On the incidence matrix construction of superimposed codes

Gohar Kyureghyan

Abstract—Superimposed codes were introduced by Kautz-Singlton [1]. These codes find a lot of applications.Macula [2] suggested the incidence matrix construction of superimposed codes with $t = \binom{n}{k}$ codewords. We present a way to extend the incidence matrix construction for an arbitrary number of codewords t. We find the smallest possible codelength for such codes with fixed size t. We also study the design and the list-decoding parameters of the incidence matrix codes.

1 Definitions and notations

Let $1 \leq s < t,\ 1 \leq k < t,\ N > 1$ be integers and $X = \|x_i(u)\|,\ i = 1, 2, \dots, N$, $u = 1, 2, \dots, t$, be a binary $(N \times t)$–matrix (code) with columns (codewords) $\mathbf{x}(1), \mathbf{x}(2), \dots, \mathbf{x}(t)$ and rows $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N$, where
$\mathbf{x}(u) = (x_1(u), x_2(u), \dots, x_N(u))$ and $\mathbf{x}_i = (x_i(1), x_i(2), \dots, x_i(t))$. Let

$$w = \min_u \sum_{i=1}^{N} x_i(u), \quad k = \max_i \sum_{u=1}^{t} x_i(u), \quad \lambda = \max_{u,v} \sum_{i=1}^{N} x_i(u) x_i(v)$$

be the *minimal weight of codewords*, the *maximal weight of rows* and the *maximal dot product of codewords*,resp.
The *Boolean sum*

$$\mathbf{y} = \bigvee_{j=1}^{s} \mathbf{y}(j) = \mathbf{y}(1) \vee \mathbf{y}(2) \vee \ldots \vee \mathbf{y}(s)$$

of columns $\mathbf{y}(1), \mathbf{y}(2), \dots, \mathbf{y}(s)$ is the binary column $\mathbf{y} = (y_1, y_2, \dots, y_N)$ with

$$y_i = \begin{cases} 0 & \text{if } y_i(1) = y_i(2) = \ldots = y_i(s) = 0 \\ 1 & \text{otherwise} \end{cases} .$$

We say columns $\mathbf{y}(1), \mathbf{y}(2), \ldots, \mathbf{y}(s)$ *cover* column $\mathbf{z}$ and wright

$$\mathbf{z} \subset \bigvee_{\mathbf{j=1}}^{\mathbf{s}} \mathbf{y}(\mathbf{j})$$

if

$$\bigvee_{j=1}^{s} \mathbf{y}(j) \vee \mathbf{z} = \bigvee_{\mathbf{j=1}}^{\mathbf{s}} \mathbf{y}(\mathbf{j}).$$

Definition 1 *The code X is called a* superimposed (s,t)-code,*if the Boolean sum of any s columns of X does not cover any other column.*

Definition 2 *The code X is called a* superimposed (s,t)-design*, if the Boolean sum of s columns are all distinct.*

Definition 3 *An $(N \times t)$–matrix X is called a superimposed (s, t, k)–code(design) of* length N,size t *and* constraint k *if*

- *code X is a superimposed (s,t)-code(design)*
- *for any $i = 1, 2, \ldots, N$ weight of row $\mathbf{x}_i \leq k$.*

Definition 4 *An $N \times t$–matrix X is called a* list-decoding superimposed (s, L, t)-code*(LDSC)* of length N, size t, strength s, *and* list-size L *if the Boolean sum of any s columns of X covers not more than $L-1$ other columns.*

Note,that the case $L = 1$ is equivalent to the definition 1.

