Matching Numbers for n-th Powers of Bipartite Graphs and Matching Capacities

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1 Introduction

History, Motivation and Background

Cartesian products are one of the most basic (but also least understood) operations in mathematics. Some well-known problems dealing with the Cartesian product of graphs have been discussed in literature. First let us mention the famous Shannon problem concerning the *zero-error capacity of a noisy channel* ([13]). The zero error capacity C_0 of a noisy channel is defined as the least upper bound of rates at which it is possible to transmit information with zero probability of error. Shannon also considered products of channels. The input alphabet of the product of two channels consists of all ordered pairs (x, x'), where x is a letter from the first channel alphabet and x' from the second one. The output alphabet is the similar set of ordered pairs of letters from the two individual output alphabets, and the transition probability from (x, x') to (y, y') is $w(y|x)w'(y'|x')$, where $w(y|x)$ is the probability of receiving letter y when letter x is sent and $w'(y'|x')$ is the probability of receiving letter y' when letter x' is sent. Products of channels are of big interest in Information Theory, because one works with the transmission of codewords x^n via a channel, in particular $x^n = (x_1, x_2, \ldots, x_n)$ is the element of a Cartesian product.

Shannon translated the problem of the *zero-error capacity of a noisy channel* into the language of Graph Theory. Every channel with identical input and output alphabet corresponds exactly to the undirected graph, where input letters are identified with the vertices of the graph $\mathcal{G}$ and vertices are adjacent, if input letters have common output with positive probability. A set of letters with zero error by transmission corresponds to a set of independent (= non-adjacent) vertices in the corresponding graph. The independence number $\alpha(\mathcal{G})$ is the maximal number of independent vertices in $\mathcal{G}$. Therefore one is interested in evaluating $\Theta(\mathcal{G}) = \lim\limits_{n\to\infty} \sqrt[n]{\alpha(\mathcal{G}^n)}$, where $(\mathcal{G})^n$ is the n-fold Cartesian Product of $\mathcal{G}$ with $\mathcal{V}(\mathcal{G}^n) = \prod\limits_{i=1}^{n} \mathcal{V}(\mathcal{G})$ and $\mathcal{E}(\mathcal{G}^n) = \{(x^n, y^n) : x^n = (x_1, \ldots, x_n),$ $y^n = (y_1, \ldots, y_n),\ x_i, y_i \in\ \mathcal{V}(\mathcal{G}^n),\ (x_i, y_i) \in\ \mathcal{E}(\mathcal{G})\}$. It is called the *Shannon capacity* of the graph $\mathcal{G}$. A surprising phenomenon occurs already here: One might expect that $\Theta(\mathcal{G})$ would be equal to $\alpha(\mathcal{G})$, but this in general is not true. We just know that $\Theta(\mathcal{G}) \geq \alpha(\mathcal{G})$ and equality does not hold in general. For the Pentagon P_5 (cycle of length 5 with vertex set $\{1,\ 2,\ 3,\ 4,\ 5\}$) $\alpha(P_5) = 2$, whereas $\alpha(\{P_5\}^2) = 5$ with independent set $\{(1,\ 1),\ (2,\ 3),\ (3,\ 5),\ (4,\ 2),\ (5,\ 4)\}$ (and two vectors are dependent, if both components are dependent).

Lovász mentioned in [10] that "the determination of the Shannon capacity is a very difficult problem even for very simple small graphs". In his article he determined the Shannon capacity for the Pentagon, a problem which was open for many years, to be $\sqrt{5}$. Moreover he generalized the method to obtain upper bounds on the capacity of arbitrary graphs and found a well-characterized function which bounds the capacity and equals the capacity in a large number of cases. He obtained several results on the capacity of special graphs.

In 1993 Ahlswede and Cai investigated extremal set partitions in Cartesian product spaces in [1]. For a given hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a finite set of vertices and $\mathcal{E}$ a system of subsets (hyperedges) of $\mathcal{V}$, and products of hypergraphs they analyzed packing numbers (that is the maximal size of a disjoint hyperedge set in a given hypergraph), covering numbers (that is the minimal number of hyperedges to cover all vertices in a given hypergraph) and partition numbers (that is the minimal size of a vertex partition of a given graph into edge sets). Let us cite one remarkable opinion given in [1]: "It seems to us that an understanding of these partition problems would be a significant contribution to an understanding of the basic, and seemingly simple, notion of Cartesian products." Partitioning and packing products with rectangles was discussed in [2] only one year later. Here Ahlswede and Cai investigated the partitioning and packing problem for the Cartesian product of hypergraphs.

In [3], which appeared in 1997, Ahlswede and Cai created a new channel model, the *matching channel*. This gives the starting point for the interest in this thesis. In the model of the matching channel the transmission of letters is ruled by a *class* of stochastic matrices. In addition to sender and receiver there is a third person called controller, who decides which matrix of the given class should govern the transmission of a letter by the sender. The controller knows which codeword the sender wants to transmit, whereas the receiver has no knowledge about the actions of the controller. The main fact is that the controller can choose (match) the state sequence of the channel. The exact definition of a matching channel is given in Section 2. The special case of the *deterministic matching channel*, where the class of matrices consists exactly of 0-1-matrices, initiates interesting connections to combinatorial Matching Theory. We will see that a code for a deterministic matching channel has just a one-element decoding set. This code forms a maximal matching in the bipartite graph, where the vertex set here has the bipartite partition into codewords and one-element decoding sets for each codeword. This bipartite graph is itself a Cartesian product of bipartite graphs, and this gives the idea of examining maximal matchings and matching numbers (the size of a maximal matching) in Cartesian products. Matchings are defined as a set of independent (non-intersecting) edges in a given graph. This reminds us somehow of the problem of determining the vertex independence number. In the *matching problem* we search for the maximal number of *independent edges* and in the *vertex independence problem* we want to calculate the maximal number of *independent vertices*. From another point of view the matching problem can be seen as a special hypergraph packing problem: The packing number for a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is the maximal size of a set of disjoint sets from $\mathcal{E}$. Thus the matching number equals the packing number, in which the underlying hypergraph really is a graph and the disjoint sets of $\mathcal{E}$ are the independent edges.

At the end of this historical sketch we return to the Shannon capacity. In [4], Alon disproved a conjecture of the work mentioned already, [13]. Shannon proved in [13] that for every two graphs $\mathcal{G}$ and $\mathcal{H}$ the inequality
$\Theta(\mathcal{G}\dot{\cup}\mathcal{H}) \geq \Theta(\mathcal{G}) + \Theta(\mathcal{H})$ holds and that equality holds for graphs with special properties. Shannon remarked in [13]: "We conjecture but have not been able to prove that the inequalities in Theorem 4 hold in general" (the inequality of our interest is $\Theta(\mathcal{G}\dot{\cup}\mathcal{H}) \geq \Theta(\mathcal{G}) + \Theta(\mathcal{H})$). Alon showed that there exists a graph $\mathcal{G}$ on 27 vertices such that $\Theta(\mathcal{G}) \leq 7$, $\Theta(\overline{\mathcal{G}}) = 3$, whereas $\Theta(\mathcal{G}\dot{\cup}\overline{\mathcal{G}}) \geq 2\sqrt{27} > 10$, where $\overline{\mathcal{G}}$ is the complement graph. This counterexample is the well known Schläfli Graph, which is used by Haemers, [8], for disproving Shannon's other conjecture (of the above mentioned theorem) in 1979. We present Alon's construction in Subsection 3.3.1.

Results for Matching Channels

Ahlswede and Cai, [3] showed that finding a maximal MC-Code for a deterministic matching channel belonging to $\mathcal{G}$ is equivalent to the construction of a maximal matching in all n-th powers of a bipartite graph $\mathcal{G}$. The matching numbers for the n-th powers are denoted by $\nu(\mathcal{G}^n)$. The authors studied the asymptotic behaviour and proved the existence of the matching capacity $\gamma(\mathcal{G}) = \lim_{n\to\infty} \frac{1}{n} \log \nu(\mathcal{G}^n)$ of a bipartite graph $\mathcal{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{E})$. Moreover they gave the exact formula $\gamma(\mathcal{G}) = \max_{(P,Q)\in\mathcal{K}(\mathcal{G})} \min(H(P), H(Q))$. The *König-Hall set* $\mathcal{K}(\mathcal{G})$ is defined by

$$\mathcal{K}(\mathcal{G}) = \left\{ (P,Q) : \begin{array}{c} P \text{ is probability distribution on } \mathcal{X}, \\ Q \text{ is probability distribution on } \mathcal{Y}, \\ \text{and } \ P(\mathcal{L}) \leq Q(\Gamma_{\mathcal{G}}(\mathcal{L})) \text{ for all } \mathcal{L} \subset \mathcal{X} \end{array} \right\},$$

where $\Gamma_{\mathcal{G}}(\mathcal{L})$ is the *neighbourhood of a subset* $\mathcal{L} \subset \mathcal{X}$ (the exact definition is given on page 12).

This König-Hall pair generalizes the well known König-Hall condition in

Theorem 1.1 (Hall's Marriage Theorem,[11]) *A bipartite*

graph

$\mathcal{G} = (\mathcal{V}_1, \mathcal{V}_2, \mathcal{E})$ *has a matching of* $\mathcal{V}_1$ *into* $\mathcal{V}_2$ *iff*

$$|\Gamma_{\mathcal{G}}(\mathcal{L})| \geq \mathcal{L} \ \ \forall \ \mathcal{L} \subset \ \mathcal{V}_1.$$

Observe that Ahlswede and Cai used probability distributions associated with $\mathcal{G}$ instead of cardinalities of sets .

Moreover, in [3] Ahlswede and Cai gave the *starred bipartite graph* (a graph consisting of independent stars; a star is the complete bipartite graph $K_{1,n}$ with colour classes of size 1 and n, for exact definition cf. p. 11) as an example, for which they exactly determined the matching numbers of the n-th powers.

New Results

This work has its origin in [3], and it starts with the example (bipartite graph consisting of independent stars) given by Ahlswede and Cai.

We investigate matchings for Cartesian products of a special class of bipartite graphs and find exact formulas for matching numbers of the n-th powers.

For two bipartite graphs $\mathcal{G}_i = (\mathcal{X}_i, \mathcal{Y}_i, \mathcal{E}_i)$, $i = 1, 2$, the direct product $\mathcal{G}_1 \otimes \mathcal{G}_2$ is defined as $\left(\prod_{i=1}^{2} (\mathcal{X}_i, \mathcal{Y}_i), \mathcal{E}\right)$ with $\mathcal{E} = \{(x^2, y^2) : (x_i, y_i) \in \mathcal{E}_i \text{ for } i = 1, 2\}$, for which $x^2 = (x_1, x_2)$ and $y^2 = (y_1, y_2)$.

$\mathcal{G}^n = \mathcal{G}^{\otimes n} = \underbrace{\mathcal{G} \otimes \mathcal{G} \otimes \ldots \otimes \mathcal{G}}_{n \ times}$ is called the n-th power of $\mathcal{G}$. Particularly in $\mathcal{V}(\mathcal{G}^n)$ we have vertices $x^n = (x_1, \ldots, x_n)$ and $y^n = (y_1, \ldots, y_n)$ with $x_i, y_i \in \mathcal{V}(\mathcal{G})$ for all $i = 1, \ldots, n$, and edges (x^n, y^n) iff $(x_i, y_i) \in \mathcal{E}(\mathcal{G})$ for all $i = 1, \ldots, n$.

We give the exact matching numbers for n-powers of $\mathcal{G} = K_{l,m} \dot{\cup} K_{p,q}$, which are disjoint unions of complete bipartite graphs $K_{l,m}$ and $K_{p,q}$. The matching numbers have the value $\nu(\mathcal{G}^n) = \sum_{k=0}^{n} \binom{n}{k} \min(l^{n-k} p^k, m^{n-k} q^k)$ (Theorem 3.2). It is nice to see how the Binomial Theorem influences the behaviour of the matching numbers: The construction of the Cartesian product for two independent complete bipartite graphs is done in the same manner as the n-fold multiplication for a sum of two terms. Therefore it is not surprising that a generalization to a graph, consisting of α complete bipartite graphs, has matching numbers according to the Polynomial Theorem

$$\nu(\mathcal{G}^n) = \sum_{k_1 + \ldots + k_\alpha = n} \binom{n}{k_1 \ k_2 \ldots k_\alpha} \min(l_1^{k_1} l_2^{k_2} \ldots l_\alpha^{k_\alpha}, m_1^{k_1} m_2^{k_2} \ldots m_\alpha^{k_\alpha})$$

with multinomial coefficients $\binom{n}{k_1 \ k_2 \ldots k_\alpha} = \frac{n!}{k_1! k_2! \ldots k_\alpha!}$ and $k_i \geq 0$ for all
$i = 1, \ldots, \alpha$. Since we have got the exact values for the matching numbers of the n-th powers of graphs $\mathcal{G} = K_{l,m} \dot{\cup} K_{p,q}$ we also have the exact value for the matching capacity $\gamma(\mathcal{G})$. According to Theorem 3.6 we have 3 possibilities for $\gamma(\mathcal{G})$ depending on the special constellation of l, m, p and q:

$$\log(l + p), \quad \log(m + q) \quad \text{or} \quad h(\lambda_{cut}) + (1 - \lambda_{cut}) \log l + \lambda_{cut} \log p$$

with $\lambda_{cut} = \frac{\log \frac{l}{m}}{\log \frac{q}{p} + \log \frac{l}{m}}$.

Beyond the technical proof we are able to interpret this result, going back to the Ahlswede-Cai formula for the matching capacity with maximization over all König-Hall pairs of probability distributions.

We see that in each case the König-Hall pair (P_λ, Q_λ) with

$$P_\lambda = (\underbrace{\frac{\lambda}{l}, \dots, \frac{\lambda}{l}}_{l \text{ times}}, \underbrace{\frac{1-\lambda}{p}, \dots, \frac{1-\lambda}{p}}_{p \text{ times}})$$

$$\text{and} \quad Q_\lambda = (\underbrace{\frac{\lambda}{m}, \dots, \frac{\lambda}{m}}_{m \text{ times}}, \underbrace{\frac{1-\lambda}{q}, \dots, \frac{1-\lambda}{q}}_{q \text{ times}}),$$

where λ takes the values $\frac{l}{l+p}$, $\frac{m}{m+q}$ or $\lambda = \frac{\log \frac{l}{m}}{\log \frac{q}{p} + \log \frac{l}{m}}$, is maximizing and determines the values for the $\gamma(\mathcal{G}) = \min(H(P_\lambda, Q_\lambda))$.

P_λ and Q_λ have the nice property that the probabilities are grouped into two parts, one part gets probability λ and the other part $1-\lambda$, and in each part there is equidistribution.

Since we have a certain analogy in the structure for matching numbers and vertex independence numbers (both are independent sets of edges respectively vertices) it is interesting to know, whether we can find inequalities for the matching capacities comparing $\gamma(\mathcal{G}) + \gamma(\mathcal{H})$ and $\gamma(\mathcal{G} \dot\cup \mathcal{H})$ (like we have the inequality $\Theta(\mathcal{G} \dot\cup \mathcal{H}) \geq \Theta(\mathcal{G}) + \Theta(\mathcal{H})$ for the Shannon capacity). Even in the class $K_{l,m} \dot\cup K_{m,l}$ we have no such inequality.

In another direction we indeed find a nice parallel between the matching numbers and the vertex independence numbers (Theorem 3.21):

For $\mathcal{G} = K_{l,m} \dot\cup K_{m,l}$ we have a regularity for the development of the matching numbers $\nu(\mathcal{G}^n)$ in the dimension n. There occurs a factor $l+m$ from an odd to an even dimension. Unfortunately there is no such factor from an even to an odd dimension. Nevertheless $l+m$ is at least an upper bound in this case.

This constant factor for the transition from one dimension to the next one gives us a flavour of the pentagon-problem Lovász investigated. He gave an exact formula for the vertex-independence number of the n-th powers of the Pentagon. In particular he observed a transition of an odd to an even dimension with factor $\frac{5}{2}$ and a factor 2 in the other case.

In Section 4 we investigate the effect of deleting edges from the graph $K_{l,m} \dot\cup K_{p,q}$. It can be proved that deleting just one edge does not harm the matching numbers of $K_{l,m} \dot\cup K_{p,q}$. In more general, we delete a complete bipartite graph $K_{c,d}$ out of the component $K_{l,m}$, and Theorem 4.3 determines the matching numbers of the n-th powers.

In particular for all six-tuples with the property $\frac{l}{m} \leq \frac{l-c}{d}$ the graph $(K_{l,m} \setminus K_{c,d}) \dot{\cup} K_{p,q}$ has the same matching capacity as $K_{l,m} \dot{\cup} K_{p,q}$ (Corollary 4.4), and again the parallel to the maximizing König-Hall pair (P_λ, Q_λ) occurs (Corollary 4.5).

After having finished the main part of this thesis Brualdi's article [5] came to our mind. He determined bounds for the term rank of the direct product of (0,1)-matrices (term rank in (0,1)-language equals the matching number in language of Graph Theory). In the last section we first present Brualdi's results and compare them with results of Ore [12] and Dulmage & Mendelsohn [7]. Finally we test Brualdi's bounds in our situation and find out that the lower bound is sharp for the class $K_{l,m} \dot{\cup} K_{p,q}$ (Theorem 5.6), whereas big gaps can occur between the matching number and the upper bound.

The sections are organized as follows: Section 1 is the Introduction. In Section 2 notations, definitions and basic results of Graph Theory, Matching Theory and Information Theory are shortly summarized, also the matching channel and the deterministic matching channel are described. Sections 3 and 4 form the main part of this thesis: Here we present our new results. Section 3 contains the calculation of the matching numbers of n-th powers of $K_{l,m} \dot{\cup} K_{p,q}$ and the matching capacity. Moreover we compare the results to the Shannon capacity and the structure of the vertex independence number of the n-th powers of the Pentagon. Also Alon's counterexample [4] is presented here. Section 4 treats the special incomplete case $(K_{l,m} \setminus K_{c,d}) \dot{\cup} K_{p,q}$ and we present the matching numbers of the n-powers. Section 5 may be seen as an appendix: Brualdi's basic bounds [5] are presented and tested in our situation.

I want to thank Prof. Ahlswede, who lead me to the interesting topic of matching channels and always gave me interesting proposals for my progress. Furthermore I am grateful to C. Deppe, L. Bäumer, B. Balkenhol and U. Tamm. They often spent their time for helpful discussions and reading earlier versions of the thesis. In particular B. Balkenhol gave many interesting hints and good ideas for the development of my thesis.

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2 Foundations

2.1 Basic Terminology

For definitions we follow the comprehensive presentation of Matching Theory by Lovász and Plummer ([11]).

An **undirected graph** (or simply **graph**) $\mathcal{G} = (\mathcal{V}(\mathcal{G}), \mathcal{E}(\mathcal{G}))$ consists of a finite nonempty set of elements $\mathcal{V}(\mathcal{G})$ called **vertices** and a set of unordered pairs of vertices $\mathcal{E}(\mathcal{G})$ called edges.

If uv is an edge in graph G, edge uv is said to **join** vertices u and v, to be **incident** with vertices u and v, and vertices u and v are said to be **adjacent**. A **loop** is an edge connecting a vertex with itself. Distinct edges connecting the same vertices are called **multiple edges**.

A **simple graph** is an undirected graph without loops and multiple edges.

The number of edges in graph $\mathcal{G}$ incident with a vertex u is called the **degree** of u (in $\mathcal{G}$) and denoted by $deg_{\mathcal{G}}(u)$. If graph $\mathcal{G}$ is understood, we shall sometimes abbreviate this to $deg(u)$. A graph in which all degrees are equal to k is said to be k-**regular** and if $\mathcal{G}$ is k-regular for some k, we simply say that $\mathcal{G}$ is **regular**.

An alternating sequence of vertices and edges, beginning and ending with vertices, is called a walk. If all edges and all vertices are distinct, the walk is a **path**.

A graph is **connected** if every two vertices are joined by a path.

A maximal connected subgraph of $\mathcal{G}$ is called a **connected component** (shortly **component**) of $\mathcal{G}$.

The **complete graph** on n vertices , denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.

The **complementary graph** $\overline{\mathcal{G}}$ of a graph $\mathcal{G}$ has the same vertices as $\mathcal{G}$. Two vertices are adjacent in $\overline{\mathcal{G}}$ iff (i.e. if and only if) they are not adjacent in $\mathcal{G}$.

If the vertex set of a graph $\mathcal{G}$ can be partitioned into two disjoint non-empty sets, $\mathcal{V}(\mathcal{G}) = \mathcal{A} \cup \mathcal{B}$, such that all edges of $\mathcal{G}$ join a vertex of $\mathcal{A}$ to a vertex of $\mathcal{B}$, we call $\mathcal{G}$ **bipartite** and refer to $\mathcal{A} \cup \mathcal{B}$ as the **bipartition** of $\mathcal{G}$, and sometimes we write $\mathcal{G} = (\mathcal{A}, \mathcal{B}, \mathcal{E})$. A bipartite graph is often referred to as a **2-colourable** graph (which means that we can partition the vertices into two classes, and the vertices in one class have the same colour, and no two points with the same colour are adjacent).

Let $\mathcal{G}$ be a bipartite graph with bipartition $\mathcal{U} \cup \mathcal{V}$, where $\mathcal{U} = \{u_1, \ldots, u_m\}$ and $\mathcal{V} = \{v_1, \ldots, v_n\}$. The **biadjacency matrix** $A = (a_{ij})$ is given by

$$a_{ij} = \begin{cases} 1, & \text{if } u_i v_j \in \mathcal{E}(\mathcal{G}), \\ 0, & \text{otherwise.} \end{cases}$$

We abbreviate the all-0-matrix of size $m \times n$ by $0_{m \times n}$ and the all-1-matrix of size $m \times n$ by $1_{m \times n}$.

A special bipartite graph is $K_{m,n}$, the **complete bipartite graph** having colour classes of size m and n and in which every vertex in each colour class is adjacent with every vertex in the other. In particular, $K_{1,n}$ is called an **n-star** (or sometimes simply a **star**).

For two bipartite graphs $\mathcal{G}_i = (\mathcal{X}_i, \mathcal{Y}_i, \mathcal{E}_i)$, $i = 1, 2$ the **direct product** $\mathcal{G}_1 \otimes \mathcal{G}_2$ is defined as $\left(\prod_{i=1}^{2} (\mathcal{X}_i, \mathcal{Y}_i), \mathcal{E} \right)$ with $\mathcal{E} = \{(x^2, y^2) : (x_i, y_i) \in \mathcal{E}_i \text{ for } i = 1, 2\}$, where $x^2 = (x_1, x_2)$ and $y^2 = (y_1, y_2)$.

$\mathcal{G}^n = \mathcal{G}^{\otimes n} = \underbrace{\mathcal{G} \otimes \mathcal{G} \otimes \ldots \otimes \mathcal{G}}_{n \text{ times}}$ is called the n-th power of $\mathcal{G}$. Especially in $\mathcal{V}(\mathcal{G}^n)$ we have vertices $x^n = (x_1, \ldots, x_n)$ and $y^n = (y_1, \ldots, y_n)$ with $x_i, y_i \in \mathcal{V}(\mathcal{G})$ for all $i = 1, \ldots, n$, and edges (x^n, y^n), iff $(x_i, y_i) \in \mathcal{E}(\mathcal{G})$ for all $i = 0, \ldots, n$.

