On the remarkable properties of the hyperbolic Whitehead link cone-manifold *

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Abstract

Denote by $W(m,n)$ the hyperbolic cone-manifold whose underlying space is the 3-sphere and singular geodesics are formed by two components of the Whitehead link with cone angles $2\pi/m$ and $2\pi/n$. The aim of the paper is to establish the Tangent and Sine Rules relating the complex lengthes of the singular geodesics and the cone angles of $W(m,n)$. An explicit upper bound for the real length of the singular geodesic is also given.

Keywords: hyperbolic orbifold, hyperbolic cone-manifold, complex length, the Tangent Rule, the Sine Rule

1. Introduction

A 3–dimensional *hyperbolic cone–manifold* is a riemannian 3–dimensional manifold of constant negative sectional curvature with cone-type singularity along simple closed geodesics. To each component of singular set we associate a real number $n \geq 1$ such that the cone-angle around the component is $2\pi/n$. The concept of the hyperbolic cone-manifold generalizes the hyperbolic manifold which appears in the partial case when all cone-angles are 2π. The hyperbolic cone-manifold is also a generalization of hyperbolic 3–orbifold which arises when all associated numbers n are integers.

Denote by $W(m,n)$ the cone-manifold whose underlying space is the 3-sphere and whose singular set consists of two components of the Whitehead link with cone angles $2\pi/m$ and $2\pi/n$ (see Fig.1).

We remark that $W(n,m) = W(m,n)$ by definition and the properties of the Whitehead link.

In the partial case when $m,n \in \{3,4,\ldots\}$ the properties of $W(m,n)$ were investigated by many authors [1], [2], [3], [4]. In this case the orbifold $W(m,n)$ can be represented in the form $W(m,n) = \mathbb{H}^3/\Gamma$, where Γ is a discrete cocompact group of the isometries of the 3 –dimensional hyperbolic space $\mathbb{H}^3$ with the following presentation

$$\Gamma = \langle S,T : S^m = T^n = 1,\ SL = LS,\ L = STS^{-1}T^{-1}ST^{-1}S^{-1}T\rangle. \tag{1.1}$$

*Supported by the RFFI (grant 99-01-00630)

H.Helling, A.Kim, J.Mennicke [1] had constructed the fundamental set for the group Γ in $\mathbb{H}^3$ in the case $m = n$ and had investigated the geometrical and arithmetical properties of 3 – dimensional hyperbolic manifolds arising as cyclic n–fold coverings of $W(n,n)$. H.M.Hilden, M.T.Lozano and J.M.Montesinos [2], [4] completely investigated the hyperbolicity and arithmeticity of the orbifold $W(m,n)$ for all $m, n \in \mathbb{N}$. In the paper [5] a simple construction is given for the fundamental set of the orbifold $W(m,n)$; that is bounded by two pairs of curvilinear faces mutually identified by S and T.

In this paper we shall prove the following theorems.

Theorem A (The Tangent Rule). *Let $\gamma_m = \gamma_m(n)$ be a complex length of the singular geodesic of the hyperbolic cone-manifold $W(m,n)$ with cone angle $2\pi/m$. Then*

$$\frac{\tanh\frac{\gamma_m}{4}}{\tanh\frac{\gamma_n}{4}} = \frac{\tan\frac{\pi}{m}}{\tan\frac{\pi}{n}}.$$

Theorem B (The Sine Rule). *Let $\gamma_m = l_m + i\,\varphi_m$ be a complex length of the singular geodesic of the hyperbolic cone-manifold $W(m,n)$ with cone angle $2\pi/m$. Then*

$$\frac{\sin\frac{\varphi_m}{2}}{\sinh\frac{l_m}{2}} = \frac{\sin\frac{\varphi_n}{2}}{\sinh\frac{l_n}{2}}.$$

Theorem C. *Let $l_m = l_m(n)$ be a real length of the singular geodesic of the hyperbolic cone-manifold $W(m,n)$ with cone angle $2\pi/m$. Then*

$$\cosh l_m(n) < 17.$$

The upper bound is sharp and $\cosh l_m(n) \to 17$ *as* $m \to 2$ *and* $n \to \infty$.

2. The complex distances and lengthes

Let $\mathbb{H}^3 = \{(z,\xi) \in \mathbb{C} \times \mathbb{R} : \xi > 0\}$ be the upper half model of the 3 -dimensional hyperbolic space endowed by the Riemannian metric $ds^2 = \dfrac{dzd\overline{z} + d\xi^2}{\xi^2}$. We indentify the group of orientation preserving isometries of $\mathbb{H}^3$ with the group $PSL(2,\mathbb{C})$ consisting of linear fractional transformations

$$A : z \in \mathbb{C} \to \frac{az+b}{cz+d}.$$

By the canonical procedure the linear transformation A can be uniquely extended to the isometry of $\mathbb{H}^3$. We prefer to deal with the matrix $\widetilde{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$ rather than the element $A \in PSL(2,\mathbb{C})$. The matrix $\widetilde{A}$ is uniquely up to sign determined by the element A. If there will be no confusions we shall use the same letter A for both A and $\widetilde{A}$.