2 The incidence matrix construction

Macula [2] suggested the following construction of superimposed (s, t, k)-codes.
Let $s \geq 2,\ l \geq 1$ be fixed integers and $n > 2s + l$ be an arbitrary integer. Let $[n] = \{1, 2, \ldots, n\}$ be the set of integers from 1 to n and $\binom{[n]}{s}$ be the family of all s-subsets of $[n]$.We define the binary code $X = \|x_B(A)\|$, where $B \in \binom{[n]}{s},\ A \in \binom{[n]}{s+l}$,whose entry $x_B(A) = 1$ iff $B \subset A$.
It is easy to see that X is constant weight code with parameters

$$t = \binom{n}{s+l}, \quad N = \binom{n}{s}, \quad k = \binom{n-s}{l}, \quad w = \binom{s+l}{s}, \quad \lambda = \binom{s+l-1}{s}$$

where t is code size,N is code length,w is weight of columns, k is weight of rows,and λ is the maximal dot product of columns.
This code is superimposed (s,t,k)-code. We refer to it as incidence matrix(IM) (s,l,n)-code,and denote by $\mathbf{x}(A)$ the column generated by $(s+l)$-set A.

Theorem 2.1 (D'yachkov-Rykov lower bound) *[3] Let $t > k \geq s \geq 2$ and $N > 1$ be integers. For any superimposed $(s-1,t,k)$-code((s,t,k)-design) X of length N,the following inequality holds*

$$N \geq \left\lceil \frac{st}{k} \right\rceil .$$

If $k \geq s+1, st = kN$ and there exists an optimal superimposed $(s-1,t,k)$-code X of length $N = st/k$,then

1. *code X is a constant weight code of weight $w = s$, for any $i = 1,2,\ldots,N$,the weight of row x_i is equal to k and the maximal dot product $\lambda = 1$,*
2. *the following inequality is true*

$$k^2 - \frac{k(k-1)}{s} \leq t.$$

This Theorem implies the optimality of the incidence matrix superimposed codes for $l = 1$. The optimality of the case $l > 1$ is unclear, the D'yachkov-Rykov bound cannot be achieved in this case.

3 On a way to generalize the incident matrix construction for an arbitrary codesize t

The above presented IM codes have codesize $t = \binom{n}{s+l}$ for some n and l. Our goal is to get a superimposed (s,t)-code for an arbitrary t.A way to do it is to find n and l,such that $t \leq \binom{n}{s+l}$, and take t codewords from IM (s,l,n)-code. Naturally,it arises the following two questions concerning this idea to get superimposed code:
(i) given n,l,which choice of t codewords gives the minimal codelength N(i.e., codewords with the maximal number of all zero rows);
(ii) which choice of the parameters n,l provides the minimal codelength N.

3.1 The optimal choice of t codewords

Here we need the following auxiliary combinatorial definitions and results,which could be found for example in [5].

Definition 5 *Let $A, B \in \binom{[n]}{k}$. Under the squashed order $A < B$ if the largest element of the symmetric difference $A \vartriangle B (= (A \setminus B) \cup (B \setminus A))$ is in B.*

Definition 6 *Let $\mathcal{A}$ be a family of k-subsets of $[1, n]$ and $r < k$. Then*

$$\triangle^{(r)}\mathcal{A} = \{B \subset [1, n] : |B| = r, B \subset A \text{ for some } A \in \mathcal{A}\}$$

is called the r-shadow of $\mathcal{A}$.

Proposition 3.1 *Given positive integers m and k,there exists a unique representation of m in the form*

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_p}{p}, \qquad (1)$$

where $a_k > a_{k-1} > \ldots > a_p \geq p \geq 1$.

This representation of m is called the k-binomial representation of m.a_k we shall call the leader coefficient of the k-binomial representation of m.Every $a_{k-i}, i \geq 0$ is chosen as the largest integer for which

$$\binom{a_{k-i}}{k-i} \leq m - \binom{a_k}{k} - \binom{a_{k-1}}{k-1} - \cdots - \binom{a_{k-i+1}}{k-i+1}.$$

Proposition 3.2 *The cardinality of the r-shadow of the first m k-sets in the squashed order is*

$$\binom{a_k}{r} + \binom{a_{k-1}}{r-1} + \cdots + \binom{a_p}{p-k+r},$$

when m is given by (1).