We can as well define the direct product in the language of biadjacency matrices. Let A be a (0,1)-matrix of size m times n and B a (0,1)-matrix of size s times t. The **direct product** or the **Kronecker product** of A with B is defined by

$$A \times B = \begin{pmatrix} a_{11}B & a_{12}B & \ldots & a_{1n}B \\ a_{21}B & a_{22}B & \ldots & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & \ldots & a_{mn}B \end{pmatrix}.$$

It is a (0,1)-matrix of size ms times nt.

A set of edges in a graph $\mathcal{G}$ is called **independent** or a **matching** if no two edges have a vertex in common. The size of a largest matching in $\mathcal{G}$ is called the **matching number** of $\mathcal{G}$ and is denoted by $\nu(\mathcal{G})$.

A matching of the bipartite graph $\mathcal{G} = (\mathcal{V}_1, \mathcal{V}_2, \mathcal{E})$ is called a **matching of $\mathcal{V}_1$ into $\mathcal{V}_2$**, if every $v \in \mathcal{V}_1$ is endpoint of an edge in the matching. A matching is called **perfect**, if it is both, a matching of $\mathcal{V}_1$ into $\mathcal{V}_2$ and vice versa.

A set of vertices in a graph $\mathcal{G}$ is said to be **independent** if no two of them are adjacent. The cardinality of a largest independent set of vertices is the **(vertex) independence number** of $\mathcal{G}$, and is written $\alpha(\mathcal{G})$.

A set of vertices $\mathcal{S}$ of $\mathcal{G}$ is a **vertex cover** of $\mathcal{G}$ if each edge of $\mathcal{G}$ has at least one endpoint in set $\mathcal{S}$. The cardinality of a smallest vertex cover is denoted by $\tau(\mathcal{G})$ and is the **point covering number** of $\mathcal{G}$.

The **neighbourhood** of a vertex $v \in \mathcal{V}$ for a given graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is
defined as the set of vertices, which are adjacent to v and we write $\Gamma_{\mathcal{G}}(v) = \{v' : (v, v') \in \mathcal{E}\}$. If $\mathcal{G}$ is understood, we also write $\Gamma(v)$ instead of $\Gamma_{\mathcal{G}}(v)$. The neighbourhood for a subset $\mathcal{L} \subset \mathcal{V}$ is defined by $\Gamma(\mathcal{L}) = \bigcup\limits_{v \in \mathcal{L}} \Gamma_{\mathcal{G}}(v)$, especially $|\Gamma_{\mathcal{G}}(v)| = deg_{\mathcal{G}}(v)$.

Furthermore we need **Entropy** and **Information Quantities**: In this context we always use finite alphabets $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ with random variables X, Y, Z, where X has values in $\mathcal{X}$, Y in $\mathcal{Y}$ and Z in $\mathcal{Z}$. $P_X,\ P_Y,\ P_Z$ are the accompanying probability distributions.

Let W, which is an $\mathcal{X} \times \mathcal{Y}$-stochastic matrix, serve as a channel with finite input alphabet $\mathcal{X}$ and finite output alphabet $\mathcal{Y}$. The joint distribution of (X, Y) on $\mathcal{X} \times \mathcal{Y}$ shall be given by

$$Pr(X = x,\ Y = y) = P(x) \cdot W(y|x).$$

The entropy of the random variable X is defined as

$$H(X) \triangleq -\sum_{x \in \mathcal{X}} P(x) \cdot \log P(x).$$

The logarithm always refers to the basis 2. Furthermore we often write with abuse of notation $H(P_X)$ instead of $H(X)$, if P_X is the distribution of X.

In particular if X is binary, where $\mathcal{X} = \{0, 1\}$, $P_X(0) = \lambda$, $P_X(1) = 1 - \lambda$, we get

$$H(X) = -\lambda log(\lambda) - (1 - \lambda) log(1 - \lambda),$$

and we also write $h(\lambda)$ instead of $H(X)$.

The conditional entropy of two random variables X and Z is defined as

$$H(X|Z) \triangleq \sum_{z\in\mathcal{Z}} P_Z(z) \cdot H(X|Z=z),$$

where $H(X|Z=z) \triangleq -\sum_{x\in\mathcal{X}} P(x|z) \cdot \log P(x|z)$.

The mutual information of two random variables X and Y is given by

$$I(X \wedge Y) \triangleq H(X) - H(X|Y) = H(Y) - H(Y|X).$$

2.2 Known Theorems

Theorem 2.1 (Hall's Marriage Theorem) *A bipartite graph $\mathcal{G} = (\mathcal{V}_1, \mathcal{V}_2, \mathcal{E})$ has a matching of $\mathcal{V}_1$ into $\mathcal{V}_2$ iff $|\Gamma_{\mathcal{G}}(\mathcal{L})| \geq |\mathcal{L}|$ for all $\mathcal{L} \subset \mathcal{V}_1$.*

Theorem 2.2 (König's Minimax Theorem) *For every bipartite graph $\mathcal{G}$ the following equality holds:*

$$\tau(\mathcal{G}) = \nu(\mathcal{G}).$$

2.3 Matching Channel

The study of maximal matchings in products of bipartite graphs goes back to the new model of the "Matching channel", introduced by Ahlswede and Cai, [3]. In this model a cooperative person knows the codeword to be sent and can choose (match) the state sequence of the channel.

2.3.1 Description of the Matching Channel

Let $\mathcal{X}$ be the input alphabet, $\mathcal{Y}$ the output alphabet and $\mathcal{S}$ any set of states. A *class* of stochastic matrices with $|\mathcal{X}|$ rows and $|\mathcal{Y}|$ columns rules the transmission of letters. In addition to sender and receiver there is a third person, called controller, who decides which matrix $W \in \mathcal{W} = \{w(\cdot| \cdot |s) : s \in \mathcal{S}\}$ shall govern the transmission of a letter by the sender. The construction is done in such a way that the controller knows the codeword the sender wants to transmit and the receiver has no knowledge about the actions of the controller.

As code concept for this situation Ahlswede and Cai introduced a matching code (MC).

Definition 2.3 *Let $\mathcal{X}, \mathcal{Y}$ be finite alphabets and $\mathcal{S}$ any set of states.*

$\{(u_i, \mathcal{D}_i) : 1 \leq i \leq M\}$ *is called an* (n, M, λ)*-MC-code, if*

$$u_i \in \mathcal{X}^n, \mathcal{D}_i \subset \mathcal{Y}^n \quad \text{for} \quad i = 1, 2, \ldots, M,$$

$$u_i \neq u_j \quad \text{and} \quad \mathcal{D}_i \cap \mathcal{D}_j = \emptyset \quad \text{for} \quad i \neq j$$

and if for every i there is a sequence $s^n(i) = (s_1(i), \ldots, s_n(i)) \in \mathcal{S}^n = \prod_1^n \mathcal{S}$ with

$$W^n(\mathcal{D}_i | u_i | s^n(i)) \geq 1 - \lambda,$$

if $W^n(y^n | x^n | s^n(i)) = \prod_{t=1}^n W(y_t | x_t | s_t(i))$ for $x^n = (x_1, \ldots, x_n) \in \mathcal{X}^n$ and
$y^n = (y_1, \ldots, y^n) \in \mathcal{Y}^n$.

Let $C(\mathcal{W})$ denote the capacity of the matching channel $\mathcal{W}$. As usual we denote by X, S, Y random variables with values in $\mathcal{X}, \mathcal{S}$ and $\mathcal{Y}$. Let P_{XS} be the joint distribution of (X, S) and P_{XSY} the joint distribution of (X, S, Y), where $P_{XSY}(x, s, y) = P_{XS}(x, s) W(y|x|s)$ for $x \in \mathcal{X}, s \in \mathcal{S}$, and $y \in \mathcal{Y}$.

Theorem 2.4 (Ahlswede, Cai) *The capacity of the matching channel is given by*

$$C(\mathcal{W}) = \max_{P_{XS}} \min \left(H(X), I(XS \wedge Y) \right).$$

Notice that the quantity $C = \max_{P_{XS}} I(XS \wedge Y)$ is the capacity of the corresponding discrete memoryless channel $(\mathcal{X} \times \mathcal{S}, \mathcal{Y}, W')$ and that therefore $C(\mathcal{W}) \leq C$.

2.3.2 The Deterministic Matching Channel

In case of the deterministic matching channel the class $\mathcal{W} = \mathcal{W}_0$ contains only 0-1-matrices. Then Theorem 2.4 has the following specialisation:

Theorem 2.5 (Ahlswede, Cai)

$$C(\mathcal{W}_0) = \max_{P_{XS}} \min\left(H(X), H(Y)\right).$$

Clearly, by the definition of an (n, M, λ)-MC-code we can assume now that $|\mathcal{D}_i| = 1$ and thus $\mathcal{D}_i = \{v_i\}$. Also if $\lambda < 1$, then it can actually be chosen to be 0. So the distinct u_i's are matched with distinct v_i's.

It is also possible to work with an equivalent formulation of the coding problem for $\mathcal{W}_0$ in terms of an associated discrete memoryless channel (DMC) W:

$$W(\cdot|x) = \sum_{s \in \mathcal{S}} Q(s) W(\cdot|x|s),$$

where Q is any probability distribution on $\mathcal{S}$ with $Q(s) > 0$ for $s \in \mathcal{S}$. Since for the DMC $W^n(y^n|x^n) = \prod_{t=1}^{n} w(y_t|x_t)$, one notices that for any $(n, M, 0)$-MC-code $\{(u_i, \mathcal{D}_i) : 1 \leq i \leq M\}$ for $\mathcal{W}_0$ the condition

$$W^n(\mathcal{D}_i|u_i|s^n(i)) \geq 1 - \lambda$$

can equivalently be described in the "dummy" formulation by

$$W^n(\mathcal{D}_i|u_i) > 0 \quad \text{for} \quad i = 1, 2, \ldots M.$$

This condition means that the decoding probability is positive. In this situation we have $|\mathcal{D}_i| = 1$ for all i. So for code constructions we search for an injective map $f : \mathcal{U} \rightarrow \mathcal{Y}^n$ such that for $u \in \mathcal{U} \quad w^n(f(u)|u) > 0$.

Such an f is called a matching and $(\mathcal{U}, f)$ is a matching code. Analyzing them concerns only the support of W^n (i.e. the set of positive entries of W^n). This set can be viewed as the edge set $\mathcal{E}_n = \mathcal{E}_n(W)$ in the bipartite graph $\mathcal{G}(W^n) = (\mathcal{X}^n, \mathcal{Y}^n, \mathcal{E}_n(W))$, where $(x^n, y^n) \in \mathcal{E}_n$ iff $w^n(y^n|x^n) > 0$.

Clearly, for the f above $\{(u, f(u)) : u \in \mathcal{U}\}$ is exactly a set of non-intersecting edges in $\mathcal{G}(W^n)$, which means that it is a matching in the terminology of Graph Theory. Conversely, such a matching is a matching in the corresponding matching code $(\mathcal{U}, f)$, where $\mathcal{U} = \{u_1, \dots, u_M\}$ is the set of vertices in $\mathcal{X}^n$, which are matched to vertices in $\mathcal{Y}^n$.

So Ahlswede and Cai have reduced the study of the matching channel $\mathcal{W}_0$ via W to the study of *maximal matchings in the bipartite graph* $\mathcal{G}(W^n)$.

In their paper [3] they presented the following result which describes the asymptotic behaviour of the matching numbers $\nu(\mathcal{G}^n)$ for any given bipartite graph $\mathcal{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{E})$. A key idea is to extend the König-Hall-condition (which is in terms of cardinalities as measure of sets) to pairs of probabiliy distributions associated with $\mathcal{G}$.

The matching capacity of $\mathcal{G}$ is

$$\gamma(\mathcal{G}) = \lim_{n \to \infty} \frac{1}{n} \log \nu(\mathcal{G}^n).$$

Ahlswede and Cai defined the set

$$\mathcal{K}(\mathcal{G}) = \{(P, Q) : P \in \mathcal{P}(\mathcal{X}),\ Q \in \mathcal{Q}(\mathcal{Y}),\ P(\mathcal{L}) \leq Q(\Gamma_{\mathcal{G}}(\mathcal{L})) \text{ for all } \mathcal{L} \subset \mathcal{X}\},$$

where $\mathcal{P}(\mathcal{X})$ is the set of probability distributions on $\mathcal{X}$ and $\mathcal{Q}(\mathcal{Y})$ the set of probability distributions on $\mathcal{Y}$.

The members of $\mathcal{K}(\mathcal{G})$ are called **König-Hall pairs of distributions**.

Theorem 2.6 (Ahlswede, Cai) *For every bipartite graph* $\mathcal{G} = (\mathcal{X}, \mathcal{Y}, \mathcal{E})$ *the matching capacity can be described by*

$$\gamma(\mathcal{G}) = \max_{(P,Q) \in \mathcal{K}(\mathcal{G})} \min(H(P), H(Q)).$$

3 Exact Formulas for Special Bipartite Graphs

After studying the asymptotic behaviour of the matching numbers for n-th powers of a given bipartite graph, the question for exact formulas and constructions of maximal matchings naturally arises. The construction of a maximal matching in an n-th power of a bipartite graph $\mathcal{G}$ is equivalent to finding a maximal MC-Code for the deterministic matching channel belonging to $\mathcal{G}$. Ahlswede and Cai have given an example, a stared bipartite graph (a graph consisting of independent stars), where they determine exactly the matching numbers of the n-th powers of $\mathcal{G}$. We now will generalize this result.

It is quite clear that a bipartite graph has a perfect matching if the biadjacency matrix has a column permutation such that the permutated matrix contains an all-1-diagonal. In this case obviously $\nu(\mathcal{G}^n) = |\mathcal{X}|^n$ and $\gamma(\mathcal{G}^n) = \log|\mathcal{X}|$.

3.1 Bipartite Graphs with a Decomposition into Complete Connected Components

We will determine the matching number in bipartite graphs consisting of complete connection components K_{l_i,m_i}. In [3] Ahlswede and Cai have found an exact formula for a stared bipartite graph, where the graph consists of stars $K_{l_i,1}$ and K_{1,m_i}.

3.1.1 Graphs with Two Connected Components

Definition 3.1 (two connected components) *Let $\mathcal{G}$ be a bipartite graph and let $l, m, p, q \in \mathbb{N}$. $\mathcal{G}$ has a decomposition into two complete graphs $K_{l,m}$ and $K_{p,q}$, if $\mathcal{G}$ is given by the biadjacency matrix*

$$A = A(\mathcal{G}) = \begin{pmatrix} 1_{l,m} & 0_{l,q} \\ 0_{p,m} & 1_{p,q} \end{pmatrix},$$

where the all-1-matrix $1_{l,m}$ is a representation of the component $K_{l,m}$ and the all-1-matrix $1_{p,q}$ represents the component $K_{p,q}$. This bipartite graph we also call "(disjoint) union of two complete bipartite graphs" and write $\mathcal{G} = K_{l,m} \dot{\cup} K_{p,q}$.

For this class of bipartite graphs we quickly find a formula for the matching numbers.

Theorem 3.2 *Let $\mathcal{G}$ be a bipartite graph having a decomposition into two complete graphs $K_{l,m}$ and $K_{p,q}$, then the matching number is*

$$\nu(\mathcal{G}^n) = \sum_{k=0}^{n} \binom{n}{k} \min(l^{n-k}p^k, m^{n-k}q^k).$$

The proof of the theorem is based on the following lemma.

Lemma 3.3 *Let $\mathcal{G}$ consist of the two components $K_{l,m}$ and $K_{p,q}$. Then the n-th power $\mathcal{G}^n$ consists exactly of 2^n components of the form $K_{r,c}$ with $r = l^{n-k}p^k$, $c = m^{n-k}q^k$ and each component $K_{r,c}$ occurs $\binom{n}{k}$ times.*

Proof of Lemma 3.3:

We will proceed by induction on n.

It is obvious for $n = 1$.

Now let $\mathcal{G}^n$ exist exactly of 2^n complete bipartite graphs $K_{r,c}$, where every graph of the size $K_{r,c}$ with $r = l^{n-k}p^k$ and $c = m^{n-k}q^k$ occurs $\binom{n}{k}$ times. The graph $\mathcal{G}^{n+1}$ is essentially constructed by "multiplication" of $\mathcal{G}^n$ with $\mathcal{G}$. Now let us speak of a bipartite graph in the language of its biadjacency matrix: Then we get $\mathcal{G}^{n+1}$ exactly in that way that we place the biadjacency matrix of $\mathcal{G}^n$ in the position where $\mathcal{G}$ has a 1 and an all-zero-matrix (with the (matrix-)size of $\mathcal{G}^n$) where $\mathcal{G}$ has a 0.

Remember that $\mathcal{G}$ consists exactly of two complete bipartite graphs $K_{l,m}$ and $K_{p,q}$. Every component of $\mathcal{G}^n$ is a complete subgraph of the size $K_{r,c}$ with $r = l^{n-k}p^k$ and $c = m^{n-k}q^k$ for $k = 0, \cdots, n$. The "multiplication" of $\mathcal{G}^n$ with $\mathcal{G}$ happens separately in every component of $\mathcal{G}^n$. A component of $\mathcal{G}^n$ is duplicated $l \times m$ times (meaning, that its biadjacency matrix is written m times side by side and l times one below the other) and $p \times q$ times (biadjacency matrix q times side by side and p times one below the other). Therefore the component $K_{l^{n-k+1}p^k, m^{n-k+1}q^k}$ $(k = 0, \ldots, n+1)$ in the graph $\mathcal{G}^{n+1}$ can be generated in two ways: it arises by $(l \times m)$-duplication of the component $K_{l^{n-k}p^k, m^{n-k}q^k}$ of $\mathcal{G}^n$ or by $(p \times q)$-duplication of the component $K_{l^{n-k+1}p^{k-1}, m^{n-k+1}q^{k-1}}$ of $\mathcal{G}^n$. Since by induction hypothesis the components $K_{l^{n-k}p^k, m^{n-k}q^k}$ occur $\binom{n}{k}$ times and $K_{l^{n-k+1}p^{k-1}, m^{n-k+1}q^{k-1}}$ $\binom{n}{k-1}$ times, the arising components $K_{l^{n-k+1}p^k, m^{n-k+1}q^k}$ in $\mathcal{G}^{n+1}$ occur exactly $\binom{n}{k} + \binom{n}{k-1}$ times, which is equal to $\binom{n+1}{k}$ because of the **Pascal Identity** for the binomial coefficients. □

Proof of Theorem 3.2:

The bipartite graph $\mathcal{G}^n$ consists of 2^n components, due to Lemma 3.3.

Each component is - following the lemma - a complete bipartite graph of an exactly determined size and each component will occur in an exact number of times.

We obtain

$\sharp$ components of the form $K_{r,c}$	$r = \sharp$ rows	$c = \sharp$ columns
$\binom{n}{k}$	$l^{n-k}p^k$	$m^{n-k}q^k$

for all $k = 0, \ldots, n$.

From this fact we derive the matching number

$$\nu(\mathcal{G}^n) = \sum_{k=0}^{n} \binom{n}{k} \cdot \min(l^{n-k}p^k, m^{n-k}q^k).$$

□

Later on we make use of a consequence of Theorem 3.2:

Corollary 3.4 *The matching number for the graph* $\mathcal{G} = K_{l,m} \dot{\cup} K_{m,l}$ *is given by*

$$\nu(\mathcal{G}^n) = \begin{cases} 2\sum_{k=0}^{\frac{n}{2}-1} \binom{n}{k} (\min(l,m))^{n-k} (\max(l,m))^k + \binom{n}{\frac{n}{2}} l^{\frac{n}{2}} m^{\frac{n}{2}} & \\ \qquad\qquad\qquad\qquad \text{for } n = 2a,\ a \in \mathbb{N} \\ 2\sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} (\min(l,m))^{n-k} (\max(l,m))^k & \\ \qquad\qquad\qquad\qquad \text{for } n = 2a+1,\ a \in \mathbb{N}. \end{cases}$$

3.2 Calculation of the Matching Capacity

Since we have established the exact formula for the matching number in this special case, we are able to calculate the exact value of the matching capacity.

Taking the formula of Theorem 3.2 we have to distinguish essentially 4 cases dependent on the ordering of the natural numbers l, m, p and q.

3.2.1 Asymptotic Estimates of Binomial Coefficients and Matching Numbers

Before discussing the different cases, we carry out some useful calculations for the later analysis. Studying the asymptotic behaviour of the matching numbers in form of the matching capacity demands searching for the biggest term of $\sum_{k=0}^{n} \binom{n}{k} min(l^{n-k}p^k, m^{n-k}q^k)$. Since we know that $\binom{n}{\lambda n} \approx 2^{h(\lambda)n}$ for $n \to \infty$ we differentiate $f(\lambda) = 2^{h(\lambda)n} l^{n-\lambda n} p^{\lambda n}$ and $g(\lambda) = 2^{h(\lambda)n} m^{n-\lambda n} q^{\lambda n}$.

Lemma 3.5 *The functions $f(\lambda) = 2^{h(\lambda)n} l^{n-\lambda n} p^{\lambda n}$ and*

$g(\lambda) = 2^{h(\lambda)n} m^{n-\lambda n} q^{\lambda n}$ have local and simultaneously global optima at $\lambda_f = \frac{p}{l+p}$ (at f) and $\lambda_g = \frac{q}{m+q}$ (for g).

