MODERATE DEVIATIONS FOR SELF-NORMALIZED SUMS

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ABSTRACT. For self-normalized sums, say S_n/V_n, we consider Linnik type zones, where the ratio $\mathbf{P}\left(S_n/V_n \geqslant x\right)/\left(1-\Phi(x)\right)$ converges to 1, and establish optimal bounds for remainder terms related to this convergence.

1. Introduction and results. Let $X, X_1, X_2 \dots,$ denote independent identically distributed (i.i.d.) random variables such that $\mathbf{E}\, X = 0$ and $0 < \sigma^2 = \mathbf{E}\, X^2 < \infty$. Write

$$S_n = \sum_{j=1}^{n} X_j, \qquad \text{and} \qquad V_n^2 = \sum_{j=1}^{n} X_j^2.$$

Define the self-normalized sum as S_n/V_n, if $V_n \neq 0$, and 0 if $V_n = 0$.

Let $\Lambda(n)$ denote a non-decreasing function satisfying $\Lambda(n) \to \infty$, as $n \to \infty$. The interval $[0, \Lambda(n)]$ is called zone of normal convergence if

$$\frac{1-\mathbf{P}\left(S_n < x\sigma\sqrt{n}\right)}{1-\Phi(x)} \to 1, \qquad \frac{\mathbf{P}\left(S_n < -x\sigma\sqrt{n}\right)}{\Phi(-x)} \to 1 \tag{1.1}$$

uniformly in $x \in [0, \Lambda(n)]$, where Φ denotes the standard Gaussian distribution function. It is well-known, assuming that the condition $\beta = \mathbf{E}\,|X|^3 < \infty$ holds, that the Berry–Esseen bound implies (1.1) with $\Lambda(n) = (1-\varepsilon)\sqrt{\log n}$, for $0 < \varepsilon < 1$. In

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this case the zone of normal convergence is of logarithmic size and is rather narrow. Consider the condition

$$\mathbf{E}\exp\left\{|X_1|^{4\alpha/(2\alpha+1)}\right\} < \infty \tag{1.2}$$

with some α such that $0 < \alpha \leqslant 1/6$. Linnik (1961) (see as well Ibragimov and Linnik (1971)) proved the following result. For any function $\rho(n)$ with $\lim_{n\to\infty}\rho(n) = \infty$ the relation (1.1) holds with $\Lambda(n) = n^\alpha/\rho(n)$. Furthermore, if (1.1) holds with $\Lambda(n) = n^\alpha\rho(n)$, then the condition (1.2) is fulfilled. This result of Linnik shows that for deterministically normalized sums the existence of exponential moments is a natural condition for the existence of power type zones of normal convergence.

Giné, Götze and Mason (1997) noticed that the tails of S_n/V_n are uniformly sub-Gaussian in cases when the sequence S_n/V_n is stochastically bounded. This observation suggests that Linnik type results for self-normalized sums have to hold under conditions considerably weaker than (1.2). Shao (1998) established that for self-normalized sums Linnik type results hold under the more natural condition of the existence of the third moment $\beta < \infty$. In this paper we shall obtain optimal explicit bounds for the remainders in the relations (1.1) for the self-normalized sums (see Theorem 1 and Corollary 1 below). The bounds will depend on n, x, σ, β only. In Theorems 3 and 4 we extend the results to the non-i.i.d. situation. In this more general setting the bounds depend on x and on Lyapunov's ratio L_n only.

Below we shall denote by θ different real valued quantities such that $|\theta| \leqslant 1$. We shall prove the following relations

$$\frac{\mathbf{P}\left(S_n/V_n > x\right)}{1-\Phi(x)} = 1 + c\theta r_n(x), \qquad \text{for } x \geqslant 0, \tag{1.3}$$

$$\frac{\mathbf{P}\left(S_n/V_n < x\right)}{\Phi(x)} = 1 + c\theta r_n(x), \qquad \text{for } x \leqslant 0, \tag{1.4}$$

where c denotes an absolute constant and $r_n(x)$ is a function which we have to specify. Notice that for $|x| > \sqrt{n}$ the left-hand sides of (1.3) and (1.4) are equal to zero since $\mathbf{P}\left(|S_n|/V_n > |x|\right) = 0$.

Theorem 1. *Assume that X is a symmetric random variable such that $\sigma^2 = 1$ and $\beta < \infty$. Then the relations* (1.3) *and* (1.4) *hold with*

$$r_n(x) = \begin{cases} \beta/\sqrt{n}, & \text{for } |x| \leqslant 1, \\ |x|^3\beta/\sqrt{n}, & \text{for } 1 \leqslant |x| \leqslant n^{1/6}/\beta^{1/3}, \\ 1, & \text{for } |x| > n^{1/6}/\beta^{1/3}. \end{cases}$$

Write

$$G(x,l) = \sup \mathbf{P}\left(S_n/V_n > x\right),\ x \geqslant 0, \quad \text{and} \quad G(x,l) = \sup \mathbf{P}\left(S_n/V_n < x\right),\ x \leqslant 0, \tag{1.5}$$

where the supremum is taken over all symmetric X such that the Lyapunov ratio $L_n := \beta/\big(\sigma^3\sqrt{n}\big)$ is fixed, that is, $L_n = l$. Let $\rho(l)$ be a function such that $\rho(l) \geqslant 1$. Consider the relations

$$\frac{G(x,l)}{1-\Phi(x)} = 1 + c\,\theta\Big(l + \frac{1}{\rho(l)}\Big), \qquad \text{for} \qquad x \geqslant 0, \tag{1.6}$$

$$\frac{G(x,l)}{\Phi(x)} = 1 + c\,\theta\Big(l + \frac{1}{\rho(l)}\Big), \qquad \text{for} \qquad x \leqslant 0. \tag{1.7}$$

Corollary 1. *Under the assumptions of Theorem* 1 *the relations* (1.6) *and* (1.7) *hold for* $|x| \leqslant 1/(l\rho(l))^{1/3}$.

In particular, Corollary 1 yields

$$\frac{\mathbf{P}\left(S_n/V_n > x\right)}{1-\Phi(x)} = 1 + o(1), \tag{1.8}$$

uniformly for $x = o\big(n^{1/6}\big)$. To see this it suffices to choose an appropriate function ρ.

Theorem 1 extends to the case of non-symmetric random variables as follows.

Theorem 2. *Let X be a random variable such that $\mathbf{E}\,X = 0$, $\sigma^2 = 1$, and $\beta < \infty$. Then the relations* (1.3) *and* (1.4) *hold with*

$$r_n(x) = \begin{cases} \beta/\sqrt{n}, & \textit{for} \quad |x| \leqslant 1, \\ M_{1n}(x), & \textit{for} \quad 1 < |x| \leqslant n^{1/6}/\beta^{1/3}, \\ \exp\left\{65\beta|x|^3/\sqrt{n}\right\}, & \textit{for} \quad n^{1/6}/\beta^{1/3} < |x| \leqslant \sqrt{n}/(2^{11}\beta), \\ x\exp\left\{x^2/2 - |x|\sqrt{n}/(2^{15}\beta^2)\right\}, & \textit{for} \quad |x| > \sqrt{n}/(2^{11}\beta), \end{cases}$$

where

$$M_{1n}(x) = \min\Big\{f(x, \beta/\sqrt{n}),\ \frac{1}{|x|} + \frac{\beta}{\sqrt{n}}\,|x|^3\Big\},$$

and

$$f(x,\tau) = x^2 \min_{\gamma\in[0,1/2]} \Big\{|x|\tau^{1-\gamma} + \mathbf{E}\,X^2\mathbf{I}\Big\{|X| > \frac{1}{6}\,\tau^{(1-\gamma)/4}\,\frac{\sqrt{n}}{|x|^{3/4}}\Big\}\Big\}.$$

By $\mathbf{I}\{A\}$ we denote the indicator function of the event A. Notice that the function $M_{1n}(x)$ admits the following simple estimate

$$M_{1n}(x) \leqslant \min\Big\{cx^2(|x|L_n)^{4/5},\ \ \frac{1}{|x|} + L_n|x|^3\Big\}, \ \ |x| \geqslant 1. \tag{1.9}$$

Corollary 2. *Let $G(x,l)$ be defined by* (1.5) *with the supremum taken over all X* (*eventually non-symmetric*) *such that $\mathbf{E}\,X = 0$ and $\beta/(\sigma^3\sqrt{n}) = l$. Then under the assumptions of Theorem* 2 *the following relations hold, for $|x| \leqslant 1/\big(l\rho(l)\big)^{1/3}$*,

$$\frac{G(x,l)}{1-\Phi(x)} = 1 + c\,\theta\Big(l^{1/10} + \frac{1}{\rho(l)}\Big), \qquad \text{for} \qquad x \geqslant 0, \tag{1.10}$$

$$\frac{G(x,l)}{\Phi(x)} = 1 + c\,\theta\Big(l^{1/10} + \frac{1}{\rho(l)}\Big), \qquad \text{for} \qquad x \leqslant 0. \tag{1.11}$$

Let us discuss some results for self-normalized sums related to the CLT and their refinements. Self-normalized sums satisfy the CLT provided that X belongs to the domain of attraction of a normal law (Maller (1981), Csörgő and Mason (1981)). Griffin and Mason (1991) proved that the condition is necessary in the symmetric case. Giné, Götze and Mason (1997) proved that the condition is necessary for non-symmetric X as well. These results indicate that the self-normalization may improve the limiting behavior of sums.

Berry–Esseen bounds of increasing generality and accuracy were obtained in Hall (1988), Slavova (1985), Sharakhmetov (1995), Bentkus and Götze (1996), Bentkus, Bloznelis and Götze (1996). The paper of Bentkus and Götze (1996) contains bounds which imply, in particular, sufficient conditions for the CLT. The bounds extend to the non-i.i.d. case, see Bentkus, Bloznelis and Götze (1996). The bounds of the last two papers imply our results for $|x| \leqslant c$.

Sub-Gaussian type behavior of tails related to self-normalized sums was observed in Logan, Mallows and Rice (1973). Giné, Götze and Mason (1997) showed that the tails of S_n/V_n are uniformly sub-Gaussian in the case when a sequence S_n/V_n, $n \geqslant 1$, is stochastically bounded. Shao (1997, 1998, 1999) considered large and moderate deviations. In particular, he proved (1.8) in the zone $x = o\big(n^{1/6}\big)$. Wang and Jing (1999) proved a bound for symmetric non-identically distributed random variables. Under the conditions of Theorem 1 it implies (1.3) and (1.4) in the i.i.d. case with

$$r_n(x) = \begin{cases} L_n, & \text{for} \quad |x| \leqslant 1, \\ |x|^4 L_n, & \text{for} \quad 1 \leqslant |x| \leqslant 5^{-1}L_n^{-1/3}, \\ |x|, & \text{for} \quad |x| > 5^{-1}L_n^{-1/3}. \end{cases} \tag{1.12}$$

Notice that the function $r_n(x)$ in (1.12) contains for $|x| \geqslant 1$ an extra factor $|x|$, compared to $r_n(x)$ from Theorem 1. Wang and Jing (1999) proved as well a bound for non-symmetric random variables which differs from the bound of Theorem 2 by an extra factor $\exp\big\{\varepsilon x^2\big\}$, $\varepsilon > 0$.

