Tame Algebras are WILD

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ABSTRACT. We consider the quivers $K(n)$ with two vertices a,b and n arrows $a \to b$; the quiver $K(2)$ is usually called the Kronecker quiver, since the finite dimensional representations of $K(2)$ are just the matrix pencils studied and classified by Kronecker. The quiver $K(2)$ is a typical tame quiver, whereas the quivers $K(n)$ with $n \geq 3$ are wild. It is known since quite a while that the category of (not necessarily finite dimensional) representations of the Kronecker quiver $K(2)$ has a full exact abelian subcategory which is equivalent to the category of representations of $K(3)$. Here we are going to present a short direct proof.

It has been shown in 1979 [R1] that *tame finite dimensional hereditary algebras are Wild,* here the small "t" in *tame* refers to tameness with respect to modules of finite length, the capital "W" in *Wild* refers to wildness with respect to arbitrary modules which are not necessarily of finite length. In order to give a proof, it is sufficient to deal with tame finite dimensional hereditary algebras with precisely two simple modules, thus with the Kronecker quiver $K(2)$, in case we assume in addition that the base field is algebraically closed. The aim of the present note is to provide a short direct proof that the Kronecker quiver $K(2)$ is Wild. We will provide an explicit full exact embedding of the category of representations of $K(3)$ into the category of representations of $K(2)$ (in general, $K(n)$ will denote the quiver with two vertices a, b and n arrows $a \to b$), and, of course, $K(3)$ is a typical strictly wild quiver. Such an embedding can be exhibited for any base field k; for the sake of simplicity, we will assume that k is infinite, but we also will outline the small changes necessary in case k is finite.

The general assertion stated in the title still has to be rated as a challenge. After all, the paper only deals with a single tame algebra, the path algebra $kK(2)$ of the Kronecker quiver. Of course, the Wildness of this algebra entails the same property for many other tame algebras: similar to the notion of a strictly wild algebra, one may call a k-algebra A *strictly tame* provided there is a full exact embedding of the category $\operatorname{Mod} kK(2)$ of all $kK(2)$-modules into the category $\operatorname{Mod} A$ of all A-modules. Clearly, the result discussed here implies that all strictly tame algebras are Wild. There is a strong belief that for all tame finite dimensional k-algebras, where k is an algebraically closed field, the one-parameter families of indecomposables can be obtained using functors $\operatorname{Mod} kK(2) \to \operatorname{Mod} A$ which are quite well-behaved also

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with respect to infinite dimensional modules. If this turns out to be true, then one will see that really all tame algebras are Wild.

On the other hand, we should stress that the Wildness property discussed here is parallel to the property of being **strictly** wild: the embedding functor which we construct yields a **full** embedding; thus a more appropriate formulation of the title would be: *strictly tame algebras are Strictly Wild* (again we use both small and capital letters). Indeed, the concept of Wild behaviour has been developed by Corner [C1] and his successors for the (tame) category of abelian groups; for the discussion of the parallel situation of $k[T]$-modules we refer to [BR]. The main difference to be remembered is the following: If A is a Strictly Wild k-algebra, then any k-algebra B can be realized as the endomorphism ring of a suitable A-module, whereas there are severe restrictions on the endomorphism rings say of $k[T]$-modules (for example, $k(T) \times k(T)$ does not occur as the endomorphism ring of a $k[T]$-module).

1. The representations $P(I)$ of $K(2)$.

Let k be an infinite field. We consider the Kronecker quiver $K(2)$

$$\begin{tikzcd} a \circ \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & \circ\, b \end{tikzcd}$$

its representations are of the form $V = (V_a, V_b; V_\alpha, V_\beta)$, where V_a, V_b are vector spaces, and $V_\alpha, V_\beta \colon V_a \to V_b$ linear transformations. Representations of a quiver Q will also be referred to as *modules* (of course, given any representation V of a quiver Q, there is a corresponding kQ-module $\rho(V)$, where kQ is the path algebra of Q, and it is quite usual not to distinguish between V and $\rho(V)$; note that the $kK(2)$-module $\rho(V)$ corresponding to the representation V of $K(2)$ has the underlying vector space $V_a \oplus V_b$). A submodule W of V is given by a pair (W_a, W_b), where W_a, W_b are subspaces of V_a, V_b, respectively, and where $V_\alpha(W_a) \subseteq W_b$ and $V_\beta(W_a) \subseteq W_b$). A module V is said to be finite dimensional, in case both spaces V_a, V_b are finite dimensional, and the pair $\mathbf{dim}\, V = (\dim V_a, \dim V_b)$ is called the dimension vector of V and $\dim V_a + \dim V_b$ the dimension of V.