Proof:

$$
\begin{aligned}
f(\lambda) &= 2^{h(\lambda)n} l^{n-\lambda n} p^{\lambda n} \\
&= e^{h(\lambda)(\ln 2)n + (1-\lambda)n(\ln l) + \lambda n \ln p}
\end{aligned}
$$

$$
\begin{aligned}
f'(\lambda) &= 2^{h(\lambda)n} l^{n-\lambda n} p^{\lambda n} \left(h'(\lambda) n \ln 2 - n \ln l + n \ln p \right) \\
&= 2^{h(\lambda)n} l^{n-\lambda n} p^{\lambda n} \\
&\quad \left(n \ln 2 \left(-\frac{\ln(\lambda)}{\ln 2} + \frac{\ln(1-\lambda)}{\ln 2} \right) - n \ln l + n \ln p \right) \\
&= 2^{h(\lambda)n} l^{n-\lambda n} p^{\lambda n} n \left(\ln \frac{1-\lambda}{\lambda} - \ln \frac{l}{p} \right)
\end{aligned}
$$

The structure of both functions $f(\lambda)$ and $g(\lambda)$ is exactly the same, only l is changed by m and p by q. Therefore we also get

$$
g'(\lambda) = 2^{h(\lambda)n} m^{n-\lambda n} q^{\lambda n} n \left(\ln \frac{1-\lambda}{\lambda} - \ln \frac{m}{q} \right)
$$

The functions have local maxima at $\lambda_f = \frac{p}{l+p}$ (for f) and at $\lambda_g = \frac{q}{m+q}$ (for g). These local optima are also globally optimal as easy calculations show,

e.g.:

$$
\begin{aligned}
h\left(\frac{p}{l+p}\right) &= -\frac{p}{l+p} \log \frac{p}{l+p} - \frac{l}{l+p} \log \frac{l}{l+p} \\
&= \log \frac{1}{\left(\frac{p}{l+p}\right)^{\frac{p}{l+p}} \left(\frac{l}{l+p}\right)^{\frac{l}{l+p}}} \\
&= \log \frac{l+p}{p^{\frac{p}{l+p}} l^{\frac{l}{l+p}}} \quad .
\end{aligned}
$$

If $l \geq m$, we get

$$l + p > l = l^{\frac{p}{l+p}} l^{\frac{l}{l+p}} = \left(\frac{l}{p}\right)^{\frac{p}{l+p}} p^{\frac{p}{l+p}} l^{\frac{l}{l+p}} \quad ,$$

therefore

$$2^{h(\frac{p}{l+p})} = \frac{l+p}{p^{\frac{p}{l+p}} l^{\frac{l}{l+p}}} > \left(\frac{l}{p}\right)^{\frac{p}{l+p}} ,$$

and this inequality is equivalent to

$$2^{h(\frac{p}{l+p})n} l^{\frac{l}{l+p}n} p^{\frac{p}{l+p}n} > l^n = \max(l^n, m^n).$$

This last inequality shows that f has a global maximum at $\lambda_f = \frac{p}{l+p}$, if $l \geq m$. If $l < m$ calculations of the same form lead to the same result. Because of the symmetry of the functions f and g, we find that g has its global maximum at $\lambda_g = \frac{q}{m+q}$. $\square$

3.2.2 Exact Formulas for Matching Capacities of Bipartite Graphs with Complete Connected Components

Theorem 3.6 (Matching capacity for $\mathbf{K_{l,m} \dot\cup K_{p,q}}$) *The Graph $\mathcal{G} = K_{l,m} \dot\cup K_{p,q}$ has the matching capacity*

$$\gamma(\mathcal{G}) = \begin{cases} \log(l+p), & \\ \qquad \text{if} \left\langle \begin{array}{ll} & m \geq l,\ q \geq p, \\ \text{or} & m < l,\ q \geq p,\ T \geq \frac{l}{p},\ T > \frac{m}{q}, \\ \text{or} & m > l,\ q < p,\ T < \frac{l}{p},\ T < \frac{m}{q}, \end{array} \right\rangle \\ \log(m+q), & \\ \qquad \text{if} \left\langle \begin{array}{ll} & m \leq l,\ q < p, \\ \text{or} & m < l,\ q \geq p,\ T < \frac{l}{p},\ T < \frac{m}{q}, \\ \text{or} & m > l,\ q < p,\ T > \frac{l}{p},\ T \geq \frac{m}{q}, \end{array} \right\rangle \\ h(\lambda_{cut}) + (1-\lambda_{cut}) \log l + \lambda_{cut} \log p, & \\ \qquad \text{if} \left\langle \begin{array}{ll} & m < l,\ q \geq p,\ T < \frac{l}{p},\ T \geq \frac{m}{q}, \\ \text{or} & m > l,\ q < p,\ T \geq \frac{l}{p},\ T < \frac{m}{q}. \end{array} \right\rangle \end{cases}$$

where $h(\lambda)$ is the binary entropy, $T = \frac{\log \frac{q}{p}}{\log \frac{l}{m}}$ and $\lambda_{cut} = \frac{\log \frac{l}{m}}{\log \frac{q}{p} + \log \frac{l}{m}}$.

The proof is given in 3.2.4.

3.2.3 A König-Hall Pair for Which the Matching Capacity is Assumed

Before we will give the technical proof, let us interpret the result. Ahlswede and Cai in Theorem 2.6 have proved that the matching capacity in this situation can be described as $\gamma(\mathcal{G}) = \max_{(P,Q)\in\mathcal{K}(\mathcal{G})} \min(H(P), H(Q))$, where $(P,Q) \in \mathcal{K}(\mathcal{G})$ means that the two probability distributions P and Q form a König-Hall pair, i.e. $Q(\Gamma_{\mathcal{G}}(\mathcal{L})) \geq P(\mathcal{L})$ for all $\mathcal{L} \subset \mathcal{X}$.

Proposition 3.7 (A special König-Hall pair) *For $K_{l,m}\dot{\cup}K_{p,q}$ and for every $\lambda \in [0,1]$ especially the pair (P_λ, Q_λ) with*

$$P_\lambda = (\underbrace{\frac{\lambda}{l}, \ldots, \frac{\lambda}{l}}_{l \text{ times}}, \underbrace{\frac{1-\lambda}{p}, \ldots, \frac{1-\lambda}{p}}_{p \text{ times}})$$

$$\text{and} \quad Q_\lambda = (\underbrace{\frac{\lambda}{m}, \ldots, \frac{\lambda}{m}}_{m \text{ times}}, \underbrace{\frac{1-\lambda}{q}, \ldots, \frac{1-\lambda}{q}}_{q \text{ times}})$$

forms a König-Hall pair.

This statement is rather obvious, because according to the decomposition of $\mathcal{G}$ we give the l-set as well as the m-set in $K_{l,m}$ probability λ equidistributed on the l respectively m vertices. Since $K_{l,m}$ is a complete bipartite graph every subset $\mathcal{M}$ of the l-set automatically has the full m-set as neighbourhood and the König-Hall condition is fulfilled. The same argument is valid for the complete bipartite subgraph $K_{p,q}$, where both sets take weight $(1-\lambda)$.

In the first group of the cases from Theorem 3.6 choose $\lambda = \frac{l}{l+p}$. Especially P_λ forms the equidistribution on $l+p$ vertices.

Proposition 3.8 (Maximizing König-Hall pair) *For $\lambda = \frac{l}{l+p}$ we get*

$$\min(H(P_\lambda), H(Q_\lambda)) = H(P_\lambda) = \log(l+p).$$

Therefore (P_λ, Q_λ) is a maximizing König-Hall pair with

$$\gamma(\mathcal{G}) = \min(H(P_\lambda), H(Q_\lambda)) = H(P_\lambda) = \log(l+p).$$

Proof:

- $\underline{m \geq l \text{ and } q \geq p}$:

  $H(Q_\lambda) = h(\frac{l}{l+p}) + \frac{l}{l+p}\log m + \frac{p}{l+p}\log q \geq h(\frac{l}{l+p}) + \frac{l}{l+p}\log l + \frac{p}{l+p}\log p = \log(l+p) = H(P_\lambda)$

- $\underline{m < l,\ q \geq p,\ T \geq \frac{l}{p},\ T > \frac{m}{q} \text{ with } T = \frac{\log\frac{q}{p}}{\log\frac{l}{m}}}$:

  We know that $\frac{\log\frac{q}{p}}{\log\frac{l}{m}} = T \geq \frac{l}{p}$, therefore $p\log\frac{q}{p} \geq l\log\frac{l}{m}$ yields. From this we can conclude $l\log m + p\log q \geq l\log l + p\log p$, and then $\frac{l}{l+p}\log m + \frac{p}{l+p}\log q \geq \frac{l}{l+p}\log l + \frac{p}{l+p}\log p$. It follows that $H(Q_\lambda) = h(\frac{l}{l+p}) + \frac{l}{l+p}\log m + \frac{p}{l+p}\log q \geq h(\frac{l}{l+p}) + \frac{l}{l+p}\log l + \frac{p}{l+p}\log p = H(P_\lambda)$.

- $\underline{m > l,\ q < p,\ T < \frac{l}{p},\ T < \frac{m}{q} \text{ with } T = \frac{\log\frac{q}{p}}{\log\frac{l}{m}}}$:

  From $\frac{\log\frac{q}{p}}{\log\frac{l}{m}} = T < \frac{l}{p}$ we can conclude $p\log\frac{q}{p} > l\log\frac{l}{m}$.

  With precisely the same calculation as in the subcase before we also get the desired result. $\square$

In the second group choose $\lambda = \frac{m}{m+q}$. Here Q_λ forms the equidistribution on $m+q$ vertices.

Proposition 3.9 *For $\lambda = \frac{m}{m+q}$ we get*

$$\min(H(P_\lambda), H(Q_\lambda)) = H(Q_\lambda) = \log(m+q).$$

Therefore (P_λ, Q_λ) is a maximizing König-Hall pair with

$$\gamma(\mathcal{G}) = \min(H(P_\lambda), H(Q_\lambda)) = H(Q_\lambda) = \log(m+q).$$

Proof:

- $\underline{m \leq l \text{ and } q < p}$:

  $H(P_\lambda) = h(\frac{m}{m+q}) + \frac{m}{m+q}\log l + \frac{q}{m+q}\log p \geq h(\frac{m}{m+q}) + \frac{m}{m+q}\log m + \frac{q}{m+q}\log q = \log(m+q) = H(Q_\lambda)$

- $m < l,\ q \geq p,\ T < \frac{l}{p},\ T < \frac{m}{q}$ with $T = \frac{\log\frac{q}{p}}{\log\frac{l}{m}}$:

  From $\frac{\log\frac{q}{p}}{\log\frac{l}{m}} = T < \frac{m}{q}$ it follows, that $\log\frac{q}{p} < m\log\frac{l}{m}$. We can conclude, that $\log m + q\log q < m\log l + q\log p$, and with this we obtain $\frac{m}{m+q}\log m + \frac{q}{m+q}\log q < \frac{m}{m+q}\log l + \frac{q}{m+q}\log p$. Finally it follows $H(P_\lambda) = h(\frac{m}{m+q}) + \frac{m}{m+q}\log l + \frac{q}{m+q}\log p > h(\frac{m}{m+q}) + \frac{m}{m+q}\log m + \frac{q}{m+q}\log q = H(Q_\lambda)$.

- $m > l,\ q < p,\ T > \frac{l}{p},\ T \geq \frac{m}{q}$ with $T = \frac{\log\frac{q}{p}}{\log\frac{l}{m}}$:

  From $\frac{\log\frac{q}{p}}{\log\frac{l}{m}} = T \geq \frac{m}{q}$ we can conclude $q\log\frac{q}{p} \leq m\log\frac{l}{m}$, and with the same same calculation as before we get the desired result. □

In the third group choose $\lambda = \lambda_{cut}$, and we get

Proposition 3.10 *For* $\lambda = \lambda_{cut} = \frac{\log\frac{l}{m}}{\log\frac{q}{p} + \log\frac{l}{m}}$ *we obtain*

$$\min(H(P_\lambda), H(Q_\lambda)) = H(P_\lambda) = h(\lambda_{cut}) + (1-\lambda)\log l + \lambda\log p.$$

Proof:

Here we have $\lambda = \lambda_{cut} = \frac{\log\frac{l}{m}}{\log\frac{l}{m} + \log\frac{q}{p}}$, there upon $\log\frac{q}{p} + \log\frac{l}{m})\lambda = \log\frac{l}{m}$ yields. It follows $(1-\lambda)\log\frac{l}{m} = \log\frac{q}{p}$, from which we can conclude $H(P_\lambda) = h(\lambda) + (1-\lambda)\log l + \lambda\log p = h(\lambda) + (1-\lambda)\log m + \lambda\log q = H(Q_\lambda)$.

Realize that because of $H(P_\lambda) = H(Q_\lambda)$ the minimum as well can be described by $H(Q_{\lambda_{cut}}))$. □

Summarizing the four propositions we see that in each case the special choice of λ leads to a König-Hall pair (P_λ, Q_λ) which gives the value of the matching capacity for $K_{l,m}\dot{\cup}K_{p,q}$ (i.e. maximizing $\min(H(P), H(Q))$ over all König-Hall pairs (P, Q)).

The theorem illustrates that one finds the matching capacity with looking for the biggest entropy of P respectively Q, and the result is as natural as it could be, because the entropy has its maximum for the equidistribution. Nothing else is to be done as we have seen. It is just necessary to group the vertices according to their component which they belong to: One group gets weight sum λ and the other $1-\lambda$, and the vertices in each group are equidistributed.

3.2.4 Technical Proof

The proof of Theorem 3.6 is more technical even though straight forward by using the tool of differentiation for finding local extrema.

Proof of Theorem 3.6:

The matching capacity for the given bipartite graph $\mathcal{G}$ having a decomposition into $K_{l,m}$ and $K_{p,q}$ can be expressed in terms of the given numbers l, m, p and q. The matching numbers are calculated as a sum of exponential terms. The biggest term in the sum - and only l, m, p, q decide- determines the matching capacity. Look at the continuous extensions of $l^{n-k}p^k$ and $m^{n-k}q^k$, which are $F : (0,1) \to \mathbb{R}$, $F(\lambda) \triangleq l^{(1-\lambda)n}p^{\lambda n}$ and $G : (0,1) \to \mathbb{R}$, $G(\lambda) \triangleq m^{(1-\lambda)n}q^{\lambda n}$. The graphs of the functions F and G may cut or not cut. Therefore we have to distinguish several cases. In Case I and Case II they do not cut. Case III and IV- when the graphs of F and G cut - have some subcases dependent on the monotonicity of F and G.

Case I: $m \geq l$ and $q \geq p$

Obviously the matching number for $\mathcal{G}^n$ equals $\nu(\mathcal{G}^n) = \sum\limits_{k=0}^{n} \binom{n}{k} l^{n-k}p^k$.
The biggest term in the continuous extension is $2^{h(\frac{p}{l+p})n} l^{\frac{l}{l+p}n} p^{\frac{p}{l+p}}$.
For the matching capacity $\gamma(\mathcal{G})$ we get

$$\begin{aligned}
\gamma(\mathcal{G}) &= \lim_{n\to\infty} \frac{1}{n} \log \nu(\mathcal{G}^n) = \lim_{n\to\infty} \frac{1}{n} \log(2^{h(\frac{p}{l+p})n} l^{\frac{l}{l+p}n} p^{\frac{p}{p+n}n}) \\
&= h(\frac{p}{l+p}) + \frac{l}{l+p} \log l + \frac{p}{l+p} \log p \\
&= -\frac{p}{l+p} \log \frac{p}{l+p} - \frac{l}{p+l} \log \frac{l}{p+l} + \frac{l}{l+p} \log l + \frac{p}{l+p} \log p \\
&= \log(l+p) \ .
\end{aligned}$$

Case II: $m \leq l$ and $q < p$

This case is in principle the same like Case I, we just have to change the roles of l and p to m and q, and we get

$$\gamma(\mathcal{G}) = h(\frac{q}{m+q}) + \frac{m}{m+q} \log m + \frac{q}{m+q} \log q = \log(m+q) \ .$$

<u>Case III: $m < l$ and $q \geq p$</u>

The graphs of the functions F and G cut for $\lambda_{cut} = \lambda_{cut}(l, m, p, q) = \frac{\log \frac{l}{m}}{\log \frac{q}{p} + \log \frac{l}{m}}$, especially $\lambda_{cut}(l, m, p, q)$ equals 1 for $p = q$, 0 for $l = m$, and $\frac{1}{2}$ for $\frac{l}{m} = \frac{q}{p}$.

For the functions f' and g' we know the zero-positions $\lambda_f = \frac{p}{l+p}$ and $\lambda_g = \frac{q}{m+q}$. We also know that $f'(\lambda) > 0$ for $0 < \lambda < \lambda_f$ and $f'(\lambda) < 0$ for $\lambda_f < \lambda < 1$, analogously for g' with respect to λ_g.

The decision whether f' and g' are positive or negative at the position λ_{cut} immediately leads to the position of $\max_k \binom{n}{k} \min(l^{n-k} p^k, m^{n-k} q^k)$. We observe that

$$\begin{aligned} f'(\lambda_{cut}) &= 2^{h(\lambda_{cut})} l^{(1-\lambda_{cut})n} p^{\lambda_{cut}} n \left(\log \frac{1 - \frac{\log \frac{l}{m}}{\log \frac{q}{p} + \log \frac{l}{m}}}{\frac{\log \frac{l}{m}}{\log \frac{q}{p} + \log \frac{l}{m}}} - \log \frac{l}{p} \right) \\ &= 2^{h(\lambda_{cut})} l^{(1-\lambda_{cut})n} p^{\lambda_{cut}} n \left(\log \frac{\log \frac{q}{p}}{\log \frac{l}{m}} - \log \frac{l}{p} \right) \end{aligned}$$

and

$$g'(\lambda_{cut}) = 2^{h(\lambda_{cut})} m^{(1-\lambda_{cut})n} q^{\lambda_{cut}} n \left(\log \frac{\log \frac{q}{p}}{\log \frac{l}{m}} - \log \frac{m}{q} \right).$$

The terms in brackets determine whether f'/g' are positive or negative. Using the abbreviation $T := \frac{\log \frac{q}{p}}{\log \frac{l}{m}}$, we have to examine the greater/smaller relation of T and $\frac{l}{p}$ (respectively $\frac{m}{q}$).

Before discussing all the subcases which in principle can occur, let us remark that the case $\frac{m}{q} \geq T \geq \frac{l}{p}$ contradicts to $m < l$ and $q \geq p$.

<u>Subcase 3.1: $m < l$, $q \geq p$, $T < \frac{l}{p}$, $T \geq \frac{m}{q}$</u>

In this situation we get $f'(\lambda_{cut}) < 0$ and $g'(\lambda_{cut}) > 0$. Therefore we observe

$$\max_{\lambda \in (0,1)} 2^{h(\lambda)n} \min(l^{(1-\lambda)n} p^{\lambda n}, m^{(1-\lambda)n} q^{\lambda n}) = f(\lambda_{cut})$$

$$= 2^{h(\lambda_{cut})n} l^{(1-\lambda_{cut})n} p^{\lambda_{cut} n},$$

and we get

$$\gamma(\mathcal{G}) = h(\lambda_{cut}) + (1 - \lambda_{cut}) \log l + \lambda_{cut} \log p.$$

Subcase 3.2: $m < l$, $q \geq p$, $T < \frac{l}{p}$, $T < \frac{m}{q}$

In this situation $f'(\lambda_{cut}) < 0$ and $g'(\lambda_{cut}) < 0$ holds. Therefore it follows

$$\max_{\lambda \in (0,1)} 2^{h(\lambda)n} \min(l^{(1-\lambda)n} p^{\lambda n}, m^{(1-\lambda)n} q^{\lambda n}) = g(\lambda_g)$$

$$= 2^{h(\frac{q}{m+q})n} m^{(\frac{m}{m+q})n} q^{\frac{q}{m+q}n},$$

and we get

$$\gamma(\mathcal{G}) \;=\; h(\frac{q}{m+q}) + (\frac{m}{q+m}) \log m + \frac{q}{m+q} \log q.$$

Subcase 3.3: $m < l$, $q \geq p$, $T \geq \frac{l}{p}$, $T > \frac{m}{q}$

In this situation we obtain $f'(\lambda_{cut}) > 0$ and $g'(\lambda_{cut}) > 0$. So we conclude

$$\max_{\lambda \in (0,1)} 2^{h(\lambda)n} \min(l^{(1-\lambda)n} p^{\lambda n}, m^{(1-\lambda)n} q^{\lambda n}) = f(\lambda_f)$$

$$= 2^{h(\frac{p}{l+p})n} l^{(\frac{l}{l+p})n} p^{\frac{p}{l+p}n},$$

and we get

$$\gamma(\mathcal{G}) \;=\; h(\frac{p}{l+p}) + (\frac{l}{p+l}) \log l + \frac{p}{l+p} \log p.$$

Case IV: $m > l$ and $q < p$

We have the same cutting point like in Case III $\lambda_{cut} = \frac{\log lm}{\log \frac{q}{p} + \log \frac{l}{m}}$, and here we get the following subcases (where the subcase $\frac{l}{p} \geq T \geq \frac{m}{q}$ is excluded because of a contradiction to $m > l$ and $q < p$):

Subcase 4.1: $m > l$, $q < p$, $T < \frac{l}{p}$, $T < \frac{m}{q}$

In this situation we observe $f'(\lambda_{cut}) < 0$ and $g'(\lambda_{cut}) < 0$, thus

$$\max_{\lambda \in (0,1)} 2^{h(\lambda)n} \min(l^{(1-\lambda)n} p^{\lambda n}, m^{(1-\lambda)n} q^{\lambda n}) = f(\lambda_f)$$

$$= 2^{h(\frac{p}{l+p})n} l^{(\frac{l}{l+p})n} p^{\frac{p}{l+p}n},$$

and we get

$$\gamma(\mathcal{G}) \;=\; h(\frac{p}{l+p}) + (\frac{l}{p+l}) \log l + \frac{p}{l+p} \log p.$$

<u>Subcase 4.2: $m > l$, $q < p$, $T > \frac{l}{p}$, $T \geq \frac{m}{q}$</u>

In this situation $f'(\lambda_{cut}) > 0$ and $g'(\lambda_{cut}) \geq 0$ holds, therefore

$$\max_{\lambda \in (0,1)} 2^{h(\lambda)n} \min(l^{(1-\lambda)n} p^{\lambda n}, m^{(1-\lambda)n} q^{\lambda n}) = g(\lambda_g)$$

$$= 2^{h(\frac{q}{m+q})n} m^{(\frac{m}{m+q})n} q^{\frac{q}{m+q}n},$$

and we get

$$\gamma(\mathcal{G}) \;=\; h(\frac{q}{m+q}) + (\frac{m}{q+m}) \log m + \frac{q}{m+q} \log q.$$

<u>Subcase 4.3: $m > l$, $q < p$, $T \geq \frac{l}{p}$, $T < \frac{m}{q}$</u>

In this situation $f'(\lambda_{cut}) = 0$ and $g'(\lambda_{cut}) < 0$ is given. It follows that

$$\max_{\lambda \in (0,1)} 2^{h(\lambda)n} \min(l^{(1-\lambda)n} p^{\lambda n}, m^{(1-\lambda)n} q^{\lambda n}) = f(\lambda_{cut})$$

$$= 2^{h(\lambda_{cut})n} l^{(1-\lambda_{cut})n} p^{\lambda_{cut} n},$$

and we get

$$\gamma(\mathcal{G}) \;=\; h(\lambda_{cut}) + (1 - \lambda_{cut}) \log l + \lambda_{cut} \log p.$$

□

At the end of this investigations concerning the matching capacity let us turn the focus once again on the special subcase $\mathcal{G} = K_{l,m} \dot\cup K_{m,l}$.

Corollary 3.11 *The matching capacity for the graph $\mathcal{G} = K_{l,m} \dot\cup K_{m,l}$ is given by*

$$\gamma(\mathcal{G}) = 1 + \frac{1}{2}(\log l + \log m)\ .$$

<u>**Proof:**</u> We are in the third group, $\lambda_{cut} = \frac{1}{2}$, and the matching capacity equals $\gamma(\mathcal{G}) = 1 + \frac{1}{2}(\log l + \log m)$.

The König-Hall pair (P_λ, Q_λ) which maximizes $\min(H(P), H(Q))$ is

$$P_\lambda = (\underbrace{\frac{1}{2l}, \ldots, \frac{1}{2l}}_{l \text{ times}}, \underbrace{\frac{1}{2p}, \ldots, \frac{1}{2p}}_{p \text{ times}}) \text{ and } Q_\lambda = (\underbrace{\frac{1}{2m}, \ldots, \frac{1}{2m}}_{m \text{ times}}, \underbrace{\frac{1}{2q}, \ldots, \frac{1}{2q}}_{q \text{ times}}).$$

□

3.3 Matching Capacity versus Shannon Capacity

The matching number $\nu(\mathcal{G})$ for a given graph $\mathcal{G}$ is defined as the maximal number of independent (i.e. non-intersecting) edges. Similarly the (vertex-) independence number $\alpha(\mathcal{G})$ is defined as the maximal number of independent vertices (i.e. no edge between every two vertices). Like the matching capacity in the examination of the asymptotic behaviour the Shannon capacity arises.

