will follow easily if we recall the inductive construction of tilting modules via tensoring with finite-dimensional modules. Suppose that λ is such that $\lambda - s_\alpha(\lambda) \in \mathbb{N}\alpha$ and $M(s_\alpha(\lambda))$ is simple. Then $T(\lambda) \simeq \hat{P}(s_\alpha(\lambda))$ by the construction of $\hat{P}(s_\alpha(\lambda))$ and hence $T(\lambda) \in \mathcal{F}(\Delta)_\alpha$. But $T(\lambda)$ is also self-dual as a tilting module in the category $\mathcal{O}$, hence $T(\lambda) \in \mathcal{F}(\bigtriangledown)_\alpha$. Finally, $T(\lambda) \in \mathcal{O}(K(\alpha))$.

Now we note, that from Lemma 8 it follows that $\mathcal{O}(K(\alpha))$ is stable under tensoring with finite-dimensional modules. In particular, it means that if we fix λ as in the previous paragraph, then $T(\lambda) \otimes F$ belongs to $\mathcal{O}(K(\alpha))$ for any finite-dimensional $\mathfrak{G}$-module F. To complete the proof we only need to recall that any $T(\mu)$ with $\mu - s_\alpha(\mu) \in \mathbb{N}\alpha$ occurs as a direct summand in $T(\lambda) \otimes F$ for some λ as in the previous paragraph and some finite-dimensional F ([CI]). □

The modules in $\mathcal{O}(K(\alpha))$ will be called *strong tilting modules*. Later we will see that they are closely related to tilting modules in $\mathcal{O}(\mathcal{P}, \Lambda)$.

Corollary 8. *The big projective module is a strong tilting module.*

Proof. Obvious. □

9 Tilting modules in $\mathcal{O}(\mathcal{P}, \Lambda)$

Let $\Lambda = \Lambda(V(\mathfrak{l}, \gamma))$ as in the previous sections. In order to introduce the notion of a tilting module in $\mathcal{O}(\mathcal{P}, \Lambda)$ we need a natural duality on $\mathcal{O}(\mathcal{P}, \Lambda)$. This can be easily done using σ for $\mathcal{O}(\mathcal{P}, \Lambda(V(\lambda, \gamma)))$ in the case, when $V(\lambda, \gamma)$ is a simple $\mathfrak{A}$-module (see [FP] or [FKM, Section 12]). The same direct construction for the case $\mathcal{O}(\mathcal{P}, \Lambda(V(\mathfrak{l}, \gamma)))$ does not work, because dualization does not preserve the bijectivity of the action of f. In fact, e acts bijectively on the dual module. There are two ways to solve this problem. The first way is to fix a non-integer x and to define a duality * on $\mathcal{O}(\mathcal{P}, \Lambda)$ as the composition of θ_x, the natural duality on $\theta_x(\mathcal{O}(\mathcal{P}, \Lambda))$, which can be constructed via σ (here everything works since both e and f act bijectively on $\theta_x(\mathcal{O}(\mathcal{P}, \Lambda)))$, and θ_{-x}. The second way is to compose σ with the natural automorphism of $\mathfrak{G}$ corresponding to the simple reflection s_α. We choose the second way and from now on for $M \in \mathcal{O}(\mathcal{P}, \Lambda)$ we will denote by M^* the corresponding dual module.

Let $\mathcal{G}(\Delta)$ (resp. $\mathcal{G}(\bigtriangledown)$) denote the full subcategory of $\mathcal{O}(\mathcal{P}, \Lambda)$ which consists of all modules having standard filtration, i.e. a filtration, whose subquotients are isomorphic to $M_{\mathcal{P}}(W)$, where W is projective in Λ (resp. a dual standard filtration, i.e. a filtration, whose subquotients are isomorphic to $M_{\mathcal{P}}(W)^*$, where W is projective in Λ). A module $M \in \mathcal{O}(\mathcal{P}, \Lambda)$ will be called a tilting module if $M \in \mathcal{G}(\Delta) \cap \mathcal{G}(\bigtriangledown)$.

