On Generic Modules for String Algebras

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ABSTRACT. We consider a string algebra A. In case A is domestic, and G,G' are non-isomorphic generic modules, then G' neither generates nor cogenerates G. On the other hand, in case A is non-domestic, we are going to construct sequences G_1,G_2,... of pairwise non-isomorphic generic modules such that there are monomorphisms $G_t \to G_{i+1}$ and epimorphisms $G_{i+1} \to G_i$, for all i. The proof of the existence of such sequences answers a question raised by Bautista.

In particular, we see that any non-domestic string algebra has generic modules whose endomorphism rings have a non-zero radical. Actually, we will show that in the non-domestic case there always do exist generic modules with nilpotent endomorphisms of arbitrary large nilpotency index. Of course, in the domestic case the nilpotency index of a nilpotent endomorphism of a generic modules is bounded; in fact, it is bounded by the nilpotency index of the radical of A.

Let k be a field and A a finite dimensional k-algebra which is a string algebra. Recall that this means that $A = kQ/\mathcal{I}$ where $Q = (Q_0, Q_1)$ is a finite quiver and $\mathcal{I}$ is an admissible ideal generated by monomials, with the following properties: every vertex of Q is endpoint of at most two arrows and starting point of at most two arrows, and second, for any arrow β, there is at most one arrow α such that $\alpha\beta$ does not belong to $\mathcal{I}$, and at most one arrow γ such that $\beta\gamma$ does not belong to $\mathcal{I}$.

The finite-dimensional indecomposable A-modules are well-known, this classication is essentially due to Gelfand and Ponomarev (see [GP]; for the notation used here and for more detailled information, we refer to [R]). There are two types of finite-dimensional indecomposable A-modules: First of all, there are the string modules $M(w)$ described by words w. Second, there are the band modules $M(w, \phi, n)$, where w is a primitive cyclic word, ϕ an irreducible polynomial in $k[T]$ different from T and n a natural number: if λ is a non-zero element of k, we write $M(w, \lambda, n)$ instead of $M(w, T-\lambda, n)$. The words considered here use as letters the arrows of the quivers (the 'direct' letters) and formal inverses of these arrows (the 'inverse' letters); such a word may be interpreted as walking around in the quiver, avoiding the given zero relations. We denote by $|w|$ the length of the word w. Recall that a word w is called *cyclic* provided it contains both direct and inverse letters and such that also $w^2 = ww$ is a word; a cyclic word is said to be *primitive* provided it is not a proper power of some other word.

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For any primitive cyclic word w (with last letter being inverse), we need the following functor

$$F_w\colon \operatorname{Mod} k[T] \to \operatorname{Mod} A,$$

the underlying vector space of the A-module $M = F_w(V, \phi)$ is the direct sum of t copies V_i of V, and the operation of the arrows of Q on M follows the word w. More precisely, write $w = l_1 \cdots l_t$ with letters l_i, and consider first an index $1 \le i < t$. If $l_i = \epsilon$ for some arrow ϵ, then consider ϵ as an element of the path algebra kQ; the multiplication by ϵ yields the identity map $V_i \to V_{i-1}$. If $l_i = \epsilon^{-1}$ for some arrow ϵ, then the multiplication by ϵ yields the identity map $V_{i-1} \to V_i$. Finally, the last letter is inverse, say $l_t = \beta^{-1}$, where β is an arrow; here, the multiplication by β yields the map $\phi\colon V_{t-1} \to V_0$. In case ϵ is an arrow of the quiver Q, and the multiplication by ϵ is not yet defined on V_i, then this just should be zero.

The functor F_w sends the $k[T]$-module $k[T]/\langle \phi^n \rangle$ to $M(w, \phi, n)$, for any irreducible polynomial ϕ, and sends $k(T)$ to a generic module; we denote this generic module by $G(w)$. A module M will be said to be *prime*, provided there is a primitive cyclic word w such that M is either of the form $M(w, \phi, 1)$, where ϕ is a primitive polynomial in $k[T]$, different from T, or of the form $G(w)$. The word w (or better, its equivalence class with respect to inversion and rotation) is said to be the type of M. Note that the prime modules are of the form $F_w(K, x)$, where $K : k$ is a simple field extension generated by the element $x \in K$ (and where we consider the pair (K, x) as the $k[T]$-module whose underlying vector space is equal to K and the scalar multiplication by T is just the multiplication in K by x).

Given an arrow α, we denote by $\mathcal{N}(\alpha)$ the set of cyclic words starting with α and ending in an inverse letter (for all the words in $\mathcal{N}(\alpha)$, the last letter is a fixed one, namely the inverse of the only arrow different from α which has the same end point as α). Clearly, $\mathcal{N}(\alpha)$ is a semigroup. Note that $\mathcal{N}(\alpha)$ is the **free** semigroup on the subset $\mathcal{N}_1(\alpha) = \mathcal{N}(\alpha) \setminus \mathcal{N}(\alpha)^2$; also, A is non-domestic if and only if there exists an arrow α such that $\mathcal{N}(\alpha)$ is (non-empty and) not cyclic (for both assertions, see [R], Proposition 2 and its proof).

Recall that a module M is said to be serial provided it has a unique composition series. For a string algebra, a serial module is either a string module (and then of the form $M(w)$ where w is a word which consists only of direct letters or only of inverse letters), or a band module $M(w, \phi, 1)$, where w is a cyclic word with precisely one inverse letter, or precisely one direct letter. A serial module of the form $M(w)$ is said to be *isolated* provided there does not exist a letter l such that wl is a cyclic word.