Now let us consider the case of non i.i.d. random variables. Henceforth $X_1, \dots, X_n$ denote independent random variables with distribution functions $F_1, \dots, F_n$ respectively. Assume that

$$\mathbf{E}\,X_j = 0, \qquad 0 < \sigma_j^2 = \mathbf{E}\,X_j^2 < \infty, \qquad \text{and} \qquad \beta_j = \mathbf{E}\,|X_j|^3$$

for all j. Write

$$B_n^2 = \sum_{j=1}^{n} \sigma_j^2, \qquad A_n = \sum_{j=1}^{n} \beta_j, \qquad L_n = A_n/B_n^3.$$

Theorem 3. *Let $X_1, \dots, X_n$ be symmetric independent random variables such that $L_n < \infty$. Then the relations* (1.3) *and* (1.4) *hold with*

$$r_n(x) = \begin{cases} L_n, & \textit{for} \quad |x| \leqslant 1, \\ |x|^3 L_n, & \textit{for} \quad 1 \leqslant |x| \leqslant L_n^{-1/3}, \\ 1, & \textit{for} \quad |x| > L_n^{-1/3}. \end{cases}$$

Corollary 3. *Let $G(x, l)$ be defined by* (1.5) *with supremum taken over all symmetric non-identically distributed random variables $X_1, \dots, X_n$ such that $L_n = l$. Then under the assumptions of Theorem* 3 *the relations* (1.6) *and* (1.7) *are valid for $|x| \leqslant 1/\big(l\rho(l)\big)^{1/3}$.*

Note that Theorem 1 is an obvious consequence of Theorem 3. We omit the evident proofs of Corollaries 1 and 3. To formulate the results in the non-symmetric case introduce the following notation. Denote

$$\widehat{A}_n = \big(\beta_1/\sigma_1\big)^2 + \dots + \big(\beta_n/\sigma_n\big)^2, \qquad \widehat{\sigma}^2 = \max_{j=1,\dots,n} \sigma_j^2, \qquad \widehat{\beta} = \max_{j=1,\dots,n} \beta_j$$

and

$$\widehat{B}_n^2 = \min_{\{l_1,\dots,l_{[n/4]}\}} \Big(\mathbf{E}\, V_n^2 - \sum_{j=1}^{[n/4]} \sigma_{l_j}^2 \Big),$$

where min is taken over all subpopulations $\{l_1, \dots, l_{[n/4]}\}$ of the population $\{1, \dots, n\}$. Introduce a parameter $\delta_n \in (0, 1/4]$ by the relation

$$\delta_n \log \frac{e}{\delta_n} = \frac{1}{4} \min\Big\{ \frac{1}{3 \cdot 2^7} \widehat{B}_n^4/(n\widehat{A}_n), 1 \Big\}. \tag{1.13}$$

Introduce also the parameters

$$R_{1n} = \frac{1}{2} \min\Big\{ B_n^2/(16\widehat{A}_n^{1/2}), B_n/(8\widehat{\beta}^{1/3}), B_n^3/(5e^2 n\widehat{\beta}), 1/(2^{10} L_n) \Big\},$$
$$R_{2n} = \min\Big\{ \widehat{B}_n^4/(2^9 \widehat{A}_n \sqrt{n}),\ \delta_n B_n/(8\widehat{\sigma}),\ \widehat{B}_n^3/(3e^3 n\widehat{\beta}) \Big\},$$

Using these parameters R_{1n} and R_{2n} we formulate our general result.

Theorem 4. *Let $X_1,\dots,X_n$ be independent random variables such that $L_n<\infty$. Then the relations* (1.3) *and* (1.4) *hold with*

$$r_n(x)=\begin{cases} L_n, & \text{for}\quad |x|\leqslant 1,\\ \min\{1,M_{2n}(x)\}, & \text{for}\quad 1<|x|\leqslant \widehat{R}_{1n},\\ \exp\big\{65L_n|x|^3\big\}, & \text{for}\quad \widehat{R}_{1n}<|x|\leqslant \max\{2,R_{1n}\},\\ x\exp\big\{x^2/2-\min\{R_{1n},R_{2n}\}x/16\big\}, & \text{for}\quad |x|>\max\{2,R_{1n}\},\end{cases}$$

where $\widehat{R}_{1n}=\max\big\{2,\min\{R_{1n},L_n^{-1/3}\}\big\}$ *and*

$$M_{2n}(x)=x^2\min_{\gamma\in[0,1/2]}\Big\{|x|L_n^{1-\gamma}+B_n^{-2}\sum_{j=1}^n \mathbf{E}\,X_j^2\,\mathbf{I}\{|X_j|>L_n^{(1-\gamma)/4}B_n/(6|x|^{3/4})\}\Big\}.$$

It is not difficult to deduce from Theorem 4

Corollary 4. *Let $G(x,l)$ be defined by* (1.5) *with the supremum taken over all random variables $X_1,\dots,X_n$ such that $R_{1n}\geqslant L_n^{-2/7}$ and $L_n=l$. Then the following relations hold, for $|x|\leqslant 1/\big(l\rho(l)\big)^{2/7}$,*

$$\frac{G(x,l)}{1-\Phi(x)}=1+c\,\theta\Big(l+\frac{1}{\big(\rho(l)\big)^{4/5}}\Big),\quad \text{for}\quad x\geqslant 0, \tag{1.14}$$

$$\frac{G(x,l)}{\Phi(x)}=1+c\,\theta\Big(l+\frac{1}{\big(\rho(l)\big)^{4/5}}\Big),\quad \text{for}\quad x\leqslant 0. \tag{1.15}$$

Notice that the zone $|x|\leqslant 1/\big(l\rho(l)\big)^{1/3}$ in Corollaries 2 and 3 is wider than the zone $|x|\leqslant 1/\big(l\rho(l)\big)^{2/7}$ of Corollary 4 if $\rho(l)\leqslant 1/l$.

Consider Student's t-statistic T_n defined by

$$T_n=\frac{1}{\sqrt{n}}\sum_{j=1}^n X_j\Big/\Big(\frac{1}{n-1}\sum_{j=1}^n (X_j-\bar{X}_n)^2\Big)^{1/2}$$

where $\bar{X}_n=S_n/n$. It is well known (see Efron, 1969) that for $x\geqslant 0$

$$\mathbf{P}\,(T_n\geqslant x)=\mathbf{P}\,\Big(\frac{S_n}{V_n}\geqslant\Big(\frac{nx^2}{n+x^2-1}\Big)^{1/2}\Big). \tag{1.16}$$

Denote $\Delta_{T_n}(x)=|\,\mathbf{P}\,(T_n<x)-\Phi(x)|$. With the help of (1.16) we obtain the following non-uniform bounds and Cramer large (moderate) deviations for T_n as an immediate consequence of Theorem 2 and 3 and (1.9).

Theorem 5. *Under the assumptions of Theorem* 3, *we have, for all positive integers* n *and real* x,

$$\Delta_{T_n}(x) \leqslant c \min\bigl\{(1+|x|^3)L_n, 1\bigr\}\Bigl(1-\Phi\Bigl(\Bigl(\frac{nx^2}{n+x^2-1}\Bigr)^{1/2}\Bigr)\Bigr).$$

Under the assumptions of Theorem 2, *we have, for all positive integers* n *and real* x *such that* $|x| \leqslant \sqrt{n}/(2^{11}\beta)$,

$$\Delta_{T_n}(x) \leqslant cM_{3n}(x)\,(1+|x|)^{-1}\exp\Bigl\{-\frac{nx^2}{2(n+x^2-1)} + 65\,\frac{\beta}{\sqrt{n}}\Bigl(\frac{nx^2}{n+x^2-1}\Bigr)^{3/2}\Bigr\},$$

where

$$M_{3n}(x) = \begin{cases} \dfrac{\beta}{\sqrt{n}}, & \text{for} \quad |x| \leqslant 1, \\ \min\bigl\{1, x^2(\beta|x|/\sqrt{n})^{4/5}, |x|^{-1} + \beta|x|^3/\sqrt{n}\bigr\}, & \text{for} \quad |x| > 1. \end{cases}$$

For $|x| > \sqrt{n}/(2^{11}\beta)$, *we have*

$$\Delta_{T_n}(x) \leqslant c \exp\Bigl\{-\frac{n|x|}{2^{15}\beta^2(n+x^2-1)^{1/2}}\Bigr\}.$$

2. Proof of Theorem 3. To prove Theorem 3 we apply the saddle-point method to the inverse formula for $\mathbf{P}\bigl(S_n/V_n < x\bigr)$ and with the help of this method obtain the optimal result in the case of symmetric random variables. Our method of the proof of Theorem 3 essentially differs from the methods used by Shao (1997, 1999), and Wang and Jing (1999).

By the Berry–Esseen inequality for self-normalized sums (see Bentkus, Bloznelis and Götze (1996)), it is sufficient to prove Theorem 3 for $x \geqslant 1$.

We have, by the inverse formula (see Petrov (1975), p. 12),

$$\mathbf{P}\Bigl(\frac{S_n}{V_n} < N\Bigr) - \mathbf{P}\Bigl(\frac{S_n}{V_n} < x\Bigr) = \lim_{T\to\infty}\frac{1}{2\pi}\int\limits_{-T}^{T}\frac{e^{-ixt}-e^{-iNt}}{it}\,\mathbf{E}\,e^{itS_n/V_n}\,dt \tag{2.1}$$

for all real x, N. Rewrite this formula, using the well-known relation

$$\mathbf{E}\,e^{itS_n/V_n} = \int\limits_{\mathbb{R}^n}\prod_{j=1}^{n}\cos\frac{tx_j}{v_n}\,dF_1(x_1)\dots dF_n(x_n) \quad (t\in\mathbb{C}), \tag{2.2}$$

where $v_n = \{x_1^2+\dots+x_n^2\}^{1/2}$ and $\mathbb{C}$ is the open complex plane. Here and in the sequel we shall assume that $0/0 = 0$. Let D denote a Borel's set of $\mathbb{R}^n$. Denote

$$I(x,N,D) = \int\limits_D \lim_{T\to\infty} \frac{1}{2\pi} \int\limits_{-T}^{T} \frac{e^{-ixt} - e^{-iNt}}{it} \prod_{j=1}^n \cos\frac{tx_j}{v_n}\, dt\, dF_1(x_1)\dots dF_n(x_n),$$

where $v_n = \{x_1^2+\dots+x_n^2\}^{1/2}$. Then with the help of (2.2) and Lebesgue's dominated convergence theorem we obtain from (2.1)

$$1 - \mathbf{P}\Big(\frac{S_n}{V_n} < x\Big) = \lim_{N\to\infty} I(x,N,\mathbb{R}^n\setminus D) + \lim_{N\to\infty} I(x,N,D). \tag{2.3}$$

The function under the integral sign on the right-hand side of (2.1) is a regular function t for all values of t and its absolute value is bounded in each strip $|\operatorname{Im} t| < a$ $(a > 0)$. Using Cauchy's theorem we obtain the formula

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{2\pi} \int\limits_{-T}^{T} \frac{e^{-ixt} - e^{-iNt}}{it} \prod_{j=1}^n \cos\frac{tx_j}{v_n}\, dt \\
&= e^{-x^2} \lim_{T\to\infty} \frac{1}{2\pi} \int\limits_{-T}^{T} \frac{e^{-ixt}}{x+it} \prod_{j=1}^n \cos\frac{(t-ix)x_j}{v_n}\, dt \\
&- ye^{-Nx} \lim_{T\to\infty} \frac{1}{2\pi} \int\limits_{-T}^{T} \frac{e^{-iNt}}{x+it} \prod_{j=1}^n \cos\frac{(t-ix)x_j}{v_n}\, dt
\end{aligned}$$

It is easy to see that

$$\sup_{(x_1,\dots,x_n)\in\mathbb{R}^n} \sup_{N\in\mathbb{R}} \Big| \lim_{T\to\infty} \int\limits_{-T}^{T} e^{-iNt} \frac{1}{x+it} \prod_{j=1}^n \cos\frac{(t-ix)x_j}{v_n}\, dt \Big| < \infty.$$

Indeed, the product of the second and third factor under the integral sign is the Fourier transform of a function f such that $0 \leqslant f(u; x_1,\dots,x_n) \leqslant A < \infty$, where A does not depend on $x_1,\dots,x_n$ and N.