So far we do not know if there is any tilting module in $\mathcal{O}(\mathcal{P}, \Lambda)$. The aim of this section is to describe all tilting module in $\mathcal{O}(\mathcal{P}, \Lambda)$. We recall that our definition of tilting module does not coincide with the general definition, used in [AHLU]. The difference is in the definition of $\mathcal{G}(\bigtriangledown)$. In [AHLU], the existence of a filtration is required, whose subquotients are isomorphic to $M_{\mathcal{P}}(W)^*$, where W is simple in Λ. Our condition is more restrictive. Taking into account the uniqueness of characteristic tilting module for standardly stratified algebras (this class includes, in particular, projectively stratified algebras) in [AHLU], we only have to show that for any simple $L = L_{\mathcal{P}}(W) \in \mathcal{O}(\mathcal{P}, \Lambda)$ there exists an indecomposable tilting module $T(L) \in \mathcal{O}(\mathcal{P}, \Lambda)$ such that the standard filtration of $T(L)$ starts with $M_{\mathcal{P}}(W)$.

Lemma 9. *For any $M \in \mathcal{G}(\Delta)$ (resp. $M \in \mathcal{G}(\bigtriangledown)$) and any submodule N occuring in a standard filtration (resp. dual standard filtration) of M holds $E(N) \subset E(M)$ and $E(M/N) \simeq E(M)/E(N)$.*

Proof. Follows from the definition of E and the fact that $M \simeq N \oplus (M/N)$ as an $\mathfrak{A}$-module. $\square$

Lemma 10. *Let T be a tilting module in $\mathcal{O}(\mathcal{P}, \Lambda)$. Then $E(T)$ is a strong tilting module in $\mathcal{O}$.*

Proof. From the definition of $K(\alpha)$ it follows immediately, that for any projective $W \in \Lambda$ holds $E(M_{\mathcal{P}}(W)) \in K(\alpha)$. Now, by Lemma 9, the standard (resp. dual standard) filtration of T is sent by E to a filtration with subquotients from $K(\alpha)$ (resp. with subquotients, dual to modules in $K(\alpha)$). This completes the proof. $\square$

Lemma 11. *For any strong tilting module $T' \in \mathcal{O}$ there exists a tilting module $T \in \mathcal{O}(\mathcal{P}, \Lambda)$ such that $E(T) \simeq T'$.*

Proof. Clearly, it is enough to prove this statement for indecomposable T', so we can suppose that $T' = T(\lambda)$. First, assume that $M(s_\alpha(\lambda))$ is a simple $\mathfrak{G}$-module. Clearly, $M(\lambda)$ belongs to the image of E, hence $M(\lambda) = E(M_{\mathcal{P}}(W))$ for some simple object $W \in \Lambda$. Let W' be the projective cover of W. From the definition of $K(\alpha)$ one immediately obtains $T(\lambda) = E(M_{\mathcal{P}}(W'))$. Now the statement follows from Lemma 3, the inductive construction of strong tilting modules as in the proof of Proposition 3 and the remark that tilting modules in $\mathcal{O}(\mathcal{P}, \Lambda)$ are self-dual. $\square$

Theorem 5. *For any simple object $L = L_{\mathcal{P}}(W) \in \mathcal{O}(\mathcal{P}, \Lambda)$ there exists a unique indecomposable tilting module $T(L) \in \mathcal{O}(\mathcal{P}, \Lambda)$ such that the standard filtration of $T(L)$ starts with $M_{\mathcal{P}}(W')$, where W' is a projective cover of W in Λ. The set $T(L)$, where L runs through simple modules in $\mathcal{O}(\mathcal{P}, \Lambda)$ is a complete set of indecomposable tilting modules in $\mathcal{O}(\mathcal{P}, \Lambda)$. Any tilting module is a finite direct sum of indecomposable tilting modules.*

Proof. Existence follows from Lemma 11. The rest follows from [AHLU, 2.1 and 2.2]. □

10 Characters of tilting modules: Analogue for $\mathcal{O}(\mathcal{P}, \Lambda)$ of Soergel's Character Formula

In Section 9 we proved the existence of tilting modules in $\mathcal{O}(\mathcal{P}, \Lambda)$. The aim of this section in to determine the formal character of a tilting module. Clearly, it is sufficient to do this for an indecomposable module $T(L)$, where L is a simple module in $\mathcal{O}(\mathcal{P}, \Lambda)$. Further, by the definition, $T(\lambda)$ has a standard filtration, hence it has a filtration by $M_{\mathcal{P}}(W)$, where $W \in \Lambda$ is a simple object. Since $M_{\mathcal{P}}(W)$ is an extension of two Verma modules (with respect to a different basis in $\mathfrak{G}$), its character is known. So the problem is to determine the multiplicities $[T(L) : M_{\mathcal{P}}(W)]$. We solve this problem by reducing it to the recently solved analogous problem for $\mathcal{O}$ (see [S3]).