1. Domestic string algebras.

In this section, *we assume that A is a domestic string algebra.* For the first two assertions, we refer to [R].

(1) *Let M be a prime module and S a simple module. Then* $\operatorname{Hom}(S, M)$ *is either zero or a simple* $\operatorname{End}(M)$*-module. Similarly,* $\operatorname{Hom}(M, S)$ *is either zero or a simple* $\operatorname{End}(M)^{\mathrm{op}}$*-module.*

(2) *Let M, M' be prime modules of types w, w' respectively. Let S be a simple A-module. If both* $\operatorname{Hom}(S, M) \neq 0$ *and* $\operatorname{Hom}(S, M') \neq 0$*, then w, w' are equivalent* *cyclic words. If both* $\operatorname{Hom}(M,S)\neq 0$ *and* $\operatorname{Hom}(M',S)\neq 0$, *then* w,w' *are equivalent cyclic words.*

(3) *Let* M,M' *be prime modules. Let* $f\colon M\to M'$ *be a non-invertible homomorphism. Then the image of* f *is contained in the radical of* M' *and the kernel of* f *contains the socle of* M. *Also,* f *can be factored through a direct sum of isolated serial modules.*

Proof. Let us assume that the socle of M is not contained in the kernel of f, thus there is a simple submodule $S=S(i)$ of M such that $f(S)\neq 0$, thus also M' has a submodule isomorphic to S. We conclude that M and M' have the same type, say $w=\alpha w'\beta^{-1}$, where α,β are arrows ending in the vertex i. Thus, we can write $M=F_w(K,x), M'=F_w(K',x')$, where $K:k$ and $K':k$ are simple field extensions, generated by x and x' respectively. The functor $F_w\colon \operatorname{Mod}k[T]\to \operatorname{Mod}A$ is a split embedding with retraction $M\mapsto w^\infty M/w^\infty 0_M$. By assumption, the map $f\colon M\to M'$ induces a non-zero map $w^\infty M/w^\infty 0_M\to w^\infty M'/w^\infty 0_{M'}$, thus we obtain a non-zero $k[T]$-linear map $\overline{f}\colon (K,x)\to(K',x')$. Such a map is an isomorphism, and therefore $F_w(\overline{f})$ is an isomorphism. It remains to observe that with $F_w(\overline{f})$ also f is an isomorphism, a contradiction.

Similarly, assume that the image of f is not contained in $\operatorname{rad}M$. There exists a simple module S and a map $\pi\colon M'\to S$ such that $\pi f\neq 0$. From $\operatorname{Hom}(M,S)\neq 0$ and $\operatorname{Hom}(M',S)\neq 0$ we conclude that M and M' have the same type, say w. As above, we can assume that $M=F_w(K,x), M'=F_w(K',x'))$ where $K:k$ and $K':k$ are simple field extensions, generated by x and x' respectively, and see that f induces a non-zero $k[T]$-linear map $\overline{f}\colon (K,x)\to(K',x')$. Again, $\overline{f}$ has to be an isomorphism of $k[T]$-modules and this then implies that also f is an isomorphism.

Since the map $f\colon M\to M'$ maps into the radical $\operatorname{rad}M'$ of M' and vanishes on the socle $\operatorname{soc}M$ of M, we can factor it via $M\to M/\operatorname{soc}M\to\operatorname{rad}M'\to M'$. Now, $M/\operatorname{soc}M$ is the direct sum of string modules M_i with simple top, whereas $N=\operatorname{rad}M'$ is the direct sum of string modules with simple socle. Let $M_i=M(w_i'w_i'')$, where all letters of w_i are direct, all letters of w_i' are invers. The canonical projections $\pi_i'\colon M_i\to M(w_i')$ and $\pi_i''M_i\to M(w_i'')$ provide an embedding $\pi_i\colon M_i\to M(w_i')\oplus M(w_i'')$ and it is easy to see that any map $f_i\colon M_i\to N$ can be extended via π_i to a map $M(w_i')\oplus M(w_i'')\to N$: Namely, consider a direct sum of string modules with simple socle. Then the canonical basis vectors of N are annihilated by all but at most one arrow. Thus, let α be the last letter of w_i' and β the last letter of $(w_i'')^{-1}$. Let e be the canonical basis element of M_i which generates M_i as an A-module, then $f_i(e)=y'+y''$, where y' is a linear combination of canonical basis elements of N which are annihilated by β, whereas y'' is a linear combination of canonical basis elements of N which are annihilated by α. We can define maps $M_i\to N$ by $e\mapsto y'$ and by $e\mapsto y''$, the first one factors through π_i', the second through π_i'', their sum is just f_i. This provides a factorization of f_i through π_i. Of course, the modules $M(w_i')$ and $M(w_i'')$ are serial string modules and it remains to be seen that they are isolated.

Assume one of them, say $M(w_i')$ is not isolated. Then $w_i'\beta^{-1}$ is a primitive cyclic word. Its top composition factor occurs also in the top of M. However, according to (2), it follows that the cyclic words w and $w_i'\beta^{-1}$ are equivalent. But this is impossible, since the length of a module of the form $M(w_i'\beta^{-1},\lambda,1)$ is equal to the length of $M(w_i')$, but this is a factor module of $M/\operatorname{soc}M$, where M is a prime module of type w.