Recall some definitions and results from the book [6]. A matrix $l \in SL(2,\mathbb{C})$ satisfying $tr(l) = 0$ will be called a (normalized) *line matrix.* We have from definition $l^2 = -I$, where I is the identity matrix. Hence any line matrix determines a half-turn about a line in $\mathbb{H}^3$, and this line determines matrix up to sign. According to [6, p.63] there exists a natural one-to-one correspondence between line matrices and oriented lines in $\mathbb{H}^3$. Hereby, if a line matrix l deternines an oriented line $[e, e']$ with and points e and e', then the line matrix $(-l)$ detrnimes the line $[e', e]$. Moreover, if matrix $f \in SL(2,\mathbb{C})$ is consided as a motion of $\mathbb{H}^3$, then the matrix lfl^{-1} determines the line $[f(e), f(e')]$.

Let A and B be oriented lines determined by line matrices a and b. A complex number μ is called a *complex distance* from A to B if it real part $\Re\,\mu$ is the distance from A to B, and it imagimary part $\Im\,\mu$ is the angle from L to M in the obvious sence. We get [6, p.68]

$$\cosh \mu = -\frac{1}{2}\, tr(ab). \tag{2.1}$$

From now on, all lines in this paper will be supposed to be oriented.

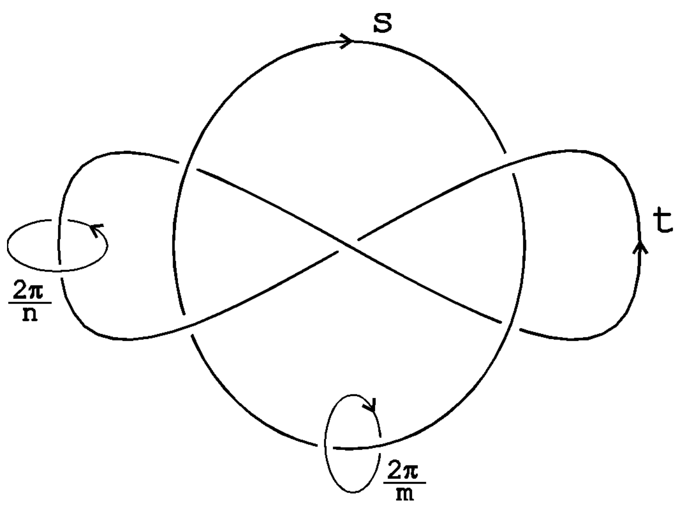

Fig.1. The Whitehead link cone-manifold $W(n, m)$

Let us investigate the hyperbolic structure of the cone-manifold $W = W(m, n)$. The singular set $\Sigma = \Sigma^1 \cup \Sigma^2$ of W consists of two components $\Sigma^1 = \Sigma^1(m)$ and $\Sigma^2 = \Sigma^2(n)$ with cone-angles $2\pi/m$ and $2\pi/n$ respectively. W defines a nonsingular but incomplete hyperbolic manifold $N = W - \Sigma$. The fundamental group of the manifold N has the following presentation

$$\Phi = \langle s, t : \; sl = ls, \; l = sts^{-1}t^{-1}st^{-1}s^{-1}t \rangle,$$

where s and t are meridians of the components Σ^1 and Σ^2 respectively, and l is a longitude of Σ^1.

The hyperbolic structure of N defines, up to congugation in $PSL(2,\mathbb{C})$, a holonomy homomorphism

$$\hat{h} = \hat{h}_{m,n} : \Phi \to PSL(2,\mathbb{C}).$$

The images $\hat{h}(s)$ and $\hat{h}(t)$ of meridians s and t are rotations in $\mathbb{H}^3$ on angles $2\pi/m$ and $2\pi/n$ respectively. It is shown in [7], see also [3] and [8], that the monodromy homomorphism of a compact orientable cone-orbifold can be lifted to $SL(2,\mathbb{C})$. Denote by $h = h_{m,n} : \Phi \to SL(2,\mathbb{C})$ this lifting homomorphism and set $\Gamma = \Gamma_{m,n} = h_{m,n}(\Phi)$. The group $\Gamma = \langle S, T\rangle$ is generated by two matricies S and T with the following properties:

$$tr(S) = 2\cos\frac{\pi}{m},\ tr(T) = 2\cos\frac{\pi}{n},$$

$$SL = LS, L = STS^{-1}T^{-1}ST^{-1}S^{-1}T. \tag{2.2}$$

Recall that subgroup G of $SL(2,\mathbb{C})$ is called *elementary* if it has a finite orbit in $\mathbb{H}^3 \cup \overline{\mathbb{C}}$. We remark the group (2.2) is non-elementary and not necassary discrete. According to [2] there is a one-to-one correspondence between hyperbolic structures on the cone-manifold $W(m,n)$ and conjugacy classes of the presentations $h_{m,n} : \Phi \to SL(2,\mathbb{C})$, where the group $\Gamma = \Gamma_{m,n}$ is non-elementary and is not conjugated to a subgroup of $SL(2,\mathbb{R})$.

Introduce the following definitions:

Definition 2.1. Let A be an isometry of $\mathbb{H}^3$ different from identity. It has two fixed points u and v in $\overline{\mathbb{C}}$ which may coincide. Consider the oriented axis $[u,v]$. A complex number $\delta(A)$ is call to be the *displacement* of A if its real part is the distance which the axis is translated and its imaginary part is the angle through which the half–planes bounded by the axis are rotated.