¿From the last two relations we deduce the following key formula

$$\lim_{N\to\infty} I(x,N,D) = e^{-x^2} \int\limits_D \prod_{j=1}^n \cosh\frac{xx_j}{v_n} \cdot \lim_{T\to\infty} I(x,T)\, dF_1(x_1)\dots dF_n(x_n) \tag{2.4}$$

for all $x > 0$, where

$$I(x,T) = \frac{1}{2\pi} \int\limits_{-T}^{T} \frac{e^{-ixt}}{x+it} \prod_{j=1}^{n} \psi_j(t;x)\,dt$$

and $\psi_j(t;x) = \cos\{(t-ix)x_j/v_n\}/\cosh\{xx_j/v_n\}$.

In the sequel we use the formulas (2.3) and (2.4) with $D = Q$, where Q is the set of points $(x_1,\dots,x_n) \in \mathbb{R}^n$ such that

$$L = L(x_1,\dots,x_n) = \frac{|x_1|^3 + \cdots + |x_n|^3}{v_n^3} \leqslant \frac{1}{8(x+1)^3}\,. \tag{2.5}$$

To estimate $I(x,N,Q)$ we need the following lemmas.

Lemma 2.1. *Let $X_1,\dots,X_n$ be independent random variables such that $\mathbf{E}\,X_j = 0$, $\sigma_j^2 > 0$, $\beta_j < \infty$, $j = 1,\dots,n$. Then, for any $\varepsilon > 0$,*

$$\mathbf{P}\left(|V_n^2 - B_n^2| > \varepsilon B_n^2\right) \leqslant \frac{4}{\varepsilon^{3/2}}\,L_n, \tag{2.6}$$

$$\mathbf{E}\,\frac{|X_1|^3 + \cdots + |X_n|^3}{V_n^3} \leqslant 15 L_n. \tag{2.7}$$

Proof. To prove (2.6) we use the von Bahr and Esseen result (see Petrov, 1975, p. 60). Let $Y_1,\dots,Y_n$ be independent random variables with zero means and finite absolute moments of order $p\,(1 \leqslant p \leqslant 2)$. Then

$$\mathbf{E}\,\Big|\sum_{k=1}^{n} Y_k\Big|^p \leqslant \Big(2 - \frac{1}{n}\Big) \sum_{k=1}^{n} \mathbf{E}\,|Y_k|^p.$$

We obtain with the help of Chebyshev's inequality and the previous bound

$$\mathbf{P}\left(|V_n^2 - B_n^2| > \varepsilon B_n^2\right) \leqslant \left(\varepsilon B_n^2\right)^{-3/2} \mathbf{E}\,\Big|\sum_{j=1}^{n}\left(X_j^2 - \sigma_j^2\right)\Big|^{3/2}$$

$$\leqslant 2 \sum_{j=1}^{n} \left(\varepsilon B_n^2\right)^{-3/2} \mathbf{E}\left|X_j^2 - \sigma_j^2\right|^{3/2} \leqslant 4\varepsilon^{-3/2} L_n$$

and (2.6) is proved.

To prove (2.7) we first note that, by Jensen's inequality, $|X_1|^3 + \cdots + |X_n|^3 \leqslant V_n^3$. Using this bound and (2.6) with $\varepsilon = 1/2$ we obtain

$$\mathbf{E}\,V_n^{-3} \sum_{j=1}^{n} |X_j|^3 \leqslant \mathbf{E}\,\mathbf{I}\{|V_n^2 - B_n^2| > B_n^2/2\} V_n^{-3} \sum_{j=1}^{n} |X_j|^3$$

$$+\, \mathbf{E}\,\mathbf{I}\{|V_n^2 - B_n^2| \leqslant B_n^2/2\} V_n^{-3} \sum_{j=1}^{n} |X_j|^3 \leqslant 2^{7/2} L_n + 2^{3/2} B_n^{-3} \sum_{j=1}^{n} \beta_j \leqslant 15 L_n,$$

which is (2.7).

Lemma 2.2. *Let $X_1,\dots,X_n$ denote independent random variables under the assumptions of Lemma 2.1. Denote by $f_n(t)$ the characteristic function of the random variable $B_n^{-1}\sum_{j=1}^n X_j$. Then*

$$\Big|f_n(t)-e^{-t^2/2}\Big|\leqslant 16L_n|t|^3e^{-t^2/3},\qquad |t|\leqslant \frac{1}{4L_n}.$$

The proof of this lemma see Petrov (1975), p. 109–111.

Lemma 2.3. *Let X be a random variable such that $\mathbf{E}\,e^{yX}<\infty$ for $0\leqslant y\leqslant a$. Then, for $x>0$ and $0\leqslant y\leqslant a$,*

$$1-\mathbf{P}\,(X<x)=e^{-yx}\lim_{T\to\infty}\frac{1}{2\pi}\int\limits_{-T}^{T}\frac{e^{-ixt}}{y+it}\,\mathbf{E}\,e^{(y+it)X}\,dt.$$

The proof of this lemma is similar to the proof of the formula (2.4) and hence will be omitted.

Corollary 2.1. *The following formula holds, for $x>0$,*

$$1-\Phi(x)=e^{-x^2/2}\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{e^{-t^2/2}}{x+it}\,dt.$$

For the proof it suffices to apply Lemma 2.3 for the standard normal random variable X.

Lemma 2.4. *The following inequality holds*

$$\frac{1}{\sqrt{2\pi}}\Big(\frac{1}{x}-\frac{1}{x^3}\Big)e^{-x^2/2}\leqslant 1-\Phi(x)\leqslant\frac{1}{\sqrt{2\pi}x}\,e^{-x^2/2},\qquad x>0.$$

The proof of this well-known inequality can be found, for instance, in Feller (1968, p. 175).

The following result is due to Pinelis (1994, p. 359).

Lemma 2.5. *Let $X_1,\dots,X_n$ be i.i.d. random variables such that $\mathbf{P}\,(X_1=\pm1)=\frac{1}{2}$ and $y_1,\dots,y_n$ be real numbers such that $y_1^2+\dots+y_n^2=1$. Then, for any $u>0$,*

$$\mathbf{P}\,\big(|y_1X_1+\dots+y_nX_n|>u\big)<2c\big(1-\Phi(u)\big),$$

where $c=2e^3/9$.

Our next aim is to prove the statement of Theorem 3 for $1\leqslant x\leqslant L_n^{-1/3}$. Let us return to the formula (2.4) and represent the interior integral $I(x,T)$, $T>\frac{1}{10}\,L^{-1}$, in the form

$$I(x,T) = \frac{1}{2\pi}\Big(\int\limits_{-\frac12 L^{-1/3}}^{\frac12 L^{-1/3}} + \int\limits_{\frac12 L^{-1/3}\leqslant|t|\leqslant\frac{1}{10}L^{-1}} + \int\limits_{\frac{1}{10}L^{-1}<|t|\leqslant T} \Big) \frac{e^{-ixt}}{x+it} \prod_{j=1}^{n} \psi_j(t;x)\,dt$$

$$:= I_1(x) + I_2(x) + I_3(x,T). \tag{2.8}$$

By (2.5), we see that

$$\frac{1}{v_n} \max_j |x_j| \leqslant L^{1/3} \leqslant \frac{1}{2(x+1)} \tag{2.9}$$

for the points $(x_1,\dots,x_n) \in Q$. In the sequel we suppose that $(x_1,\dots,x_n) \in Q$.

Estimate now the integral $I_1(x)$, using (2.9). Note that $|\tanh y| \leqslant y$ for all real y and $|\tan y| \leqslant 2|y|$ for real y such that $|y| \leqslant 1$, and, by (2.9),

$$\max\{x,|t|\}\,\frac{|x_j|}{v_n} \leqslant \max\{x,\,\frac12 L^{-1/3}\}L^{1/3} \leqslant \frac12, \qquad 2|t|x\,\frac{x_j^2}{v_n^2} \leqslant L^{-1/3}xL^{2/3} \leqslant \frac12$$

for $t \in [-\frac12 L^{-1/3}, \frac12 L^{-1/3}]$. With the help of the series expansion for the function $\log(1+z)$, $|z|<1$, we easily deduce the following relation for the functions $\psi_j(t;x)$

$$\begin{aligned}\psi_j(t;x) &= \cos\frac{tx_j}{v_n}\Big(1 + i\tanh\frac{xx_j}{v_n}\tan\frac{tx_j}{v_n}\Big)\\ &= \cos\frac{tx_j}{v_n}\exp\Big\{i\tanh\frac{xx_j}{v_n}\tan\frac{tx_j}{v_n} + 3\theta\frac{t^2x^2x_j^4}{v_n^4} + 4i\theta\frac{t^3x^3x_j^6}{v_n^6}\Big\}\end{aligned} \tag{2.10}$$

for $t \in [-\frac12 L^{-1/3}, \frac12 L^{-1/3}]$. Recall that $\tanh y = -i\tan(iy)$. By the inequalities $|\tanh y - y| \leqslant |y|^3/2$ for all real y and $|\tan y - y| \leqslant |y|^3$ for real y such that $|y| \leqslant 1$, we have, for the same t,

$$\tanh\frac{xx_j}{v_n}\tan\frac{tx_j}{v_n} = tx\,\frac{x_j^2}{v_n^2} + \theta tx\,\frac{x_j^4}{v_n^4}\Big(t^2 + \frac12 x^2 + \frac12 t^2x^2\frac{x_j^2}{v_n^2}\Big). \tag{2.11}$$