Corollary *Let M, M' be prime modules of types w, w' respectively. If M' generates or cogenerates M, then w and w' are equivalent.*

Proof. If w and w' are non-equivalent, then any homomorphism $f\colon M' \to M$ is non-invertible, thus the image of f is contained in the radical of M. This shows that M' cannot generate M. Similarly, the kernel of any homomorphism $g\colon M \to M'$ contains the socle of M, thus M' cannot cogenerate M.

The assertion can be strengthened. Write $\operatorname{Ann} M = \{a \in A \mid aM = 0\}$, this is a twosided ideal of A, called the *annihilator ideal* of M. Two ideals I, I' of A are said to be *comparable* provided $I \subseteq I'$ or $I' \subseteq I$.

(4) *Let M, M' be prime modules of types w, w' respectively. If the annihilator ideals $\operatorname{Ann} M$, $\operatorname{Ann} M'$ are comparable, then then w and w' are equivalent.*

Proof: Let us assume that $\operatorname{Ann} M' \subseteq \operatorname{Ann} M$. Let a be a vertex of the quiver Q such that $S(a)$ occurs in the socle of M. Note that this implies that a is the endpoint of two different arrows α, β in Q, and we can assume that w starts with the letter α and ends with the letter β^{-1}. In particular, the elements α, β of kQ do not annihilate M, thus also not M'. As a consequence, both α and β (or their inverses) occur in w'. Up to rotation and inversion, we can assume that either w' is of the form $\alpha u \beta^{-1} v$ or $\alpha u \beta v$, with suitable words u, v.

First, consider the case $w' = \alpha u \beta^{-1} v$. Let us assume that w' is not equivalent to w, thus $S(a)$ cannot lie in the socle of M'. It follows that the first letter of v is inverse, and the last letter of v is direct. We form the word $w'' = \alpha u \beta^{-1} w$ (this is a cyclic word, since w starts with α and ends with β^{-1}). Note that w', w'' cannot be powers of a fixed word, since the first letter of v is inverse, whereas the first letter of w is direct. This shows that $\mathcal{N}(\alpha)$ is not cyclic, a contradiction.

Second, consider the case that $w' = \alpha u \beta v$. Again, we assume that $S(a)$ does not occur in the socle of M'. Here we conclude that the last letters of u and v both are direct, say $u = u'\gamma, v = v'\delta$. Of course, then γ, δ are arrows starting in a and they have to be different. Now we form the cyclic word $w'' = \alpha u v^{-1} \beta^{-1}$. Again, w', w'' cannot be powers of a fixed word, thus $\mathcal{N}(\alpha)$ is not cyclic. This contradiction shows that $S(a)$ occurs in the socle of M', thus w and w' have to be equivalent.

(5) *Let $n(A)$ be the nilpotency index of the radical of A. Let M be a prime module. Then $n(\operatorname{End}(M)) \le n(A)$.*

Proof: Consider a product $f_1 \cdot \ldots \cdot f_{n(A)}$ of nilpotent endomorphisms f_i of M. Consider nilpotent endomorphisms f_i of M. As we have seen, any f_i vanishes on the socle of M, thus a product of t nilpotent endomorphisms will vanish on the t-th socle $\operatorname{soc}_t M$ of M (it is defined inductively by $\operatorname{soc}_0 = 0$ and $\operatorname{soc}_{t+1} M/\operatorname{soc}_t M = \operatorname{soc}(M/\operatorname{soc}_t M)$). On the other hand, $\operatorname{soc}_{n(A)} M = M$ for any A-module M, thus any product of $n(A)$ nilpotent endomorphisms of M is zero.

2. Serial modules and band modules.

We need the following general result which may be deduced from the work of Krause [K].

Lemma 1. *Let M be a prime module, and I a serial module. If $f\colon I \to M$ is a homomorphism, then either f is an isomorphism, or else the image of f is contained in the radical of M.*

Here is a direct proof: First, consider the case where I is a string module, say $I = M(\alpha_t \cdots \alpha_1)$, where $\alpha_1, \dots, \alpha_t$ are arrows. Denote by a the starting vertex of α_1. If α_1 is the only arrow starting in a, then the simple module $S(a)$ does not occur in the top of a band module, thus the image of $f\colon I \to M$ has to be contained in the radical of M. Also, in case there is a second arrow, say β, which starts at a, then the image of I under f is generated by an element of M which belongs to the kernel of the multiplication by β; again: such an element belongs to the radical of M.

Next, let I be a band module, say $I = M(\alpha_t \cdots \alpha_1\beta^{-1}, \lambda, 1)$, where $\alpha_1, \dots, \alpha_t, \beta$ are arrows and λ is a non-zero element of k. Consider a canonical generator x of I, this is an element of I such that $\beta x = \lambda\alpha_t \cdots \alpha_1 x$. Assume that the image x' of x does not belong to the radical of M. As before, we see that $\beta x' \neq 0$. But $\beta x'$ can be a multiple of $\alpha_t \cdots \alpha_1 x'$ only in case the type of M is just $\alpha_t \cdots \alpha_1 \beta^{-1}$ and f is an isomorphism.

Here is the dual assertion:

Lemma 1*. *Let M be a prime module, and I a serial module. If $f\colon M \to I$ is a homomorphism, then either f is an isomorphism, or else the kernel of f contains the socle of M.*

3. Rich primitive cyclic words.

A primitive cyclic word w will be called *rich* provided there exists a word c which contains both direct and inverse letters, with $|c| < |w|$ and such that w^2 has a subword of the form $\alpha c\beta^{-1}$ as well as one of the form $\gamma^{-1}c\delta$ or $\gamma^{-1}c^{-1}\delta$.