The finite dimensional indecomposable representations of $K(2)$ have been classified by Kronecker, we recall some of the features (as presented for example in [R2]): If V is finite dimensional and indecomposable, then $\mathbf{dim}\, V$ is of the form $(n-1, n)$, (n, n) or $(n, n-1)$, for some $n \in \mathbb{N}_1$; an indecomposable module is said to be *preprojective*, or *regular* or *preinjective*, provided its dimension vector is of the form $(n-1, n)$, or (n, n), or $(n, n-1)$, respectively. The preprojective and the preinjective indecomposables are uniquely determined (up to isomorphism) by their dimension vectors; the regular ones not. In fact, representatives of the isomorphism classes of indecomposables with dimension vector $(1, 1)$ are the modules $S(\lambda) = (k, k; 1, \lambda)$, where $\lambda \in k$, and, in addition, $S(\infty) = (k, k; 0, 1)$. Note that in this way, we have exhibited all the finite dimensional indecomposables with endomorphism ring k: these are the preprojectives, the preinjectives and those with dimension vector $(1, 1)$.

Let $\mathcal{R}_1$ be the full subcategory of all representations of $K(2)$ which are direct sums of copies of the various $S(\lambda)$, with $\lambda \in k \cup \{\infty\}$. Note that this is an abelian exact subcategory which is semisimple (any object is a direct sum of simple objects); the simple objects in $\mathcal{R}_1$ are the modules of the form $S(\lambda)$. Given a subset I of $k \cup \{\infty\}$, let $S(I) = \bigoplus_{\lambda \in k} S(\lambda)$.

One infinite dimensional representation of $K(2)$ is of special interest, the so called *generic* module $G = (k(T), k(T); 1, T)$, where $k(T)$ is the field of rational functions in the variable T. Its endomorphism ring is $k(T)$. Given a subset I of k, consider the subspace $V(I)$ of $k(T)$ generated by the elements $\frac{1}{T-\lambda}$ with $\lambda \in I$. We are interested in the submodule

$$P(I) = (V(I), V(I) + k1)$$

of $k(T)$ (here, $k1$ is the one-dimensional subspace of $k(T)$ generated by the element 1). In particular, $P = P(\emptyset)$ is the one-dimensional projective representation of $K(2)$. Of course, if $J \subseteq I$, then $P(J) \subseteq P(I)$, and we have

$$P(I)/P(J) \simeq S(I \setminus J).$$

(For the proof, we note that 1 and the various elements $\frac{1}{T-\lambda}$ are linearly independent; this is easy to see and is part of the well-known partial fraction theorem. Also we use that $T \cdot \frac{1}{T-\lambda} = 1 + \lambda \cdot \frac{1}{T-\lambda}$.) In particular,

$$P(I)/P \simeq S(I).$$

Instead of $P(\{\lambda\})$, we just write $P(\lambda)$.

We list some properties of the modules $P(I)$. Here, I and J are subsets of k.

(1) *If I is a finite set, then $P(I)$ is a preprojective indecomposable module* (with dimension vector $(n, n+1)$, where n is the cardinality of I). Proof: For a finite dimensional module V, the difference $\dim V_a - \dim V_b$ is called its *defect*. The dimension vector of $P(I)$ is $(n, n+1)$, thus its defect is -1. Also, any non-zero submodule of $P(I)$ has negative defect, since this is true for G. But a decomposable module of defect -1 has a non-zero submodule of non-negative defect.