We obtain from (2.10) and (2.11)

$$\prod_{j=1}^{n}\psi_j(t;x) = e^{f_1(t;x)+if_2(t;x)}e^{itx}\prod_{j=1}^{n}\cos\frac{tx_j}{v_n},$$

where

$$f_1(t;x) = 3\theta t^2x^2\,\frac{1}{v_n^4}\sum_{j=1}^{n}x_j^4,$$

$$f_2(t;x) = \theta tx(t^2 + \frac12 x^2)\,\frac{1}{v_n^4}\sum_{j=1}^{n}x_j^4 + \frac92\theta t^3x^3\,\frac{1}{v_n^6}\sum_{j=1}^{n}x_j^6.$$

The function $\exp\{f_1(t;x)+if_2(t;x)\}$ admits the following bound

$$\begin{aligned}\big|e^{f_1(t;x)+if_2(t;x)}-1\big| &\leqslant \big|e^{f_1(t;x)}-1\big|+2e^{f_1(t;x)}\sin^2\frac{f_2(t;x)}{2}+e^{f_1(t;x)}\big|\sin f_2(t;x)\big|\\ &\leqslant \big|e^{f_1(t;x)}-1\big|+\frac{1}{2}e^{f_1(t;x)}f_2^2(t;x)+e^{f_1(t;x)}|f_2(t;x)|.\end{aligned}$$

By (2.9), we have

$$\begin{aligned}&|f_1(t;x)|\leqslant \frac{3}{2}t^2xL\leqslant\frac{1}{4}t^2, &&\big|e^{f_1(t;x)}-1\big|\leqslant \big|f_1(t;x)\big|e^{|f_1(t;x)|}\leqslant ct^2xLe^{t^2/4},\\ &|f_2(t;x)|\leqslant c|t|(t^2+x^2)L, &&f_2^2(t;x)\leqslant c\frac{t^2(t^2+x^2)^2}{(1+x)^3}L\end{aligned}$$

for $t\in[-\frac{1}{2}L^{-1/3},\frac{1}{2}L^{-1/3}]$. Using these bounds we arrive at the inequality

$$\big|e^{f_1(t;x)+if_2(t;x)}-1\big|\leqslant c\Big(t^2x+|t|(t^2+x^2)+\frac{t^2(t^2+x^2)^2}{(1+x)^3}\Big)Le^{t^2/4} \tag{2.12}$$

for the same t. By Lemma 2.2, note that

$$\Big|\prod_{j=1}^n\cos\frac{tx_j}{v_n}-e^{-t^2/2}\Big|\leqslant 16L|t|^3e^{-t^2/3} \tag{2.13}$$

for $|t|\leqslant\frac{1}{4L^{1/3}}$. Since

$$e^{-itx}\prod_{j=1}^n\psi_j(t;x)=e^{-t^2/2}+\Big(\prod_{j=1}^n\cos\frac{tx_j}{v_n}-e^{-t^2/2}\Big)+\Big(e^{f_1(t;x)+if_2(t;x)}-1\Big)\prod_{j=1}^n\cos\frac{tx_j}{v_n},$$

we can write the integral $I_1(x)$ with the help of (2.12) and (2.13) in the form

$$I_1(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-t^2/2}}{x+it}\,dt+\theta(r_1+r_2+r_3). \tag{2.14}$$

Here the non-negative quantities r_1,r_2, and r_3 admit the bounds

$$\begin{aligned}r_1&\leqslant\frac{8L}{\pi x}\int_{-\infty}^{\infty}|t|^3e^{-t^2/3}\leqslant\frac{c}{x}L,\\ r_2&\leqslant\frac{cL}{x}\int_{-\infty}^{\infty}\Big(t^2x+|t|(t^2+x^2)+t^2(t^2+x^2)^2(1+x)^{-3}\Big)e^{-t^2/12}\,dt\leqslant cxL,\\ r_3&\leqslant\frac{1}{2\pi x}\int_{|t|>\frac{1}{2}L^{-1/3}}e^{-t^2/2}\,dt\leqslant\frac{cL}{x}.\end{aligned} \tag{2.15}$$

Let us estimate the integral $I_2(x)$. First we determine Lyapunov's ratio L_1 for the random variables Y_j, $j = 1, \dots, n$, with characteristic functions $\psi_j(t;x)$, $j = 1, \dots, n$, respectively. Since

$$\mathbf{E}\, Y_j = \frac{x_j}{v_n} \tanh \frac{x x_j}{v_n}, \qquad \operatorname{Var} Y_j = \frac{x_j^2}{v_n^2} \Big(1 - \tanh^2 \frac{x x_j}{v_n}\Big),$$
$$\mathbf{E}\, |Y_j - \mathbf{E}\, Y_j|^3 = \frac{|x_j|^3}{v_n^3} \Big(1 - \tanh^4 \frac{x x_j}{v_n}\Big),$$

we have

$$L_1 = \Big(\sum_{j=1}^n \Big(1 - \tanh^2 \frac{x x_j}{v_n}\Big) \frac{x_j^2}{v_n^2}\Big)^{-3/2} \sum_{j=1}^n \Big(1 - \tanh^4 \frac{x x_j}{v_n}\Big) \frac{|x_j|^3}{v_n^3}.$$

By (2.9), we see that

$$\frac{15}{16} L \leqslant L_1 \leqslant \Big(\frac{4}{3}\Big)^{3/2} L. \tag{2.16}$$

In addition we note that

$$1 \geqslant \sum_{j=1}^n \operatorname{Var} Y_j \geqslant 1 - \frac{x^2}{v_n^4} \sum_{j=1}^n x_j^4 = 1 - \frac{1}{2} |\theta| x L. \tag{2.17}$$

By Lemma 2.2 and (2.16), (2.17), we then obtain

$$\Big|\prod_{j=1}^n \psi(t;x)\Big| \leqslant c\big(1+t^2\big) e^{-(1-xL/2)t^2/3} \leqslant c\big(1+t^2\big) e^{-t^2/4}$$

for $|t| \leqslant \frac{1}{10} L^{-1}$. This bound yields the inequality

$$|I_2(x)| \leqslant \frac{c}{x} \int\limits_{|t| > \frac{1}{2} L^{-1/3}} \big(1+t^2\big) e^{-t^2/4}\, dt \leqslant \frac{c}{x} L. \tag{2.18}$$

Consider now the integral $I_3(x,T)$ for $T > \frac{1}{10} L^{-1}$. Represent it in the form

$$\begin{aligned} I_3(x,T) = {}& \frac{1}{2\pi} \int\limits_{\frac{1}{10} L^{-1} < |t| \leqslant T} \frac{e^{-ixt}}{it} \prod_{j=1}^n \psi_j(t;x)\, dt \\ & - \frac{x}{2\pi} \int\limits_{\frac{1}{10} L^{-1} < |t| \leqslant T} \frac{e^{-ixt}}{it(x+it)} \prod_{j=1}^n \psi_j(t;x)\, dt := I_{3,1}(x,T) + I_{3,2}(x,T). \end{aligned}$$

It is easy to see that

$$|I_{3,2}(x,T)| \leqslant \frac{x}{2\pi} \int\limits_{|t|>\frac{1}{10}L^{-1}} \frac{dt}{|t|\sqrt{x^2+t^2}} \leqslant \frac{x}{2\pi} \int\limits_{|t|>\frac{1}{10}L^{-1}} \frac{dt}{t^2} \leqslant \frac{5}{\pi} xL. \tag{2.19}$$

Rewrite $I_{3,1}(x,T)$ in the form

$$\frac{1}{2\pi}\int\limits_{-T}^{T} \frac{e^{-ixt}}{it} \prod_{j=1}^{n} \big(\psi_j(t;x) - e^{-t^2/2}\big)\,dt - \frac{1}{2\pi}\int\limits_{|t|\leqslant\frac{1}{10}L^{-1}} \frac{e^{-ixt}}{it} \prod_{j=1}^{n} \big(\psi_j(t;x) - e^{-t^2/2}\big)\,dt$$
$$+\frac{1}{2\pi}\int\limits_{\frac{1}{10}L^{-1}<|t|\leqslant T} \frac{e^{-ixt}}{it} e^{-t^2/2}\,dt := I_{3,1,1}(x,T) - I_{3,1,2}(x) + I_{3,1,3}(x,T). \tag{2.20}$$

First we note that

$$\lim_{T\to\infty} I_{3,1,1}(x,T) = \Phi(x) - \mathbf{P}\,(Y<x),$$

where $Y=\sum_{j=1}^n Y_j$ and, by the Berry–Esseen inequality and (2.16), (2.17),

$$|\lim_{T\to\infty} I_{3,1,1}(x,T)| \leqslant cL_1 + \big|\Phi(x/\sqrt{\operatorname{Var} Y}) - \Phi(x)\big| \leqslant cL.$$

By Lemma 2.2 and (2.16), (2.17), we get

$$|I_{3,1,2}(x)| \leqslant \frac{1}{2\pi}\int\limits_{|t|\leqslant\frac{1}{10}L^{-1}} \Big|\prod_{j=1}^{n}\psi_j(t;x) - e^{-\operatorname{Var} Y t^2/2}\Big| \frac{dt}{|t|}$$
$$+\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \Big|e^{-t^2/2} - e^{-\operatorname{Var} Y t^2/2}\Big| \frac{dt}{|t|} \leqslant cxL_1 \int\limits_{-\infty}^{\infty} \big(|t|+t^2\big)e^{-t^2/4}\,dt \leqslant cxL_1 \leqslant cxL.$$

In addition we have $|I_{3,1,3}(x,T)| \leqslant cL$. In view of the last three bounds, we obtain from (2.20) the inequality

$$\Big|\lim_{T\to\infty} I_{3,1}(x,T)\Big| \leqslant cxL. \tag{2.21}$$

Comparing (2.19) and (2.21) we finally deduce

$$\Big|\lim_{T\to\infty} I_{3}(x,T)\Big| \leqslant cxL. \tag{2.22}$$

Let us return now to formula (2.14). By Corollary 2.1 and (2.14), (2.15), we have

$$I_1(x) = \big(1 - \Phi(x)\big)e^{x^2/2} + \theta cxL. \tag{2.23}$$

Then, in view of (2.18), (2.22), and (2.23), we obtain from (2.8)

$$\lim_{T\to\infty} I(x,T) = \big(1 - \Phi(x)\big)e^{x^2/2} + \theta cxL. \tag{2.24}$$

Using (2.9) and the series expansion for $\log(1 + z)$, $|z| < 1$, we see that, for $(x_1, \dots, x_n) \in Q$,

$$\begin{aligned}\log\cosh\frac{xx_j}{v_n} &= \log\Big\{1 + \frac{1}{2}\left(\frac{xx_j}{v_n}\right)^2 + \frac{1}{4!}\left(\frac{xx_j}{v_n}\right)^4\Big(1 + \frac{\theta}{20}\left(\frac{xx_j}{v_n}\right)^2\Big)\Big\}\\ &= \frac{1}{2}\left(\frac{xx_j}{v_n}\right)^2 - \frac{1}{12}\left(\frac{xx_j}{v_n}\right)^4\Big(1 + 2\theta\Big(\frac{xx_j}{v_n}\Big)^2\Big).\end{aligned}$$