Let us consider prime modules of rich type in more detail.

Proposition 1. *Let M be a prime module of type w. Let c be a word with $|c| < |w|$ and such that w^2 has a subword of the form $\alpha c\beta^{-1}$ as well as one of the form $\gamma^{-1}c\delta$ or $\gamma^{-1}c^{-1}\delta$ (where $\alpha, \beta, \gamma, \delta$ are arrows). Then there is an endomorphism f of M with image $M(c)$ and the following additional properties:*

Case 1: Assume that c contains a subword of the form $\epsilon\eta^{-1}$, where ϵ, η are arrows. Then the image of f is not contained in the radical of M (and the socle of $M(c)$ is not simple).

Case 2: Assume that c contains a subword of the form $\epsilon^{-1}\eta$, where ϵ, η are arrows. Then the kernel of f does not contain the socle of M (and the top of $M(c)$ is not simple).

In both cases, f cannot be factored through a direct sum of serial modules.

Proof. Let $M = F_w(K, x)$, where $K : k$ is a simple field extension generated by the element $x \in K$. The subword $\alpha c\beta^{-1}$ of w shows that actually $|c| \leq |w| - 2$, since otherwise both αc and $c\beta^{-1}$ would be obtained from w by rotation, but this is impossible. Thus $\alpha c\beta^{-1}$ is a subword of some word w' obtained from w by rotation, and therefore $M(c)\otimes_k K$ is a canonical factor module of $F_{w'}(K, x)$, thus also a factor module of the isomorphic module M. Also, if $\gamma^{-1}c\delta$ is a subword of w^2, then this is a subword of some word w'' obtained from w by rotation, and therefore $M(c)\otimes_k K$ is a canonical submodule of $F_{w''}(K, x)$, thus also a submodule of the isomorphic module M. Similarly, if $\gamma^{-1}c^{-1}\delta$ is a subword of w^2, then $\delta^{-1}c\gamma$ is a subword of w^{-2}, thus a subword of some word w'' obtained from w by rotation and inversion. It follows that $M(c) \otimes_k K$ is a canonical submodule of $F_{w''}(K, x)$, thus also a submodule of the isomorphic module M. It remains to note that $M(c) \otimes_k K$ is a direct sum of copies of $M(c)$, thus we obtain $M(c)$ as the image of an endomorphism of M.

Finally, we are going to show that f cannot be factored through a direct sum of serial modules. We observe that M itself is not serial: otherwise, also any submodule of M would be serial; but the module $M(c)$ is isomorphic to a submodule of M and, as we have noted already, is not serial. Now, let us assume that $f\colon M \to M$ factors through a direct sum $N = \bigoplus N_i$ of serial modules N_i, say $f = gh$, where $h\colon M \to N$, $g\colon N \to M$. According to section 2, the map h vanishes on the socle of M, and the image of the map g is contained in the radical of M. But as we know, in the first case the image of f is not contained in the radical of M, and in the second case the kernel of f does not contain the socle of M. This gives a contradiction.

Remark 1. We will see below a method for the construction of rich primitive cyclic words w such that the prime modules of type w have nilpotant endomorphisms with large nilpitency index. The following recipe for obtaining rich primitive cyclic words is easier: *Let u, v be different elements of $\mathcal{N}_1(\alpha)$. Then all the words $u^n v^m$ for $n, m \geq 1$ are primitive, and all the words $u^n v^m$ for $n, m \geq 2$ are rich.*

Proof. Assume we have $u^n v^m = w^t$ for some $t \geq 2$. Write $w = w_1 \cdots w_s$ with all $w_i \in \mathcal{N}_1(\alpha)$. Since we deal with free generators of a semigroup, the factorizations $u^n v^m = (w_1 \cdots w_s) \cdots (w_1 \cdots w_s)$ have to coincide, in particular, we have $n+m = st$. If $s \leq n$, then all $u = w_i$, since these are the first s factors. It follows that the given word is a power of u, thus also $v = u$, a contradiction. If $s > n$, then $n+m = st \geq 2{\cdot}s$ implies that $s < m$; now consider the last s factors: we see that $v = w_i$ for all i, thus again we obtain a contradiction.

Next, we have to show that the word $u^n v^m$ with $n, m \geq 2$ are rich. We can assume that $u < w$. First, consider the case where $u^{-1} > v^{-1}$. We are going to show that in this case any word $w = u^n v^m$ with $n, m \geq 1$ is rich. Since $u < v$, we can write $u = au'$, $v = av'$, where u' is empty or starts with a direct letter, and v' starts with an inverse letter. Similarly, $u^{-1} > v^{-1}$ means that we can write $u^{-1} = bu''$, $v^{-1} = bv''$, where this times v'' is empty or starts with a direct letter, and u'' starts with an inverse letter. We want to show that we can take $c = b^{-1}a$. Note that the first letter of a is α, thus a direct letter, whereas the last letter of b is an inverse letter (since it is the commen last letter of all the words in $\mathcal{N}(\alpha)$. This shows that c is neither direct nor inverse.

We can write uv as follows:

$$u \cdot v = (u'')^{-1} b^{-1} \cdot av' = (u'')^{-1} \cdot c \cdot v',$$

where $(u'')^{-1}$ ends with a direct letter, and v' starts with an inverse letter. In particular, we see that the length of the word c is smaller than $|uv|$. Similarly, we write vu in the form

$$v \cdot u = (v'')^{-1} b^{-1} \cdot au' = (v'')^{-1} \cdot c \cdot u',$$

here $(v'')^{-1}$ is empty or ends with an inverse letter, and u' is empty or starts with a direct letter. Now the copy of vu inside $w^2 = u^n v^m u^n v^m$ is surrounded by words which start with a direct letter and end with an inverse letter, thus the copy of c inside vu has as predecessor an inverse letter, as successor a direct letter.