(2) *If $f\colon M \to P(I)$ is a homomorphism, where M is a finitely generated module, then there exists a finite subset I' of I such that $f(M) \subseteq P(I')$.*

(3) *If $I \setminus J$ is infinite, then* $\operatorname{Hom}(P(I), P(J)) = 0$. Proof: Let $f\colon P(I) \to P(J)$ be a homomorphism. Let I' be any finite subset of I. According to (2), there exists some finite subset J' of J with $f(P(I')) \subseteq P(J')$, let n be the cardinality of J'. We denote by f' the induced map

$$f'\colon S(I \setminus I') = P(I)/P(I') \longrightarrow P(J)/P(J') = S(J \setminus J')$$

Take $\lambda \in I$ which does not belong to $I' \cup J$. Then $f'(S(\lambda)) = 0$, since $S(\lambda)$ is not a subobject of $S(J \setminus J')$, thus f maps $P(I' \cup \{\lambda\})$ into $P(J')$. Since $I \setminus J$ is infinite, we may enlarge I' by elements of $I \setminus (I' \cup J)$ in order to obtain a subset I'' of cardinality $n+1$. Then f maps $P(I'') = P(I') + \sum_{\lambda \in I'' \setminus I'} P(\lambda)$ into $P(J')$; in particular, f cannot be a monomorphism, since the dimension vector of $P(I'')$ is $(n+1, n+2)$ whereas that of $P(J')$ is $(n, n+1)$. However, every non-zero homomorphism between indecomposable preprojective modules is a monomorphism, thus $f(P(I'')) = 0$. In particular, the restriction of f to $P(I')$ is zero. Since I' was an arbitrary finite subset of I, we see that $f=0$.

(4) *For any I, we have* $\operatorname{End}(P(I)) = k$. Proof: Let f be a non-zero endomorphism of $P(I)$. Again, take a finite subset I' of I and consider $f(P(I'))$. We may assume that $f(P(I'))$ is non-zero (otherwise, we may enlarge I' accordingly). Using (2), we find a finite subset J' of I with $f(P(I')) \subseteq P(J')$, and we denote by f' the induced map

$$f' \colon S(I \setminus I') = P(I)/P(I') \longrightarrow P(I)/P(J') = S(I \setminus J').$$

Note again that $f'(S(\lambda)) = 0$ for $\lambda \in J'$, so that $f(P(I' \cup \{\lambda\})) \subseteq f(J')$. Thus $f(P(I' \cup J')) \subseteq P(J')$. If $I' \not\subseteq J'$, then f cannot be a monomorphism. But this implies that the restriction of f to $P(I')$ is zero, a contradiction. This shows that $I' \subseteq J'$, and thus, the restriction of f to $P(I') = P(J')$ is an endomorphism, thus a scalar multiplication, say using the scalar $c_{I'}$. However, if I', I'' are different finite subsets of I, then $c_{I'} = c_{I''}$, since we may consider $I''' = I' \cup I''$: we have $c_{I'} = c_{I'''}$, since the multiplication by $c_{I'}$ is the restriction of the multiplication by $c_{I'''}$ to $P(I')$; and similarly $c_{I''} = c_{I'''}$.

We obtain in this way large sets of pairwise orthogonal bricks; recall that a module is said to be a *brick* provided its endomorphism ring is a division ring; and two bricks X, Y are said to be orthogonal provided $\operatorname{Hom}(X, Y) = 0 = \operatorname{Hom}(Y, X)$:

(5) *Assume that the sets I_i with $i \in \mathbb{N}$ are pairwise disjoint infinite subsets of k. Then the modules $P(I_i)$ have the following properties:* $\operatorname{End}(P(I_i)) = k$ *for all i and* $\operatorname{Hom}(P(I_i), P(I_j)) = 0$ *for $i \neq j$.*

Remark 1. For the definition of $P(I)$, we have started with a subset I of k. In contrast, our modules $S(I)$ are defined for all subsets I of $k \cup \{\infty\}$. Actually, we also may define $P(I)$ in this more general situation. Namely, there is the submodule $P(\infty) = (k1, k1 + kT)$ of G. Of course, we have also here $P \subset P(\infty)$, and $P(\infty)/P \simeq S(\infty)$. In general, for any subset I of k, let $P(I \cup \{\infty\}) = P(I) + P(\infty)$ (as submodules of G).