Theorem 3.3 (Kruskal-Katona) *Let $\mathcal{A}$ be a family of k-sets and $r < k$. If*

$$|\mathcal{A}| = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_p}{p}, a_k > a_{k-1} > \ldots > a_p \geq p \geq 1,$$

then

$$|\triangle^{(r)}\mathcal{A}| \geq \binom{a_k}{r} + \binom{a_{k-1}}{r-1} + \cdots + \binom{a_p}{p-k+r}.$$

In our case we want to choose t codewords from IM (s,l,n)-code,such that to have the smallest codelength. Thus,in the presented above terminology we have to choose $t \quad (s+l)$-subsets from $[1,n]$ with the minimal s-shadow.The Kruskal-Katona theorem stated above says:to minimize the shadow of a family of $t \quad (s+l)$-sets,where t is given,take the first $t \quad (s+l)$-sets in the squashed order.
Hence,it is possible to generalize the incidence matrix construction of superimposed codes for an arbitrary t in the following way.
Let t,l are given,and

$$t = \binom{a_{s+l}}{s+l} + \binom{a_{s+l-1}}{s+l-1} + \cdots + \binom{a_p}{p} < \binom{a_{s+l}+1}{s+l}.$$

Let $A_1, A_2, \ldots, A_t$ be the first $t \quad (s+l)$-sets in the squashed order of $(s+l)$-sets. To get superimposed (s,t)-code we take $\mathbf{x}(A_1), \mathbf{x}(A_2), \ldots, \mathbf{x}(A_t)$ from IM $(s,l,a_{s+l}+1)$-code and we delete the all-zero rows from them.

Remark 1. The suggested construction does not depend on n.

3.2 The generalized incidence matrix construction

Let $s \geq 2, l \geq 1$ and t be integers and

$$t = \binom{a_{s+l}}{s+l} + \binom{a_{s+l-1}}{s+l-1} + \cdots + \binom{a_p}{p} < \binom{a_{s+l}+1}{s+l}$$

$$(a_{s+l} > a_{s+l-1} > \ldots > a_p \geq p \geq 1 \text{ and } a_{s+l} > 2s+l).$$

Let $S_{s+l}(t)$ be the family of the first t $(s+l)$-sets in the squashed ordering of $(s+l)$-sets.We define the binary code $X = \|x_B(A)\|$ where $A \in S_{s+l}(t)$ and $B \in \triangle^{(s)} S_{s+l}(t)$, whose entry $x_B(A) = 1$ iff $B \subset A$.
We denote this code by GIM (s,l,t)-code.

Remark 2.GIM (s,l,t)-code coincides with IM (s,l,n)-code,when $t = \binom{n}{s+l}$.

It is clear,that GIM (s,l,t)-code is constant weight code with parameters

$$N = \binom{a_{s+l}}{s} + \binom{a_{s+l-1}}{s-1} + \cdots + \binom{a_p}{p-l},$$

$$w = \binom{s+l}{s}, k \leq \binom{a_{s+l}-s+1}{l}, \lambda = \binom{s+l-1}{s}.$$

3.3 The optimal choice of the parameter l for GIM (s,t)-code

In this section we show,that for given t and s the codelength N of GIM (s,l,t)-code is minimal for l,which is the greatest integer, such that $\lceil \frac{a_{s+l}}{2} \rceil \geq s+l$, where a_{s+l} is the leader coefficient of $(s+l)$-binomial representation of t.For convenience,here we denote the leader coefficient of $(s+l)$-binomial representation of t by λ_{s+l}.