Second, we also may have $u^{-1} < v^{-1}$. Consider a word of the form $w = u^n v^m$ with $n, m \geq 2$. As before, we write $u = au'$, $v = av'$, where u' is empty or starts with a direct letter, and v' starts with an inverse letter. Since $u^{-1} < v^{-1}$, we write $u^{-1} = bu''$, $v^{-1} = bv''$, where u'' is empty or starts with a direct letter, and v'' starts with an inverse letter. Again, let $c = b^{-1}a$, a word which is neither direct nor inverse. Write

$$v^2 = (v'')^{-1}b^{-1} \cdot av' = (v'')^{-1} \cdot c \cdot v',$$

where $(v'')^{-1}$ ends with a direct letter and v' starts with an inverse letter. Similarly, write

$$u^2 = (u'')^{-1}b^{-1} \cdot au' = (u'')^{-1} \cdot c \cdot u',$$

where $(u'')^{-1}$ is empty or ends with a direct letter and u' is empty or starts with an inverse letter. Now there are copies of u^2 inside $w^2 = u^n v^m u^n v^m$ which are surrounded by words which start with a direct letter and end with an inverse letter, thus a copy of c inside such a subword u^2 has as predecessor an inverse letter, as successor a direct letter. This completes the proof.

Remark 2. The previous remark indicates that for a non-domestic string algebra nearly all of the primitive cyclic words will be rich. But let us stress that there do exist non-domestic string algebras which have infinitely many primitive cyclic words which are not rich. As an example, take the quiver with three vertices a, b, c, two arrows $\beta, \gamma \colon a \to b$, two arrows $\alpha, \delta \colon b \to c$, and with relations $\delta\beta = 0 = \alpha\gamma$. Consider $\mathcal{N}(\alpha)$. The words $u = \alpha\delta^{-1}$ and $v = \alpha\beta\gamma^{-1}\delta^{-1}$ belong to $\mathcal{N}(\alpha)$ and all the words $u^t v$ are primitive, but not rich.

Remark 3. *Assume there exists a rich primitive cyclic word w in $\mathcal{N}(\eta)$. Then $\mathcal{N}(\eta)$ is not cyclic.*

Proof: Let c be a word which is neither direct nor inverse, with $|c| < |w|$ and such that w^2 has subwords of the form $\alpha c \beta^{-1}$ and $\gamma^{-1} c \delta$, say $w^2 = x\alpha c\beta^{-1}x' = y\gamma^{-1}c\delta y'$.

Let us first assume that c contains a subword of the form $\epsilon^{-1}\eta$, where ϵ and η are arrows, say $c = z\epsilon^{-1}\eta z'$. Then

$$w^2 = x\alpha z\epsilon^{-1}\eta z'\beta^{-1}x' = y\gamma^{-1}z\epsilon^{-1}\eta z'\delta y'.$$

We see that the words

$$\eta z'\beta^{-1}x'x\alpha z\epsilon^{-1} \quad \text{and} \quad \eta z'\delta y'y\gamma^{-1}z\epsilon^{-1}$$

belong to $\mathcal{N}(\eta)$, and this shows that $\mathcal{N}(\eta)$ cannot be cyclic. But

$$x\alpha z\epsilon^{-1}\mathcal{N}(\eta)\eta z'\beta^{-1}x'$$

is a subset of $\mathcal{N}(\eta)$, thus $\mathcal{N}(\eta)$ cannot be cyclic.

Now assume that c does not contain any subword of the form $\epsilon^{-1}\eta$, where ϵ and η are arrows, thus it is of the form $c = c_1c_2$, where c_1 is a direct word and c_2 is an inverse word, both being non-empty. If such a word c is a subword of w^2, then it is already a subword of w (since w starts with a direct letter and ends with an inverse letter). Let us assume that w has the following form $w = w_1cw_2cw_3$, where the last letter of w_1 and the first letter of w_3 are direct, and the first and the last letters of w_2 are inverse (in case this is not true, one may consider w^{-1} instead). Let γ^{-1} be the first letter of c_2, and δ the last letter of c_1. We denote by $\mathcal{N}^*(\gamma^{-1})$ the set of cyclic words with first letter γ^{-1} and last letter δ. Consider the words $u = c_2w_1c_1$ and $v = c_2w_3w_1c_1$ which belong to $\mathcal{N}^*(\gamma^{-1})$. They are cyclic words and not powers of the same word, since w is primitive. Of course, this set $\mathcal{N}^*(\gamma^{-1})$ is again a free semigroup, and as we now see, not cyclic. Note that

$$w_1c_1\mathcal{N}^*(\gamma^{-1})c_2w_3$$

is a subset of $\mathcal{N}(\eta)$, thus $\mathcal{N}(\eta)$ is not cyclic.

4. Construction of rich primitive cyclic words.

Lemma 2. *Let c be a word which contains at least one direct and one inverse letter. If the words $u'c$ and cv' exist, then also $u'cv'$ is a word.*

Proof. Since c is neither direct nor inverse, we can factor it as $c = c_1c_2$ such that the last letter of c_1 and the first letter of c_2 are not both direct and also not both inverse. Now, $u'c_1$ and c_2v' are words, then also $u'cv' = u'c_1 \cdot c_2v'$ is a word.