Theorem 6. *Let W_1 and W_2 be simple objects in Λ. Denote by $l(W_2')$ the length of the projective cover W_2' of W_2 in Λ. Then*

$$[T(L_{\mathcal{P}}(W_1)) : M_{\mathcal{P}}(W_2)] = l(W_2')[E(T(L_{\mathcal{P}}(W_1))) : E(M_{\mathcal{P}}(W_2))].$$

Proof. Set $m = [T(L_{\mathcal{P}}(W_1)) : M_{\mathcal{P}}(W_2)]$. Then $m = l(W_2')[T(L_{\mathcal{P}}(W_1)) : M_{\mathcal{P}}(W_2')]$ and by Lemma 9, $[T(L_{\mathcal{P}}(W_1)) : M_{\mathcal{P}}(W_2')] = [E(T(L_{\mathcal{P}}(W_1))) : E(M_{\mathcal{P}}(W_2'))]$. If $l(W_2') = 1$ then $W_2 = W_2'$ and we are done. Otherwise, it follows from the definition of $K(\alpha)$, that the number of Verma modules in a Verma flag of $E(M_{\mathcal{P}}(W_2'))$ equals 2, moreover, $[E(M_{\mathcal{P}}(W_2')) : E(M_{\mathcal{P}}(W_2))] = 1$. This completes the proof. □

Remark 4. *According to Lemma 10, $E(T(L_{\mathcal{P}}(W_1)))$ is a strong tilting module. In particular, it is a tilting module in $\mathcal{O}$. Furthermore, $E(M_{\mathcal{P}}(W_2))$ is a Verma module in $\mathcal{O}$, hence, the multiplicity $[E(T(L_{\mathcal{P}}(W_1))) : E(M_{\mathcal{P}}(W_2))]$ can be computed by Soergel's Theorem, [S3, Theorem 5.12 and Theorem 6.7].*

Remark 5. *Applying the functors θ_x one extends the above results to an arbitrary category $\mathcal{O}(\mathcal{P}, \Lambda(V(l, \gamma)))$.*

Finally, if one looks at the proof of [S3, Theorem 2.1], one sees that it implies another interesting result for the principal block $\mathcal{O}(\mathcal{P}, \Lambda)_{triv}$ of $\mathcal{O}(\mathcal{P}, \Lambda)$. We keep the notation from [S3]. Let $S = S_\delta$ denote the semi-regular bimodule, associated with a semi-infinite character δ. As it was shown in [S3], the functor $S \otimes_{U(\mathfrak{G})} \_$ maps an indecomposable projective $P(\lambda)$, $\lambda \in \mathfrak{H}^*$ into an indecomposable tilting module $T(w_0(\lambda)) \in \mathcal{O}$, wher w_0 is the longest element of the Weyl group. Comparing Proposition 2 with the definition of strong tilting module we see that for any indecomposable projective module $P(L) \in \mathcal{O}(\mathcal{P}, \Lambda)$ the module $S \otimes_{U(\mathfrak{G})} E(P(L))$ is an indecomposable strong tilting module. If we recall Lemma 10 and Lemma 11 and the fact that $S \otimes_{U(\mathfrak{G})} \_$ is an equivalence of certain categories ([S3, Section 2]) which preserves short exact sequences, we obtain the following result:

Theorem 7. *The projectively stratified algebra of $\mathcal{O}(\mathcal{P}, \Lambda)_{triv}$ is its own Ringel dual (see [R, KK] for detail).*

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References

[AHLU] *I.Agoston, D.Happel, E.Lukacs and L.Unger*, Standardly stratified algebras and tilting, to appear.