Definition 3.12 (Shannon Capacity) *The Shannon capacity for a graph $\mathcal{G} = (V, E)$ is defined by*

$$\Theta(\mathcal{G}) = \lim_{n\to\infty} (\alpha(\mathcal{G}^n))^{\frac{1}{n}}.$$

The limit exists (cf. e.g. Van Lint/ Wilson, [14]). Realize that the matching capacity measures the logarithmic asymptotic behaviour of the average matching number per dimension, and the Shannon capacity measures the absolute asymptotics of the average independence number per dimension.

In [13] Shannon proved for every two graphs $\mathcal{G}$ and $\mathcal{H}$ the inequality $\Theta(\mathcal{G}\dot\cup\mathcal{H}) \geq \Theta(\mathcal{G}) + \Theta(\mathcal{H})$. Moreover he conjectured that in fact equality always holds. Alon recently proved in [4] that Shannon's conjecture is wrong by giving an example where the sharp inequality holds.

3.3.1 Alon's Counterexample to Shannon's Conjecture

The following theorem was recently proved by Alon, [4].

Theorem 3.13 (Alon) *For every k there is a graph G so that the Shannon capacity of the graph and that of its complement $\overline{\mathcal{G}}$ satisfy $\Theta(\mathcal{G}) \leq k$, $\Theta(\overline{\mathcal{G}}) \leq k$, whereas $\Theta(\mathcal{G}\dot\cup\overline{\mathcal{G}}) \geq k^{1+o(1))\frac{\log k}{8\log\log k}}$ and the $o(1)$-term tends to zero as k tends to infinity.*

The proof contains an explicit description of $\mathcal{G}$. Alon also gives a more simple, small counterexample to the conjecture of Shannon. He uses similar ideas like in the proof of Theorem 3.3.1 together with a construction of Haemers [8], [9] and obtains

Theorem 3.14 (Alon) *There exists a graph G on 27 vertices so that $\Theta(\mathcal{G}) \leq 7$, $\Theta(\overline{\mathcal{G}}) = 3$, whereas $\Theta(\mathcal{G}\dot{\cup}\overline{\mathcal{G}}) \geq 2\sqrt{27}(> 10)$.*

For proving this theorem Alon uses the following proposition, which supplies a lower bound for the Shannon capacity of the union of any graph with its complement.

Proposition 3.15 (Alon) *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph on $|\mathcal{V}| = m$ vertices, and let $\overline{\mathcal{G}}$ denote its complement. Then $\Theta(\mathcal{G}\dot{\cup}\overline{\mathcal{G}}) \geq 2\sqrt{m}$.*

Proof of Proposition 3.15:

Denote the set of vertices of $\mathcal{G}\dot{\cup}\overline{\mathcal{G}}$ by $\mathcal{A}\cup\mathcal{B}$ where $\mathcal{A} = \{a_1, a_2, \ldots, a_m\}$ and $\mathcal{B} = \{b_1, b_2, \ldots, b_m\}$ are the vertex sets of the copy of $\mathcal{G}$ and of that of $\overline{\mathcal{G}}$, respectively, and where $a_i a_j$ is an edge iff $b_i b_j$ is not an edge. Thus there are no edges of the form $a_i b_j$. Let n be a positive integer, and let $\mathcal{M}$ denote the set of all vectors $v = (v_1, v_2, \ldots, v_{2n})$ of length $2n$ of $\mathcal{G}\dot{\cup}\overline{\mathcal{G}}$ satisfying two properties:

1. $|\{i : v_i \in \mathcal{A}\}| = |\{j : v_j \in \mathcal{B}\}| = n.$
2. For each index i, $1 \leq i \leq n$, if a_r is the i-th leftmost coordinate of v that belongs to $\mathcal{A}$, and b_s is the leftmost coordinate of v that belongs to $\mathcal{B}$, then $r = s$.

$|\mathcal{M}| = \binom{2n}{n} m^n$, since there are $\binom{2n}{n}$ ways to choose the location of the coordinates of $\mathcal{A}$ in v, and once these are chosen there are m^n many ways to choose the actual coordinates, thus determining the coordinates of $\mathcal{B}$ as well. In addition $\mathcal{M}$ is an independent set in $\mathcal{G}^n$. Indeed, if v and $u = (u_1, u_2, \ldots, u_{2n})$ are two distinct members of $\mathcal{M}$, then if there is a coordinate t such that $u_t \in \mathcal{A}$ and $v_t \in \mathcal{B}$ (or vice versa), then clearly u and v are non adjacent in $(\mathcal{G}\dot{\cup}\overline{\mathcal{G}})^n$. Otherwise, there must be r and s between 1 and $2n$ and two distinct indices $1 \leq i, j \leq n$ such that $u_r = a_i$, $u_s = b_i$ and $v_r = a_i$, $v_s = b_j$. Since a_i and a_j are adjacent iff b_i and b_j are not, it follows that in this case as well u and v are not adjacent. Therefore for every n, $\alpha((\mathcal{G}\dot{\cup}\overline{\mathcal{G}})^{2n}) \geq |\mathcal{M}| = \binom{2n}{n} m^n$, implying that

$$\Theta(\mathcal{G}\dot{\cup}\overline{\mathcal{G}}) \geq \lim_{n\to\infty} \left[\binom{2n}{n} m^n\right]^{1/(2n)} = 2\sqrt{m}.$$

$\square$

Proof of Theorem 3.14:

In [8] Haemers describes explicitly a graph $\mathcal{G}$ on $m = 27$ vertices satisfying $\Theta(\mathcal{G}) \leq 7$ and $\Theta(\overline{\mathcal{G}}) = 3$. This graph is the complement of the so-called Schläfli graph. The vertices in $\mathcal{G}$ are all edges of the complete graph on $\{1, 2, \ldots, 8\}$ besides the edge 12. Call an edge a *type 1 edge* iff it contains either 1 or 2, otherwise, call it a *type 2 edge*. The adjacency relations are the following: two vertices represented by edges as above are adjacent iff either they are of the same type and do not contain a common end or they are of different types and contain a common end. In [8] the author computes the eigenvalues of $\mathcal{G}$ and deduces from them the bounds mentioned above for its capacity and the capacity of its complement. The assertion of Theorem 3.14 thus follows from Proposition 3.15. $\square$

Let us conclude with an observation Alon made: In a subsequent paper [9] Haemers gave an infinite class of examples of graphs $\mathcal{G}$ on n vertices for which $\Theta(\mathcal{G})\Theta(\overline{\mathcal{G}}) < n$, and many of these can replace the Schläfli graph and provide examples similar to the one in Theorem 3.14. In all these examples the capacity of the union is still smaller than the sum of the capacities raised to the power 3/2, whereas Alon's construction in Theorem 3.3.1 shows that the gap can in fact be much bigger.

3.3.2 No Inequality for the Matching Capacity

We can ask the same question for the existence of an inequality for the matching capacity of a union.

The answer is negative as the following examples for the special class of bipartite graphs $\{K_{l,m} \dot{\cup} K_{m,l}\}_{l,m \in \mathbb{N}}$ show:

$$\gamma(K_{3,2} \dot{\cup} K_{2,3}) = \frac{3}{2} + \frac{1}{2} \log 3 > 2 = \gamma(K_{3,2}) + \gamma(K_{2,3}),$$

whereas

$$\gamma(K_{5,4} \dot{\cup} K_{4,5}) = 2 + \frac{1}{2} \log 5 < 4 = \gamma(K_{5,4}) + \gamma(K_{4,5}).$$

Moreover we can easily classify the graphs of $\{K_{l,m} \dot{\cup} K_{m,l}\}_{l,m \in \mathbb{N}}$:

Proposition 3.16 *Let $l \geq m$.*

The inequality $\gamma(K_{l,m} \dot{\cup} K_{m,l}) \geq \gamma(K_{l,m}) + \gamma(K_{m,l})$ holds iff $l \geq \frac{m^3}{4}$.

Proof:

$$\log(2(lm)^{\frac{1}{2}}) = 1 + \frac{1}{2}(\log l + \log m) = \gamma(K_{l,m}\dot{\cup}K_{m,l})$$
$$\geq \gamma(K_{l,m}) + \gamma(K_{m,l}) = 2\log m$$

This inequality yields iff $2(lm)^{\frac{1}{2}} \geq m^2$, which is equivalent to $l \geq \frac{m^3}{4}$. □

3.4 Generalizations

Now we present three generalizations of Theorem 3.2.

The first generalization considers the case that the two components occur more than once.

Theorem 3.17 (Generalization 1) *Let $\mathcal{G}$ have a decomposition into A complete bipartite subgraphs $K_{l,m}$ and B complete bipartite subgraphs $K_{p,q}$, $A, B \in \mathbb{N}$, then the matching number is*

$$\nu(\mathcal{G}^n) = \sum_{k=0}^{n} \binom{n}{k} A^{n-k}B^k \min(l^{n-k}p^k, m^{n-k}q^k).$$

Proof: The graph $\mathcal{G}^n$ consists exactly of $(A+B)^n$ components of the same size like in Theorem 3.2.

We have the following result:

♯ components of the form $K_{r,c}$	r= ♯ rows	c= ♯ columns
$\binom{n}{k}A^{n-k}B^k$	$l^{n-k}p^k$	$m^{n-k}p^k$

for all $k = 0, \ldots, n$.

The proof is done by induction in the same manner as before. □

The main formula for Theorem 3.2 is the well known **Binomial Theorem**. For the second generalization we can make use of the **Polynomial Theorem**, a generalization of the Binomial Theorem for more than two terms.

Theorem 3.18 (Polynomial Theorem) *The n-th power of a sum with α summands can be calculated by*

$$(l_1 + \ldots + l_\alpha)^n = \sum_{m_1+\ldots+m_\alpha=n} \binom{n}{m_1\ m_2 \ldots m_\alpha} l_1^{m_1} l_2^{m_2} \ldots l_\alpha^{m_\alpha}.$$

Theorem 3.19 (Generalization 2) *Let $\mathcal{G}$ have a decomposition into α independent complete subgraphs K_{l_i,m_i}, $l_i, m_i \in \mathbb{N}$, for all $i = 1, \ldots, \alpha$. Every component occurs exactly once. Then the matching number is*

$$\nu(\mathcal{G}^n) = \sum_{k_1+\ldots+k_\alpha=n} \binom{n}{k_1\ k_2 \ldots k_\alpha} \min(l_1^{k_1} l_2^{k_2} \ldots l_\alpha^{k_\alpha}, m_1^{k_1} m_2^{k_2} \ldots m_\alpha^{k_\alpha}).$$

__Proof:__ The graph $\mathcal{G}^n$ consists exactly of α^n components, which are complete subgraphs of the form $K_{r,c}$ with $r = {l_1}^{k_1} {l_2}^{k_2} \cdots {l_\alpha}^{k_\alpha}$ and $c = {m_1}^{k_1} {m_2}^{k_2} \cdots {m_\alpha}^{k_\alpha}$, where $\sum_{i=1}^{\alpha} k_i = n$. Each component occurs exactly $\binom{n}{k_1\ k_2 \ldots k_\alpha}$ times. Therefore the formula for the matching number looks like the Polynomial Theorem with minimization over each term. $\square$

Combining the two previous generalizations we get the third one.

Theorem 3.20 (Generalization 3) *Let $\mathcal{G}$ have a decomposition into A_i complete bipartite subgraphs K_{l_i,m_i} with $A_i, l_i, m_i \in \mathbb{N}$ for all $i = 1, 2, \ldots, \alpha$. The matching number for $\mathcal{G}^n$ can be calculated by*

$$\nu(\mathcal{G}^n) = \sum_{k_1+\ldots+k_\alpha=n} \binom{n}{k_1\ k_2 \ldots k_\alpha} \min(l_1^{k_1} l_2^{k_2} \ldots l_\alpha^{k_\alpha}, m_1^{k_1} m_2^{k_2} \ldots m_\alpha^{k_\alpha}).$$

__Proof:__

The graph $\mathcal{G}^n$ consists exactly of $(\sum_{i=1}^{\alpha} A_i)^n$ complete bipartite subgraphs. The formula for $\nu(\mathcal{G}^n)$ is a combination of the first and the second generalization.

$\square$

3.5 Association to the Pentagon-Problem

We did not succeed in finding a parallel between the Matching capacity and the Shannon capacity, as seen in 3.3. But because of the similarity of the problem - on the one hand the maximal number of independent vertices and on the other hand the maximal number of independent edges - it will be not surprising to find other relations.

Therefore we take a look at the bipartite graph $\mathcal{G}$ having a decomposition into $K_{3,2}$ and $K_{2,3}$. The biadjacency matrix has the structure

$$A = \begin{pmatrix} 1_{3,2} & 0_{3,3} \\ 0_{2,2} & 1_{2,3} \end{pmatrix}.$$

The matching number of the n-th power of $\mathcal{G}$ is determined by Theorem 3.2

$$\begin{aligned}\nu(\mathcal{G}^n) &= \sum_{k=0}^{n} \binom{n}{k} \cdot \min(3^{n-k}2^k, 2^{n-k}3^k) \\ &= \begin{cases} 2\sum_{k=0}^{\frac{n}{2}-1} \binom{n}{k} 2^{n-k}3^k + \binom{n}{\frac{n}{2}} 2^{\frac{n}{2}} 3^{\frac{n}{2}}, & \text{for } n = 2a,\ a \in \mathbb{N}, \\ 2\sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} 2^{n-k}3^k, & \text{for } n = 2a+1,\ a \in \mathbb{N}. \end{cases}\end{aligned}$$

The matching capacity of the given graph $\mathcal{G}$ is $\gamma(\mathcal{G}) = \frac{3}{2} + \log 3$ (Corollary 3.11).

The table of matching numbers for small dimensions n

dim n	matching number $\nu(\mathcal{G}^n)$
1	4
2	20
3	88
4	440
5	1984
6	9920
7	45280
8	226400
9	1041280

gives the idea that we have in this special case a factor 5 going from an odd to an even dimension. Unfortunately there is no such factor from an even to an odd dimension. Nevertheless we get a bound in this case.

The constant factor for the transition from one dimension to the next gives us a flavour of the pentagon-problem Lovász investigated. He gave an exact formula for the vertex-independence number of the

n-th powers of the pentagon. In particular he observed a transition of an odd to an even dimension with factor $\frac{5}{2}$ and a factor 2 in the other case.

The next theorem formalizes our observations and gives a bound for the even-odd- transition (which is the same factor as in the odd-even-transition).

Theorem 3.21 *Let $\mathcal{G}$ be the bipartite graph $\mathcal{G} = K_{(l,m)} \dot{\cup} K_{(m,l)}$. Then the following equality and inequality hold:*

$\nu(\mathcal{G}^n) = (l+m)\,\nu(\mathcal{G}^{n-1})$ *for* $n = 2a,\ a \in \mathbb{N}$,

$\nu(\mathcal{G}^n) \leq (l+m)\,\nu(\mathcal{G}^{n-1})$ *for* $n = 2a+1,\ a \in \mathbb{N}$.

Proof:

w.l.o.g. (without loss of generality) let $l \geq m$.

Then $\nu(\mathcal{G}^n) = \sum_{k=0}^{n} \binom{n}{k} \min(l^{n-k}m^k, m^{n-k}l^k) =$

$$\begin{cases} \sum_{k=0}^{\frac{n}{2}-1} 2\binom{n}{k} m^{n-k} l^k + \binom{n}{\frac{n}{2}} m^{\frac{n}{2}} l^{\frac{n}{2}}, & n = 2a, a \in \mathbb{N} \\ \sum_{k=0}^{\frac{n+1}{2}} 2\binom{n}{k} m^{n-k} l^k, & n = 2a+1, a \in \mathbb{N} \end{cases}$$

First step: Proof of the equality. Let $n = 2a,\ a \in \mathbb{N}$.

We obtain

$$\begin{aligned}
(l+m)\,\nu(\mathcal{G}^{n-1}) &= (l+m) \sum_{k=0}^{\frac{n-2}{2}} 2\binom{n-1}{k} m^{n-1-k} l^k \\
&= 2\sum_{k=0}^{a-1} \binom{2a-1}{k} m^{2a-1-k} l^{k+1} + 2\sum_{k=0}^{a-1} \binom{2a-1}{k} m^{2a-k} l^k \\
&= 2\sum_{k=1}^{a} \binom{2a-1}{k-1} m^{2a-k} l^k + 2\sum_{k=0}^{a-1} \binom{2a-1}{k} m^{2a-k} l^k \\
&= 2\binom{2a-1}{0} m^{2a} l^0 + 2\sum_{k=1}^{a-1} \left(\binom{2a-1}{k-1} + \binom{2a-1}{k}\right) m^{2a-k} l^k + 2\binom{2a-1}{a-1} m^a l^a \\
&= 2m^{2a} + 2\sum_{k=1}^{a-1} \binom{2a}{k} m^{2a-k} l^k + 2 \cdot \frac{(2a-1)!}{a!(a-1)!} m^a l^a \\
&= 2\binom{2a}{0} m^{2a} l^0 + 2\sum_{k=1}^{a-1} \binom{2a}{k} m^{2a-k} l^k + \binom{2a}{a} m^a l^a \\
&= 2\sum_{k=0}^{\frac{n}{2}-1} \binom{n}{k} m^{n-k} l^k + \binom{n}{\frac{n}{2}} m^{\frac{n}{2}} l^{\frac{n}{2}} = \nu(\mathcal{G}^n)\ .
\end{aligned}$$

Second step: Proof of the inequality. Let $n = 2a + 1,\ a \in \mathbb{N}$,

then we observe

$$(m+l)\nu(\mathcal{G}^{n-1}) = (m+l)\left(\sum_{k=0}^{\frac{n-1}{2}-1} 2\binom{n-1}{k} m^{n-k-1} l^k + \binom{n-1}{\frac{n-1}{2}} m^{\frac{n-1}{2}} l^{\frac{n-1}{2}}\right)$$

$$= (m+l)\left(\sum_{k=0}^{a-1} 2\binom{2a}{k} m^{2a-k} l^k + \binom{2a}{a} m^a l^a\right)$$

$$= 2\sum_{k=0}^{a-1} \binom{2a}{k} m^{2a-k+1} l^k + 2\sum_{k=0}^{a-1} \binom{2a}{k} m^{2a-k} l^{k+1} + (m+l)\binom{2a}{a} m^a l^a$$

$$= 2\sum_{k=0}^{a-1} \binom{2a}{k} m^{2a+1-k} l^k + 2\sum_{k=1}^{a} \binom{2a}{k-1} m^{2a+1-k} l^k + (m+l)\binom{2a}{a} m^a l^a$$

$$= 2\binom{2a}{0} m^{2a+1} l^0 + 2\sum_{k=1}^{a-1} \left(\binom{2a}{k} + \binom{2a}{k-1}\right) m^{2a+1-k} l^k + 2\binom{2a}{a-1} m^{a+1} l^a +$$
$$(m+l)\binom{2a}{a} m^a l^a$$

$$= 2m^{2a+1} + 2\sum_{k=1}^{a-1} \binom{2a+1}{k} m^{2a+1-k} l^k + 2\binom{2a}{a-1} m^{a+1} l^a + (m+l)\binom{2a}{a} m^a l^a$$

$$\geq 2\sum_{k=0}^{a-1} \binom{2a+1}{k} m^{2a+1-k} l^k + 2\binom{2a}{a-1} m^{a+1} l^a + 2\left(\binom{2a+1}{a} - \binom{2a}{a-1}\right) m^{a+1} l^a$$
$$= 2\sum_{k=0}^{a} \binom{2a+1}{k} m^{2a+1-k} l^k = 2\sum_{k=0}^{\frac{n-1}{2}} \binom{n}{k} m^{n-k} l^k = \nu(\mathcal{G}^n)\ . \qquad \square$$

This regularity does not work in the general case of Theorem 3.2 as one can see in the following simple example. For $\mathcal{G} = K_{3,2} \dot\cup K_{1,4}$ we get matching numbers $\nu(\mathcal{G}) = 3,\ \nu(\mathcal{G}^2) = 11,\ \nu(\mathcal{G}^3) = 45,\ \nu(\mathcal{G}^4) = 191,\ \nu(\mathcal{G}^5) = 728, \ldots$ We find out that there is no factor 5 or any other constant factor from a transition from one dimension to the next.

4 Graphs with Incomplete Connected Components

In the previous section we have examined the development of matching numbers for bipartite graphs having a decomposition into complete connected components. Now we omit the requirement of completeness. Questions which naturally arise are:

- How many edges in maximum can we delete from a complete connected component such that we do not violate the existing matching structure in the n-th powers and get the same values for matching numbers?

- If we have deleted more than this maximal number of edges, what can we say about the structure of a maximal matching and especially about the matching numbers?

4.1 A Special Class of Graphs with One Incomplete Connected Component

From now on we discuss graphs where the special structure is given by the biadjacency matrix

$$A = \begin{pmatrix} \begin{pmatrix} 1_{(l-c),(m-d)} & 1_{(l-c),d} \\ 1_{c,(m-c)} & 0_{c,d} \end{pmatrix} & 0_{l,q} \\ 0_{p,m} & 1_{p,q} \end{pmatrix}$$

with $1 \leq c \leq l-1$ and $1 \leq d \leq m-1$.

Let us abbreviate $\mathcal{G}$ with $(K_{l,m} \backslash K_{c,d}) \dot{\cup} K_{p,q}$. $\mathcal{G}$ has a decomposition into two connecting components and can be viewed as the graph $K_{l,m} \dot{\cup} K_{p,q}$, where in component $K_{l,m}$ a complete bipartite subgraph of size $K_{c,d}$ is deleted. In this context we simply call this component the **incomplete $(\mathbf{l}, \mathbf{m})$-component**.

Graphs of this type $(K_{l,m} \backslash K_{c,d}) \dot{\cup} K_{p,q}$ consist of two independent connected components of the type

{1}

$\sharp$ columns	with	$\sharp$ entries
d		$(l-c)$
$(m-d)$		l

{2}

$\sharp$ columns	with	$\sharp$ entries
q		p

$\mathcal{G}^2$ consists of 4 independent connected components of the type

{11}

$\sharp$ columns	with	$\sharp$ entries
d^2		$(l-c)^2$
$2 \cdot d(m-d)$		$l(l-c)$
$(m-d)^2$		l^2

$\{12\}$

$\sharp$ columns	with	$\sharp$ entries
qd		$p(l-c)$
$q(m-d)$		pl

$\{21\}$

$\sharp$ columns	with	$\sharp$ entries
qd		$p(l-c)$
$q(m-d)$		pl

$\{22\}$

$\sharp$ columns	with	$\sharp$ entries
q^2		p^2

4.1.1 Naming of the Components

The components of $\mathcal{G}^2$ are constructed by duplication of the graph $\mathcal{G}$ with itself. Therefore we get components

$$\begin{aligned}
\{11\} &= \text{duplication of component } \{1\} \text{ with component } \{1\}\,, \\
\{12\} &= \text{duplication of component } \{1\} \text{ with component } \{2\}\,, \\
\{21\} &= \text{duplication of component } \{2\} \text{ with component } \{1\}\,, \\
\{22\} &= \text{duplication of component } \{2\} \text{ with component } \{2\}\,.
\end{aligned}$$

Notice that the arising components $\{12\}$ and $\{21\}$ are identical except of row and column permutation. For the matching problem we consider them as identical, and we treat them as if had two independent components $\{12\}$.