The isometry A without an orientation of its axis determines $\delta(A)$ only up to sign. By [6, p.46] for the isometry is given by matrix $A \in SL(2,\mathbb{C})$ we have

$$2\cosh\delta(A) = \mathrm{tr}(\mathrm{A}^2) = \mathrm{tr}^2(\mathrm{A}) - 2. \tag{2.3}$$

Definition 2.2. A *complex length* $\gamma_m = \gamma_m(n)$ of the singular component $\Sigma^1 = \Sigma^1(m)$ of the cone-manifold $W(m,n)$ is defined as displacement of the isometry L of $\mathbb{H}^3$, where $L = STS^{-1}T^{-1}ST^{-1}S^{-1}T$ is represented by the longitude of Σ^1.

Similary, the a *complex lenght* $\gamma_n = \gamma_n(m)$ of the singular component $\Sigma^2 = \Sigma^2(n)$ of the cone-manifold $W(m,n)$ is defined as displacement of the isometry V of $\mathbb{H}^3$, where $V = TST^{-1}S^{-1}TS^{-1}T^{-1}S$ is represented by the longitude of Σ^2,

By [6, p.46] we have

$$\cosh\gamma_m = \frac{1}{2}\,\mathrm{tr}\,(L^2) \tag{2.4}$$

and

$$\cosh\gamma_n = \frac{1}{2}\,\mathrm{tr}\,(V^2). \tag{2.5}$$

We note [9, p.38] that the meridian-longitude pair {s, l} is uniquely determined up to a common conjugating element of the group Φ. Hence, the complex length

$\gamma_m = l_m + i\ \varphi_m$ is uniquely determined up to sign and (mod $2\pi i$) by the above definition.

We need two conventions to choose correctly real and imaginary parts of γ_m. The first convention is the following. Since $\Sigma^1(m)$ does not shrink to the point, $l_m \neq 0$. Hence, we choose γ_m in such a way that $l_m = \Re\gamma_m > 0$. The second convention is concerned with imaginary part $\varphi_m = \Im\gamma_m$. We want to choose φ_m such that the following identity holds

$$\cosh\frac{\gamma_m}{2} = -\frac{1}{2}\,\mathrm{tr}(L) \tag{2.6}$$

By virtue of identity $\mathrm{tr}(L)^2 - 2 = \mathrm{tr}(L^2)$ equality (2.4) is a consiquence of (2.6). The converse, in general, is true only up to sign. Under the second convention (2.4) and (2.6) are equivalent.

Remark 2.3. There is a natural geometrical reason for choosing minus in (2.6). Indeed, it will be shown in Lemma 2.7 that $L = -D^{-2}$, where D is a matrix in $SL(2,\mathbb{C})$ such that $DSD^{-1} = S$ and $DTD^{-1} = T^{-1}L$. Geometrically speaking, this means that D defines an order two isometry of the cone-manifold $W(m,n)$ corresponding to rotation along the component $\Sigma^1(m)$. Moreover, the axis of D coincides with common axis of S and L. Consider an orientable normal to axis D determined by line matrix a . The motion of $\mathbb{H}^3$ defined by matrix D moves it to the oriented line determined by DaD^{-1}. The displacement of this motion is $-\gamma_m/2$, where γ_m is displacement of the motion defined by matrix L. Hence, $-\gamma_m/2$ is the complex distance from oriented line determined by line matrix a to oriented line determined by line matrix DaD^{-1}. By formula (2.1) we get

$$\cosh(-\frac{\gamma_m}{2}) = -\frac{1}{2}\,\mathrm{tr}(a\cdot DaD^{-1}). \tag{2.7}$$

Since $a\cdot DaD^{-1} = aaD^{-1}D^{-1}$, $a^2 = -I$, and $(D^{-1})^2 = -L$ the equality (2.7) gives (2.6).

The two above conventions lead to convinient analytic formulas for calculation of γ_m and l_m. More precisely, there are simlple relations between these numbers and eigenvalues of matrix L. Recall that $\det L = 1$ and axis of L coincides with axis of S. Hence matrix L is not parabolic and has two eigenvalues f and $1/f$. We choose f so that $|f| > 1$. The case $|f| = 1$ is impossible because in this case the matrix L is elliptic and $l_m = 0$. We have from (2.6)

$$\frac{e^{\frac{\gamma_m}{2}} + e^{-\frac{\gamma_m}{2}}}{2} = \frac{-f - f^{-1}}{2},$$

where $|e^{\frac{\gamma_m}{2}}| = e^{\frac{l_m}{2}} > 1$ and $|f| > 1$. Hence

$$f = -e^{\frac{\gamma_m}{2}} \tag{2.8}$$

and

$$|f| = e^{\frac{l_m}{2}}. \tag{2.9}$$

Respectively, if g be an eigenvalue of matrix V such that $|g| > 1$, then

$$g = -e^{\frac{\gamma_n}{2}} \tag{2.10}$$

and

$$|g| = e^{\frac{l_n}{2}}. \tag{2.11}$$

Now we want to find a canonical representation for holonomy $h_{m,n}$ in each class of conjugacy.