The words in $\mathcal{N}_1(\alpha)$ are totally ordered as follows: Let $x \neq y$ be elements of $\mathcal{N}_1(\alpha)$. First case: assume that $x = zx'$ and $y = zy'$, where x' and y' are words of length at least 1, with different first letters. If the first letter say of x' is direct, then the first one of y' ihas to be inverse, and we write $x < y$. Second case: One of the words is a left factor of the other, say $xx' = y$ for some word x'. In this case, again let $x < y$. The last definition is motivated by the following observation: *the word x' with $xx' = y$ has to start with an inverse letter* (thus we use a similar rule as in the first case, but now applied to xx and xx'). Namely, assume that x' starts with a direct letter. Since both words x and xx' belong to $\mathcal{N}(\alpha)$, this implies that also x' belongs to $\mathcal{N}(\alpha)$, but then $y = xx'$ belongs to $\mathcal{N}(\alpha)^2$.

If the words in $\mathcal{N}(\alpha)$ have last letter β^{-1}, then $w \mapsto w^{-1}$ yields an anti-isomorphism $\mathcal{N}(\alpha) \to \mathcal{N}(\beta)$. In particular, w belongs to $\mathcal{N}_1(\alpha)$ if and only if w^{-1} belongs to $\mathcal{N}_1(\beta)$. Thus, given two different words u, v in $\mathcal{N}_1(\alpha)$, we can assume that $u < v$ and we also may compare the inverses u^{-1} and v^{-1}; we see that we have to distinguish two cases: $u^{-1} < v^{-1}$ and $u^{-1} > v^{-1}$.

Lemma 3. *Let A be a non-domestic string algebra. Then there exist non-serial words x, y, z such that yxy and zxz are words, and such that the first and the last letters of y are direct letters, whereas the first and the last letters of z are inverse letters.*

Proof: Since A is non-domestic, there exists an arrow α with two words $u < v$ in $\mathcal{N}_1(\alpha)$. Note that both words ends with the same inverse letter, say β^{-1}. Since $u < v$, we can write $u = au'$, $v = av'$, where u' is empty or starts with a direct letter, and v' starts with an inverse letter.

Consider first the case $u^{-1} > v^{-1}$. This means that we can write $u^{-1} = bu''$, $v^{-1} = bv''$, where this times v'' is empty or starts with a direct letter, and u'' starts with an inverse letter. Taking inverses, we have $u = (u'')^{-1}b^{-1}$ and $v = (v'')^{-1}b^{-1}$ and $(u'')^{-1}$ ends with a direct letter, whereas $(v'')^{-1}$ is empty or ends with an inverse letter. Let

$$x = b^{-1}a,\ y = u'u(u'')^{-1},\ z = v'v(v'')^{-1}.$$

The word yxy exists, since this is a subword of u^6. Similarly, the word zxz exists, it is a subword of v^6. Since the last letter of b^{-1} is β^{-1} and the first letter of a is α, we see that x is non-uniserial. Since u is a subword of y and v is a subword of z, clearly z and z are non-serial. The first letter of y is direct, since either u' is non-empty and its first letter is direct; or else the first letter of u is the first letter of u and this is α. The last letter of y is that of $(u'')^{-1}$ and this is a direct letter. Similarly, one sees that the first letter as well as the last one of z are inverse letters.

We also have to consider the case $u^{-1} < v^{-1}$, thus we can write $u^{-1} = bu''$, $v^{-1} = bv''$, where u'' is empty or starts with a direct letter, and v'' starts with an inverse letter. Taking inverses, we have $u = (u'')^{-1}b^{-1}$ and $v = (v'')^{-1}b^{-1}$, where $(u'')^{-1}$ is empty or ends with an inverse letter and $(v'')^{-1}$ ends with a direct letter. This time, let

$$x = b^{-1}a,\ y = u'u(v'')^{-1},\ z = v'u(u'')^{-1}.$$

Now, yxy is a subword of u^2vu^2v and zxz is a subword of vu^2vu^2. The remaining arguments are the same as in the first case.

Lemma 4. *Consider a primitive cyclic word of the form $w = (xy)^n xz$ with $n \geq 1$, and assume that y starts and ends with direct letters, whereas z starts and ends with inverse letters. Let M be a prime module of type w. Then M has an endomorphism ϕ such that for $1 \leq t \leq n$, the image of ϕ^t is of the form $M((xy)^{n-t}x)$, whereas $\phi^{n+1} = 0$.*

Proof. Let $|x| = a, |y| = b, |z| = c$, thus $t = |w| = (n+1)a + nb + c$. Since M is a prime module of type w, we know that $M = F_w(K, x)$, where $K : k$ is a simple field extension generated by the element $x \in K$ and where we consider the pair (K, x) as the $k[T]$-module whose underlying vector space is equal to K and the scalar multiplication by T is just the multiplication in K by x. The underlying vector space of M is the direct sum of copies V_i of K, where $0 \leq i < t$, and the operation of the arrows of Q on M follows the word w. We define $\phi\colon M \to M$ as follows: For $a+b \leq i \leq (n+1)a+nb$, the restriction of ϕ to V_i shall be the identity map $V_i \to V_{i-a-b}$, wheras $\phi|V_i = 0$ for the remaining indices i. Thus, the kernel of ϕ is just the vectorspace sum $U = \bigoplus_{i<a+b} V_i \oplus \bigoplus_{i>(n+1)a+nb} V_i$. Note that U is indeed a submodule, since the last letter of y is direct and the first letter of z is invers, and M/U is isomorphic to $M((xy)^{n-1}x) \otimes K$. The image I of ϕ is the vectorspace sum $I = \bigoplus_{i=0}^{na+(n-1)b}$; also this is a submodule of M (here we use that the last letter of z is inverse and the first letter of y is direct), again isomorphic to $M((xy)^{n-1}x) \otimes K$. Also, it is obvious that ϕ is A-linear: here we use that the operation of the arrows of Q on M follows the word w and that all but the last letter of $w = l_1 \cdots l_t$ yield identity maps $V_i \to V_{i-1}$ or $V_{i-1} \to V_i$ (the last letter is part of z, thus concerns only the kernel U of ϕ). Finally, we observe that the image of the t-th power ϕ^t is isomorphic to $M((xy)^{n-t}x) \otimes K$, for $1 \leq t \leq n$ and that $\phi^{n+1} = 0$. This completes the proof.