Remark 2. In case k is a finite field, it is not sufficient to deal with subsets I of k or of $k \cup \{\infty\}$, since we need infinite sets I. In this case, but also in general, one should proceed as follows: Let Φ be the set of (monic) irreducible polynomials with coefficients in k. For $\varphi \in \Phi$, let $d(\varphi)$ be its degree and denote by $V(\varphi)$ the subspace of $k(T)$ generated by the elements $\frac{T^i}{\varphi}$ with $0 \le i < d(\varphi)$. We consider the submodule

$$P(\varphi) = (V(\varphi), V(\varphi) + k1)$$

of G and the factor module

$$S(\varphi) = P(\varphi)/P.$$

Note that the set of modules $S(\varphi)$ together with $S(\infty)$ furnish a complete list of all regular finite dimensional bricks. Given now a subset I of $\Phi \cup \{\infty\}$, we define

$$P(I) = \sum_{\varphi \in \Phi \cup \{\infty\}} P(\varphi).$$

Since we may identify k with the set of elements of the form $T-\lambda$ in Φ with $\lambda \in k$, our previous definitions coincide with the new ones, in case I is a subset of $k \cup \{\infty\}$.

Remark 3. The considerations above may be adjusted in order to deal with arbitrary tame finite dimensional hereditary algebras. As we have mentioned, it is sufficient do deal with the case of precisely two simple modules (the so called bimodule case). However, in the general situation, we can be less explicit and have to rely on the general structure theory as presented in [R1]. There always is just one *generic* module G, this is the only infinite dimensional brick which is endofinite (the latter means: finitely generated when considered as a module over its endomorphism ring). In G, we choose a cyclic submodule P such that G/P has no non-zero finite dimensional direct summand (in the terminology of [R1], we choose a *peg*). Then G/P is the direct sum of infinitely many indecomposable modules, these are the so called Prüfer modules. Any such Prüfer module X has a filtration $X_1 \subset X_2 \subset \cdots \subset \bigcup X_i = X$, where all these modules X_i are indecomposable and regular, and where we can assume that this sequence cannot be refined. As a replacement of the index set $\Phi \cup \{\infty\}$ used in Remark 2, we now use the set of possible Prüfer modules X, and we consider instead of $P(\varphi)$ the submodule $P(X_1)$ of G which contains P, such that $P(X_1)/P = X_1$.

2. Extensions.

Proposition *Let I, J be disjoint infinite subsets of k. Then there exists an exact sequence*

$$0 \to P(J)^3 \to M \to P(I) \to 0$$

with $\operatorname{Hom}(M, P(J)) = 0$.

Proof. Write I as the disjoint union of four infinite subsets $I(1), I(2), I(3), I(4)$. We are going to define a certain submodule M' of $S(k)^4 = \bigoplus_{\lambda \in k} S(\lambda)^4$. For any module X, let $\iota_{123}(X) = X \oplus X \oplus X \oplus 0 \subseteq X^4$, and $\iota_4(X) = 0 \oplus 0 \oplus 0 \oplus X$. Also, given an integer $1 \le i \le 3$, let $\delta_{i4}(X)$ be the submodule of X^4 given by all elements of the form $(x_1, \dots, x_4)$ with $x_i \in X$, such that $x_i = x_4$ and $x_t = 0$ for $t \notin \{i, 4\}$. By definition,

$$M' = \iota_{123} S(J) \ + \ \sum_{i=1}^{3} \delta_{i4} S(I(i)) \ + \ \iota_4 S(I(4)).$$

Now, form the induced exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P^4 & \longrightarrow & P(k)^4 & \longrightarrow & S(k)^4 & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & P^4 & \longrightarrow & M & \longrightarrow & M' & \longrightarrow & 0 \end{array}$$

Clearly, the intersection $U = M \cap \iota_{123} P(k)$ is just the direct sum of three copies of $P(J)$; here we use that the sets $I(1), I(2), I(3), I(4)$ are disjoint. Also, M/U is just the module $P(I)$, since I is the disjoint union of $I(1), I(2), I(3), I(4)$. This establishes the exact sequence mentioned above.

It remains to be seen that $\operatorname{Hom}(M, P(J)) = 0$. Let $f\colon M \to P(J)$ be a homomorphism. Let $U' = M \cap \iota_4 P(k)$. Since $U' = P(I(4))$ and $I(4) \setminus J = I(4)$ is infinite, it follows that the restriction of f to U' is zero, thus f induces a map $f'\colon M/U' \to P(J)$. Next, we note that $M/U' = \bigoplus_{i=1}^{3} P(J \cup I(i))$. Again, $\operatorname{Hom}(P(J \cup I(i)), P(J)) = 0$, since $(J \cup I(i)) \setminus J = I(i)$ is infinite. But this shows that $f' = 0$, and therefore also $f = 0$.