Proposition 2.4. *Let $\Gamma = \langle S, T \rangle$ be a group whose generators S and T satisfy to conditions (2.2). Suppose that the group Γ is non-elementary and is not conjugated to a subgroup of $SL(2,\mathbb{R})$. Then, up to cojugation Γ in the group $SL(2,\mathbb{C})$, its generators S and T can be chosen thus that*

$$S = \begin{pmatrix} \cos\mu & i\,e^{\frac{\rho}{2}}\sin\mu \\ i\,e^{-\frac{\rho}{2}}\sin\mu & \cos\mu \end{pmatrix}, \quad T = \begin{pmatrix} \cos\nu & i\,e^{-\frac{\rho}{2}}\sin\nu \\ i\,e^{\frac{\rho}{2}}\sin\nu & \cos\nu \end{pmatrix}, \tag{2.12}$$

where $\mu = \pi/m$, $\nu = \pi/n$, ρ is a complex distance between axis of S and *T, and $u = \cosh\rho$ is a non-real root of the complex distance equation*

$$u^3 - MNu^2 + \frac{M^2N^2 + M^2 + N^2 - 1}{2}u + MN = 0, \tag{2.13}$$

$M = \cot\mu$, and $N = \cot\nu$.

Proof. After a suitable conjugation in the group $SL(2,\mathbb{C})$, one can assume that the axes of elliptic elements S and T are lines with endpoints $\{\pm e^{\frac{\rho}{2}}\}$ and $\{\pm e^{-\frac{\rho}{2}}\}$ respectively. Since $\mathrm{tr}(S) = 2\cos\mu$ and $\mathrm{tr}(T) = 2\cos\nu$, the matrices S and T are given by the formulas (2.5) . Check that ρ coincides with the complex distance $\rho(S,T)$ between oriented axes $[-e^{\frac{\rho}{2}}, e^{\frac{\rho}{2}}]$ and $[-e^{-\frac{\rho}{2}}, e^{-\frac{\rho}{2}}]$ of S and T. Indeed, the line matrices p, q, corresponding to these lines can be obtained from the matricies S,T by taking $\mu = \nu = \frac{\pi}{2}$.

We have $p = \begin{pmatrix} 0 & ie^{\frac{\rho}{2}} \\ ie^{-\frac{\rho}{2}} & 0 \end{pmatrix}$, $q = \begin{pmatrix} 0 & ie^{-\frac{\rho}{2}} \\ ie^{\frac{\rho}{2}} & 0 \end{pmatrix}$. By [6, p.68] we get $\cosh\rho(S,T) = -\frac{1}{2}\,tr(pq) = \cosh\rho$.

Now we show that $u = \cosh\rho$ is a non-real root of equation (2.13). For this purpose we have to use the condition $SL = LS$ from (2.2).

Let a be a line matrix corresponding to common normal to axes of S and T. If matrices S and T are represented in the form (2.12) then one can take $a = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. It is not difficult to verify that $aSa^{-1} = S^{-1}$ and $aTa^{-1} = T^{-1}$.

The following lemma gives simple criteria for matrices S and L to be permutable.

Lemma 2.5. *The following conditions are equivalent:* (i) $SL = LS$; (ii) $aLa^{-1} = L^{-1}$; (iii) $\mathrm{tr}(aL) = 0$.

Proof. In the first we show that (i) and (ii) are equivalent. Indeed, since $L = STS^{-1}T^{-1}ST^{-1}S^{-1}T$ we get $aLa^{-1} = S^{-1}T^{-1}STS^{-1}TST^{-1} = S^{-1}L^{-1}S$. Hence

(ii) holds if and only if S and L^{-1} are permutable. The last property is equivalent to (i). Because of $a^2 = -I$ the condition (ii) can be rewritten in the form $aLaL = -I$; that is equivalent to (iii).

By lemma 2.5 and direct calculations we have

$$\mathrm{tr}(aL) = \frac{16\sinh\rho}{(1+M^2)^2(1+N^2)^2}$$

$$\times(\cosh^3\rho - MN\cosh^2\rho + \frac{M^2N^2+M^2+N^2-1}{2}\cosh\rho + MN) = 0. \tag{2.14}$$

If $\sinh\rho = 0$ then the axes S and T coincide, and the group Γ is elementary. This contradicts to our assumption. Consiquently, $u = \cosh\rho$ is a root of equation (2.13).

Suppose that $u = \cosh\rho$ is a real root. Let

$$R(a,b,c,d) = \frac{(c-a)(d-b)}{(c-b)(d-a)}$$

be the cross ratio of four points. Then $R(-e^{\frac{\rho}{2}}, e^{\frac{\rho}{2}}, -e^{-\frac{\rho}{2}}, e^{-\frac{\rho}{2}}) = \dfrac{\cosh\rho - 1}{\cosh\rho + 1} \in \mathbb{R}\cup\{\infty\}$. We have that the axes $[-e^{\frac{\rho}{2}}, e^{\frac{\rho}{2}}]$ and $[-e^{-\frac{\rho}{2}}, e^{-\frac{\rho}{2}}]$ of S and T lie in a common plane. If the axes intersect then the group $\Gamma = \langle S, T\rangle$ has a fixed point and is elementary. If they do not intersect, Γ is conjugated to a subgroup of $SL(2,\mathbb{R})$. By assumption both cases are impossible.