Lemma 5. *If $\mathcal{N}(\alpha)$ is not cyclic, then for any $w \in \mathcal{N}(\alpha)$ there exists a $w' \in \mathcal{N}(\alpha)$ such that ww' is primitive.*

Proof. Let $w \in \mathcal{N}(\alpha)$. Since $\mathcal{N}_1(\alpha)$ contains at least two elements, we can choose an element $v \in \mathcal{N}_1(\alpha)$ such that w is not a power of v. Clearly, for large t, the word wv^t is primitive.

Theorem 1. *Let A be a non-domestic string algebra and a natural number n. Then there exist rich primitive cyclic words w such that any prime module M of type w has a nilpotent endomorphism of nilpotency index n.*

Proof. According to Lemma 3, there exist non-serial words x, y, z' such that yxy and $z'xz'$ are words, and such that the first and the last letters of y are direct letters, whereas the first and the last letters of z' are inverse letters. Since z' is non-serial, we may write $z' = z_1z_2$ where z_1, z_2 are non-empty words, such that the last letter of z_1 and the first letter of z_2 are not both direct and not both inverse. Up to duality, we may assume that the first letter of z_2 is direct, say equal to α. The two words z_2xz_1 and z_2xyxz_1 belong to $\mathcal{N}(\alpha)$ (the word z_2xyxz_1 exists and is cyclic, according to Lemma 2), and since the first letters of y and z_1 are different, these words z_2xz_1 and z_2xyxz_1 cannot be powers of some word u, thus $\mathcal{N}(\alpha)$ is not cyclic. According to Lemma 5, there exists a word w' in $\mathcal{N}(\alpha)$ such that $z_2(xy)^nxz_1w'$ is primitive. Of course, then also $(xy)^nxz_1w'z_2$ is primitive, thus we can apply Lemma 4 (with $z = z_1w'z_2$).

5. Generic modules as submodules and as factor modules of direct sums of copies of a string module.

Proposition 2. *Let $w = d\alpha c\beta^{-1}$ be a primitive cyclic word, let $z' = c\beta^{-1}d\alpha c$ and $z'' = d\alpha c\beta^{-1}d$. Let V be a vector space and ϕ an endomorphism of V. Then $F_w(V, \phi)$ is a submodule of the module $M(z')\otimes_k V$ and a factor module of the module $M(z'') \otimes_k V$.*

Proof: We factor F_w through the category of Kronecker modules as follows: The Kronecker modules are the representations of the Kronecker quiver, this quiver has precisely two vertices: a sink and a source, and two arrows going from the source to the sink; its path algebra will be denoted by B. Thus, a Kronecker module is of the form (V', V'', ϕ, ψ), where V', V'' are vector spaces and $\phi, \psi\colon V' \to V''$ are linear maps. There is the well-known full embedding functor

$$\iota\colon \operatorname{Mod} k[T] \to \operatorname{Mod} B \quad \text{given by} \quad \iota(V, \phi) = (V, V, \phi, 1).$$

Given the factorization $w = d\alpha c\beta^{-1}$, we define a functor

$$F\colon \operatorname{Mod} B \to \operatorname{Mod} A$$

as follows: Again, let $r = |c| + 1$, and $s = |d| + 1$. Let (V', V'', ϕ, ψ) be a Kronecker module, we are going to define an A-module $M = F(V', V'', \phi, \psi)$. The underlying vector space of M is the direct sum of r copies of V'', they are labelled V_i, where $0 \le i < r$ and s copies of V'', also labelled V_i, but now $r \le i < r + s$. The operation of the arrows of Q on M follows the word $w = d\alpha c\beta^{-1}$. More precisely, write $w = l_1 \cdots l_{r+s}$ with letters l_i, and consider first an index i different form r and $r + s$. If $l_i = \epsilon$ for some arrow ϵ, then consider ϵ as an element of the path algebra kQ; on V_i, the multiplication by ϵ will be the identity map $V_i \to V_{i-1}$. If $l_i = \epsilon^{-1}$ for some arrow ϵ, then on V_{i-1} the multiplication by ϵ will be the identity map $V_{i-1} \to V_i$. The letter with index r is $l_r = \alpha$, and we define on V_r the multiplication by α as follows: note that V_r is a copy of V', whereas V_{r-1} is a copy of V'', thus we can (and will) take the map $\psi\colon V' \to V''$. Finally, the last letter of w is $l_{r+s} = \beta^{-1}$. On V_{r+s-1} the multiplication by β will be given by the map $\phi\colon V' \to V''$, considered as a map $V_{r+s-1} \to V_0$; (here we use the identifcation of V_{r+s-1} with V' and of V_0 with V''). In case ϵ is an arrow of the quiver Q, and the multiplication by ϵ is not yet defined on V_i, then this part of the multiplication map just should be the zero map.