Remark 4. A constructive way for obtaining M will be as follows: we start with five pairwise disjoint subsets $I(0), I(1), I(2), I(3), I(4)$ of k. For example, if the characteristic of k is zero, then we may take as set $I(i)$ the integers z with $z \equiv i$ modulo 5. Let $J = I(0)$ and $I = I(1) \cup I(2) \cup I(3) \cup I(4)$.

Remark 5. We may reformulate the statement above as follows: the vector space $\operatorname{Ext}^1(P(I), P(J))$ has dimension at least 3. Actually, the dimension of this vector space is infinite, even uncountable.

3. The category of representations of $K(3)$.

There is the following well-known characterization of the category of representations of $K(3)$.

Proposition. *Let $\mathcal{A}$ be a Grothendieck k-category. Let S, T be objects in $\mathcal{A}$ with* $\operatorname{End}(S) = \operatorname{End}(T) = k$ *and* $\operatorname{Hom}(S, T) = 0 = \operatorname{Hom}(T, S)$. *Assume M is an object in $\mathcal{A}$ with an exact sequence*

$$0 \to S^3 \to M \to T \to 0$$

and such that $\operatorname{Hom}(M, S) = 0$. *Let $\mathcal{B}$ be the full subcategory whose objects are the cokernels of maps of the form $X \to Y \oplus Z$, where X and Z are direct sums of copies of S and Y is a direct sum of copies of M. Then $\mathcal{B}$ is equivalent to the category of representations of $K(3)$.*

Outline of proof. Let us consider the class $\mathcal{P}'$ of all maps of the form $\phi\colon X \to Y \oplus Z$, where X and Z are direct sums of copies of S and Y is a direct sum of copies of M. An element $\phi\colon X \to Y \oplus Z$ in $\mathcal{P}'$ will be said to be *reduced* provided ϕ is a monomorphism and the image of ϕ is contained in the subobject $Y \oplus 0$ of $Y \oplus Z$. Let $\mathcal{P}$ be the category of all reduced elements in $\mathcal{P}'$; given two reduced maps $\phi\colon X \to Y \oplus Z$ and $\phi'\colon X' \to Y' \oplus Z'$, a map from ϕ to ϕ' in the category $\mathcal{P}$ is (by definition) a corresponding commutative diagram, thus a pair (σ, τ) of maps $\sigma\colon X \to X'$ and $\tau\colon Y \oplus Z \to Y' \oplus Z'$ such that $\tau\phi = \phi'\sigma$.

For the proof of Proposition, one first shows that any object in $\mathcal{B}$ actually is the cokernel of a reduced map. This shows that the functor $\mathcal{P} \to \mathcal{B}$ which attaches to a reduced map ϕ its cokernel is dense. Second, one shows that this functor is also fully faithful, thus an equivalence. In this way, one sees that $\mathcal{B}$ is equivalent to $\mathcal{P}$.

It is easy to see that $\mathcal{P}$ may be identified with the category of minimal projective presentations of $kK(3)$-modules; and, of course, the category of minimal projective presentations of $kK(3)$-modules is equivalent to the category of $kK(3)$-modules (again, using the cokernel functor: here we deal just with a special case of the considerations above). This completes the proof.

Remark 6. The old discussion of the difference of tameness and wildness usually was formulated in terms of the four-subspace and the five-subspace problem [B, C2]. It is easy to adjust to this setting: Since $kK(3)$ is a strictly wild algebra, there is a full embedding of the category $\mathcal{S}(5)$ of representations of the five-subspace-quiver into the category $\operatorname{Mod} kK(3)$ (and such an embedding can easily be written down), of course, there also is a full exact embedding of the category $\operatorname{Mod} kK(2)$ into $\mathcal{S}(4)$. Thus, altogether, we deal with the composition

$$\mathcal{S}(5) \longrightarrow \operatorname{Mod} kK(3) \longrightarrow \operatorname{Mod} kK(2) \longrightarrow \mathcal{S}(4)$$

of full exact embeddings.

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