We have shown that u is a non-real root of equation (2.13).

Remark 2.6. Set $z = \mathrm{tr}(S^{-1}T)$. From the presentation (2.12) in Proposition 2.4 we have

$$z = 2\,(\cos\mu\cos\nu - \sin\mu\sin\nu\cosh\rho). \tag{2.15}$$

In terms of z the complex distance equation (2.13) is equivalent to the following trace equation

$$st - (s^2+t^2-2)z + stz^2 - z^3 = 0, \tag{2.16}$$

where $s = \mathrm{tr}(S) = 2\cos\mu$ and $t = \mathrm{tr}(T) = 2\cos\nu$, which was found early in [2]. Also, it was shown in this paper that the Whitehead link cone-manifold is hyperbolic if and only if the trace equation (2.9) and, hence, the complex distance equation (2.6), have a non-real root. This takes a place in the region V outside of discriminant curve of the equation (2.9) which is shown in [2, Fig.2]. In particular, $W(m,n)$ is hyperbolic if $n > 2.507$ and $m > 2.507$.

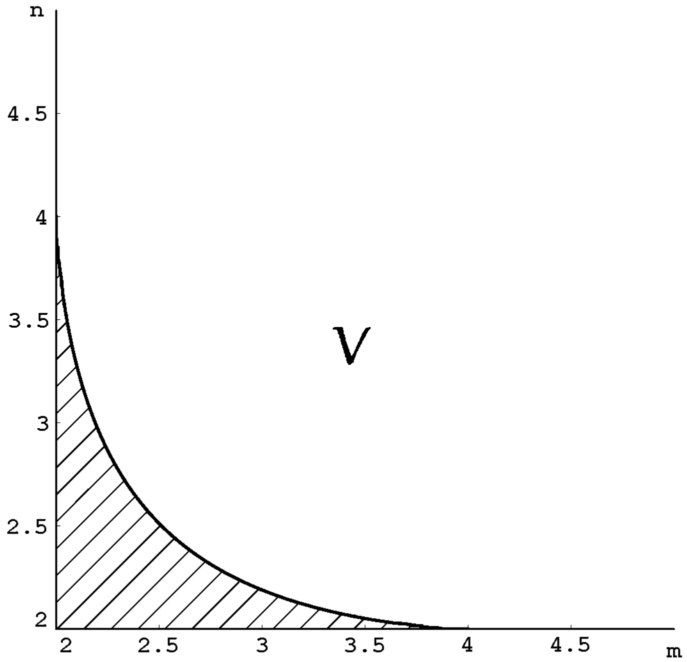

Fig.2. The region of hyperbolicity of the cone-manifold $W(m,n)$.

The next step is to find relations between values u, f and γ_m. They are given by the following two assertions.

Lemma 2.7 *Let $\Gamma = \langle S, T\rangle$ be the same as in Proposition 2.4. Suppose that $u = \cosh \rho$, where ρ is a complex distance between axes S and T chosen thus that $\Im u > 0$, f is an eigenvalue of the matrix L such that $|f| > 1$, and $N = \cot \frac{\pi}{n}$. Then u and f are related by the following formulas*

$$u = iN\frac{f-1}{f+1} \tag{2.17}$$

and

$$f = \frac{Ni+u}{Ni-u}. \tag{2.18}$$

Proof. After a suitable conjugation by isometry of $\mathbb{H}^3$ one can assume that the fixed points of S are 0 and ∞. In this case the generators S and T of the group Γ are given by the following matricies in $SL(2,\mathbb{C})$.

$$S = \begin{pmatrix} e^{i\frac{\pi}{m}} & 0 \\ 0 & e^{-i\frac{\pi}{m}} \end{pmatrix}, T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.19}$$

where $tr(T) = a + d = 2\cos\frac{\pi}{n}$.

We note [1] that the group Γ has the following outer automorphism of order 2

$$\theta : S \to S, \quad T \to T^{-1}L, \quad L \to L. \tag{2.20}$$

By the Rigidity theorem for cone-manifolds proved by S. Kojima [8] there exists an element $D \in SL(2, \mathbb{C})$ such that

$$DSD^{-1} = S, \quad DTD^{-1} = T^{-1}L, \quad DLD^{-1} = L. \tag{2.21}$$

Our next goal is to find matrix D explicitly. We get from (2.2), (2.20), and (2.21)

$$\begin{array}{l} D^2\ S\ D^{-2} = L^{-1}S\ L, \\ D^2\ T\ D^{-2} = L^{-1}T\ L, \\ D^2\ L\ D^{-2} = L^{-1}L\ L. \end{array} \tag{2.22}$$

We recall [10, p.90] that centralizer of non-elementary subgroup of $SL(2, \mathbb{C})$ consists of $\{\pm I\}$. Hence by virtue of (2.22) we have

$$D^2 = \pm\ L^{-1}. \tag{2.23}$$

Since $DSD^{-1} = S$ the matrix D has the form $\begin{pmatrix} k & 0 \\ 0 & k' \end{pmatrix}$ or the form $\begin{pmatrix} 0 & ki \\ k'i & 0 \end{pmatrix}$, where $kk' = 1$. It follows from (2.23) that $D^2 \neq \pm I$, hence

$$D = \begin{pmatrix} k & 0 \\ 0 & k' \end{pmatrix} \tag{2.24}$$

for some $k \in \mathbb{C}$ and $kk' = 1$.