For higher dimensions the regularity continues: $\mathcal{G}^n$ consists of 2^n independent connected components

$$\{\underbrace{11\ldots1}_{n\ times}\},\ \{\underbrace{11\ldots1}_{n-1\ times}\ 2\}\ldots\ \{\underbrace{1\ldots1}_{n-k\ times}\ \underbrace{2\ldots2}_{k\ times}\}\ldots\ \{\underbrace{22\ldots2}_{n\ times}\}.$$

It is convenient to use the abbreviation

$$[n-k,k] \triangleq \{\underbrace{1\ldots1}_{n-k\ times}\ \underbrace{2\ldots2}_{k\ times}\} \quad \text{for } k=0,\ldots n.$$

For $k=0,\ldots,n$ each component $[n-k,k]$ occurs - except of row permutation- $\binom{n}{k}$ times.

4.1.2 Subset Structure of the Column Entries

The column entries for a given column in the biadjacency matrix represent a set of edges incident to a given vertex (because the biadjacency matrix has an entry "1", when two vertices are connected by an edge and a "0" otherwise) In case of a "1" we speak of an entry/edge, and in case of a "0" we speak of no entry/edge). For the given graph $\mathcal{G}$ component $\{1\}$ has b columns with $(l-c)$ identical entries, and the other columns consist of l entries. We call the columns with $(l-c)$ entries an $(l-c)$-block, because they all have $(l-c)$ entries at the same positions, and the other columns an l-block.

Definition 4.1 (l-block) *Let A be a matrix with i column vectors*

$$(0,\dots,0,\underbrace{1,\dots,1}_{l\ times},0,\dots,0)^t, \quad i \in \mathbb{N}, i \geq 1.$$

Then we call them an l-block of A.

In particular the $(l-c)$-columns have edges which are incident to a certain set of vertices which is a subset of the set of vertices reached by the edges of the l-block. In this situation we shortly say *that the $(l-c)$-block is a subset of the l-block.* To find a maximal matching we are interested in the subset structure of the columns, and especially in the number of edges which are new by taking a new block. The number of new edges is also called the increasing number (per *one* new block). Therefore it is more convenient to work with the tables of edge increasings (shortly: increasings) instead of the number of total column entries (per block). In this special case above we have an increasing of a edges (from $(l-c)$ to l edges).

For higher dimensions we have to be aware *which* is the underlying subset from which we get edge increasings by taking the new blocks.

For $\mathcal{G}$ we get the following tables (where the rows in each table are ordered in monotonically increasing numbers of entries):

	$\sharp$blocks	with	$\sharp$ columns	$\sharp$ entries	$\sharp$ increasings
$\{1\}$	1		d	$(l-c)$	$(l-c)$
	1		$(m-d)$	l	c

$\{2\}$

♯blocks	with	♯ columns	♯ entries	♯ increasings
1		q	p	p

For $\mathcal{G}^2$ we get the tables (rows ordered in the ♯ entries):

$\{11\}$

♯blocks	with	♯ columns	♯ entries	♯ increasings
1		d^2	$(l-c)^2$	$(l-c)^2$
2		$d(m-d)$	$l(l-c)$	$c(l-c)$
1		$(m-d)^2$	l^2	c^2

$\{12\}$

♯blocks	with	♯ columns	♯ entries	♯ increasings
1		q	$p(l-c)$	$p(l-c)$
1		$q(m-b)$	pl	pc

$\{22\}$

♯blocks	with	♯ columns	♯ entries	♯ increasings
1		q^2	p^2	p^2

It is indispensable to explain in more detail the edge increasing structure of component $\{11\}$.

We take the block with the minimal number of entries first and put it at the first level.

At the second level we take all the blocks which have the second minimal number of entries and count the new edges in each block (separately!) not occurring at the first level.

At the third level we take all the columns which have the third minimal number of entries and count the new edges in each block (separately!) not occurring at the union of the first and second levels.

For $\mathcal{G}^2$ we have the following increasing structures:

- Component $\{11\}$

$$\begin{tikzcd}
 & l^2 & & \\
l(l-c) \arrow[ur] & & l(l-c) \arrow[ul] & \arrow[u, "+c^2"'] \\
 & (l-c)^2 \arrow[ul] \arrow[ur] & & \arrow[u, "+c(l-c)"']
\end{tikzcd}$$

- Component $\{12\}$

$$\begin{array}{c} pl \\ \uparrow \\ p(l-c) \end{array} \qquad\qquad \Big\uparrow +pc$$

Realize that for component $\{12\}$ we have in principle the same increasing structure as component $\{1\}$ except a factor q in the number of columns and a factor p in the number of entries/increasings. These factors occur because $\{12\}$ results by duplication of $\{1\}$ with $\{2\}$ and component $\{2\}$ consists of q columns with p entries.

For $\mathcal{G}^n$ we get for all $a = 0, \ldots, n$ the tables

$[n-a, a]$

$\sharp$blocks	with	$\sharp$ columns	$\sharp$ entries	$\sharp$ increasings
$\binom{n-a}{i}$		$q^a(m-d)^i d^{n-a-i}$	$p^a l^i (l-c)^{n-a-i}$	$p^a c^i (l-c)^{n-a-i}$

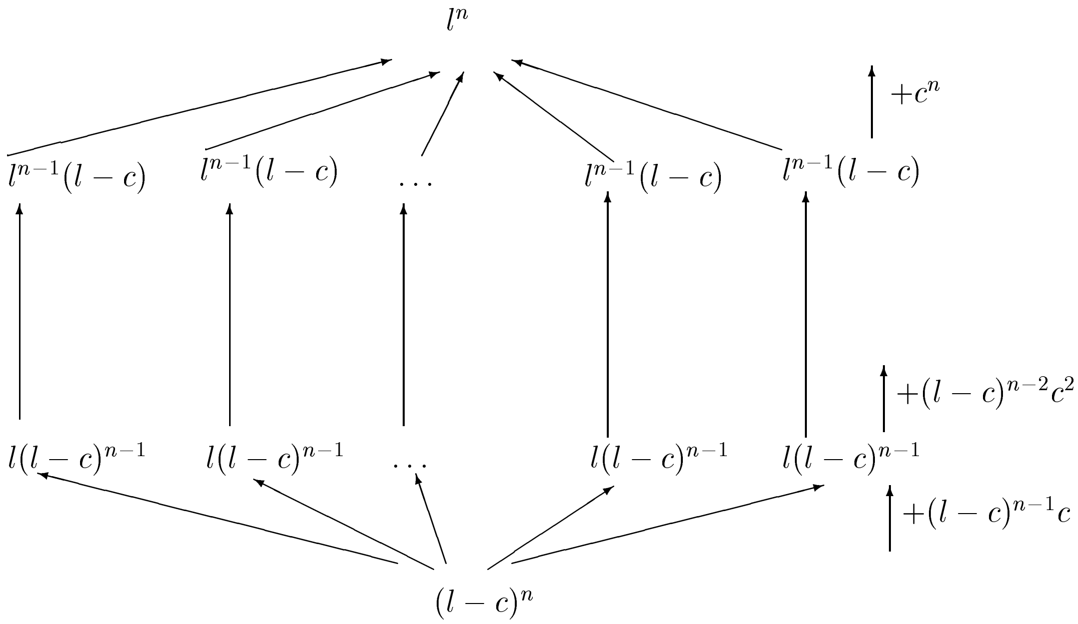
for all $i = 0, \ldots, n-a$.

Proposition 4.2 *1. The union of all entries belonging to the blocks of level i is a subset of the union of all entries belonging to the blocks of level $i+1$ for all $i = 0, \ldots, n-1$.*

2. For every $a = 0, \ldots, n$ the sequences of number of columns and number of increasings are monotone. Both sequences are geometric sequences with a factor $\frac{m-d}{d}$ (sequence of number of columns) or $\frac{c}{l-c}$ (sequence of number of increasings).

Taking a closer look at the increasing structure of each component we realize that the increasings split up in the same way like above (for $\mathcal{G}^2$).

For component $[n, 0]$ we get the "splitting up diagram"



We have $n+1$ increasing levels. We take the block with the minimal number of entries at the first level. At the i^{th} level, $i = 2, \dots, n+1$, we take all the blocks which have the i^{th} minimal number of entries and count the new edges in each column (separately) not occurring at the union of the lower $(1^{st}, 2^{nd}, \dots (i-1)^{th})$ levels. At the i^{th} level we have $\binom{n}{i-1}$ blocks, where each block consists of $(m-d)^{i-1}d^{n-i+1}$ columns, each column having $l^{i-1}(l-c)^{n-i+1}$ entries.

For component $[n-a, a]$, $a = 1, \dots, n-1$, we have in principle the same increasing structure like in $[n-a, 0]$ except a factor q^a in the number of columns and a factor p^a in the number of entries/increasings. These factors occur because in component $[n-a, a]$ we have an a-fold duplication of component $[n-a, a]$ in the manner of $\{2\}$.

An example for this notation

Before discussing the matching process we give an example for this new notation. Let $\mathcal{G}$ be the graph $\mathcal{G} = (K_{3,2} \backslash K_{2,1}) \dot{\cup} K_{4,5}$. The biadjacency matrix A is of the form

$$A = \left(\begin{array}{cc|cccc} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right).$$

$\mathcal{G}$ has two connected components, which are described by the table [1,0]

♯ blocks	with	♯ columns	♯ entries	♯ increasings
1		1	1	1
1		1	3	2

,

and by the table [0,1]

♯ blocks	with	♯ columns	♯ entries	♯ increasings
1		5	4	4

.

$\mathcal{G}^2$ consists of 4 connected components. One component is described by the table [2,0]

♯ blocks	with	♯ columns	♯ entries	♯ increasings
1		1	1	1
2		1	3	2
1		1	9	4

,

two components are given by the table [1,1]

♯ blocks	with	♯ columns	♯ entries	♯ increasings
1		5	4	4
1		5	12	8

,

and one component is determined by the table [0,2]

♯ blocks	with	♯ columns	♯ entries	♯ increasings
1		25	16	16

,

The graph $\mathcal{G}^3$ has a decomposition into 8 connected components. One component is given by the table [3,0]

♯ blocks	with	♯ columns	♯ entries	♯ increasings
1		1	1	1
3		1	3	2
3		1	9	4
1		1	27	8

,

three components are described by the table [2,1]

♯ blocks	with	♯ columns	♯ entries	♯ increasings
1		5	4	4
2		5	12	8
1		5	36	16

,

furthermore we have three components of the form [1,2]

♯ blocks	with	♯ columns	♯ entries	♯ increasings
1		25	16	16
1		25	48	32

,

and one component is given by the table [0,3]

♯ blocks	with	♯ columns	♯ entries	♯ increasings
1		125	64	64

.

4.2 Matching in this Special Class of Bipartite Graphs

$\mathcal{G}$ is a subgraph of the bipartite graph $K_{l,m}\dot{\cup}K_{p,q}$. Therefore the calculated matching number $\nu((K_{l,m}\dot{\cup}K_{p,q})^n) = \sum\limits_{k=0}^{n} \binom{n}{k} \min(l^{n-k}p^k, m^{n-k}q^k)$ forms an upper bound for $\nu(\mathcal{G}^n)$. An interesting question is, if this upper bound is reached for special constellations of six-tuples (l, m, p, q, c, d) under the conditions $1 \leq c \leq l-1$ and $1 \leq d \leq m-1$.

This question will be answered in the following and moreover we determine the matching numbers for all possible constellations of the six-tuples (l, m, p, q, c, d) with $1 \leq c \leq l-1$ and $1 \leq d \leq m-1$.

The process of getting the matching numbers works as follows:

As before we calculate the matching numbers of each component and sum them up afterwards.

In each component we start with the columns having minimal entry number (standing in the first row of each $[n-a,a]$-table) and try to match each of the $q^a d^{n-a}$ columns of the first block with the $p^a(l-c)^{n-a}$ entries. If each column can be matched, we will go on with the next row and keep in mind that we might not have exhausted all column entries. The number of entries we do not use for matching - which we will call the "surplus" - in this step has the value $p^a(l-c)^{n-a} - q^a d^{n-a}$. This surplus can be used later on in the proceeding to match columns with entries (especially when we have a deficiency of the number of entries belonging to one column block, i.e. $\sharp$ entries $<$ $\sharp$ columns).

4.2.1 Matching Number in Each Component

Every $[n-a,a]$-table describes a bipartite graph $\mathcal{G}^{(n)}_{[n-a,a]} \triangleq (\mathcal{A},\mathcal{B},\mathcal{E})$ with $|\mathcal{A}| = p^a l^{n-a}$ and $|\mathcal{B}| = q^a m^{n-a}$. $\mathcal{G}^n$ has a decomposition into subgraphs $\mathcal{G}^{(n)}_{[n-a,a]}$ with $a \in \{0,\ldots,n\}$, where each $\mathcal{G}^{(n)}_{[n-a,a]}$ occurs exactly $\binom{n}{a}$ times.

For the calculation of the matching numbers for each subgraph $\mathcal{G}^{(n)}_{[n-a,a]}$ we have to regard several cases separately depending on the constellation of l,m,p,q,c and d and we get the result.

Theorem 4.3 *Each connected component $\mathcal{G}^{(n)}_{[n-a,a]}$ of $\mathcal{G}^n$, $a \in \{0,\ldots,n\}$, has the matching number*

$$\nu(\mathcal{G}^{(n)}_{[n-a,a]}) = \begin{cases} \sum\limits_{k=0}^{\lceil i_{cut}\rceil -1} \binom{n-a}{k} p^a c^k (l-c)^{n-a-k} \\ + \sum\limits_{k=\lceil i_{cut}\rceil}^{n-a} \binom{n-a}{k} q^a (m-d)^k d^{n-a-k}, \\ \qquad\qquad \text{if } (l,m,p,q,c,d) \in \mathcal{M}, \\ \min(p^a l^{n-a}, q^a m^{n-a}) \quad \text{otherwise,} \end{cases}$$

where $i_{cut} \triangleq \dfrac{\log\left(\left(\frac{p}{q}\right)^a \left(\frac{l-c}{d}\right)^{n-a}\right)}{\log\left(\frac{(m-d)(l-c)}{dc}\right)}$.

Here a six-tuple $(l,m,p,q,c,d) \in \mathcal{M}$, iff the two following conditions are fulfilled:

a) $l,m,p,q \in \mathbb{N}$, $l,m \geq 2$ and $1 \leq c \leq l-1$ and $1 \leq d \leq m-1$,

b) With $T_{m,l} \triangleq \frac{\log \frac{m}{l}}{\log \frac{p}{q} + \log \frac{m}{l}}$ and $T_{d,l-c} \triangleq \frac{\log \frac{d}{l-c}}{\log \frac{p}{q} + \log \frac{d}{l-c}}$ the condition

$$\begin{cases} & \frac{l}{m} > 1 > \frac{p}{q} = \frac{l-c}{d} \quad \text{and} \quad a < nT_{m,l}, \\ \text{or} & \frac{l}{m} = \frac{p}{q} \geq 1 > \frac{l-c}{d} \quad \text{and} \quad a < nT_{d,l-c}, \\ \text{or} & \frac{p}{q} \geq 1 > \frac{l}{m} > \frac{l-c}{d} \quad \text{and} \quad nT_{m,l} < a < nT_{d,l-c}, \\ \text{or} & \frac{p}{q} > \frac{l}{m} \geq 1 > \frac{l-c}{d} \quad \text{and} \quad a < nT_{d,l-c}, \\ \text{or} & \frac{l}{m} > \frac{p}{q} \geq 1 > \frac{l-c}{d} \quad \text{and} \quad a < nT_{d,l-c}, \\ \text{or} & \frac{l}{m} \geq 1 > \frac{l-c}{d} > \frac{p}{q} \quad \text{and} \quad a \leq nT_{m,l}, \\ \text{or} & \frac{l}{m} > \frac{l-c}{d} \geq 1 > \frac{p}{q} \quad \text{and} \quad nT_{d,l-c} < a \leq nT_{m,l}, \\ \text{or} & \frac{l}{m} \geq 1 > \frac{l-c}{d} > \frac{p}{q} \quad \text{and} \quad a < nT_{m,l} \end{cases}$$

holds.

The proof of this theorem is given in 4.2.4.

Corollary 4.4 *For constellations (l, m, p, q, c, d) which obey $\frac{l}{m} \leq \frac{l-c}{d}$, the n-th powers of $\mathcal{G}$ have the same matching numbers like in the complete case.*

Proof:

Obviously every of the 8 subcases in the "bad case" (the case with $(l, m, p, q, c, d) \in \mathcal{M}$) fulfills the condition $\frac{l}{m} > \frac{l-c}{d}$. We can exclude these subcases by assuming the opposite. □

Corollary 4.5 (Maximizing König-Hall pair) *The pair of probability distributions (P_λ, Q_λ) with*

$$_\lambda = (\underbrace{\frac{\lambda}{l}, \ldots, \frac{\lambda}{l}}_{l\ times}, \underbrace{\frac{1-\lambda}{p}, \ldots, \frac{1-\lambda}{p}}_{p\ times})$$

$$\text{and} \quad Q_\lambda = (\underbrace{\frac{\lambda}{m}, \ldots, \frac{\lambda}{m}}_{m\ times}, \underbrace{\frac{1-\lambda}{q}, \ldots, \frac{1-\lambda}{q}}_{q\ times}),$$

forms a maximizing König-Hall pair for $(K_{l,m} \backslash K_{c,d}) \dot\cup K_{p,q}$ for every six-tuple (l, m, p, q, c, d) with $\frac{l}{m} \leq \frac{l-c}{d}$.

Proof:

(P_λ, Q_λ) forms a maximizing König-Hall pair for the complete case $K_{l,m} \dot{\cup} K_{p,q}$ (3.2.3).

Here component $K_{l,m} \backslash K_{c,d}$ consists of two vertex sets $\mathcal{X} = \{x_1, \ldots, x_l\}$ and $\mathcal{Y} = \{y_1, \ldots, y_m\}$. c vertices in $\mathcal{X}$, say $\{x_1, \ldots, x_c\}$ do not have edges incident to d vertices in $\mathcal{Y}$, say $\{y_{m-d+1} \ldots, y_m\}$, all other x_i's $(i = c+1, \ldots, l)$ have the full neighbourhood in $\mathcal{Y}$. The König-Hall condition for the "critical" subset $\{x_1, \ldots, x_c\}$ implies the inequality $\frac{(m-d)\lambda}{m} = Q_\lambda(\{y_1, \ldots, y_{m-d}\}) = Q_\lambda(\Gamma(\{x_1, \ldots, x_c\})) \geq P_\lambda(\{x_1, \ldots, x_c\}) = \frac{c\lambda}{l}$, which is equivalent to $\frac{l}{m} \leq \frac{l-c}{d}$.

□

Like in 3.2.3 we get the idea of a deeper correlation between Ahlswede and Cai's description of the matching capacity with König-Hall pairs on the one hand and calculation by matching numbers on the other hand.

Before proving Theorem 4.3, we will exactly define the expression of a surplus function and state a helpful lemma.

4.2.2 Surplus Function

For matching it is important to know, whether we have a positive surplus of column entries. This surplus possibly can be used for matching "further down", when we are going through the $[n-a, a]$-table.

Definition 4.6 (surplus function) *For every row r $(r = 0, \ldots, n-a)$ of the $[n-a, a]$-table we define the surplus function*

$$s(r) = \sum_{j=0}^{r} \binom{n-a}{j} \Delta(j)$$

with $\Delta(j) \triangleq p^a c^j (l-c)^{n-a-j} - q^a (m-d)^j d^{n-a-j}$.

Remark 4.7 The surplus function is the summation of differences $\Delta(j)$ of the number of increasings and the number of or each row j, $0 \leq j \leq r$. Every positive value $s(r)$ implies a surplus of column entries, which is not used up to the r-th step in going through the $[n-a, a]$-table, and it can be completely exhausted in the following steps of the matching process.

Remark 4.8 For the arguments 0 and $n-a$ the values of s have a simple structure:

$$\begin{aligned} s(0) &= \Delta(0) = p^a(l-c)^{n-a} - q^a d^{n-a}, \\ \Delta(n-a) &= p^a c^{n-a} - q^a (m-d)^{n-a}, \\ s(n-a) &= p^a l^{n-a} - q^a m^{n-a}. \end{aligned}$$

The last equality follows because of

$$\begin{aligned} & s(n-a) \\ = & \sum_{j=0}^{n-a} \binom{n-a}{j} (p^a c^j (l-c)^{n-a-j} - q^a (m-d)^j d^{n-a-j}) \\ = & p^a \sum_{j=0}^{n-a} \binom{n-a}{j} c^j (l-c)^{n-a-j} - q^a \sum_{j=0}^{n-a} \binom{n-a}{j} (m-d)^j d^{n-a-j} \\ = & p^a l^{n-a} - q^a m^{n-a}. \end{aligned}$$

Remark 4.9 The finite geometric sequence of the number of is monotonically with the factor $\frac{m-d}{d}$, and the finite geometric sequence of the number of increasings is monotone with the factor $\frac{c}{l-c}$. Because of the monotonicity of both finite geometric sequences the difference sequence $(\Delta(j))_{j=0,\ldots,n-a}$ changes its sign at most once, and the surplus function as well.

Therefore we obtain

Remark 4.10 The behaviour of function s has four possibilities:

- <u>Case I:</u> $s(0) \geq 0$ and $s(n-a) \geq 0$.

  This formulation is equivalent to $s(0) \geq 0$ for all $r = 0, \ldots, n-a$.

- <u>Case II:</u> $s(0) \geq 0$ and $s(n-a) < 0$,

  which means it exists r_{min}, $r_{min} \in \{1 \ldots, n-a\}$, with $s(r) \geq 0$ for all $r \in \{1, \ldots, r_{min} - 1\}$ and $s(r) < 0$ for all $r \in \{r_{min}, \ldots, n-a\}$.

  r_{min} represents the *turn around* of the surplus function.

- <u>Case III:</u> $s(0) < 0$ and $s(n-a) \leq 0$.

  It follows that $s(r) \leq 0$ for all $r = 0, 1, \ldots, n-a$.

- Case IV: $s(0) < 0$ and $s(n-a) > 0$,

  which implies that it exists r_{min}, $r_{min} \in \{1 \ldots, n-a\}$ with $s(j) < 0$ for all $j \in \{1, \ldots, r_{min} - 1\}$ and $s(j) \geq 0$ for all $j \in \{r_{min}, \ldots, n-a\}$, where r_{min} again is the *point of turn around* of the surplus function.

In Case II and Case IV the surplus function changes the sign between 0 and $n-a$. This function is a summation of differences $\Delta(j)$. Because of Remark 4.9 we know that the sequence $(\Delta(j))_{j=0,\ldots,n}$ changes its sign at most once. A change of the sign of the surplus function necessarily implies a change of the sign for Δ. In this situation, it exists $i \in \{1, \ldots, n-a\}$ with $\Delta(i-1) \geq 0$ and $\Delta(i) < 0$ (Case II), respectively $\Delta(i-1) < 0$ and $\Delta(i) > 0$. The next proposition determines this point of *turn around* for Δ.