If V is a vector space and ϕ an endomorphism of V, then clearly $F\iota(V,\phi) = F_w(V,\phi)$.

On the other hand, let us consider the two indecomposable 3-dimensional representations of B. One of them is the indecomposable projective module $P = (k, k^2, \iota_1, \iota_2)$ where ι_1 and ι_2 are the canonical inclusion maps. The other is the indecomposable injective module $I = (k^2, k, \pi_1, \pi_2)$, where π_1, π_2 are the canonical projection maps. Note that

$$F(P) = M(z'') \quad \text{and} \quad F(I) = M(z').$$

It is obvious that the functor F commutes with (arbitrary) direct sums, in particular with forming tensor products $- \otimes_k V$, where V is a vector space. Now, given a vector space V and an endomorphism ϕ of V, the projective cover of the B-module $\iota(V,\phi)$ is of the form $P \otimes_k V$, the injective envelope of $\iota(V,\phi)$ is of the form $I \otimes_k V$. Thus we have a surjective map

$$\pi\colon P \otimes_k V \to \iota(V,\phi)$$

and an injective map

$$\mu\colon \iota(V,\phi) \to I \otimes_k V$$

Under the functor F, we obtain a surjective map

$$F(\pi)\colon M(z'') \otimes_k V = F(P \otimes_k V) \longrightarrow F\iota(V,\phi) = F_w(V,\phi)$$

and an injective map

$$F(\mu)\colon F_w(V,\phi) = F\iota(V,\phi) \longrightarrow F(I \otimes_k V) = M(z') \otimes_k V.$$

This completes the proof.

6. Chains of generic modules.

Theorem 2. *Let w be a rich primitive cyclic word. Then there exists a rich primitive cyclic word w' with the following property: Let $K : k$ be a simple field extension generated by the element $x \in K$, let $M = F_w(K,x)$ and $M' = F_{w'}(K,x)$. Then M' has a submodule and also a factor module which both are isomorphic to M.*

Proof of Theorem 2. Let w belong to $\mathcal{N}(\eta)$. Since w is rich, there exists a word c which is neither direct nor inverse, with $|c| < |w|$ and such that we can write $w^2 = x\alpha c\beta^{-1}x'$ and also $w^2 = y\gamma^{-1}c\delta y'$.

Now $|c| < |w|$ implies that we even have $|c| \le |w| - 2$. Namely, otherwise both αc and $c\beta^{-1}$ would be obtained from w by rotation, but this is impossible, since α is a direct letter and β^{-1} an inverse letter.

First of all, we conclude that there is a word d such that $d\alpha c\beta^{-1}$ is obtained form w by rotation and therefore $G(d\alpha c\beta^{-1})$ is isomorphic to $G(w)$. According to Proposition 2, we may embed $G(d\alpha c\beta^{-1})$ into $M(c\beta^{-1}d\alpha c)\otimes_k k(T)$,

Similarly, we see that there is a word d' such that $d'\gamma^{-1}c\delta$, and therefore also $c\delta d'\gamma^{-1}$ is obtained from w by rotation, thus $G(w)$ is isomorphic to $G(c\delta d'\gamma^{-1})$. Again, we apply Proposition 2 in order to see that $G(c\delta d'\gamma^{-1})$ is a factor module of $M(c\delta d'\gamma^{-1}c)\otimes_k k(T)$.

According to Lemma 2, the following words do exist

$$w_1 = y\gamma^{-1}c\beta^{-1}d\alpha c\delta y'$$
$$w_2 = x\alpha c\delta d'\gamma^{-1}c\beta^{-1}x'$$

and both belong to $\mathcal{N}(\eta)$. Since there exists a rich primitive cyclic word in $\mathcal{N}(\eta)$, we know that $\mathcal{N}(\eta)$ is not cyclic, see Remark 3. Thus we can use Lemma 2 in order to obtain a word w_3 such that $w' = w^2w_1w_2w_3$ is primitive. This is the required word. Of course, it is rich: just look at the factor w^2. The subword $\gamma^{-1}c\beta^{-1}d\alpha c\delta$ of w' shows that the generic module $G(w')$ has a submodule isomorphic to $M(c\beta^{-1}d\alpha c)\otimes_k k(T)$, thus also a submodule isomorphic to $G(w)$. The subword $\alpha c\delta d'\gamma^{-1}c\beta^{-1}$ of w' shows that the generic module $G(w')$ has a factor module isomorphic to $M(c\delta d'\gamma^{-1}c)\otimes_k k(T)$, thus also a factor module isomorphic to $G(w)$. This completes the proof.

Corollary. *Let A be a non-domestic string algebra. Then there are generic modules G_i with $i\in\mathbb{N}$ and a chain of monomorphisms*

$$G_1 \to G_2 \to \dots,$$

as well as a chain of epimorphisms

$$G_1 \leftarrow G_2 \leftarrow \dots.$$

Final Remark. Seeing the behaviour of special biserial algebras and the corresponding behaviour of tubular algebras, one may wonder whether the following is true: *A tame algebra should be non-domestic if and only if there exists a pair of non-isomorphic generic modules M, M' such that M' generates or cogenerates M,* and also *if and only if there exists a pair of non-isomorphic generic modules M, M' with comparable annihilator ideals.*

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