Using again (2.9) we obtain, for $(x_1, \dots, x_n) \in Q$,

$$\prod_{j=1}^n \cosh\frac{xx_j}{v_n} = \exp\Big\{\frac{x^2}{2} - \frac{x^4}{12}\sum_{j=1}^n\left(\frac{x_j}{v_n}\right)^4 + \frac{\theta x^6}{6}\sum_{j=1}^n\left(\frac{x_j}{v_n}\right)^6\Big\} = e^{x^2/2}\Big(1 - \frac{1}{2}\,|\theta|x^3L\Big). \tag{2.25}$$

Applying (2.24) and (2.25) to (2.4), we deduce with the help of (2.7) and Lemma 2.4 the relation

$$\begin{aligned}&\lim_{N\to\infty} I(x,N,Q)\\ &= \big(1 - \Phi(x)\big)\int\limits_Q dF_1(x_1)\dots dF_n(x_n) + c\theta x^2e^{-x^2/2}\int\limits_Q \theta L\,dF_1(x_1)\dots dF_n(x_n)\\ &= \big(1 - \Phi(x)\big)\int\limits_Q dF_1(x_1)\dots dF_n(x_n) + c\theta x^2e^{-x^2/2}L_n.\end{aligned} \tag{2.26}$$

With the help of (2.5) and (2.7) we note that

$$\int\limits_{\mathbb{R}^n\setminus Q} dF_1(x_1)\dots dF_n(x_n) \leqslant 8(1+x)^3 \int\limits_{\mathbb{R}^n\setminus Q} L\,dF_1(x_1)\dots dF_n(x_n) \leqslant cx^3L_n. \tag{2.27}$$

Then, applying (2.27) to (2.26), we arrive, in view of Lemma 2.4, at the relation

$$\lim_{N\to\infty} I(x,N,Q) = 1 - \Phi(x) + c\theta x^2e^{-x^2/2}L_n = \big(1 - \Phi(x)\big)\big(1 + c\theta x^3L_n\big). \tag{2.28}$$

To estimate $I(x,N,\mathbb{R}^n\setminus Q)$ we use Lemma 2.5 with $y_j = x_j/v_n$ and (2.1), (2.27). We have

$$|\lim_{N\to\infty} I(x,N,\mathbb{R}^n\setminus Q)| \leqslant c\big(1-\Phi(x)\big)\int\limits_{\mathbb{R}^n\setminus Q} dF_1(x_1)\dots dF_n(x_n) \leqslant cx^3L_n\big(1-\Phi(x)\big). \tag{2.29}$$

The statement of Theorem 3 for $1 \leqslant x \leqslant L_n^{-1/3}$ follows from (2.3) with $D = Q$ and (2.28), (2.29).

The statement of Theorem 3 for $x > L_n^{-1/3}$ immediately follows from Lemma 2.5 and the formulas (2.1) and (2.2).

3. Proof of Theorems 2 and 4. In this section we prove Theorem 2 and 4 and their corollaries. To this end we need some auxiliary results. We use here the notation from Section 1. Lemma 3.3 gives an upper bound for $\mathbf{P}\big(|S_n|/V_n > x\big)$ for large positive x. We will prove this lemma with the help of Lemma 3.1.

Lemma 3.1. *Let $X_1,\dots,X_n$ be random variables, satisfying the assumptions of Theorem 4. Then*

$$\mathbf{E}\,\exp\big\{yS_n/V_n\big\} \leqslant c\exp\big\{y^2/2 + 2^6 n\widehat{\beta}|y|^3/\widehat{B}_n^3\big\}, \quad y\in[-R_{2n},R_{2n}]. \tag{3.1}$$

If in addition $X_1,\dots,X_n$ are i.i.d. random variables and $\sigma_1^2 = 1$, then

$$\mathbf{E}\,\exp\big\{yS_n/V_n\big\} \leqslant c\exp\big\{y^2/2 + 2^6 L_n|y|^3\big\},\ y\in[-\sqrt{n}/(2^9\beta_1^2), \sqrt{n}/(2^9\beta_1^2)]. \tag{3.2}$$

To prove this lemma we need the following auxiliary lemma.

Lemma 3.2. *The following inequality holds*

$$\mathbf{P}\,\big(V_n^2 \leqslant B_n^2/2\big) \leqslant \exp\big\{-(3\cdot 2^7)^{-1}B_n^4\big/\widehat{A}_n\big\}.$$

Proof. Consider the truncated random variables $\widehat{X}_j^2 = X_j^2\mathbf{I}\{|X_j| \leqslant 12\beta_j/\sigma_j^2\}$ for $j = 1,\dots,n$. Let us show that $\mathbf{E}\,\widehat{X}_j^2 \geqslant 3\sigma_j^2/4$. By Chebychev's inequality, we have

$$\mathbf{P}\,(|X_j| > x) \leqslant \beta_j x^{-3}, \qquad x > 0. \tag{3.3}$$

Integrating by parts and using (3.3), we obtain the bound

$$\mathbf{E}\,X_j^2\mathbf{I}\big\{|X_j| > 12\beta_j/\sigma_j^2\big\} \leqslant \big(12\beta_j/\sigma_j^2\big)^2\,\mathbf{P}\,\big(|X_j| > 12\beta_j/\sigma_j^2\big) + 2\int\limits_{12\beta_j/\sigma_j^2}^{\infty}\mathbf{P}\,(|X_j|>x)x\,dx \leqslant \sigma_j^2/4$$

which provides the necessary bound.

Since $A \subset B$, where $A = \{X_1^2 + \cdots + X_n^2 \leqslant B_n^2/2\}$ and $B = \{\widehat{X}_1^2 + \cdots + \widehat{X}_n^2 \leqslant B_n^2/2\}$, it suffices to estimate $\mathbf{P}\,(B)$. It is easy to see that

$$\mathbf{P}\,(B) \leqslant \mathbf{P}\,\Big(\sum_{j=1}^{n}\big(\mathbf{E}\,\widehat{X}_j^2 - \widehat{X}_j^2\big) \geqslant B_n^2/4\Big) \leqslant e^{-t/4}\prod_{j=1}^{n}\mathbf{E}\,e^{tB_n^{-2}(\mathbf{E}\,\widehat{X}_j^2 - \widehat{X}_j^2)}.$$

Since

$$\begin{aligned}\mathbf{E}\,e^{tB_n^{-2}(\mathbf{E}\,\widehat{X}_j^2 - \widehat{X}_j^2)} &= e^{tB_n^{-2}\,\mathbf{E}\,\widehat{X}_j^2}\big(1 - tB_n^{-2}\,\mathbf{E}\,\widehat{X}_j^2 + \theta t^2 B_n^{-4}\,\mathbf{E}\,\widehat{X}_j^4/2\big)\\ &\leqslant \exp\big\{6t^2 B_n^{-4}\beta_j^2/\sigma_j^2\big\},\end{aligned}$$

we obtain the simple bound

$$\mathbf{P}\,(B) \leqslant \exp\big\{-t(1 - 24tB_n^{-4}\widehat{A}_n)/4\big\}.$$

Putting here $t = B_n^4/(48\widehat{A}_n)$ we arrive at the assertion of the lemma.

Proof of Lemma 3.1. Without loss of generality it suffices to prove Lemma 3.1 for $y \geqslant 0$. In the first step we establish (3.1). Represent $\mathbf{E}\,e^{yS_n/V_n}$ in the form

$$\begin{aligned}\mathbf{E}\,e^{y\frac{S_n}{V_n}} &= \mathbf{E}\prod_{j=1}^{n}\Big(\frac{yX_j}{V_n} + g\Big(\frac{yX_j}{V_n}\Big)\Big) = \mathbf{E}\sum_{k=0}^{n} y^k \sum_{\{l_1,\dots,l_k\}}\prod_{j=1}^{k}\frac{X_{l_j}}{V_n}\prod_{l\neq l_1,\dots,l_k} g\Big(\frac{yX_l}{V_n}\Big)\\ &= \mathbf{E}\Big(\sum_{k=0}^{n_1} + \sum_{k=n_1+1}^{n}\Big)y^k\sum_{\{l_1,\dots,l_k\}}\prod_{j=1}^{k}\frac{X_{l_j}}{V_n}\prod_{l\neq l_1,\dots,l_k} g\Big(\frac{yX_l}{V_n}\Big) := \mathbf{E}\,\widehat{S}_1 + \mathbf{E}\,\widehat{S}_2,\end{aligned} \tag{3.4}$$

where $g(y) = e^y - y$, $n_1 = [\delta_n n]$, and the summation in the interior sum on the right-hand side of (3.4) is taken over all subpopulations $\{l_1, \dots, l_k\}$ of size k of the population $\{1, \dots, n\}$. Recall that the parameter δ_n is defined by the relation (1.13). The function $g(y)$ admits the simple bound

$$g(y) = \sum_{m=0}^{\infty}\frac{y^{2m}}{(2m)!} + y^3\sum_{m=0}^{\infty}\frac{y^{2m}}{(2m+3)!} := g_1(y) + y^3 g_2(y) \leqslant \big(1 + y^3\big)e^{y^2/2}, \quad y \geqslant 0. \tag{3.5}$$

In addition for any event A the following inequality holds

$$\mathbf{E}\,\mathbf{I}\{A\}\exp\Big\{\frac{y}{V_n}\sum_{j=1}^{n}|X_j|\Big\} = \mathbf{E}\,\mathbf{I}\{A\}\prod_{j=1}^{n}\Big(\frac{y|X_j|}{V_n} + g\Big(\frac{y|X_j|}{V_n}\Big)\Big) \leqslant \mathbf{P}\,(A)\exp\{y\sqrt{n}\}. \tag{3.6}$$

In view of (3.6) and Lemma 3.2, we obtain, for $0 \leqslant y \leqslant B_n^4/(3\cdot 2^7 \widehat{A}_n\sqrt{n})$,

$$\mathbf{E}\,\mathbf{I}\{V_n^2 \leqslant B_n^2/2\}\exp\Big\{\frac{y}{V_n}\sum_{j=1}^n |X_j|\Big\} \leqslant \exp\big\{y\sqrt{n} - B_n^4/(3\cdot 2^7\widehat{A}_n)\big\} \leqslant 1. \qquad (3.7)$$

Since $|\widehat{S}_2| \leqslant \exp\big\{y(|X_1|\cdots+|X_n|)/V_n\big\}$, it follows from (3.7)

$$\mathbf{E}\,\mathbf{I}\{V_n^2 \leqslant B_n^2/2\}|\widehat{S}_2| \leqslant 1 \qquad (3.8)$$

for the same y. Let us estimate $\mathbf{E}\,\mathbf{I}\{V_n^2 > B_n^2/2\}|\widehat{S}_2|$. Using (3.5) we see that, for $0 \leqslant k \leqslant n$ and $y \geqslant 0$,

$$\begin{aligned} &y^k\Big|\mathbf{E}\,\mathbf{I}\{V_n^2 > B_n^2/2\}\prod_{j=1}^k \frac{X_{l_j}}{V_n}\prod_{l\neq l_1,\dots,l_k} g\Big(\frac{yX_l}{V_n}\Big)\Big| \\ &\leqslant e^{y^2/2}\Big(\frac{\sqrt{2}y}{B_n}\Big)^k\prod_{j=1}^k \mathbf{E}\,|X_{l_j}|\prod_{l\neq l_1,\dots,l_k}\mathbf{E}\,\Big(1+\frac{2\sqrt{2}|X_l|^3y^3}{B_n^3}\Big) \\ &\leqslant e^{y^2/2}\prod_{j=1}^k\frac{\sqrt{2}\sigma_{l_j}y}{B_n}\prod_{l\neq l_1,\dots,l_k}\Big(1+\frac{2\sqrt{2}\beta_l y^3}{B_n^3}\Big) \leqslant e^{y^2/2+2\sqrt{2}L_ny^3}\prod_{j=1}^k\frac{\sqrt{2}\sigma_{l_j}y}{B_n}. \end{aligned} \qquad (3.9)$$