Proposition 4.11 *Let* $\frac{m-d}{d} \neq \frac{c}{l-c}$. *The real function* Δ *with* $\Delta(x) \triangleq p^a c^x (l-c)^{n-a-x} - q^a (m-d)^x d^{n-a-x}$ *has a zero position for*

$$i_{cut} = \frac{\log\left(\left(\frac{p}{q}\right)^a \left(\frac{l-c}{d}\right)^{n-a}\right)}{\log\left(\frac{(m-d)(l-c)}{dc}\right)}.$$

Proof: With $i_{cut} = \frac{\log\left(\left(\frac{p}{q}\right)^a \left(\frac{l-c}{d}\right)^{n-a}\right)}{\log\left(\frac{(m-d)(l-c)}{dc}\right)}$ it follows

$i_{cut} \log\left(\frac{(m-d)(l-c)}{dc}\right) = \log\left(\left(\frac{p}{q}\right)^a \left(\frac{l-c}{d}\right)^{n-a}\right)$, therefore $\left(\frac{\frac{m-d}{d}}{\frac{c}{l-c}}\right)^{i_{cut}} = \left(\frac{p}{q}\right)^a \left(\frac{l-c}{d}\right)^{n-a}$ and $q^a d^{n-a} \left(\frac{m-d}{d}\right)^{i_{cut}} = p^a (l-c)^{n-a} \left(\frac{c}{l-c}\right)^{i_{cut}}$. From this we obtain $s(i_{cut}) = 0$.

□

Lemma 4.12 (Matching with surplus) *Let* $\mathcal{G}^{(n)}_{[n-a,a]}$ *be described by an*

$[n-a, a]$*-table. The matching number is given by*

$$\nu(\mathcal{G}^{(n)}_{[n-a,a]}) = \begin{cases} q^a m^{n-a}, & \text{if } s(0) \geq 0 \text{ and } s(n-a) \geq 0, \\ p^a l^{n-a}, & \text{if } s(0) \geq 0 \text{ and } s(n-a) < 0 \\ & \text{or if } s(0) < 0 \text{ and } s(n-a) \leq 0, \\ \sum_{j=0}^{\lceil i_{cut} \rceil - 1} \binom{n-a}{j} p^a c^j (l-c)^{n-a-j} + \sum_{j=\lceil i_{cut} \rceil}^{n-a} \binom{n-a}{j} q^a m^{n-a-j}, & \\ & \text{if } s(0) < 0 \text{ and } s(n-a) > 0, \end{cases}$$

where $i_{cut} \in \mathbb{R}$, $0 < i_{cut} \leq n-a$ *with* $\Delta(i_{cut}) = 0$.

Thus if the matching number has value $p^a l^{n-a}$ or $q^a m^{n-a}$, this exactly is $\min(p^a l^{n-a}, q^a m^{n-a})$, which equals the result for bipartite graphs with complete components. We have only one "bad case" , when the matching number is smaller than in the complete case (starting with $\mathcal{G} = K_{l,m} \dot{\cup} K_{p,q}$).

Proof:

Case I: $s(0) \geq 0$ and $s(n-a) \geq 0$

In this case the surplus function is not negative, $s(r) \geq 0$ for all $r = 0, \ldots, n-a$.

All columns can be matched, we always have a surplus of column entries. Therefore the matching number of the component $[n-a, a]$ is $q^a m^{n-a}$.

Because of $s(n-a) \geq 0$ we obtain $\nu(\mathcal{G}^n_{[n-a,a]}) = q^a m^{n-a} =$

$min(p^a l^{n-a}, q^a m^{n-a})$.

Case II: $s(0) \geq 0$ and $s(n-a) < 0$

It exists an r_{min} with $s(r) \geq 0$ for all $r = 0, \ldots, r_{min}-1$ and $s(r) < 0$ for all $r = r_{min}, \ldots, n-a$. Here we get a matching structure of size

$$\begin{aligned}
& \sum_{j=0}^{r_{min}-1} \binom{n-a}{j} q^a (m-d)^j d^{n-a-j} + s(r_{min}-1) \\
& + \sum_{j=r_{min}}^{n-a} \binom{n-a}{j} p^a c^j (l-c)^{n-a-j} \\
= & \sum_{j=0}^{r_{min}-1} \binom{n-a}{j} q^a (m-d)^j d^{n-a-j} + \sum_{j=r_{min}}^{n-a} \binom{n-a}{j} p^a c^j (l-c)^{n-a-j} \\
& + \sum_{j=0}^{r_{min}-1} \binom{n-a}{j} (p^a c^j (l-c)^{n-a-j} - q^a (m-d)^j d^{n-a-j}) \\
= & \sum_{j=0}^{n-a} \binom{n-a}{j} p^a c^j (l-c)^{n-a-j} \\
= & p^a l^{n-a}.
\end{aligned}$$

Since $s(n-a) < 0$ we get the inequality $p^a l^{n-a} - q^a m^{n-a} < 0$ and thus $\nu(\mathcal{G}^n_{[n-a,a]}) = p^a l^{n-a} = \min(p^a l^{n-a}, q^a m^{n-a})$.

Case III: $s(0) < 0$ and $s(n-a) \leq 0$

Here we have $s(r) \leq 0$ for all $r = 0, 1, \ldots, n-a$.

All entries can be matched because we always have a non-positive surplus of column entries. Therefore the matching number of the component $[n-a, a]$ is $p^a l^{n-a} = \min(p^a l^{n-a}, q^a m^{n-a})$.

Case IV: $s(0) < 0$ and $s(n-a) > 0$

The change of the sign for function s necessarily implies a change of the sign for function Δ. We know, that $\Delta(i_{cut}) = 0$, thus $\Delta(\lceil i_{cut}\rceil - 1) < 0$ and $\Delta(\lceil i_{cut}\rceil) \geq 0$.

The differences $(\Delta(j))_{j=0,\ldots,n-a}$ for the first positions are - different to Case II - negative, so that we cannot use it for matching, thus

$$\nu(\mathcal{G}^{(n)}_{[n-a,a]}) = \sum_{j=0}^{\lceil i_{cut}\rceil - 1} \binom{n-a}{j} p^a c^j (l-c)^{n-a-j}$$

$$+ \sum_{j=\lceil i_{cut}\rceil}^{n-a} \binom{n-a}{j} q^a (m-d)^j d^{n-a-j}.$$

This number is smaller than $p^a l^{n-a}$ as well smaller than $q^a m^{n-a}$. Therefore we get $\nu(\mathcal{G}^{(n)}_{[n-a,a]}) < \min(l^a p^{n-a}, m^a q^{n-a})$ in this case . $\square$

Lemma 4.12 comprises a differentiation into 4 cases according to the signs of $s(0)$ and $s(n-a)$. We just have one "bad case": In Case IV ($s(0) < 0$ and $s(n-a) > 0$) we get smaller matching numbers in comparison to the complete case. Before proving Theorem 4.3, we state - according to this differentiation - two propositions.

Proposition 4.13 (0-condition) *a) For* $\frac{p}{q} = \frac{l-c}{d}$

$$s(0) \geq 0, \quad \text{iff} \quad \frac{p}{q} \geq 1.$$

For $\frac{p}{q} \neq \frac{l-c}{d}$ *define* $T_{d,l-c} \triangleq \frac{\log \frac{d}{l-c}}{\log \frac{p}{q} + \log \frac{d}{l-c}}$.

b) For $\frac{p}{q} > \frac{l-c}{d}$

$$s(0) \geq 0, \textit{ iff } \left\{ \begin{array}{lll} a \geq nT_{d,l-c} & \text{for } \frac{d}{l-c} > 1, & \frac{p}{q} \geq 1 \\ a > n & \text{for } \frac{d}{l-c} > 1, & \frac{p}{q} < 1 \\ a \geq 0 & \text{for } \frac{d}{l-c} \leq 1, & \frac{p}{q} \geq 1 \end{array} \right\}.$$

c)For $\frac{p}{q} < \frac{l-c}{d}$

$$s(0) \geq 0, \textit{ iff } \left\{ \begin{array}{lll} a \leq nT_{d,l-c} & \text{for } \frac{d}{l-c} \leq 1, & \frac{p}{q} < 1 \\ a \leq n & \text{for } \frac{d}{l-c} \leq 1, & \frac{p}{q} \geq 1 \\ a < 0 & \text{for } \frac{d}{l-c} > 1, & \frac{p}{q} < 1 \end{array} \right\}.$$

The theoretically situations $q = 0$ and $l = c$ do not occur because of the definitions ($q \in \mathbb{N}$ and $1 \leq c \leq l-1$).

Proof:

$s(0) \geq 0$, there upon $p^a(l-c)^{n-a} \geq q^a d^{n-a}$ and it follows $(\frac{p}{q})^a \geq \frac{d}{l-c}^{n-a}$. We can conclude $a \log \frac{p}{q} \geq (n-a) \log \frac{d}{l-c}$, and we obtain $a(\log \frac{p}{q} + \log \frac{d}{l-c}) \geq n \log \frac{d}{l-c}$.

We now have to distinguish three cases:

Case A: $\frac{p}{q} = \frac{l-c}{d}$

$s(0) \geq 0$ is equivalent to $\log \frac{d}{l-c} \leq 0$, which yields, iff $d \leq l-c$ and $q \leq p$.

We recognize, that this condition for $s(0) \geq 0$ is independent of a.

For Cases B and C we define $T_{d,l-c} \triangleq \frac{\log \frac{d}{l-c}}{\log \frac{p}{q} + \log \frac{d}{l-c}}$.

Case B: $\frac{p}{q} > \frac{l-c}{d}$

We obtain $s(0) \geq 0$ iff $a \geq T_{d,l-c} n$. For simplicity we call it $T_{d,l-c}$-condition.

Now we have to distinguish some subcases:

Subcase B1: $\frac{d}{l-c} > 1$, $\frac{p}{q} \geq 1$

$0 < T_{d,l-c} \leq 1$ and we get $s(0) \geq 0$ for all $a \in \{0, \ldots, n\}$ with $a \geq T_{d,l-c} n$.

Subcase B2: $\frac{d}{l-c} > 1$, $\frac{p}{q} < 1$

Here the $T_{d,l-c}$-condition implies $T_{d,l-c} > 1$, which would demand $a > n$ for $s(0) \geq 0$. Thus $s(0) \geq 0$ does not hold for any $a \in \{0, \ldots, n\}$.

Subcase B3: $\frac{d}{l-c} \leq 1$, $\frac{p}{q} \geq 1$

The $T_{d,l-c}$-condition implies $T_{d,l-c} \leq 0$, and so $s(0) \geq 0$ for all

$a \in \{0, \ldots, n\}$.

Realize that a subcase with $\frac{d}{l-c} \leq 1$ and $\frac{p}{q} < 1$ is not possible because

$\frac{l-c}{d} \geq 1 > \frac{p}{q}$ contradicts to the assumption of Case B.

Case C: $\frac{p}{q} < \frac{l-c}{d}$

We obtain $s(0) \geq 0$ iff $a \leq T_{d,l-c} n$. We again call it $T_{d,l-c}$-condition, and have the following distinction:

Subcase C1: $\frac{d}{l-c} \leq 1$, $\frac{p}{q} < 1$

$0 \leq T_{d,l-c} < 1$ and we get $s(0) \geq 0$ exactly for all $a \in \{0, \dots, n\}$ with $a \leq T_{d,l-c}n$.

Subcase C2: $\frac{d}{l-c} \leq 1$, $\frac{p}{q} \geq 1$

Here the $T_{d,l-c}$-condition implies $T_{d,l-c} \geq 1$, and thus $s(0) \geq 0$ for any

$a \in \{0, \dots, n\}$.

Subcase C3: $\frac{d}{l-c} \leq 1$, $\frac{p}{q} \geq 1$

Here the $T_{d,l-c}$-condition implies $T_{d,l-c} < 0$, which would demand $a < 0$ for $s(0) \geq 0$. Thus $s(0) \geq 0$ does not hold for any $a \in \{0, \dots, n\}$.

Realize that a subcase with $\frac{d}{l-c} > 1$ and $\frac{p}{q} \geq 1$ is not possible because
$\frac{l-c}{d} < 1 \leq \frac{p}{q}$ contradicts to the assumption of Case C. □

The following proposition has exactly the same structure where l takes the role of $l - c$ and m the role of d.

Proposition 4.14 ((n-a)-condition) *a) For* $\frac{p}{q} = \frac{l}{m}$

$$s(n-a) > 0, \quad \text{iff} \quad \frac{p}{q} > 1.$$

For $\frac{p}{q} \neq \frac{l}{m}$ *define* $T_{m,l} \triangleq \frac{\log \frac{m}{l}}{\log \frac{p}{q} + \log \frac{m}{l}}$.

b) For $\frac{p}{q} > \frac{l}{m}$

$$s(n-a) > 0, \text{ iff } \left\{ \begin{array}{lll} a > nT_{m,l} & \text{for } \frac{m}{l} > 1, & \frac{p}{q} \geq 1 \\ a > n & \text{for } \frac{m}{l} > 1, & \frac{p}{q} < 1 \\ a \geq 0 & \text{for } \frac{m}{l} \leq 1, & \frac{p}{q} \geq 1 \end{array} \right\}.$$

c)For $\frac{p}{q} < \frac{l}{m}$

$$s(n-a) > 0, \text{ iff } \left\{ \begin{array}{lll} a < nT_{m,l} & \text{for } \frac{m}{l} \leq 1, & \frac{p}{q} < 1 \\ a \leq n & \text{for } \frac{m}{l} \leq 1, & \frac{p}{q} \geq 1 \\ a < 0 & \text{for } \frac{m}{l} > 1, & \frac{p}{q} < 1 \end{array} \right\}.$$

4.2.3 An Illustrating Example

We come back to the example mentioned in 4.1.2 with $\mathcal{G} = (K_{3,2} \backslash K_{2,1}) \dot{\cup} K_{4,5}$ to illustrate the definitions (used in Theorem 4.3) and the process of finding the maximal matching numbers. The description of the components in tables was introduced in 4.1.2.

First we give the table of matching numbers for small dimensions for the graph $\mathcal{G}$ compared to the "complete" case $K_{3,2} \dot{\cup} K_{4,5}$.

dim n	$\nu((K_{3,2} \dot{\cup} K_{4,5})^n)$	$\nu(\mathcal{G}^n)$
1	6	6
2	40	38
3	276	252
4	1800	1730
5	12296	11591
6	84112	76766
7	560104	512602
8	3866720	3480034

The six-tuple $(l, m, p, q, c, d) = (3, 2, 4, 5, 2, 1)$ determines the behaviour of the matching numbers. Here we are in the "bad case", because $\frac{l}{m} = \frac{3}{2} > 1 = \frac{l-c}{d} > \frac{4}{5} = \frac{p}{q}$, and this condition implies $(l, m, p, q, c, d) \in \mathcal{M}$.

In particular we have $T_{m,l} = T_{2,3} = \frac{\log \frac{2}{3}}{\log \frac{4}{5} + \log \frac{2}{3}} \approx 0.645019921$ and $T_{d,l-c} = T_{1,1} = 0$.

<u>n=1:</u>

The first component, a subgraph of $K_{3,2}$, is described by the [1,0]-table

$\sharp$ blocks	with	$\sharp$ columns	$\sharp$ entries	$\sharp$ increasings	Δ	surplus
1		**1**	1	1	0	0
1		**1**	3	2	1	1

,

and the second component, exactly the graph $K_{4,5}$, by [0,1]

$\sharp$ blocks	with	$\sharp$ columns	$\sharp$ entries	$\sharp$ increasings	Δ	surplus
1		5	4	**4**	-1	-1

.

The numbers in bold contribute to the matching number. Thus $\nu(\mathcal{G}_{[1,0]}) = 2 = \min(3, 2)$, $\nu(\mathcal{G}_{[0,1]}) = 4 = \min(5, 4)$ and we obtain $\nu(\mathcal{G}) = 2 + 4 = 6$.

<u>n=2:</u>

The graph $\mathcal{G}^2$ consists of 4 connected components: 1 component [2,0], 2 components [1,1] and 1 component [0,2].

Table [2,0] describes one component, a subgraph of $K_{9,4}$, which is given by

$\sharp$ blocks	with	$\sharp$ columns	$\sharp$ entries	$\sharp$ increasings	Δ	surplus
1		**1**	1	1	0	0
2		**1**	3	2	1	2
1		**1**	9	4	3	5

,

and we obtain $\nu(\mathcal{G}_{[2,0]}) = 4 = \min(9, 4)$.

Table [1,1] represents subgraphs of $K_{12,10}$, which are given by

$\sharp$ blocks	with	$\sharp$ columns	$\sharp$ entries	$\sharp$ increasings	Δ	surplus
1		5	4	**4**	-1	-1
1		**5**	12	8	3	2

,

and we obtain $\nu(\mathcal{G}_{[1,1]}) = 4 + 5 = 9 < \min(12, 10)$; here one edge "drops out" of the matching compared to the complete case.

The fourth component, which equals the graph $K_{16,25}$, is given by [0,2]

$\sharp$ blocks	with	$\sharp$ columns	$\sharp$ entries	$\sharp$ increasings	Δ	surplus
1		25	16	**16**	-9	-9

,

thus we get $\nu(\mathcal{G}_{[0,2]}) = 16 = \min(16, 25)$ and $\nu(\mathcal{G}^2) = 4 + 2 \cdot 9 + 16 = 38$.

<u>n=3:</u>

The graph $\mathcal{G}^3$ consists of 8 connected components: 1 component [3,0], 3 components [2,1], 3 components [1,2] and 1 component [0,3].

Table [3,0] describes a subgraph of $K_{27,8}$, which is given by

$\sharp$ blocks	with	$\sharp$ columns	$\sharp$ entries	$\sharp$ increasings	Δ	surplus
1		**1**	1	1	0	0
3		**1**	3	2	1	3
3		**1**	9	4	3	12
1		**1**	27	8	7	19

,

and we obtain $\nu(\mathcal{G}_{[3,0]}) = 8 = \min(27, 8)$.

Table [2,1] illustrates three components, subgraphs of $K_{36,20}$, which are given by

♯ blocks	with	♯ columns	♯ entries	♯ increasings	Δ	surplus
1		5	4	**4**	-1	-1
2		**5**	12	8	3	5
1		**5**	36	16	11	16

,

and we obtain $\nu(\mathcal{G}_{[2,1]}) = 4 + 2 \cdot 5 + 5 = 19 < \min(36, 20)$.

Table [1,2] describes another three components, subgraphs of $K_{48,50}$, which are given by

♯ blocks	with	♯ columns	♯ entries	♯ increasings	Δ	surplus
1		25	16	**16**	-9	-9
1		**25**	48	32	7	-2

,

and we obtain $\nu(\mathcal{G}_{[1,2]}) = 16 + 25 = 41 < \min(48, 50)$; some edges "drop out" of the matching compared to the complete case.

The last component ($K_{64,125}$) is given by [0,3]

♯ blocks	with	♯ columns	♯ entries	♯ increasings	Δ	surplus
1		125	64	**64**	-61	-61

,

thus we get $\nu(\mathcal{G}_{[0,3]}) = 64 = \min(64, 125)$ and $\nu(\mathcal{G}^3) = 8 + 3 \cdot 19 + 3 \cdot 41 + 64 = 252$.

4.2.4 Proof of Theorem 4.3

Following Lemma 4.12 , we observe that the matching numbers are the same as in the complete case $\mathcal{G} = K_{l,m} \dot{\cup} K_{p,q}$ except one "bad case", where the matching structure is disturbed. This "bad case" occurs, when $s(0) < 0$ and $s(n-a) > 0$. Proposition 4.13 and Proposition 4.14 provide the constellations of (l, m, p, q, c, d) for the "bad case".

$s(0) < 0$ holds in the following constellations:

$$
\begin{array}{lll}
\frac{p}{q} = \frac{l-c}{d} & \text{and} & \frac{p}{q} < 1 \quad \text{for all } a = 0, \ldots, n, \\
\frac{p}{q} > \frac{l-c}{d} & \text{and} & \left\{ \begin{array}{l} \frac{d}{l-c} > 1, \quad \frac{p}{q} \geq 1 \text{ for all } a < nT_{d,l-c} \\ \frac{d}{l-c} > 1, \quad \frac{p}{q} < 1 \text{ for all } a \leq n \\ \frac{d}{l-c} \leq 1, \quad \frac{p}{q} \geq 1 \text{ for all } a < 0 \end{array} \right\}, \\
\frac{p}{q} < \frac{l-c}{d} & \text{and} & \left\{ \begin{array}{l} \frac{d}{l-c} \leq 1, \quad \frac{p}{q} < 1 \text{ for all } a > nT_{d,l-c} \\ \frac{d}{l-c} \leq 1, \quad \frac{p}{q} \geq 1 \text{ for all } a > n \\ \frac{d}{l-c} > 1, \quad \frac{p}{q} < 1 \text{ for all } a \geq 0 \end{array} \right\},
\end{array}
$$

with $T_{d,l-c} = \frac{\log \frac{d}{l-c}}{\log \frac{p}{q} + \log \frac{d}{l-c}}$,

and $s(n-a) > 0$ holds for

$$\begin{array}{ll}
\frac{p}{q} = \frac{l}{m} \text{ and } \frac{p}{q} > 1 & \text{for all } a = 0, \ldots, n, \\
\frac{p}{q} > \frac{l}{m} \text{ and } & \left\{ \begin{array}{ll} \frac{m}{l} > 1, & \frac{p}{q} \geq 1 \text{ for all } a < nT_{m,l} \\ \frac{m}{l} > 1, & \frac{p}{q} < 1 \text{ for all } a > n \\ \frac{m}{l} \leq 1, & \frac{p}{q} \geq 1 \text{ for all } a \geq 0 \end{array} \right\}, \\
\frac{p}{q} < \frac{l}{m} \text{ and } & \left\{ \begin{array}{ll} \frac{m}{l} \leq 1, & \frac{p}{q} < 1 \text{ for all } a < nT_{m,l} \\ \frac{m}{l} \leq 1, & \frac{p}{q} \geq 1 \text{ for all } a \leq n \\ \frac{m}{l} > 1, & \frac{p}{q} < 1 \text{ for all } a < 0 \end{array} \right\},
\end{array}$$

with $T_{m,l} = \frac{\log \frac{m}{l}}{\log \frac{p}{q} + \log \frac{m}{l}}$.

Combining both propositions we obtain 8 possible constellations for (l, m, p, q, c, d), and the set $\mathcal{M}$ is defined by six-tuples belonging to one of the following cases.

Case I: $\frac{l-c}{d} = \frac{p}{q} < 1 < \frac{m}{l}$

$s(0) < 0$ and $s(n-a) > 0$ holds for all $a < nT_{m,l}$.

Case II: $\frac{l}{m} = \frac{p}{q} > 1 > \frac{l-c}{d}$

$s(0) < 0$ and $s(n-a) > 0$ holds for all $a < nT_{d,l-c}$.

Case III: $\frac{p}{q} \geq 1 > \frac{l-c}{d}$ and $1 > \frac{l}{m}$

$s(0) < 0$ and $s(n-a) > 0$ holds for all a with $nT_{m,l} < a < nT_{d,l-c}$. This case occurs for $T_{m,l} < T_{d,l-c}$. This inequality holds iff $\frac{l-c}{d} < \frac{l}{m}$.

Case IV: $\frac{p}{q} > \frac{l}{m} \geq 1 > \frac{l-c}{d}$.

$s(0) < 0$ and $s(n-a) > 0$ holds for all $a < nT_{d,l-c}$.

Case V: $\frac{l}{m} > \frac{p}{q} \geq 1 > \frac{l-c}{d}$

$s(0) < 0$ and $s(n-a) > 0$ holds for all $a < nT_{d,l-c}$.