Lemma 3.4 *Let λ_{s+i} be the leader coefficient of $(s+i)$-binomial representation of t.Then there exists $p \geq 0$, such that*

1. $\lceil \frac{\lambda_{s+i}}{2} \rceil \geq s+i$ *for* $i \leq p$ *and* $\lceil \frac{\lambda_{s+i}}{2} \rceil < s+i$ *for* $i > p$
2. $\lambda_{s+1} \geq \lambda_{s+2} \geq \ldots \geq \lambda_{s+p}$ *and* $\lambda_{s+p} \leq \lambda_{s+p+1} \leq \ldots$.

Proof. Suppose for some i $\lceil \frac{\lambda_{s+i}}{2} \rceil \geq s+i$,then

$$\binom{\lambda_{s+i}}{s+i-1} \leq \binom{\lambda_{s+i}}{s+i} \leq t,$$

thus by the rule of the choice $\lambda_{s+i-1} \geq \lambda_{s+i}$. Clearly,$\frac{\lambda_{s+i-1}}{2} \geq \frac{\lambda_{s+i}}{2} \geq s+i-1$,so $\lceil \frac{\lambda_{s+i-1}}{2} \rceil \geq s+i-1$.
Now let for some i $\lceil \frac{\lambda_{s+i}}{2} \rceil < s+i$,then

$$\binom{\lambda_{s+i}}{s+i+1} \leq \binom{\lambda_{s+i}}{s+i} \leq t,$$

thus $\lambda_{s+i+1} \geq \lambda_{s+i}$.And by virtue of the previous case $\lceil \frac{\lambda_{s+i+1}}{2} \rceil < s+i+1$.

Theorem 3.5 *Let $p \geq 0$ be the greatest integer,such that $\lceil \frac{\lambda_{s+p}}{2} \rceil \geq s+p$,where λ_{s+p} is the leader coefficient of $(s+p)$-binomial representation of t. Let N_i be the length of GIM (s,i,t)-code. Then*

$$N_{s+1} \geq N_{s+2} \geq \ldots \geq N_{s+p} \text{ and } N_{s+p} \leq N_{s+p+1} \leq \ldots.$$

Proof.Let

$$t = \binom{\lambda_{s+i}}{s+i} + \binom{a_{s+i-1}}{s+i-1} + \cdots + \binom{a_p}{p}$$

and

$$t = \binom{\lambda_{s+i+1}}{s+i+1} + \binom{b_{s+i}}{s+i} + \cdots + \binom{b_r}{r}.$$

Thus

$$N_{s+i} = \binom{\lambda_{s+i}}{s} + \binom{a_{s+i-1}}{s-1} + \cdots + \binom{a_p}{p-i}$$

and

$$N_{s+i+1} = \binom{\lambda_{s+i+1}}{s} + \binom{b_{s+i}}{s-1} + \cdots + \binom{b_r}{r-i-1}.$$

Case 1. Suppose $i < p$ and $\lambda_{s+i} > \lambda_{s+i+1}$, then

$$N_{s+i+1} = \binom{\lambda_{s+i+1}}{s} + \binom{b_{s+i}}{s-1} + \cdots + \binom{b_r}{r-i-1} \leq$$

$$\binom{\lambda_{s+i+1}}{s} + \binom{\lambda_{s+i+1}-1}{s-1} + \cdots + \binom{\lambda_{s+i+1}-s+r-i-1}{r-i-1}$$

$$\leq \binom{\lambda_{s+i+1}+1}{s} \leq \binom{\lambda_{s+i}}{s} \leq N_{s+i}.$$

The case $i > p$ and $\lambda_{s+i} < \lambda_{s+i+1}$ is analogous.

Case 2. Let $i > p, \lambda_{s+i} = \lambda_{s+i+1} = \lambda$ and $a_{s+i-1} \neq b_{s+i}$.Since here $\lceil \frac{\lambda_{s+i}}{2} \rceil < s+i$,

$$a_{s+i-1} < \lambda_{s+i} < 2(s+i) = 2s+2i-1, \text{therefor } a_{s+i-1} \leq 2(s+i-1)-1. \qquad (*)$$

Our goal to show that $a_{s+i-1} < b_{s+i}$, since having it $N_{s+i-1} \leq N_{s+i}$ could be proved as in case 1.