We note that

$$DTD = \begin{pmatrix} k^2 a & b \\ c & k'^2 d \end{pmatrix}$$

and

$$-T^{-1} = \begin{pmatrix} -d & b \\ c & -a \end{pmatrix}. \tag{2.25}$$

From $DTD^{-1} = T^{-1}SL$ and (2.23) we have

$$DTD = \pm T^{-1}. \tag{2.26}$$

It is easy to check from (2.23) and (2.25) that the case $DTD = T^{-1}$ is impossible. Indeed, in this case $b = c = 0$ and the axes of S and T coinside. Hence

$$DTD = -T^{-1}. \tag{2.26}$$

Substituting (2.10) and (2.15) in (2.17) we get $k^2 a = -d$. The case $a = d = 0$ is excluded since $a + d = 2\cos\dfrac{\pi}{n} > 0$. So $k^2 = -d/a$ and we have from (2.23) and (2.24)

$$L^{-1} = -D^2 = \begin{pmatrix} d/a & 0 \\ 0 & a/d \end{pmatrix}.$$

Hence the eigenvalues of the matrix L are d/a and a/d.

Let ρ be a complex distance between axes of S and T. For given non-singular matrix A we set $norm(A) = \frac{1}{\sqrt{det(A)}} A$. Consider the normalized matricies $p = norm(S - S^{-1})$ and $q = norm(T - T^{-1})$ corresponding to half–turns about axes of S and T respectively. We have from (2.19) that $det(S - S^{-1}) = 4\sin^2 \frac{\pi}{m}$ and

$$det(T - T^{-1}) = -(a+d)^2 + 4(ad - bc) = 4\sin^2 \frac{\pi}{n}.$$

Hence $p = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $q = \frac{1}{2\sin\frac{\pi}{n}} \begin{pmatrix} a-d & 2b \\ 2c & d-a \end{pmatrix}$. From (2.1) we obtain

$$\cosh\rho = -\frac{1}{2}\, tr(pq) = -\frac{i}{2\sin\frac{\pi}{n}}(a-d) = iN\frac{d-a}{2\cos\frac{\pi}{n}}, \tag{2.27}$$

where $N = \cot\frac{\pi}{n}$. Since $a + d = 2\cos\frac{\pi}{n}$ we get from (2.27)

$$\cosh\rho = iN\,\frac{d/a - 1}{d/a + 1}. \tag{2.28}$$

We remark that the linear fractional transformation $u = iN\frac{f-1}{f+1}$ maps the region $|f| > 1$ on the extended complex plane onto the upper half–plane $\Im\, u > 0$. Taking $u = \cosh\rho$ and $f = d/a$ we deduce from the formula (2.28) the assertion of the lemma.

As an immediate consiquence of this lemma and formula (2.9) we have the following result.

Corollary 2.8 *Suppose that $u = \cosh\rho$, where ρ is a complex distance between axes S and T chosen thus that $\Im\, u > 0$, and γ_m is a complex length of the singular component of cone-manifold $W(m,n)$ with cone-angle $2\pi/m$. Then*

$$u = \cot\frac{\pi}{n}\coth\frac{\gamma_m}{2}. \tag{2.29}$$

3. The Tangent Rule

In this section we shall prove the following

Theorem A (The Tangent Rule). *Let $\gamma_m = \gamma_m(n)$ be a complex length of the singular geodesic of the hyperbolic cone-manifold $W(m,n)$ with cone angle $2\pi/m$. Then*

$$\frac{\tanh\frac{\gamma_m}{4}}{\tanh\frac{\gamma_n}{4}} = \frac{\tan\frac{\pi}{m}}{\tan\frac{\pi}{n}}.$$

Proof. By Corollary 2.8 we get

$$u = \cot\frac{\pi}{n}\coth\frac{\gamma_m}{2}.$$

Similarly, taking into account that $u = u(m,n) = u(n,m)$, we obtain

$$u = \cot \frac{\pi}{m} \coth \frac{\gamma_n}{2}.$$

Comparing the last two formulas we have the assertion of the theorem.

4. The Sine Rule

The real and imaginary parts of singular lengthes of the cone-manifold $W(m,n)$ are related to each others by the following theorem.

Theorem B (The Sine Rule). *Let $\gamma_m = l_m + i\,\varphi_m$ be a complex length of the singular geodesic of the hyperbolic cone-manifold $W(m,n)$ with cone angle $2\pi/m$. Then*

$$\frac{\sin \frac{\varphi_m}{2}}{\sinh \frac{l_m}{2}} = \frac{\sin \frac{\varphi_n}{2}}{\sinh \frac{l_n}{2}}.$$

Proof. For given m, n we put for simplicity $l = l_m$, $\varphi = \varphi_m$, and $\gamma = \gamma_m = l + i\,\varphi$. By Lemma 2.8 we get

$$-i\frac{u}{N} = \coth \frac{\gamma}{4}, \qquad i\frac{\bar{u}}{N} = \coth \frac{\overline{\gamma}}{4}. \tag{4.1}$$