Case VI: $\frac{l}{m} \geq 1 > \frac{l-c}{d} > \frac{p}{q}$

$s(0) < 0$ and $s(n-a) > 0$ holds for all $a < nT_{m,l}$.

Case VII: $\frac{l-c}{d} \geq 1 > \frac{p}{q}$ and $\frac{m}{l} \leq 1$

$s(0) < 0$ and $s(n-a) > 0$ holds for all a with $nT_{d,l-c} < a < nT_{m,l}$. This case occurs for $T_{d,l-c} < T_{m,l}$. This inequality holds iff $\frac{l-c}{d} < \frac{l}{m}$.

Case VIII: $\frac{l}{m} \geq 1 > \frac{l-c}{d} > \frac{p}{q}$

$s(0) < 0$ and $s(n-a) > 0$ holds for all $a < nT_{m,l}$. $\square$

4.2.5 Deleting One Edge

Now we look at the special case, when we delete only one edge out of component $K_{l,m}$.

Corollary 4.15 *For $\mathcal{G} = (K_{l,m} \backslash K_{1,1}) \dot{\cup} K_{p,q}$ the matching numbers are given by*

$$\nu(\mathcal{G}^n) = \sum_{k=0}^{n} \binom{n}{k} \min(l^{n-k}p^k, m^{n-k}q^k),$$

which are exactly the matching numbers for $K_{l,m} \dot{\cup} K_{p,q}$.

<u>Proof:</u>

We apply Theorem 4.3 to this special constellation with $c = d = 1$ and want to exclude the "bad cases". If there does not exist any "bad case", we will have the matching numbers for every component $\mathcal{G}^{(n)}_{[n-a,a]}$ of $\mathcal{G}^n$ like in the complete case.

Here we have $\frac{l-c}{d} = l - 1$. In all cases - except Case VII - the condition $l - 1 < 1$ is involved, which would demand here $l < 2$, a contradiction.

In Case VII $l-1 \geq 1$ holds, but the condition $l-1 < \frac{l}{m}$ is equivalent to $l < \frac{m}{m-1}$, with this we get $1 \leq \frac{l}{m} < \frac{m}{m(m-1)} = \frac{1}{m-1}$, which desires for $m < 2$, again a contradiction.

Thus the set $\mathcal{M}$ which consists of the "bad cases" is empty, and we get the assertion. □

4.2.6 Deleting Stars

We have seen that we can omit one edge without destroying the structure of a maximal matching. The next question which arises is: "Can we delete a star and again obtain for all n-powers the matching numbers of the complete case?"

Corollary 4.16 *a) Let $c \leq l - \frac{l}{m}$. For $\mathcal{G} = (K_{l,m} \backslash K_{c,1}) \dot{\cup} K_{p,q}$ the matching numbers are given by*

$$\nu(\mathcal{G}^n) = \sum_{k=0}^{n} \binom{n}{k} \min(l^{n-k}p^k, m^{n-k}q^k),$$

which are exactly the matching numbers for $K_{l,m}\dot{\cup}K_{p,q}$.

b) Let $d \leq m - \frac{m}{l}$. *For* $\mathcal{G} = (K_{l,m}\backslash K_{1,d})\dot{\cup}K_{p,q}$ *the matching numbers are given by*

$$\nu(\mathcal{G}^n) = \sum_{k=0}^{n} \binom{n}{k} \min(l^{n-k}p^k, m^{n-k}q^k).$$

<u>Proof:</u>

We apply Corollary 4.4 for the Case $d = 1$, and the condition $\frac{l}{m} > \frac{l-c}{d}$ is equivalent to $c \leq l - \frac{l}{m}$.

For $c = 1$ the condition is equivalent to $d \leq m - \frac{m}{l}$. □

5 Brualdi's Basic Bounds

After having finished the main part of this thesis, Brualdi's paper [5] has come to our attention. Brualdi found upper and lower bounds for the **term rank** of the direct product of matrices, where the term rank of a $(0,1)$-matrix A is defined as the maximal number of 1's of A with no two of the 1's on the same row or column. The term rank of matrix A is denoted by $\rho(A)$. For the biadjacency matrix A of a bipartite graph $\mathcal{G}$ the term rank $\rho(A)$ is exactly the matching number $\nu(\mathcal{G})$. This concept of the term rank appeared first in [12], 1955.

Brualdi's main idea was to decompose given (0,1)-matrices A, B into normal forms. Significant parameters of the normal forms of A and B determine upper and lower bounds for $\rho(A \times B)$. With a special normal form he derived a significant lower bound for $\rho(A \times B)$ and as corollary he obtained necessary and sufficient conditions in order that the term rank is multiplicative on $A \times B$.

With another normal form which bases on the concepts of essential rows and essential columns he obtained an upper bound for $\rho(A \times B)$.

Before we present Brualdi's bounds, let us focus on König's Minimax Theorem, which has the following formulation in terms of (0,1)-matrices (cf.[6]).

In this context it is convenient to refer to either a row or a column of the matrix as a **line**. By a **covering** of a (0,1)-matrix A we mean all the lines of A that together contain all the 1's of A.

Theorem 5.1 (König) *Let A be a $(0,1)$-matrix of size m by n. The minimal number of lines in A that cover all the 1's in A is equal to the term rank.*

5.1 Brualdi's Lower Bound

5.1.1 Decomposition of (0,1)-Matrices

Let A be a (0,1)-matrix of size m by n.

In A select $\rho(A)$ 1's with respect to the condition that no two of them lie on the same line. In general this can be done in many ways, but once we have made our choice we do not alter it. The rows and columns in which these 1's lie must contain all the 1's of A, for otherwise we contradict the definition of the term rank $\rho(A)$ of A. Denote by A_0 that submatrix of A consisting of those elements of A that lie in the intersection of the rows belonging to these $\rho(A)$ 1's with the columns belonging to these $\rho(A)$ 1's. Let A_1 be that submatrix of A consisting of those elements that lie within the rows belonging to the selected $\rho(A)$ 1's of A but not within their columns. Similarly, let A_2 be a submatrix of A consisting of those elements that lie within the columns of the selected $\rho(A)$ 1's but not within their rows. Thus if we permute the rows and columns of A so that these $\rho(A)$ 1's occupy the first $\rho(A)$ positions on the main diagonal of A, we obtain

$$\left(\begin{array}{c|c} A_0' & A_1' \\ \hline A_2' & 0 \end{array}\right) ,$$

where the prime sign on a matrix here denotes a matrix obtained from it by permuting rows and columns, and where 0 denotes all 0's. We proceed by induction. If we have formed $A_{i_1 \ldots i_k}$ with $i_j = 1$ or 2 for $j = 1, 2, \ldots, k$, then select $\rho(A_{i_1 \ldots i_k})$ 1's of $A_{i_1 \ldots i_k}$ with no two of them on the same line. We define $A_{i_1 \ldots i_k 1}$ to be that submatrix of $A_{i_1 \ldots i_k}$ consisting of those elements that lie within their columns. Likewise we define $A_{i_1 \ldots i_k 2}$ to be that submatrix of $A_{i_1 \ldots i_k}$ consisting of those elements that lie within the columns belonging to the chosen $\rho(A_{i_1 \ldots i_k})$ 1's but not within their rows. It is possible that the matrix $A_{i_1 \ldots i_k i_{k+1}}$ is vacuous, in which case we define

$$\rho(A_{i_1 \ldots i_k i_{k+1}}) = 0.$$

This happens in particular, if $A_{i_1 \dots i_k}$ consists entirely of 0's. Brualdi called the collection of submatrices of A,

$$\{A_0; A_1, A_2; A_{11}, A_{12}, A_{21}, A_{22}; \dots\}$$

a decomposition of A. Realize that different decompositions of A may be obtained by altering the choice of 1's in the different steps.

5.1.2 A Lower Bound for the Term Rank

Let A and B be two matrices of 0's and 1's. Form a decomposition

$$\{A_0; A_1, A_2; A_{11}, A_{12}, A_{21}, A_{22}; \dots\}$$

of A and a decomposition

$$\{B_0; B_1, B_2; B_{11}, B_{12}, B_{21}, B_{22}; \dots\}$$

of B. (Of course, within each of the matrices in these decompositions we have a prescribed set of 1's with no two on the same line.) For $k \geq 1$, let i_k denote either the integer 1 or 2 and let $j_k \triangleq 3 - i_k$. Brualdi then obtained

Theorem 5.2 (Brualdi)

$$\rho(A \times B) \geq \max \left\{ \rho(A)\rho(B) + \sum_{k \in \mathbb{N}} \sum_{(i_1, i_2, \dots, i_k)} \rho(A_{i_1 i_2 \dots i_k}) \rho(B_{j_1 j_2 \dots j_k}) \right\},$$

where $\sum_{(i_1, i_2, \dots, i_k)}$ *denotes the sum taken over all ordered k-tuples* $(i_1, i_2, \dots, i_k)$ *with* $i_s = 1$ *or* 2 *for* $s = 1, 2, \dots, k$ *and where the maximum is taken over all decompositions of A and all decompositions of B.*

5.2 Brualdi's Upper Bound

5.2.1 Decomposition of (0,1)-Matrices

Let A be an arbitrary (0,1)-matrix with term-rank $\rho(A)$. A minimal covering of A consists of $\rho(A)$ lines. Brualdi defined an **essential row** of the matrix A to be a row that is a member of *every* minimal covering of A. Similarly, an **essential column** of A is a column that belongs to *every* minimal covering of A.

Theorem 5.3 (Brualdi) *Let A be a (0,1)-matrix of size m by m' with term rank $\rho(A)$. Let A have a_1 essential rows and a_2 essential columns. Choose a set of $\rho(A)$ 1's of A with no two 1's on the same line. Then the rows and columns of A can be permuted to yield*

$$
\begin{array}{rl}
 & \qquad\qquad\quad\;\; \overbrace{\hphantom{\qquad\qquad}}^{a_2} \\
a_1 \left\{ \vphantom{\begin{matrix}1\\ \ddots\\ 1\end{matrix}} \right. & \left(\begin{array}{c|c|c|c}
\begin{matrix}1 & & * \\ & \ddots & \\ * & & 1\end{matrix} & * & * & A_1 \\ \hline
0 & \begin{matrix}1 & & * \\ & \ddots & \\ * & & 1\end{matrix} & 0 & 0 \\ \hline
0 & * & \begin{matrix}1 & & * \\ & \ddots & \\ * & & 1\end{matrix} & 0 \\ \hline
0 & A_2 & 0 & 0
\end{array}\right)
\end{array}
$$

with the chosen set of $\rho(A)$ 1's of A on the main diagonal and A_i of size a_i by a_i for $i = 1, 2$. The a_1 rows (the first row-block) correspond to the essential rows of A, while the a_2 columns (the second column block) correspond to the essential columns of A. Here 0 denotes all 0's, while $*$ denotes arbitrary elements.

Brualdi goes on inductively. By decomposition of A_1 and A_2 we obtain A_{11}, A_{12}, A_{21}, A_{22}, and $A_{i_1 i_2 \ldots i_{k+1}}$ is determined by the decomposition of $A_{i_1 i_2 \ldots i_k}$.

This normal form, which Brualdi created here, is similar to Ore's normal form, [12], 1955. Brualdi said more, because if we strike out a_1 essential rows and a_2 essential columns of the matrix A , then the remaining 1's are contained in $a_3 \triangleq \rho(A) - a_1 - a_2$ rows of A. Likewise they are contained in a_3 columns of A. Brualdi called these a_3 rows **semi-essential rows** and these a_3 columns **semi-essential columns** of A.

5.2.2 Comparison to Ore's Normal Form

Brualdi's normal form for the upper bound in principle is given approximately 10 years earlier in the context of bipartite graphs and matching theorems. In [12] Ore discussed *associated subsets, minimal sets with maximal deficiencies* and *complement sets without deficiencies.*

He uses the following decomposition of vertex sets:

Theorem 5.4 (Ore) *Each of the vertex sets of a bipartite graph can be decomposed into three disjoint subsets*

$$\begin{aligned} V &= N + V(N') + R, \\ V' &= N' + V'(N) + R'. \end{aligned}$$

If we denote by $\mathcal{G}(R, R')$ the bipartite graph consisting of the vertices in R and R' and the edges in $\mathcal{G}$ joining them, then $\mathcal{G}(R, R')$ has no deficiency. Furthermore any set of maximal deficiency in v has the form $N + A$ where A is a subset of R of zero deficiency in $\mathcal{G}(R, R')$ and similarly in V'.

For a matrix M Ore finally got the following normal form

	R'	$N' - D'$	$V'(N)$	D'
R	A_{00}	0	A_{02}	0
$V(N')$	A_{10}	A_{11}	A_{12}	A_{13}
$N - D$	0	0	A_{22}	0
D	0	0	A_{32}	0

where A_{00}, A_{11}, A_{22} are square matrices with non-vanishing diagonal terms.

By row and column permutation we obtain:

	$N' - D'$	$V'(N)$	R'	D'
$V(N')$	A_{11}	A_{12}	A_{10}	A_{13}
$N - D$	0	A_{22}	0	0
R	0	A_{02}	A_{00}	0
D	0	A_{32}	0	0

We observe that Brualdi exactly works with Ore's normal form for obtaining an upper bound for the term rank of the direct product of matrices.

5.2.3 Result of Dulmage and Mendelsohn

In 1958 Dulmage and Mendelsohn came to Brualdi's normal form for the upper bound in the context of Graph Theory, [7]. They did this first by proving some theorems of minimal covers. They introduced the concept of an exterior covering of a bipartite graph. In terms of this concept it is possible to decompose any graph into two parts, an *inadmissible part* and a *core*. It is also possible to decompose the core into irreducible parts and thus obtain a canonical reduction of the graph. The concept of irreducibility is expressed in terms of exterior coverings. The role of the inadmissible edges of a graph is to obstruct certain natural coverings of the graph.

Ore [12] studied graphs using the notion of a set of maximal deficiency. For finite graphs a set of maximal deficiency in Ore's sense becomes the complement of a first member of an exterior pair defined by Dulmage and Mendelsohn. Because of this a number of theorems obtained by Dulmage and Mendelsohn is equivalent to theorems of Ore when the graph is finite. For infinite graphs the situation is quite different since Ore's finiteness condition and the one of Dulmage and Mendelsohn can never be satisfied simultaneously.

5.2.4 An Upper Bound for the Term Rank

At first Brualdi stated the

Theorem 5.5 (Essential lines) *Let A and B be (0,1)-matrices with term ranks $\rho(A)$ and $\rho(B)$ respectively. Suppose A has a_1 essential rows and a_2 essential columns and B has b_1 essential rows and b_2 essential columns. Then the direct product $A \times B$ has at least*

$$a_1(\rho(B) - b_2) + b_1(\rho(A) - a_2) - a_1 b_1$$

essential rows and at least

$$a_2(\rho(B) - b_1) + b_2(\rho(A) - a_1) - a_2 b_2$$

essential columns.

He obtained an upper bound for the term rank $\rho(A \times B)$ of the direct product of $A \times B$ by constructing a covering Γ of the 1's of $A \times B$. He included the essential rows and columns of $A \times B$ which were found in Theorem 5.5. These essential lines have to be stroked out from $A \times B$ and the remaining 1's are contained in certain direct products of submatrices of the decompositions of A and B.

Finally after proceeding inductively Brualdi obtained a covering Γ of $A \times B$ using the following number of lines:

$$
\begin{aligned}
&\rho(A)\rho(B) + \rho(A_1)b_2 + \rho(A_2)b_1 \\
&\quad + a_{11}\rho(B_2) + \rho(A_{11})b_{22} + a_{22}\rho(B_1) + \rho(A_{22})b_{11} + \ldots \\
&\quad + a_{\underbrace{1\ldots1}_{k}}\rho(B_{\underbrace{2\ldots2}_{k-1}}) + \rho(A_{\underbrace{1\ldots1}_{k}})b_{\underbrace{2\ldots2}_{k}} + a_{\underbrace{2\ldots2}_{k}}\rho(B_{\underbrace{1\ldots1}_{k-1}}) \\
&\quad + \rho(A_{\underbrace{2\ldots2}_{k}})b_{\underbrace{1\ldots1}_{k}} \\
&\quad + \ldots
\end{aligned}
$$

Hence by Theorem 5.1 this term furnishes an upper bound for $\rho(A \times B)$.

5.3 Brualdi's Bound for Our Class of Bipartite Graphs

We are interested in the answer to the question how good Brualdi's bounds are in our class of graphs $K_{l,m}\dot{\cup}K_{p,q}$. We start with an illustrating example.

Let $\mathcal{G}$ be the bipartite graph $K_{10,3}\dot{\cup}K_{2,7}$. The biadjacency matrix A is of the form

$$
\begin{pmatrix} 1_{10,3} & 0 \\ 0 & 1_{2,7} \end{pmatrix},
$$

and Brualdi's normal forms look like

$$
\left(\begin{array}{c|c|c} 1_{2,2} & 0 & 1_{2,5} \\ \hline 0 & 1_{3,3} & 0 \\ \hline 0 & 1_{7,3} & 0 \end{array}\right),
$$

where a 0 represents a suitable matrix with all 0-entries.

We have to be aware of the fact that both normal forms - the normal form for the lower bound and that one for the upper bound - are nearly identical here: For the lower bound our decomposition starts with $A_1 = \begin{pmatrix} 1_{2,5} \\ 0 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & 1_{7,3} \end{pmatrix}$.

The decomposition for the upper bound contains submatrices $A_1 = 1_{2,5}$ and $A_2 = 1_{7,3}$. The submatrices in the decomposition for the lower bound contain some additional 0's compared to the corresponding submatrices of the other decomposition. The remaining parts are exactly the submatrices of the decomposition for the upper bound. Only that remaining part which is an all-1-matrix determines the term-rank (the 0's do not contribute), and only the term ranks of the submatrices influence the lower bound. For simplicity we work with the submatrices of the decomposition for the upper bound (because the missing 0's would not have any effect on the lower bound, if they occurred).

We observe that $A_1 = 1_{2,5}$, $A_2 = 1_{7,3}$, $\rho(A_1) = 2$, $\rho(A_2) = 3$, $a_1 = 2$ and $a_2 = 3$. Further decompositions of A_1 and A_2 provide $A_{11} = 1_{2,3}$, $A_{111} = 1_{2,1}$, $A_{1112} = 1_{1,1}$, $A_{22} = 1_{4,3}$, $A_{222} = 1_{1,3}$, $A_{2221} = 1_{1,2}$ and $A_{22211} = 1_{1,1}$.

Other submatrices determining the decomposition are empty.

According to the decomposition we obtain $a_{11} = 2$, $\rho(A_{11}) = 2$, $a_{111} = 2$, $\rho(A_{111}) = 1$, $\rho(A_{1112}) = 1$, $a_{22} = 3$, $\rho(A_{22}) = 3$, $a_{222} = 3$, $\rho(A_{222}) = 1$ and $\rho(A_{2221}) = 1$.

Brualdi's lower bound for $\rho(A \times A)$ is $\rho^2(A) + 2(\rho(A_1)\rho(A_2) + \rho(A_{11})\rho(A_{22}) + \rho(A_{111})\rho(A_{222}) + \rho(A_{1112})\rho(A_{2221})) = 25 + 2(2\cdot 3 + 2\cdot 3 + 1 + 1) = 53$, whereas the upper bound equals $\rho^2(A) + \rho(A_1)a_2 + \rho(A_2)a_1 + a_{11}(\rho(A_2) + \rho(A_{22})) + a_{22}(\rho(A_1) + \rho(A_{11})) + a_{111}(\rho(A_{22}) + \rho(A_{222})) + a_{222}(\rho(A_{11}) + \rho(A_{111})) = 25 + 2\cdot 3 + 2\cdot 2 + 2\cdot(3+3) + 3\cdot(2+2) + 2\cdot(3+1) + 3\cdot(2+1) = 78$.

The term rank for $A \times A$ (=matching number of $\mathcal{G} \otimes \mathcal{G}$) equals, according to the main part of this paper, $9 + 2\cdot 20 + 4 = 53$.

This example seems to be typical: Both normal forms are identical (except some missing 0's, which does not have any effect on the bounds) and the lower bound is sharp, whereas the upper bound leaves open a big gap compared with the exact value.

The upper bound can be very bad, as we have seen in the example above. We get a big gap if many terms occur in the sum which determines the bound. The sum for the upper bound is dependent on the fact that non-empty submatrices of the form $A_{\underbrace{1\ldots1}_{k\ times}}$ and $A_{\underbrace{2\ldots2}_{k\ times}}$ occur. We observe that this is the case for $l \geq km$ and $q \geq kp$ ($k \in \mathbb{N}$).

The fact that the lower bound is sharp had been suspected, but already Brualdi [5] had given a counterexample with two graphs, for which the lower bound is not sharp. For our special class $K_{lm}\dot{\cup}K_{p,q}$ this really is the case.

5.3.1 The Lower Bound is Sharp for $\mathbf{K_{l,m}\dot{\cup}K_{p,q}}$

For each constellation of l, m, p and q both normal forms of Brualdi for the biadjacency matrix A are nearly identical (where the submatrices occurring in the decomposition for the lower bound have some additional 0's which we drop out, because they do not influence the lower bound) .

From the main part of this paper we know that the exact value of $\rho(A \times A)$ equals $(\min(l,m))^2 + 2\min(lp, mq) + (\min(p,q))^2$.

Thus we obtain

Theorem 5.6 *For $\mathcal{G} = K_{l,m}\dot{\cup}K_{p,q}$ Brualdi's lower bound has the value*

$$(\min(l,m))^2 + 2\min(lp, mq) + (\min(p,q))^2.$$

The proof is given on page 74.

First we treat the special case $\mathbf{l \geq m}$, $\mathbf{p < q}$, because this really is the important part for proving Theorem 5.6.

From the main part of this paper we know that the exact value of $\rho(A \times A)$ equals $m^2 + 2\min(lp, mq) + p^2$.

Both normal forms are identical and they are of the form

$$\left(\begin{array}{c|c|c} 1_{p,p} & 0 & 1_{p,q-p} \\ \hline 0 & 1_{m,m} & 0 \\ \hline 0 & 1_{l-m,m} & 0 \end{array}\right).$$

Therefore $A_1 = 1_{p,q-p}$ and $A_2 = 1_{l-m,m}$.

We use Brualdi's notation for the submatrices occurring in the decomposition (cf. Theorem 5.2). Let us call the submatrices $A_{i_1 i_2 \ldots i_k}$ and $A_{j_1 j_2 \ldots j_k}$ **complementary submatrices** ($i_r + j_r = 3$ for all $r = 0, \ldots, k$).