By (3.9) and Stirling's formula, we have, for $0 \leqslant y \leqslant \delta_n B_n/(8\widehat{\sigma})$,

$$\begin{aligned} &e^{-y^2/2-2\sqrt{2}L_ny^3}\,\mathbf{E}\,\mathbf{I}\{V_n^2 > B_n^2/2\}|\widehat{S}_2| \leqslant \sum_{k=n_1+1}^n\sum_{\{l_1,\dots,l_k\}}\prod_{j=1}^k\frac{\sqrt{2}\sigma_{l_j}y}{B_n} \\ &\leqslant \sum_{k=n_1+1}^n\binom{n}{k}\Big(\frac{\sqrt{2}\widehat{\sigma}y}{B_n}\Big)^k \leqslant \sum_{k=n_1+1}^n\binom{n}{k}\Big(\frac{\delta_n}{4}\Big)^k \leqslant \sum_{k=n_1+1}^n\frac{1}{k!}\Big(\frac{n\delta_n}{4}\Big)^k \leqslant c. \end{aligned} \qquad (3.10)$$

Let us estimate $|\,\mathbf{E}\,\widehat{S}_1|$. First we introduce the random variable

$$\widehat{\mathbf{I}} = \prod_{m=0}^{n_1}\prod_{\{l_1,\dots,l_m\}}\mathbf{I}_{l_1,\dots,l_m},$$

where $\mathbf{I}_{l_1,\dots,l_m} = \mathbf{I}\Big\{\big(V_n^2-\sum_{j=1}^m X_{l_j}^2\big) > \frac{1}{2}\big(B_n^2-\sum_{j=1}^m\sigma_{l_j}^2\big)\Big\}$ and $\{l_1,\dots,l_m\}$ are all subpopulations of size m of the population $\{1,\dots,n\}$. By Lemma 3.2, we note that

$$\mathbf{E}\,\big(1-\mathbf{I}_{l_1,\dots,l_m}\big) \leqslant \exp\Big\{-\big(B_n^2-\sum_{j=1}^m\sigma_{l_j}^2\big)^2/(3\cdot 2^7\widehat{A}_n)\Big\} \leqslant \exp\Big\{-\widehat{B}_n^4/(3\cdot 2^7\widehat{A}_n)\Big\}.$$

Therefore, recalling (1.13) and using Stirling's formula, it is not difficult to show

$$\begin{aligned}\mathbf{E}\,(1-\widehat{\mathbf{I}}) &\leqslant \sum_{m=0}^{n_1}\binom{n}{m}\exp\big\{-\widehat{B}_n^4/(3\cdot 2^7\widehat{A}_n)\big\}\\ &\leqslant c\binom{n}{n_1}\exp\big\{-\widehat{B}_n^4/(3\cdot 2^7\widehat{A}_n)\big\}\leqslant c\exp\big\{-2^9\widehat{B}_n^4/\widehat{A}_n\big\}.\end{aligned}\tag{3.11}$$

We have with the help of (3.11), for $0\leqslant y\leqslant \widehat{B}_n^4/(2^9\widehat{A}_n\sqrt{n})$,

$$\begin{aligned}\mathbf{E}\,(1-\widehat{\mathbf{I}})\prod_{j=1}^{n}\Big(\tfrac{y|X_j|}{V_n}+g\Big(\tfrac{y|X_j|}{V_n}\Big)\Big) &= \mathbf{E}\,(1-\widehat{\mathbf{I}})\exp\big\{y(|X_1|+\cdots+|X_n|)/V_n\big\}\\ &\leqslant c\exp\big\{y\sqrt{n}-2^9\widehat{B}_n^4/\widehat{A}_n\big\}\leqslant 1.\end{aligned}\tag{3.12}$$

¿From (3.12) it follows

$$\mathbf{E}\,(1-\widehat{\mathbf{I}})|\widehat{S}_1|\leqslant\sum_{k=0}^{n_1}y^k\sum_{\{l_1,\dots,l_k\}}\mathbf{E}\,\big(1-\widehat{\mathbf{I}}\big)\prod_{j=1}^{k}\frac{|X_{l_j}|}{V_n}\prod_{l\neq l_1,\dots,l_k}g\Big(\tfrac{y|X_l|}{V_n}\Big)\leqslant 1\tag{3.13}$$

for the same y.

Now we estimate the expectation of every summand of $\widehat{S}_1$. For simplicity consider the summand

$$G_{1,\dots,k}(y)=\prod_{j=1}^{k}\frac{X_j}{V_n}\prod_{l=k+1}^{n}g\Big(\frac{yX_l}{V_n}\Big).$$

One can estimate the expectation of the rest summands of the sum $\widehat{S}_1$ in a similar way. Write

$$\begin{aligned}\prod_{l=k+1}^{n}g\Big(\frac{yX_l}{V_n}\Big) &= \prod_{l=k+1}^{n}\Big\{g_1\Big(\frac{yX_l}{V_n}\Big)+\Big(\frac{yX_l}{V_n}\Big)^3 g_2\Big(\frac{yX_l}{V_n}\Big)\Big\}\\ &=\sum_{q=0}^{n-k}y^{3q}\sum_{\{l_1,\dots,l_q\}}G(y;q,l_1,\dots,l_q),\end{aligned}\tag{3.14}$$

where

$$G(y;q,l_1,\dots,l_q)=\prod_{s=1}^{q}\Big(\frac{X_{l_s}}{V_n}\Big)^3 g_2\Big(\frac{yX_{l_s}}{V_n}\Big)\prod_{m\neq l_1,\dots,l_q}g_1\Big(\frac{yX_m}{V_n}\Big)$$

and the functions g_1 and g_2 are defined in (3.5). The summation in the second sum on the right-hand side of (3.14) is taken over all subpopulations $\{l_1,\dots,l_q\}$ of size q

of the population $\{k+1,\dots,n\}$. Fixing the parameter q and the set $\{l_1,\dots,l_q\}$, we get

$$G(y;q,l_1,\dots,l_q)=$$
$$\sum_{\{p_{k+1},\dots,p_n\}}\prod_{s=1}^{q}\frac{1}{(2p_{l_s}+3)!}\prod_{m\neq l_1,\dots,l_q}\frac{1}{(2p_m)!}K(q;p_{k+1},\dots,p_n)y^{2(p_{k+1}+\dots+p_n)}, \qquad (3.15)$$

where the summation is taken over all sets $\{p_{k+1},\dots,p_n\}$ of non-negative integers and
$$K(q;p_{k+1},\dots,p_n)=\prod_{s=1}^{q}\Big(\frac{X_{l_s}}{V_n}\Big)^{3+2p_{l_s}}\prod_{m\neq l_1,\dots,l_q}\Big(\frac{X_m}{V_n}\Big)^{2p_m}.$$

Following Giné, Götze and Mason (1997) apply the elementary identity

$$\int_0^\infty \lambda^{a-1}e^{-\lambda b}\,d\lambda=\frac{\Gamma(a)}{b^a}, \qquad a>0,b>0,$$

with $b=V_n^2$ and $a=a(q,k)=k/2+p_{k+1}+\dots+p_n+3q/2$, to obtain

$$\mathbf{E}\,\mathbf{I}_{1,\dots,k}\prod_{j=1}^{k}\frac{X_j}{V_n}K(q;p_{k+1},\dots,p_n)$$
$$=\frac{1}{\Gamma\big(a(q,k)\big)}\,\mathbf{E}\int_0^\infty \lambda^{a(q,k)-1}\mathbf{I}_{1,\dots,k}\prod_{j=1}^{k}X_j\prod_{s=1}^{q}X_{l_s}^{3+2p_{l_s}}\prod_{m\neq l_1,\dots,l_q}X_m^{2p_m}e^{-\lambda V_n^2}\,d\lambda \qquad (3.16)$$

in the case when $a(q,k)>0$. The order of the integrals can be reversed and the right-hand side of (3.16) can be rewritten in the form

$$\frac{1}{\Gamma\big(a(q,k)\big)}\int_0^\infty \lambda^{a(q,k)-1}\Big(\prod_{j=1}^{k}\mathbf{E}\,X_je^{-\lambda X_j^2}\Big)f(\lambda)\,d\lambda, \qquad (3.17)$$

where

$$f(\lambda)=\mathbf{E}\,\mathbf{I}_{1,\dots,k}V_n^{2a(q,k)-k}K(q;p_{k+1},\dots,p_n)\exp\big\{-\lambda V_{k,n}^2\big\}, \quad V_{k,n}^2=\sum_{j=k+1}^{n}X_j^2.$$

Since $\mathbf{E}\,X_1=\dots=\mathbf{E}\,X_n=0$, it is easy to see that

$$|\,\mathbf{E}\,X_je^{-\lambda X_j^2}\,|\leqslant\lambda\beta_j, \qquad j=1,\dots,n.$$

Applying the last estimates to (3.17) we conclude

$$\Big|\mathbf{E}\,\mathbf{I}_{1,\dots,k}\prod_{j=1}^{k}\frac{X_j}{V_n}K(q;p_{k+1},\dots,p_n)\Big|$$
$$\leqslant \prod_{j=1}^{k}\beta_j\,\mathbf{E}\,\mathbf{I}_{1,\dots,k}V_n^{2a(q,k)-k}|K(q;p_{k+1},\dots,p_n)|\int_0^\infty \lambda^{a(q,k)+k-1}e^{-\lambda V_{k,n}^2}\,d\lambda$$
$$\leqslant \prod_{j=1}^{k}\frac{\beta_j}{\widehat{B}_n^3}\,\frac{\Gamma\big(a(q,k)+k\big)}{\Gamma\big(a(q,k)\big)}\,\mathbf{E}\,\mathbf{I}_{1,\dots,k}\prod_{s=1}^{q}\Big(\frac{|X_{l_s}|}{V_{k,n}}\Big)^{3+p_{l_s}}\prod_{m\neq l_1,\dots,l_q}\Big(\frac{|X_m|}{V_{k,n}}\Big)^{2p_m}. \tag{3.18}$$