Since $l > 0$ we have $\cosh \frac{l}{2} > 1$ and $\cosh \frac{l}{2} - \cos \frac{\varphi}{2} > 0$. We set $z = \frac{\gamma}{4}$ and remark that

$$\coth \bar{z} - \coth z = \frac{\sinh(z - \bar{z})}{\sinh z \sinh \bar{z}} = \frac{2i \sin \frac{\varphi}{2}}{\cosh \frac{l}{2} - \cos \frac{\varphi}{2}} \tag{4.2}$$

and

$$\coth \bar{z} + \coth z = \frac{\sinh(\bar{z} + z)}{\sinh z \sinh \bar{z}} = \frac{2 \sinh \frac{l}{2}}{\cosh \frac{l}{2} - \cos \frac{\varphi}{2}} \tag{4.3}$$

We have also

$$-i\frac{\bar{u}}{N} + i\frac{u}{N} = \frac{2i\Re\, u}{N} \tag{4.4}$$

and

$$i\frac{\bar{u}}{N} - i\frac{u}{N} = \frac{2\Im\, u}{N}. \tag{4.5}$$

Taking into account (4.1) we get from (4.2)-(4.5)

$$\frac{\sin \frac{\varphi}{2}}{\cosh \frac{l}{2} - \cos \frac{\varphi}{2}} = \frac{\Re\, u}{N} \tag{4.6}$$

$$\frac{\sinh \frac{l}{2}}{\cosh \frac{l}{2} - \cos \frac{\varphi}{2}} = \frac{\Im\, u}{N} \tag{4.7}$$

Dividing (4.6) on (4.7) we obtain

$$\frac{\sin \frac{\varphi}{2}}{\sinh \frac{l}{2}} = \frac{\sin \frac{\varphi_m}{2}}{\sinh \frac{l_m}{2}} = \frac{\Re\, u}{\Im\, u}. \tag{4.8}$$

By similar arguments observing $u=u(m,n)=u(n,m)$ we have

$$\frac{\sin\frac{\varphi_n}{2}}{\sinh\frac{l_n}{2}}=\frac{\Re\, u}{\Im\, u}. \tag{4.9}$$

Comparing the formulas (4.8) and (4.9) finish the proof of the theorem.

5. An example and further development

We illustrate the Theorems A and B by the following example.

Example 5.1. The orbifold $\mathbf{W(3,4)}$. In this case $m=3$ and $n=4$. The root $u, \Im u>0$ of the complex distance equation

$$u^3-\frac{1}{\sqrt{3}}\,u^2+\frac{1}{3}\,u+\frac{1}{\sqrt{3}}=0$$

is given by $u=\frac{1}{\sqrt{3}}+i\sqrt{\frac{2}{3}}$. By formulas (2.9) and (2.18) we have

$$e^{\frac{\gamma_3}{2}}=-f=\frac{u+Ni}{u-Ni}=i(\sqrt{3}+\sqrt{2}),$$

$$e^{\frac{\gamma_4}{2}}=-g=\frac{u+Mi}{u-Mi}=\frac{1+i}{\sqrt{2}}\,(\sqrt{2}+1).$$

Hence, taking into account that $\gamma=l+i\,\varphi$ we obtain

$$\frac{\tanh\frac{\gamma_3}{4}}{\tanh\frac{\gamma_4}{4}}=\frac{\tan\frac{\pi}{3}}{\tan\frac{\pi}{4}}=\sqrt{3},$$

$$\frac{\sin\frac{\varphi_3}{2}}{\sinh\frac{l_3}{2}}=\frac{\sin\frac{\varphi_4}{2}}{\sinh\frac{l_4}{2}}=\frac{1}{\sqrt{2}}.$$

Remark 5.2. The key lemma for proving of Theorems A and B is Lemma 2.7. For the proof of this lemma we have used only two facts— the existence of automorphism $\theta: S\to S,\, T\to T^{-1}L,\, L\to L$, induced by order two rotation of the singular link along the longitude L and Kojima's Rigidity theorem for cone-manifolds. Both these results hold for infinite family of the two bridge link cone-manifolds $\frac{p}{q}\,(m,n)$ with the slope

$$\frac{p}{q}=2+\frac{1}{r+1/2}=\frac{4r+1}{2r+1},$$

where r is a positive integer. Here, the underlying space of cone-manifold is the three-sphere, and the singular set consists of two components of $\frac{p}{q}$–link with cone-angles $2\pi/m$ and $2\pi/n$. In particular, for $r=1$ we get the Whitehead link and for $r=2,3,4,5$ the links 6_3^2 ,$7_3^2, 8_6^2$, and 9_{10}^2 in notation of [11] respectively. Hence, the theorems similar to Theorems A and B can be obtaned also for cone-manifolds $\dfrac{p}{q}(m,n)$.

Remark 5.3. The Theorem A fails for the link 6^2_2 with the slope $\frac{10}{3}$. Indeed, denote by σ the singular geodesic of the orbifold $\frac{10}{3}(2,n)$ labbeled by 2. Then $\frac{10}{3}(2,n)$ is hyperbolic for $n \geq 4$ because its two–fold covering branched over σ is the Figure eight orbifold $\frac{5}{2}(n)$. An immediate consiquence of Theorem A is that the complex length $\gamma_2(n)$ of σ is pure imaginary. This is a contradiction since σ is not a point.