Thus

$$t = \binom{\lambda}{s+i} + \binom{a_{s+i-1}}{s+i-1} + \cdots + \binom{a_p}{p}$$

and

$$t = \binom{\lambda}{s+i+1} + \binom{b_{s+i}}{s+i} + \cdots + \binom{b_r}{r}.$$

and

$$t - \binom{\lambda}{s+i} < t - \binom{\lambda}{s+i+1},$$

since $\binom{\lambda}{s+i} > \binom{\lambda}{s+i+1}$. Using (*),

$$\binom{a_{s+i-1}}{s+i} \leq \binom{a_{s+i-1}}{s+i-1} \leq t - \binom{\lambda}{s+i} < t - \binom{\lambda}{s+i+1}.$$

Thus, by the rule of the choice of b_{s+i} we get $b_{s+i} \geq a_{s+i-1}$,and since by assumption $b_{s+i} \neq a_{s+i-1}$,so $b_{s+i} > a_{s+i-1}$.

Case 3. Suppose $i < p, \lambda_{s+i} = \lambda_{s+i+1} = \lambda$ and $a_{s+i-1} \neq b_{s+i}$.Now we want

to show,that $a_{s+i-1} > b_{s+i}$.
We have

$$t = \binom{\lambda}{s+i} + \binom{a_{s+i-1}}{s+i-1} + \cdots + \binom{a_p}{p}$$

and

$$t = \binom{\lambda}{s+i+1} + \binom{b_{s+i}}{s+i} + \cdots + \binom{b_r}{r}.$$

and $\binom{\lambda}{s+i} \leq \binom{\lambda}{s+i+1}$,since $\lceil\frac{\lambda}{2}\rceil \geq s+i$.
To prove $a_{s+i-1} > b_{s+i}$,it is enough to show,that from

$$\binom{b_{s+i}}{s+i} \leq t - \binom{\lambda}{s+i+1}$$

it follows

$$\binom{b_{s+i}}{s+i-1} \leq t - \binom{\lambda}{s+i}.$$

If $\lceil\frac{b_{s+i}}{2}\rceil \geq s+i$, then

$$\binom{b_{s+i}}{s+i-1} \leq \binom{b_{s+i}}{s+i} \leq t - \binom{\lambda}{s+i+1} \leq t - \binom{\lambda}{s+i}.$$

If $\lceil\frac{b_{s+i}}{2}\rceil < s+i$, then it is enough to show,that

$$\binom{b_{s+i}}{s+i-1} - \binom{b_{s+i}}{s+i} \leq \binom{\lambda}{s+i+1} - \binom{\lambda}{s+i}.$$

The last enequality follows from the following two simple claims.

Claim 3.6 *For any m and $k, m > 2k-2$,it is*

$$\binom{m+1}{k} - \binom{m+1}{k-1} > \binom{m}{k} - \binom{m}{k-1}.$$

Proof.

$$\binom{m+1}{k} - \binom{m+1}{k-1} = \binom{m}{k-1}\frac{m+1}{m-k+2}\Big(\frac{m+2-k}{k} - 1\Big) >$$

$$\binom{m}{k-1}\Big(\frac{m+1-k}{k} - 1\Big) = \binom{m}{k} - \binom{m}{k-1}.$$

Claim 3.7 *For any m and k, $\quad m \geq 2k+1$ it is*

$$\binom{m}{k+1} - \binom{m}{k} > \binom{m-1}{k}\binom{m-1}{k-1}.$$

Proof.

$$\binom{m}{k+1} - \binom{m}{k} = \binom{m-1}{k-1}\frac{m}{k}\Big(\frac{m-k}{k+1} - 1\Big)$$

and

$$\binom{m-1}{k}\binom{m-1}{k-1} = \binom{m-1}{k-1}\Big(\frac{m-k}{k} - 1\Big).$$