Let us return to the formulas (3.14), (3.15) and estimate the absolute value of $\mathbf{E}\,\mathbf{I}_{1,\dots,k}\prod_{j=1}^{k}\frac{X_j}{V_n}G(y;q,l_1,\dots,l_q)$ with the help of (3.18). Using Stirling's formula we derive the relation

$$\frac{\Gamma\big(a(q,k)+k\big)}{\Gamma\big(a(q,k)\big)}\leqslant c\Big(\frac{3e}{2}\Big)^k\frac{\Gamma\big(2a(q,k)+1\big)}{\Gamma\big(2a(q,k)-k+1\big)}.$$

It is not difficult to conclude, using (3.14), (3.15), and (3.18), that

$$\Big|\mathbf{E}\,\mathbf{I}_{1,\dots,k}\prod_{j=1}^{k}\frac{X_j}{V_n}\prod_{l=k+1}^{n}g\Big(\frac{yX_l}{V_n}\Big)\Big|\leqslant c\Big(\frac{3e}{2}\Big)^k\prod_{j=1}^{k}\frac{\beta_j}{\widehat{B}_n^3}\,f_{1,\dots,k}^{(k)}(y) \tag{3.19}$$

for $y\geqslant 0$, where

$$f_{1,\dots,k}(y)=y^k\,\mathbf{E}\,\mathbf{I}_{1,\dots,k}\prod_{l=k+1}^{n}g\Big(\frac{y|X_l|}{V_{k,n}}\Big).$$

Consider the function $f_{1,\dots,k}(z)$ as a function of the complex variable z. It is easy to see that $f_{1,\dots,k}(z)$ is regular for all values of z and the coefficients of the Maclaurin series of $f_{1,\dots,k}(z)$ are non-negative. Hence $M(y;f_{1,\dots,k})=f_{1,\dots,k}(y)$, where $M(y;f_{1,\dots,k})$ is the maximum modulus of $f_{1,\dots,k}(z)$ for $|z|=y$. Applying the Cauchy inequality for the derivatives of regular functions to $f(z)=f_{1,\dots,k}(y+z)$ with parameter $y\geqslant 0$, we obtain the estimate

$$f_{1,\dots,k}^{(k)}(y)\leqslant f_{1,\dots,k}(y+\rho)\rho^{-k}k! \tag{3.20}$$

for $y\geqslant 0$ and $\rho>0$. To apply (3.20) we need to estimate $f_{1,\dots,k}(y+\rho)$. Using (3.5) we obtain, for $y\geqslant 0$,

$$f_{1,\dots,k}(y)\leqslant y^k e^{y^2/2}\prod_{l=k+1}^{n}\mathbf{E}\,\Big(1+\frac{2\sqrt{2}|X_l|^3y^3}{\widehat{B}_n^3}\Big)$$
$$\leqslant y^k e^{y^2/2}\prod_{l=k+1}^{n}\mathbf{E}\,\Big(1+\frac{2\sqrt{2}\beta_l^3y^3}{\widehat{B}_n^3}\Big)\leqslant y^k e^{y^2/2+2\sqrt{2}A_ny^3/\widehat{B}_n^3}. \tag{3.21}$$

In view of (3.21) and Stirling's formula, and choosing in (3.20) $\rho = k/y$ for $0 \leqslant k \leqslant y^2$ and $\rho = y$ for $k > y^2$, we arrive at the relations, for $y > 0$,

$$f^{(k)}_{1,\dots,k}(y) \leqslant M(k,y) = \begin{cases} c(2e)^k y^{2k} e^{y^2/2+16\sqrt{2}A_n y^3/\widehat{B}_n^3}, & \text{for} \quad k \leqslant y^2, \\ 2^k e^{2y^2+16\sqrt{2}A_n y^3/\widehat{B}_n^3} k!, & \text{for} \quad k > y^2. \end{cases} \tag{3.22}$$

In addition we will use the following estimate

$$\Big|\mathbf{E}\left(1-\mathbf{I}_{1,\dots,k}\right)\prod_{j=1}^{k}\frac{X_j}{V_n}\prod_{l=k+1}^{n} g\Big(\frac{yX_l}{V_n}\Big)\Big| \leqslant \mathbf{E}\left(1-\widehat{\mathbf{I}}\right)\prod_{j=1}^{k}\frac{|X_j|}{V_n}\prod_{l=k+1}^{n} g\Big(\frac{y|X_l|}{V_n}\Big). \tag{3.23}$$

The inequalities (3.19), (3.22), and (3.23) provide a bound of $|\mathbf{E}\, G_{1,\dots,k}(y)|$. In a similar way one obtains the same bound for the expectation of the remaining summands of $\widehat{S}_1$. In view of (3.13), (3.23) and (3.19), (3.22), we obtain

$$|\mathbf{E}\,\widehat{S}_1| \leqslant 1 + \Big(\sum_{k=0}^{[y^2]} + \sum_{k=[y^2]+1}^{n_1}\Big)\binom{n}{k} c\Big(\frac{3ey}{2}\Big)^k\Big(\frac{\widehat{\beta}}{\widehat{B}_n^3}\Big)^k M(k,y) := 1 + \textstyle\sum_1 + \sum_2 \tag{3.24}$$

for $0 \leqslant y \leqslant \widehat{B}_n^4/(2^9\widehat{A}_n\sqrt{n})$. By (3.22), we deduce, for $y > 0$,

$$\begin{aligned} \textstyle\sum_1 &\leqslant c e^{\frac12 y^2+16\sqrt{2}A_n y^3/\widehat{B}_n^3} \sum_{k=0}^{[y^2]} \binom{n}{k}\Big(\frac{3e^2\widehat{\beta}y^3}{\widehat{B}_n^3}\Big)^k \\ &\leqslant c e^{\frac12 y^2+16\sqrt{2}A_n y^3/\widehat{B}_n^3} \sum_{k=0}^{[y^2]} \frac{1}{k!}\Big(\frac{3e^2 n\widehat{\beta}y^3}{\widehat{B}_n^3}\Big)^k \leqslant c e^{\frac12 y^2+2^6 n\widehat{\beta}y^3/\widehat{B}_n^3}. \end{aligned} \tag{3.25}$$

By (3.22), we have, for $0 < y \leqslant \widehat{B}_n^3/(3e^3 n\widehat{\beta})$,

$$\begin{aligned} \textstyle\sum_2 &\leqslant e^{2y^2+16\sqrt{2}A_n y^3/\widehat{B}_n^3 y^3} \sum_{k=[y^2]+1}^{n_1} \Big(\frac{3en\widehat{\beta}y}{\widehat{B}_n^3}\Big)^k \\ &\leqslant c e^{2y^2+16\sqrt{2}A_n y^3/\widehat{B}_n^3}\Big(\frac{3en\widehat{\beta}y}{\widehat{B}_n^3}\Big)^{[y^2]+1} \leqslant c e^{16\sqrt{2}A_n y^3/\widehat{B}_n^3}. \end{aligned} \tag{3.26}$$

Finally the assertion (3.1) of the lemma immediately follows from (3.8), (3.10), and (3.24)–(3.26).

Let us prove (3.2), repeating the arguments used above. In the case of i.i.d. $X_1, \dots, X_n$ the formula (3.4) has the form

$$\mathbf{E}\, e^{yS_n/V_n} = \mathbf{E}\Big(\sum_{k=0}^{n_1} + \sum_{n_1+1}^{n}\Big)\binom{n}{k}\prod_{j=1}^{k}\frac{X_j}{V_n}\prod_{l\neq 1,\dots,k} g\Big(\frac{yX_l}{V_n}\Big) = \mathbf{E}\,\widehat{S}_1 + \mathbf{E}\,\widehat{S}_2.$$

We assume that $n_1 = [n/4]$. Repeating the argument which we used to prove (3.8) and (3.10), we see that these relations remain true for $0 \leqslant y \leqslant 2^{-9}\beta_1^{-2}\sqrt{n}$. To estimate $|\,\mathbf{E}\,\widehat{S}_1|$ we introduce the random variables $\widehat{\mathbf{I}} = \mathbf{I}_{1,\dots,n_1}$ and obtain the bound (3.19) for $|\,\mathbf{E}\,\widehat{\mathbf{I}}G_{1,\dots,k}(y)|$ and (3.23) for $|\,\mathbf{E}\,(1-\widehat{\mathbf{I}})G_{1,\dots,k}(y)|$ in the same way, as in the proof of (3.19) and (3.23) for $|\,\mathbf{E}\,\mathbf{I}_{1,\dots,k}G_{1,\dots,k}(y)|$ and $|\,\mathbf{E}\,(1-\mathbf{I}_{1,\dots,k})G_{1,\dots,k}(y)|$ respectively. Then we conclude in the same way, as in the proof of (3.24)–(3.26), that (3.24)–(3.26) hold for $0 < y \leqslant 2^{-9}\beta_1^{-2}\sqrt{n}$. The assertion (3.2) of the lemma follows now from (3.8), (3.10) and (3.24)–(3.26).

As a simple consequence of Lemma 3.1 we obtain

Lemma 3.3. *Under the assumptions of Theorem 4, the following bound holds*

$$\mathbf{P}\left(|S_n|/V_n \geqslant x\right) \leqslant \begin{cases} c\exp\left\{-x^2/2 + 2^6 n\widehat{\beta}x^3/\widehat{B}_n^3\right\}, & \textit{for} \quad 0 \leqslant x \leqslant R_{2n}, \\ c\exp\left\{-Rx/16\right\}, & \textit{for} \quad x > R, \ \ 0 < R \leqslant R_{2n}. \end{cases} \tag{3.27}$$

If in addition $X_1, \dots, X_n$ are i.i.d. and $\sigma_1^2 = 1$, then (3.27) *holds with the parameter $R_{2n} = \sqrt{n}/(2^9\beta_1^2)$.*

Proof. Using Lemma 3.1 we see that the assertion of the lemma for $0 \leqslant x \leqslant R_{2n}$ follows from the chain of the inequalities

$$\begin{aligned} &\mathbf{P}\left(|S_n|/V_n \geqslant x\right) \leqslant e^{-x^2}\,\mathbf{E}\,e^{x|S_n|/V_n}\mathbf{I}\{|S_n|/V_n \geqslant x\} \\ &\leqslant c\exp\left\{-x^2 + x^2/2 + 2^6 n\widehat{\beta}x^3/\widehat{B}_n^3\right\} = c\exp\left\{-x^2/2 + 2^6 n\widehat{\beta}x^3/\widehat{B}_n^3\right\}. \end{aligned}$$

For all $R \in (0, R_{2n}]$ and real x such that $x > R$ we have, by Lemma 3.1,

$$\begin{aligned} \mathbf{P}\left(|S_n|/V_n \geqslant x\right) &\leqslant e^{-Rx/8}\,\mathbf{E}\,e^{R|S_n|/(8V_n)} \leqslant e^{-Rx/16}e^{-R^2(1/2-1/16-n\widehat{\beta}R/\widehat{B}_n^3)/8} \\ &\leqslant e^{-Rx/16}e^{-R^2(1/2-1/16-1/(3e^3))/8} \leqslant e^{-Rx/16}. \end{aligned}$$

¿From the last two estimates we obtain (3.27). In order to prove the assertion of the lemma in the case of i.i.d. random variables we should repeat the argument above, applying Lemma 3.1 to this case.