Let us mention some important facts, which we afterwards combine and thus get the desired result:

Proposition 5.7 *For $K_{l,m}\dot{\cup}K_{p,q}$ the decomposition has the following properties:*

1. *All non-empty submatrices occurring in the decomposition are all-one- matrices of the form $1_{v,w}$ with $v, w \in \mathbb{N}$.*

2. *For $A_{i_1 \dots i_k} = 1_{v,w}$ with $v, w \in \mathbb{N}$ we obtain $A_{i_1 \dots i_k 1} = 1_{v,w-v}$ for $w > v$, $A_{i_1 \dots i_k 2} = 1_{v-w,w}$ for $w < v$ and $A_{i_1 \dots i_k 1} = A_{i_1 \dots i_k 2}$ the empty matrix for $v = w$.*

3. *The size of $A_{i_1 \dots i_k}$ with $i_1 = 1$ depends only on p and q, whereas in the case $i_1 = 2$ it depends only on l and m. We describe $A_{1 i_2 \dots i_k}$ by $1_{\alpha_k,\beta_k}$ and $A_{2 i_2 \dots i_k}$ by $1_{\gamma_k,\delta_k}$, where α_k, β_k depend on p and q and γ_k, δ_k depend on l and m.*

4. *$A_{1 i_2 \dots i_k 1}$ is not empty iff $\alpha_k < \beta_k$, $A_{1 i_2 \dots i_k 2}$ is not empty iff $\alpha_k > \beta_k$, $A_{2 i_2 \dots i_k 1}$ is not empty iff $\gamma_k < \delta_k$, $A_{2 i_2 \dots i_k 2}$ is not empty iff $\gamma_k > \delta_k$.*

5. *At least one of the submatrices $A_{i_1 \dots i_k 1}$ and $A_{i_1 \dots i_k 2}$ is empty.*

6. *The decomposition process stops at step $k+1$, i.e. it exists $k \in \mathbb{N}$, for which $A_{1 i_2 \dots i_{k+1}}$ or $A_{2 j_2 \dots j_{k+1}}$ is empty and k is minimal.*

7. *In every step $0, \dots, k$ of the decomposition we exactly obtain two complementary submatrices. In step $k+1$, when the decomposition stops, the two matrices are not complementary (otherwise the decomposition would continue).*

8. *For all $r = 1, \dots, k-1$ the sizes of the submatrices are related as follows:*

$$\alpha_r < \beta_r \text{ and } \gamma_r > \delta_r$$
$$\text{or}$$
$$\alpha_r > \beta_r \text{ and } \gamma_r < \delta_r.$$

   Moreover there exist $\Phi_r, \Upsilon_r, \Omega_r \in \mathbb{N}$, $\Psi_r \in \mathbb{N}_0$ for all $r = 1, \dots, k-1$ with the properties

$$\alpha_r = \Phi_r p - \Psi_r q, \qquad \beta_r = \Upsilon_r q - \Omega_r q$$
$$\text{and}$$
$$\beta_r = \Upsilon_r l - \Omega_r m, \qquad \delta_r = \Phi_r m - \Psi_r l.$$

In this situation we call the constellation of the indices "normal situation".

9. *In the k-th step of the decomposition the sizes of the submatrices (not complementary) are related as follows*

$$\alpha_k \le \beta_k \text{ and } \gamma_k \le \delta_k$$
$$or$$
$$\alpha_k > \beta_k \text{ and } \gamma_k > \delta_k.$$

In this situation we call the constellation of the indices "reverse situation".

Here also $\Phi_k, \Upsilon_k, \Omega_k \in \mathbb{N}$, $\Psi_k \in \mathbb{N}_0$ exist with the properties

$$\alpha_k = \Phi_k p - \Psi_k q, \qquad \beta_k = \Upsilon_k q - \Omega_k p$$
$$and$$
$$\beta_k = \Upsilon_k l - \Omega_k m, \qquad \delta_r = \Phi_k m - \Psi_k l.$$

10. *If $A_{i_1 \dots i_k}$ is not the empty matrix, then every submatrix $A_{x_1 \dots x_k x_{k+1} \dots x_{k+u}}$ is of the form $A_{i_1 \dots i_k x_{k+1} \dots x_{k+u}}$ or $A_{j_1 \dots j_k x_{k+1} \dots x_{k+u}}$ with $u \in \mathbb{N}$ and $x_i \in \{1, 2\}$ for all $i = 1, \dots, k+u$.*

Let the decomposition stop at step $k+1$. Then we get the following decomposition for A:

$A_1 = 1_{p,q-p} = 1_{\alpha_1,\beta_1}$	$A_2 = 1_{l-m,m} = 1_{\gamma_1,\delta_1}$
$A_{1i_2} = 1_{\alpha_2,\beta_2}$	$A_{2j_2} = 1_{\gamma_2,\delta_2}$
$\vdots$	$\vdots$
$A_{1i_2 \dots i_k} = 1_{\alpha_k,\beta_k}$	$A_{2j_2 \dots j_k} = 1_{\gamma_k,\delta_k}$
$A_{1i_2 \dots i_k i_{k+1}}$	$A_{2j_2 \dots j_k j_{k+1}}$

Because of Proposition 5.7 Brualdi's formula (Theorem 5.2) for the lower bound of $\rho(A \times A)$ changes into

$$\rho^2(A) + 2(\rho(A_1)\rho(A_2)$$
$$+\rho(A_{1i_2})\rho(A_{2j_2})\rho(A_{1i_2i_3})\rho(A_{2j_2j_3}) + \dots + \rho(A_{1i_2 \dots i_k})\rho(A_{2j_2 \dots j_k}))$$

where k is the the maximal number, for which $A_{1i_2 \dots i_k}$ and $A_{2j_2 \dots j_k}$ are both not empty.

We start the calculation of this sum with the term

$$\rho(A_1)\rho(A_2) + \rho(A_{1i_2})\rho(A_{2j_2})\rho(A_{1i_2i_3})\rho(A_{2j_2j_3}) + \dots$$
$$+\rho(A_{1i_2\dots i_k})\rho(A_{2j_2\dots j_k}))\ .$$

The occurring submatrices are completely described by

complementary submatrices	describing parameters
$A_{1i_2\dots i_{k-1}}$ & $A_{2j_2\dots j_{k-1}}$	α_{k-1} β_{k-1} γ_{k-1} δ_{k-1}
$A_{1i_2\dots i_k}$ & $A_{2j_2\dots j_k}$	α_k β_k γ_k δ_k

For $r = 0, \dots, k-1$ let us use the abbreviation

$$T_{k-r} \triangleq \rho(A_{1i_2\dots i_{k-r}})\rho(A_{2j_2\dots j_{k-r}}) + \dots + \rho(A_{1i_2\dots i_k})\rho(A_{2j_2\dots j_k}).$$

The main idea for the proof of Theorem 5.6 is given in

Lemma 5.8 *For all $r = 0, \dots, k-1$ we obtain*

$$T_{k-r} = \begin{cases} \alpha_{k-r}\gamma_{k-r} & \textit{iff} \quad \rho(A_{1i_2\dots i_k})\rho(A_{2j_2\dots j_k}) = \alpha_k\gamma_k\ , \\ \beta_{k-r}\delta_{k-r} & \textit{iff} \quad \rho(A_{1i_2\dots i_k})\rho(A_{2j_2\dots j_k}) = \beta_k\delta_k\ . \end{cases}$$

The value of T_{k-r} is only dependent on the value of the last term $\rho(A_{1i_2\dots i_k})\rho(A_{2j_2\dots j_k})$.

Proof of Lemma 5.8:

We prove by induction with respect to r, $r \in \{0, \dots, k-1\}$. Part I contains the statement for $r = 0$ and $r = 1$ and in Part II the induction step is done.

Part I:

For $r = 0$ the sum only consists of the term $\rho(A_{1i_2\dots i_k})\rho(A_{2j_2\dots j_k})$. For the submatrices $A_{1i_2\dots i_k}$ and $A_{2j_2\dots j_k}$ we are in the reverse situation, therefore T_k has the value $\alpha_k\gamma_k$ or $\beta_k\delta_k$.

For $r = 1$ we have to calculate the term ranks of $\rho(A_{1i_2\dots i_k})\rho(A_{2j_2\dots j_k})$ and $\rho(A_{1i_2\dots i_{k-1}})\rho(A_{2j_2\dots j_{k-1}})$. The sizes of these four matrices depend on α_{k-1}, β_{k-1}, γ_{k-1}, δ_{k-1} and α_k, β_k, γ_k, δ_k. For the sizes with index $k-1$ we have normal situation and for sizes with index k we have reverse situation. Therefore the following cases occur:

Case I: $\alpha_k \leq \beta_k$, $\gamma_k \leq \delta_k$

Subcase 1.1: $\alpha_{k-1} < \beta_{k-1}$, $\gamma_{k-1} > \delta_{k-1}$

We can conclude that $\alpha_k = \alpha_{k-1}$, $\beta_k = \beta_{k-1} - \alpha_{k-1}$, $\gamma_k = \gamma_{k-1} - \delta_{k-1}$ and $\delta_k = \delta_{k-1}$. Here we obtain

$$\begin{aligned} & T_{k-1} \\ = \ & \rho(A_{1i_2\ldots i_{k-1}})\rho(A_{2j_2\ldots j_{k-1}}) + \rho(A_{1i_2\ldots i_k})\rho(A_{2j_2\ldots j_k}) \\ = \ & \alpha_{k-1}\delta_{k-1} + \alpha_k\gamma_k \\ = \ & \alpha_{k-1}\delta_{k-1} + \alpha_{k-1}(\gamma_{k-1} - \delta_{k-1}) \\ = \ & \alpha_{k-1}\gamma_{k-1}. \end{aligned}$$

Subcase 1.2: $\alpha_{k-1} > \beta_{k-1}$, $\gamma_{k-1} < \delta_{k-1}$

We can conclude that $\alpha_k = \alpha_{k-1} - \beta_{k-1}$, $\beta_k = \beta_{k-1}$, $\gamma_k = \gamma_{k-1}$ and $\delta_k = \delta_{k-1} - \gamma_{k-1}$. Here we get

$$\begin{aligned} & T_{k-1} \\ = \ & \rho(A_{1i_2\ldots i_{k-1}})\rho(A_{2j_2\ldots j_{k-1}}) + \rho(A_{1i_2\ldots i_k})\rho(A_{2j_2\ldots j_k}) \\ = \ & \beta_{k-1}\gamma_{k-1} + \alpha_k\gamma_k \\ = \ & \beta_{k-1}\gamma_{k-1} + (\alpha_{k-1} - \beta_{k-1})\gamma_{k-1} \\ = \ & \alpha_{k-1}\gamma_{k-1}. \end{aligned}$$

In both subcases we obtain $T_{k-1} = \alpha_{k-1}\gamma_{k-1}$.

Case II: $\alpha_k > \beta_k$, $\gamma_k > \delta_k$

Subcase 2.1: $\alpha_{k-1} < \beta_{k-1}$, $\gamma_{k-1} > \delta_{k-1}$

We can conclude that $\alpha_k = \alpha_{k-1}$, $\beta_k = \beta_{k-1} - \alpha_{k-1}$, $\gamma_k = \gamma_{k-1} - \delta_{k-1}$ and $\delta_k = \delta_{k-1}$. Here we get

$$\begin{aligned} & T_{k-1} \\ = \ & \rho(A_{1i_2\ldots i_{k-1}})\rho(A_{2j_2\ldots j_{k-1}}) + \rho(A_{1i_2\ldots i_k})\rho(A_{2j_2\ldots j_k}) \\ = \ & \alpha_{k-1}\delta_{k-1} + \beta_k\delta_k \\ = \ & \alpha_{k-1}\delta_{k-1} + (\beta_{k-1} - \alpha_{k-1})\delta_{k-1} \\ = \ & \beta_{k-1}\delta_{k-1}. \end{aligned}$$

Subcase 2.2: $\alpha_{k-1} > \beta_{k-1}$, $\gamma_{k-1} < \delta_{k-1}$

We can conclude that $\alpha_k = \alpha_{k-1} - \beta_{k-1}$, $\beta_k = \beta_{k-1}$, $\gamma_k = \gamma_{k-1}$ and $\delta_k = \delta_{k-1} - \gamma_{k-1}$. Here we get

$$\begin{aligned} & T_{k-1} \\ = \ & \rho(A_{1i_2\ldots i_{k-1}})\rho(A_{2j_2\ldots j_{k-1}}) + \rho(A_{1i_2\ldots i_k})\rho(A_{2j_2\ldots j_k}) \\ = \ & \beta_{k-1}\gamma_{k-1} + \beta_k\delta_k \\ = \ & \beta_{k-1}\gamma_{k-1} + \beta_{k-1}(\delta_{k-1} - \gamma_{k-1}) \\ = \ & \beta_{k-1}\delta_{k-1}. \end{aligned}$$

In both subcases we obtain $T_{k-1} = \gamma_{k-1}\delta_{k-1}$.

Part II:

Choose $r \in \{1, \ldots, k-1\}$ and let

$$T_{k-r} = \begin{cases} \alpha_{k-r}\gamma_{k-r} & \text{for} \quad \rho(A_{1i_2\ldots i_k})\rho(A_{2j_2\ldots j_k}) = \alpha_k\gamma_k, \\ \beta_{k-r}\delta_{k-r} & \text{for} \quad \rho(A_{1i_2\ldots i_k})\rho(A_{2j_2\ldots j_k}) = \beta_k\delta_k. \end{cases}$$

We want to show that under this induction hypothesis also

$$T_{k-r-1} = \begin{cases} \alpha_{k-r-1}\gamma_{k-r-1} & \text{iff} \quad \rho(A_{1i_2\ldots i_k})\rho(A_{2j_2\ldots j_k}) = \alpha_k\gamma_k \ , \\ \beta_{k-r-1}\delta_{k-r-1} & \text{iff} \quad \rho(A_{1i_2\ldots i_k})\rho(A_{2j_2\ldots j_k}) = \beta_k\delta_k \ . \end{cases}$$

For our calculation we use the equality

$$T_{k-r-1} = \rho(A_{1i_2\ldots i_{k-r-1}})\rho(A_{2j_2\ldots j_{k-r-1}}) + T_{k-r}.$$

We also use the fact that the constellation of the indices describing the complementary matrices $A_{1i_2\ldots i_{k-r-1}}$, $A_{2j_2\ldots j_{k-r-1}}$ is normal.

<u>Case I: $T_{k-r} = \alpha_{k-r}\gamma_{k-r}$</u>

Following the induction hypothesis T_{k-r} equals $\alpha_{k-r}\gamma_{k-r}$.

<u>Subcase 1.1: $\alpha_{k-r-1} < \beta_{k-r-1}$, $\gamma_{k-r-1} > \delta_{k-r-1}$</u>

We conclude that

$\alpha_{k-r} = \alpha_{k-r-1}$, $\beta_{k-r} = \beta_{k-r-1} - \alpha_{k-r-1}$, $\gamma_{k-r} = \gamma_{k-r-1} - \delta_{k-r-1}$ and $\delta_{k-r} = \delta_{k-r-1}$.

Here we get

$$\begin{aligned} & T_{k-r-1} \\ = \; & \rho(A_{1i_2\ldots i_{k-r-1}})\rho(A_{2j_2\ldots j_{k-r-1}}) + T_{k-r} \\ = \; & \alpha_{k-r-1}\delta_{k-r-1} + \alpha_{k-r}\gamma_{k-r} \\ = \; & \alpha_{k-r-1}\delta_{k-r-1} + \alpha_{k-r-1}(\gamma_{k-r-1} - \delta_{k-r-1}) \\ = \; & \alpha_{k-r-1}\gamma_{k-r-1}. \end{aligned}$$

<u>Subcase 1.2: $\alpha_{k-r-1} > \beta_{k-r-1}$, $\gamma_{k-r-1} < \delta_{k-r-1}$</u>

We observe that

$\alpha_{k-r} = \alpha_{k-r-1} - \beta_{k-r-1}$, $\beta_{k-r} = \beta_{k-r-1}$, $\gamma_{k-r} = \gamma_{k-r-1}$ and $\delta_{k-r} = \delta_{k-r-1} - \gamma_{k-r-1}$.

It follows

$$\begin{aligned}
& T_{k-r-1} \\
= \; & \rho(A_{1i_2 \ldots i_{k-r-1}})\rho(A_{2j_2 \ldots j_{k-r-1}}) + T_{k-r} \\
= \; & \beta_{k-r-1}\gamma_{k-r-1} + \alpha_{k-r}\gamma_{k-r} \\
= \; & \beta_{k-r-1}\gamma_{k-r-1} + (\alpha_{k-r-1} - \beta_{k-r-1})\gamma_{k-r-1} \\
= \; & \alpha_{k-r-1}\gamma_{k-r-1}.
\end{aligned}$$

For both subcases of Case I we obtain $T_{k-r-1} = \alpha_{k-r-1}\gamma_{k-r-1}$.

Case II: $T_{k-r} = \beta_{k-r}\delta_{k-r}$

Following the induction hypothesis T_{k-r} equals $\beta_{k-r}\delta_{k-r}$.

Subcase 2.1: $\alpha_{k-r-1} < \beta_{k-r-1}$, $\gamma_{k-r-1} > \delta_{k-r-1}$

We can conclude that

$\alpha_{k-r} = \alpha_{k-r-1}$, $\beta_{k-r} = \beta_{k-r-1} - \alpha_{k-r-1}$, $\gamma_{k-r} = \gamma_{k-r-1} - \delta_{k-r-1}$ and $\delta_{k-r} = \delta_{k-r-1}$.

Here we obtain

$$\begin{aligned}
& T_{k-r-1} \\
= \; & \rho(A_{1i_2 \ldots i_{k-r-1}})\rho(A_{2j_2 \ldots j_{k-r-1}}) + T_{k-r} \\
= \; & \alpha_{k-r-1}\delta_{k-r-1} + \beta_{k-r}\delta_{k-r} \\
= \; & \alpha_{k-r-1}\delta_{k-r-1} + (\beta_{k-r-1} - \alpha_{k-r-1})\delta_{k-r-1} \\
= \; & \beta_{k-r-1}\delta_{k-r-1}.
\end{aligned}$$

Subcase 2.2: $\alpha_{k-r-1} > \beta_{k-r-1}$, $\gamma_{k-r-1} < \delta_{k-r-1}$

We conclude that

$\alpha_{k-r} = \alpha_{k-r-1} - \beta_{k-r-1}$, $\beta_{k-r} = \beta_{k-r-1}$, $\gamma_{k-r} = \gamma_{k-r-1}$ and $\delta_{k-r} = \delta_{k-r-1} - \gamma_{k-r-1}$.

It follows

$$\begin{aligned}
& T_{k-r-1} \\
= \; & \rho(A_{1i_2 \ldots i_{k-r-1}})\rho(A_{2j_2 \ldots j_{k-r-1}}) + T_{k-r} \\
= \; & \beta_{k-r-1}\gamma_{k-r-1} + \beta_{k-r}\delta_{k-r} \\
= \; & \beta_{k-r-1}\gamma_{k-r-1} + \beta_{k-r-1}(\delta_{k-r-1} - \gamma_{k-r-1}) \\
= \; & \beta_{k-r-1}\delta_{k-r-1}.
\end{aligned}$$

For each subcase of Case II we obtain $T_{k-r-1} = \beta_{k-r-1}\delta_{k-r-1}$. □

Corollary 5.9 *For $l \geq m$, $p < q$. For $\mathcal{G} = K_{l,m} \dot{\cup} K_{p,q}$ Brualdi's lower bound for $\rho(A \times A)$ equals $m^2 + 2\min(lp, mq) + p^2$, where A is the biadjacency matrix of $\mathcal{G}$.*

Proof of 5.9:

Brualdi's lower bound is dependent on the constellation of α_k, β_k, γ_k and δ_k.

For $\alpha_k \leq \beta_k$ and $\gamma_k \leq \delta_k$ we observe $\Phi_k p - \Psi_k q \leq \Upsilon_k q - \Omega_k p$ and $\Phi_k l - \Psi_k m \leq \Upsilon_k m - \Omega_k l$ for $\Phi_k, \Upsilon_k, \Omega_k \in \mathbb{N}$, $\Psi_k \in \mathbb{N}_0$. In this case we obtain $lp \leq qm$ and thus the lower bound is given by $m^2 + 2lp + p^2$.

For $\alpha_k > \beta_k$ and $\gamma_k > \delta_k$ we get $lp > mq$ and the lower bound $m^2 + 2mq + p^2$.

$\square$

Proof of Theorem 5.6:

For the case $l \geq m$, $p \geq q$ the normal forms of the biadjacency matrix

$$A = \begin{pmatrix} 1_{l,m} & 0_{l,q} \\ 0_{p,m} & 1_{p,q} \end{pmatrix}$$

look like

$$\left(\begin{array}{c|c} 1_{m,m} & 0 \\ \hline 0 & 1_{q,q} \\ \hline 1_{l-m,m} & 0 \\ \hline 0 & 1_{p-q,q} \end{array}\right).$$

In this case the submatrix A_1 is empty and $a_1 = 0$. Brualdi's lower and upper bound have the value $\rho^2(A) = (m+q)^2 = \rho(A \times A)$.

The second case $l < m$ and $p < q$ is symmetric to the first case. Here A_2 is empty and $a_2 = 0$. Lower and upper bound have the value $\rho^2(A) = (l+p)^2 = \rho(A \times A)$.

The third case $l \geq m$ and $p < q$ is desribed above and Corollary 5.9 gives the desired result.

The case $l < m$ and $p \geq q$ is symmetric to the third case, therefore Brualdi's lower bound equals $l^2 + \min(lp, mq) + q^2$. $\square$

5.3.2 An Example for Determining the Lower Bound

We finish by giving an example, which reflects the important facts occurring in the proof.

Let $l \geq m$ and $p < q$ and let the decomposition for A be given by:

$A_1 = 1_{\mathbf{p},\ q-p}$	$A_2 = 1_{l-m,\ \mathbf{m}}$
$A_{11} = 1_{p,\ \mathbf{q-2p}}$	$A_{22} = 1_{\mathbf{l-2m},\ m}$
$A_{112} = 1_{3p-q,\ \mathbf{q-2p}}$	$A_{221} = 1_{\mathbf{l-2m},\ 3m-l}$
$A_{1122} = 1_{\mathbf{5p-2q},\ q-2p}$	$A_{2211} = 1_{l-2m,\ \mathbf{5m-2l}}$
$A_{11221} = 1_{\mathbf{5p-2q},\ 3q-7p}$	$A_{22112} = 1_{\mathbf{3l-7m},\ 5m-2l}$
$A_{112211} = 1_{5p-2q,\ 5q-12p}$	$A_{221121} = 1_{3l-7m,\ 12m-5l}$

A bold number shall indicate the smaller index, e.g. $1_{\mathbf{p},q-p}$ indicates $p < q - p$.

The decomposition stops at step 6, because A_{112211} and A_{221121} is not a pair of complementary matrices.

Therefore in this example Brualdi's lower bound has the value

$$\begin{aligned}
& \rho^2(A) + 2(\rho(A_1)\rho(A_2) + \rho(A_{11})\rho(A_{22}) + \rho(A_{112})\rho(A_{221}) \\
& +\rho(A_{1122})\rho(A_{2211}) + \rho(A_{11221})\rho(A_{22112}) \\
= \ & (m+p)^2 + 2(pm + (q-2p)(l-2m) \\
& +(q-2p)(l-2m) + \underbrace{(5p-2q)(5m-2l) + (5p-2q)(3l-7m)}_{(5p-2q)(l-2m)}) \\
= \ & (m+p)^2 + 2(pm + (q-2p)(l-2m) \\
& + \underbrace{(q-2p)(l-2m) + (5p-2q)(l-2m)}_{(3p-q)(l-2m)}) \\
= \ & (m+p)^2 + 2(pm + \underbrace{(q-2p)(l-2m) + (3p-q)(l-2m)}_{p(l-2m)}) \\
= \ & (m+p)^2 + 2(pm + p(l-2m)) \\
= \ & (m+p)^2 + 2(p(l-m)) \\
= \ & m^2 + 2lp + p^2,
\end{aligned}$$

where $pl = \min(lp, mq)$ because of $5p - 2q < 3q - 7p$ and $3l - 7m < 5m - 2l$.

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