Thus we have to show,that

$$\frac{m}{k}\Big(\frac{m-k}{k+1} - 1\Big) > \frac{m-k}{k} - 1.$$

The last inequality is equvalent to

$$m(m-2k-1) > (m-2k)(k+1).$$

Since $2k < m$,it is enough to prove

$$2k(m-2k-1) > (m-2k)(k+1).$$

It is easy to see,that the last quadratic enequality is true for $0 < k < \frac{m}{2}$.

To complete the proof of the theorem we need to prove it for the case $a_{s+i-1} = b_{s+i}, \ldots, a_{s+i-j} = b_{s+i-j+1}$ for some $j > 1$,which is an easy consequence of the considered above three cases.

Corollary 3.8 *The optimal* $l = \lceil\frac{m}{2}\rceil - s$,*where* m *meets* $\binom{m}{\lceil\frac{m}{2}\rceil} \le t < \binom{m+1}{\lceil\frac{m+1}{2}\rceil}$.

4 Design and list-decoding parameters of generalized incidence matrix superimposed codes

The following Lemma is true for any superimposed (s,t)-code.

Lemma 4.1 *Let* $X = \|x_i(u)\|,\ i = 1,\ldots,N,\ u = 1,\ldots,t$ *be superimposed* (s,t)*–code and* $\mathbf{x}(v) \subset \bigvee_{i=1}^{s+1}\mathbf{x}(u_i)$*. Then for any column* $\mathbf{x}(u_i)$ *there is at least one row* $\mathbf{x}_{j(i)}$*, such that* $x_{j(i)}(u_i) = x_{j(i)}(v) = 1$ *and* $x_{j(i)}(u_l) = 0$ *for* $l \ne i$*, i.e.*

$$\begin{array}{cccccc}
* & \mathbf{x}(u_1) & \mathbf{x}(u_2) & \ldots & \mathbf{x}(u_{s+1}) & \mathbf{x}(v) \\
\mathbf{x}_{j(1)} & 1 & 0 & \ldots & 0 & 1 \\
\mathbf{x}_{j(2)} & 0 & 1 & \ldots & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{x}_{j(s+1)} & 0 & 0 & \ldots & 1 & 1
\end{array}.$$

Proof
Suppose the proposition of the Lemma does not hold for some $\mathbf{x}(u_j), j = 1,2,\ldots,s+1$. Then the Boolean sum of s columns $\mathbf{x}(u_i), i \neq j, i = 1,2,\ldots, s+1$ still covers $\mathbf{x}(v)$, which is contradiction.

Now let consider GIM (s,l,t)-code,where

$$t = \binom{a_{s+l}}{s+l} + \binom{a_{s+l-1}}{s+l-1} + \cdots + \binom{a_p}{p} < \binom{a_{s+l}+1}{s+l} \qquad (1)$$

$$(a_{s+l} > a_{s+l-1} > \ldots > a_p \geq p \geq 1 \text{ and } a_{s+l} > 2s+l).$$

Let the Boolean sum of $s+1$ columns $\mathbf{x}(A_i),\ i = 1,2,\ldots,s+1$ covers column $\mathbf{x}(A)$, Lemma 4.1 implies the existence of a s-subset B_i for every A_i,such that

$$B_i \subset A_i \cap A \qquad \text{and} \qquad B_i \not\subseteq A_j, j \neq i, \qquad (2)$$

i.e.

$$\begin{array}{cccccc} * & A_1 & A_2 & \ldots & A_{s+1} & A \\ B_1 & 1 & 0 & \ldots & 0 & 1 \\ B_2 & 0 & 1 & \ldots & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ B_{s+1} & 0 & 0 & \ldots & 1 & 1 \end{array} .$$

Lemma 4.2 *The Boolean sum of* $\mathbf{x}(A_1), \mathbf{x}(A_2), \ldots, \mathbf{x}(A_{s+1})$ *covers* $\mathbf{x}(A)$, $A \neq A_i,\ i = 1,2,\ldots,s+1$ *iff each element* $a \in A$ *belongs at least to* s *of* $A_i,\ i = 1,2,\ldots,s+1$.