6. The upper bound for the real length of singular geodesics

Denote by $W(m,\infty)$ the hyperbolic cone-manifold with a cone-angle $2\pi/m$ on the one component of the Whitehead link and with the complete hyperbolic cusp on the another.

Lemma 6.1. *For any $m > 2$ the cone-manifold $W(m,\infty)$ is hyperbolic and*

$$\cosh l_m(\infty) = \frac{M^4 + 10M^2 + 17}{(M^2+1)^2},$$

where $M = \cot \frac{\pi}{m}$.

Proof. The hyperbolicity of $W(m,\infty)$ for $m > 2$ follows from [2]. We set $U = u/N$, where u and N are the same as in Proposition 2.4, By Lemma 2.7 the eigenvalue f of the matrix L is given by $f = \dfrac{i+U}{i-U}$

To find $l(m,\infty)$ we remark that the number $U = u/N$, $\Im U > 0$ is a complex root the equation

$$U^3 - MU^2 + (\frac{M^2+1}{2} + \frac{M^2-1}{2}N^2)\,U + \frac{M}{N^2} = 0, \tag{6.1}$$

$M = \cot \frac{\pi}{m}$ and $N = \cot \frac{\pi}{n}$. As $N \to \infty$ the last equation reduces to

$$U^3 - MU^2 + \frac{M^2+1}{2}U = 0, \tag{6.2}$$

Hence $U = \frac{1}{2}(M + i\sqrt{M^2+2})$ and

$$|f| = |\frac{i+U}{i-U}| = \frac{\sqrt{M^2+2}+1}{\sqrt{M^2+2}-1}. \tag{6.3}$$

Finaly, taking into account formula (2.4) we get from (6.3)

$$\cosh l_m(\infty) = \frac{|f|^2 + |f|^{-2}}{2} = \frac{M^4 + 10M^2 + 17}{(M^2+1)^2}.$$

Lemma 6.2. *Given $m > 2$ the function $n \to \cosh l_m(n)$ is strictly increasing and*

$$\cosh l_m(n) < \cosh l_m(\infty) = \frac{M^4 + 10M^2 + 17}{(M^2+1)^2},$$

where $M = \cot \frac{\pi}{m}$.

Proof. By formulas (2.9) and (2.18) we have

$$l_m(n) = 2\,\log|f| = 2\,\log|\frac{Ni+u}{Ni-u}|.$$

The first assertion of the lemma is a routine consequence of this formula and analytic properties of linear fractional function $u \to \dfrac{Ni+u}{Ni-u}$. The contour plot of the function $\cosh l_m(n)$ is shown on the Fig.3.

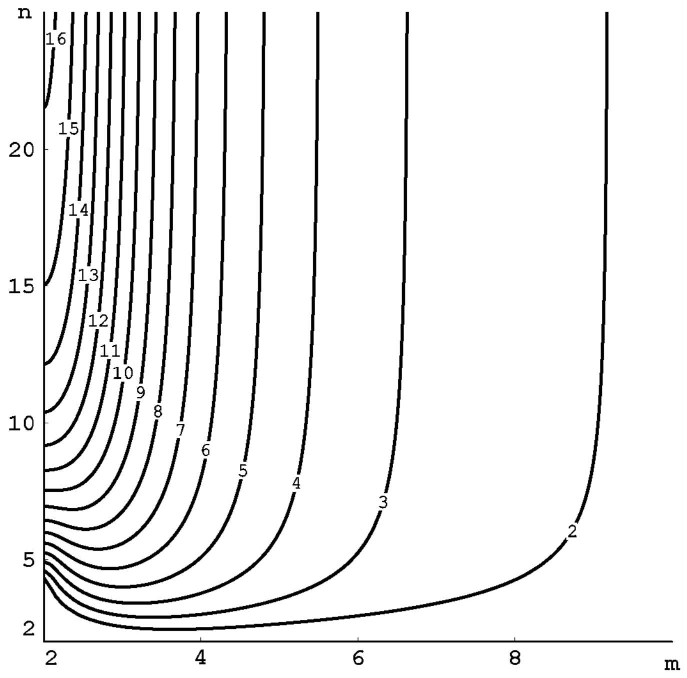

Fig.3. Contour plot of the function $\cosh l_m(n)$

The second assertion of the lemma immediately follows from the first one and Lemma 6.1.

The upper bound for the real length of singular geodesics of the cone-manifold $W(m,n)$ is given by the following theorem.

Theorem C. *Let* $l_m = l_m(n)$ *be a real length of the singular geodesic of the hyperbolic cone-manifold* $W(m,n)$ *with cone angle* $2\pi/m$. *Then*

$$\cosh l_m(n) < 17.$$

The upper bound is sharp and $\cosh l_m(n) \to 17$ *as* $m \to 2$ *and* $n \to \infty$.

Proof. By Lemma 2.8 we have

$$\cosh l_m(n) < \cosh l_m(\infty) = \frac{M^4 + 10M^2 + 17}{(M^2+1)^2} < 17,$$

Since $M = \cot \frac{\pi}{m}$, $\cosh l_m(\infty) \to 17$ as $m \to \infty$. Hence, by continuity of function $l = l_m(n)$, we obtain

$$\cosh l_m(n) \to 17$$

as $m \to 2$ and $n \to \infty$.

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