[BB] *A. Beilinson and I. Bernstein,* Localisation de $\mathfrak{G}$–modules. C. R. Acad. Sci. Paris 292, 15–18 (1981).

[BGG] *I.N.Bernstein, I.M.Gelfand, S.I.Gelfand*, On certain category of $\mathfrak{G}$-modules, Funkt. anal. i ego prilozh., 10 (1976), 1-8.

[BK] *J.-L.Brylinski and M.Kashiwara*, Kazhdan-Lusztig conjecture and holonomic systems. Invent. Math. 64 (1981), 387-410.

[CPS] *E.Cline, B.Parshall, L.Scott*, Finite dimensional algebras and highest weight categories, J. Reine Angew Math. 391 (1988), 85-99.

[CF] *A.J.Coleman, V.Futorny*, Stratified L-modules, Journal of Algebra 163 (1994), 219-234.

[CI] *D.H.Collingwood and R.S.Irving*, A decomposition theorem for certain self-dual modules in the category $\mathcal{O}$, Duke Math. J. 58 (1989), 89-102.

[D] *J.Dixmier*, Algèbres Enveloppantes, Paris, 1974.

[FKM] *V.Futorny, S.König and V.Mazorchuk*, Categories of induced modules and projectively stratified algebras, Preprint 99-024, Bielefeld University.

[FP] *V.Futorny, D.Pollack*, A new category of Lie algebra modules satisfying the BGG-reciprocity principle, Comm. in Algebra, 22(1), 1994, 213-227.

[FM1] *V.Futorny, V.Mazorchuk*, Structure of α-stratified modules for finite-dimensional Lie algebras. I. J. Algebra 183 (1996), no. 2, 456-482.

[FM2] *V.Futorny, V.Mazorchuk*, Highest weight categories of Lie algebra modules, J. Pure and Appl. Alg., to appear.

[J] *J.C.Jantzen*, Einhüllende Algebren halbeinfacher Lie Algebren, Springer-Verlag, Berlin, 1983

[KM] *A.Khomenko and V.Mazorchuk*, On multiplicities of simple subquotients in generalized Verma modules, to appear.

[KK] *M.Klucznik and S.König*, Characteristic tilting modules over quasi-hereditary algebras, course notes, available from: http://www.mathematik.uni-bielefeld.de/~koenig/

[K] *B.Kostant*, On the tensor product of a finite and infinite dimensional representations, Journal of Func. Analysis. 20 (1975), 257-285.

[M] *O.Mathieu*, Classification of irreducible weight modules, Preprint, Strasbourg, 1997.

[R] *C.M.Ringel*, The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences, Math. Z. 208 (1991), 209-233.

[S1] *W.Soergel*, Combinatorics of Harish-Chandra modules, in Proceedings of "Representation theorie and algebraic geometry" (Montreal, PQ, 1997, 401-412, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 514, Kluwer Acad. Publ., Dordrecht, 1998.

[S2] *W. Soergel*, Kategorie $\mathcal{O}$, perverse Garben und Moduln über den Koinvarianten zur Weylgruppe. (in German) [Category $\mathcal{O}$, perverse sheaves and modules over the coinvariants for the Weyl group], J. Amer. Math. Soc. 3 (1990), no. 2, 421-445.

[S3] *W.Soergel*, Character formulas for tilting modules over Kac-Moody algebras. Represent. Theory 2 (1998), 432-448 (electronic).

Vyacheslav Futorny, Universidade de Sao Paulo, Instituto de Matematica e Estatistica, Caixa Postal 66281 - CEP 05315-970, Sao Paulo, Brasil, e-mail: `futorny@ime.usp.br`

Steffen König, Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501, Bielefeld, FRG, e-mail: `koenig@mathematik.uni-bielefeld.de`

Volodymyr Mazorchuk, Mechanics and Mathematics Department, Kyiv Taras Shevchenko University, 64 Volodymyrska st., 252033, Kyiv, Ukraine,
e-mail: `mazorchu@uni-alg.kiev.ua`
Address until 31 December 1999: Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501, Bielefeld, FRG, e-mail: `mazor@mathematik.uni-bielefeld.de`