Proof *Necessity.*Suppose in the contrary $\mathbf{x}(A) \subset \bigvee_{i=1}^{s+1} \mathbf{x}(A_i)$ and there is an element $a \in A$,which belongs to $s' < s$ of $A(i), i = 1,2,\ldots,s+1$. Without loss of generality we can assume,that $a \notin A_1, A_2$. Let B_i be as in (2). Then $|B_1 \cup a| = s+1$ and $B_1 \cup a \subseteq A$, implying that every s-subset of $B_1 \cup a$, hence every $(s-1)$-subset of B_1, is contained in some $A_i, i = 3,\ldots,s+1$. The number of $(s-1)$-subsets of B_1 is s,while the number of $A_i, i = 3,\ldots,s+1$ is $s-1$. Hence there are two distinct $(s-1)$-subsets of B_1 D and C,that are contained in the same,say, A_3. Thus

$$B_1 = C \cup D \subset A_3,$$

which is contradiction with (2).
*Sufficiency.*Suppose each $a \in A$ belongs to at least s of A_i.Let $B = \{a_1, a_2, \ldots, a_s\}$ be arbitrary s-subset of A.Thus our goal is to show the existence of an $i' \in \{1,2,\ldots,s+1\}$, such that $B \subset A_{i'}$.For each $a_j \in B$ there is

family $\mathcal{A}_j = \{A_i | a_j \in A_i\}$ with $|\mathcal{A}_j| = s$.Now observe that any family from $\binom{[s+1]}{s}$ with cardinality less than $s+1$, is intersecting,i.e. $\bigcap_{A\in\mathcal{F}} A \neq \varnothing$. So $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_s\}$ is intersecting, i.e. $\bigcap_{j=1}^{s} \mathcal{A}_i \neq \varnothing$. Thus there is $A_{i'} \in \bigcap_{j=1}^{s} \mathcal{A}_j$ for some $i' = 1, 2, \ldots, s+1$, and for this $A_{i'}$
$B \subset A_{i'}$ holds.This completes the proof of the Lemma.

Theorem 4.3 *Let t given by (1) and $\left\lfloor \frac{(s+1)(s+l)}{s} \right\rfloor < a_{s+l}$, then GIM (s,l,t)-code is a list-decoding superimposed $(s+1, L(s,l)+1, t)$-code, where*

$$L(s,l) = \binom{s+l+1+\left\lfloor \frac{l}{s} \right\rfloor}{s+l} - s + l - 1 - s\left\lfloor \frac{l}{s} \right\rfloor.$$

Proof Our goal is to define the largest number $L(s,l)$ of columns of GIM (s,l,t)-code,which can be covered by the Boolean sum of $s+1$ other columns. By Lemma 4.1 the Boolean sum of $s+1$ columns $x(A_i), i = 1, 2, \ldots, s+1$ covers the largest number of other columns,when the number of elements,belonging to at least s of A_i,is the largest.It is clear that the number of s time repeated elements in $s+1$ $(s+l)$-sets is at most

$$R = \left\lfloor \frac{(s+1)(s+l)}{s} \right\rfloor = s+l+1+\left\lfloor \frac{l}{s} \right\rfloor.$$

Thus

$$L(s,l) = \binom{R}{s+l} - P,$$

where P is the number of $(s+l)$-sets A_i,which consist only from the s time repeated elements. Hence we should minimize P.It is the case, when remaining $(s+1)(s+l) - sR = l - s\left\lfloor \frac{l}{s} \right\rfloor < s$ elements(not necessary distinct) are distributed in the distinct A_i.Thus in this case

$$P = s+1-l+s\left\lfloor \frac{l}{s} \right\rfloor.$$

So we obtain

$$L(s,l) = \binom{R}{s+l} - s - 1 + l - s\left\lfloor \frac{l}{s} \right\rfloor.$$

Corollary 4.4 *If $l < s$,then GIM (s,l,t)-code is a list-decoding superimposed $(s+1, 2l+1, t)$-code.*

Corollary 4.5 *IM(s,l,n)-code is a list decoding superimposed* $(s+1, L(s,l)+1, \binom{n}{s+l})$*-code with*

$$L(s,l) = \begin{cases} \binom{s+l+1+\left\lfloor \frac{l}{s} \right\rfloor}{s+l} - s + l - 1 - s\left\lfloor \frac{l}{s} \right\rfloor & \text{if } \left\lfloor \frac{(s+1)(s+l)}{s} \right\rfloor < n \\ \binom{n}{s+l} - s - 1 & \text{otherwise} \end{cases}.$$

Proof The first case follows from the Theorem 3.3. Suppose $\left\lfloor \frac{(s+1)(s+l)}{s} \right\rfloor \geq n$. Set $f = (s+1)(n-s-l)$. Let $B_i = [1,f] \setminus \{j | i+j = 1 (\text{mod } s+1)\}, i = 1, 2, \ldots, s+1$ and $A_i = B_i \cup \{f+1, f+2, \ldots, n\}$. Obviously,$|A_i| = |B_i| + n - f = s(n-s-l) + n - (s+1)(n-s-l) = s+l$. It is easy to see,that any element $a \in [1,n]$ belongs at least to s of A_i.Hence by Lemma 4.2 any $\mathbf{x}(A), A \in \binom{[n]}{s+l}$ is covered by the Boolean sum of $\mathbf{x}(A_1), \ldots, \mathbf{x}(A_{s+l})$. Consequently,

$$L(s,l) = \binom{n}{s+l} - s - 1.$$

There is the following relations between superimposed codes and superimposed designs.

superimposed $(s+1,t)$-design $\Longrightarrow$
superimposed (s,t)-code $\Longrightarrow$
superimposed (s,t)-design.

Hence GIM (s,l,t)-code is superimposed (s,t)-design, and it is interesting,if GIM (s,l,t)-code is superimposed $(s+1,t)$-design.

Proposition 4.6 *GIM* (s,l,t)*-code is not a superimposed* $(s+1,t)$*-design.*

Proof Let B be $[1, s+1]$, $B_i, i = 1, 2, \ldots, s+1$ be $[1, s+1] \backslash i$, and C be any disjoint from B l-subset of $[1, a_{s+l}]$. Set $A_i = B_i \cup C$, and $A = B \cup C'$, where C' is a $l-1$-subset of C. Then using Lemma 4.1 it is easy to show that

$$\bigvee_{i=1}^{s+1} \mathbf{x}(A_i) = \bigvee_{i=1}^{s} \mathbf{x}(A_i) \vee \mathbf{x}(A).$$

References

[1] W.H.Kautz, R.C.Singleton, Nonrandom binary superimposed codes,*IEEE Transactions in Information Theory*, 10, No.4, 363-377, 1964.

[2] A.J.Macula, Simple Construction of d-disjunct Matrices with Certain Constant Weights,*Discrete Mathematics*,vol. 162,pp. 311-312, 1996.

[3] A.G.D'yachkov,A.J.Macula V.V.Rykov,On Optimal Parameters of a Class of Superimposed Codes and Design(to appear)

[4] A.G.D'yachkov,V.V.Rykov,A Survey of Superimposed Code Theory,*Problems of Control and Inform.Theory*, vol.12,no.4,pp.229-242,1983.

[5] B.Bollobas,Combinatorics,Cambrige University Press,1986.