Let $\gamma \in [0, 1/2]$. Introduce the random variables

$$Y_j = X_j\mathbf{I}\{|X_j| \leqslant B_n/x\}, \ \ j = 1, \dots, n, \quad S_n^* = \sum_{j=1}^n Y_j, \quad V_n^{*2} = \sum_{j=1}^n Y_j^2$$

$$Z_j = X_j\mathbf{I}\Big\{|X_j| \leqslant \frac{1}{6}\,L_n^{(1-\gamma)/4}B_n x^{-3/4}\Big\}, \ \ j = 1, \dots, n, \quad \widehat{V}_n^2 = \sum_{j=1}^n Z_j^2.$$

We shall need the following lemmas.

Lemma 3.4. *The following bound is valid*

$$\mathbf{P}\left(S_n \geqslant xV_n, |X_k| \geqslant y\right) \leqslant \mathbf{P}\left(|X_k| \geqslant y\right)\mathbf{P}\left(S_n - X_k \geqslant x\sqrt{1-x^{-2}}\sqrt{V_n^2 - X_k^2}\right).$$

for all $x \geqslant 1$, $y \geqslant 0$ *and* $1 \leqslant k \leqslant n$.

Lemma 3.4 is Lemma 4 of Wang and Jing (1999).

Lemma 3.5. *Let* $\gamma \in [0,1/2]$. *For* $6^4 \leqslant x \leqslant L_n^{-1/3}$, *the following inequality holds*

$$\mathbf{P}\left(|\widehat{V}_n^2 - B_n^2| \geqslant L_n^{(1-\gamma)/2} B_n^2 x^{1/2}\right) \leqslant cL_n^2 \exp\{-x^2/2\}.$$

Proof. Let us prove that

$$\mathbf{P}\left(\widehat{V}_n^2 - B_n^2 \leqslant -L_n^{(1-\gamma)/2} B_n^2 x^{1/2}\right) \leqslant cL_n^2 \exp\{-x^2/2\}. \tag{3.28}$$

We see, using (3.3), that, for $6^4 \leqslant x \leqslant L_n^{-1/3}$,

$$\sum_{j=1}^n \mathbf{E}\, X_j^2 \mathbf{I}\{|X_j| > \frac{1}{6} L_n^{(1-\gamma)/4} B_n x^{-3/4}\} \leqslant 18 L_n^{(3+\gamma)/4} B_n^2 x^{3/4} \leqslant \frac{1}{2} L_n^{(1-\gamma)/2} B_n^2 x^{1/2}. \tag{3.29}$$

Using Chebyshev's inequality and (3.29) we have, for any $t > 0$,

$$\begin{aligned}\mathbf{P}\Big(\sum_{j=1}^n Z_j^2 - B_n^2 \leqslant -L_n^{(1-\gamma)/2} B_n^2 x^{1/2}\Big) &\leqslant \mathbf{P}\Big(\sum_{j=1}^n \big(\mathbf{E}\, Z_j^2 - Z_j^2\big) \geqslant \frac{1}{2} L_n^{(1-\gamma)/2} B_n^2 x^{1/2}\Big)\\ &\leqslant e^{-\frac{1}{2} t L_n^{(1-\gamma)/2} x^{1/2}} \prod_{j=1}^n \mathbf{E}\, e^{tB_n^{-2}(\mathbf{E}\, Z_j^2 - Z_j^2)}.\end{aligned} \tag{3.30}$$

Taking into account the bound $|Z_j| \leqslant \frac{1}{6} L_n^{(1-\gamma)/4} B_n x^{-3/4}$ we get

$$\begin{aligned}\mathbf{E}\, e^{-tB_n^{-2} Z_j^2} &\leqslant \Big(1 - tB_n^{-2}\, \mathbf{E}\, Z_j^2 + \frac{1}{2} t^2 B_n^{-4}\, \mathbf{E}\, Z_j^4 e^{tL_n^{(1-\gamma)/2}/(36x^{3/2})}\Big)\\ &\leqslant \exp\Big\{-tB_n^{-2}\, \mathbf{E}\, Z_j^2 + \frac{1}{12} t^2 L_n^{(1-\gamma)/4} x^{-3/4} B_n^{-3} \beta_j e^{tL_n^{(1-\gamma)/2}/(36x^{3/2})}\Big\}.\end{aligned} \tag{3.31}$$

¿From (3.31) it follows that

$$\prod_{j=1}^n \mathbf{E}\, e^{tB_n^{-2}(\mathbf{E}\, Z_j^2 - Z_j^2)} \leqslant \exp\Big\{\frac{1}{12} t^2 x^{-3/4} L_n^{(5-\gamma)/4} e^{tL_n^{(1-\gamma)/2}/(36x^{3/2})}\Big\}. \tag{3.32}$$

Choosing in (3.32) $t = 2L_n^{-(1-\gamma)/2} x^{3/2}(1 - 4x^{-2}\log L_n)$ and applying the resulting bound to (3.30) with the same t, we obtain the inequality (3.28).

In the same way we prove the inequality

$$\mathbf{P}\left(\widehat{V}_n^2 - B_n^2 \geqslant \frac{1}{6} L_n^{(1-\gamma)/2} B_n^2 x^{1/2}\right) \leqslant cL_n^2 \exp\{-x^2/2\}. \tag{3.33}$$

The inequalities (3.28) and (3.33) give the assertion of the lemma.

Lemma 3.6. *Let $X_1, \dots, X_n$ satisfy the assumptions of Theorem 4. Suppose that f is a real-valued Borel function such that the random variables $W_1 = f(X_1), \dots, W_n = f(X_n)$ satisfy the conditions*

$$\Bigl| e^{yW_j/B_n} - 1 - \frac{y^2\sigma_j^2}{2B_n^2} \Bigr| \leqslant \frac{bx^3\beta_j}{B_n^3}; \tag{3.34}$$

$$\Bigl| \mathbf{E}\, W_j e^{yW_j/B_n} - \frac{y\sigma_j^2}{B_n} \Bigr| \leqslant \frac{bx^2\beta_j}{B_n^2}, \qquad \Bigl| \mathbf{E}\, W_j^2 e^{yW_j/B_n} - \sigma_j^2 \Bigr| \leqslant \frac{bx\beta_j}{B_n},$$
$$\mathbf{E}\, |W_j|^3 e^{yW_j/B_n} \leqslant b\beta_j, \tag{3.35}$$

for $y \in [9x/10, 11x/10]$ and all j, where $b \geqslant 1$, and

$$1 \leqslant x \leqslant x_1 = \min\bigl\{ B_n/(2b^{1/3}\widehat{\beta}^{1/3}), \bigl(2^4 bL_n\bigr)^{-1} \bigr\}.$$

Then

$$\mathbf{P}\Bigl(\sum_{j=1}^n W_j \geqslant yB_n \Bigr) = \bigl(1 - \Phi(y)\bigr) e^{\theta b L_n x^3} \bigl(1 + r(y)\bigr), \qquad y \in [9x/10, 11x/10],$$

where $|r(y)| \leqslant cb\bigl(x^3 + b^3 L_n^{1/3}\bigr) L_n$.

Note that the parameter x_1 satisfies $x_1 \geqslant L_n^{-1/3}/(2b^{1/3})$ if $L_n \leqslant \bigl(2^{9/2} b\bigr)^{-1}$.

Proof. To prove the lemma we use the same idea as in the proof of Theorem 3. Without loss of generality, we assume that $L_n \leqslant c$, where c is a sufficiently small absolute constant. By (3.34), $\mathbf{E}\, e^{yW_j/B_n} < \infty$ for $0 \leqslant y \leqslant 11x/10$ and all j. Therefore the same property holds for the sum $\mathbf{S}_n = \bigl(W_1 + \dots + W_n\bigr)/B_n$. Using Lemma 2.3 we obtain the formula

$$1 - \mathbf{P}\bigl(\mathbf{S}_n < y\bigr) = e^{-y^2} \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-iyt}}{y + it} \prod_{j=1}^n \mathbf{E}\, e^{(y+it)W_j/B_n}\, dt, \quad 0 \leqslant y \leqslant 11x/10. \tag{3.36}$$

Consider independent random variables $\widehat{W}_1, \dots, \widehat{W}_n$ with characteristic functions $\varphi(t; \widehat{W}_1), \dots, \varphi(t; \widehat{W}_n)$ respectively given by

$$\varphi(t; \widehat{W}_j) = \mathbf{E}\, e^{(y+it)W_j/B_n} \big/ \mathbf{E}\, e^{yW_j/B_n}.$$

In the sequel we denote

$$\widehat{a}_j = \mathbf{E}\, \widehat{W}_j, \ \widehat{\sigma}_j^2 = \operatorname{Var} \widehat{W}_j, \ \widehat{\beta}_j = \mathbf{E}\, |\widehat{W}_j - \widehat{a}_j|^3, \ \mathfrak{B}_n^2 = \widehat{\sigma}_1^2 + \dots + \widehat{\sigma}_n^2.$$

Let $\mathfrak{L}_n$ be Lyapunov's ratio for the random variables $\widehat{W}_1, \dots, \widehat{W}_n$.

In the next step we calculate $\widehat{a}_j$, $\widehat{\sigma}_j^2$, and $\widehat{\beta}_j$. Using (3.34), the condition $x \in [1, x_1]$, and the well-known inequality

$$\sigma_j^2/B_n^2 \leqslant \beta_j^{2/3}/B_n^2 \leqslant L_n^{2/3}, \tag{3.37}$$

we see that

$$\mathbf{E}\, e^{yW_j/B_n} = 1 + \frac{y^2\sigma_j^2}{2B_n^2} + \frac{1}{6}\theta b\,\frac{x^3\beta_j}{B_n^3} = 1 + r_1(y;x,n),$$

where

$$|r_1(y;x,n)| \leqslant \frac{1}{2}\Big(\frac{11x\beta_j^{1/3}}{10B_n}\Big)^2 + \frac{b}{6}\Big(\frac{x\beta_j^{1/3}}{B_n}\Big)^3 \leqslant \frac{1}{2}\Big(\frac{11}{20}\Big)^2 + \frac{1}{6}\Big(\frac{1}{2}\Big)^3 \leqslant \frac{1}{5} \tag{3.38}$$

for $1 \leqslant y \leqslant 11x/10$. In view of (3.37), (3.38), and the condition $x \in [1, x_1]$, we conclude, using the series expansion for $\log(1+t)$, $|t| < 1$, that

$$\log \mathbf{E}\, e^{yW_j/B_n} = \frac{y^2\sigma_j^2}{2B_n^2} + \frac{\theta}{6}\,\frac{x^3\beta_j}{B_n^3} + \frac{5\theta}{4}\Big(\frac{y^2\sigma_j^2}{2B_n^2} + \frac{\theta}{6}\,\frac{x^3\beta_j}{B_n^3}\Big)^2 = \frac{y^2\sigma_j^2}{2B_n^2} + \theta b\,\frac{x^3\beta_j}{B_n^3} \tag{3.39}$$

for $1 \leqslant y \leqslant 11x/10$. In the same way we get

$$\Big(\mathbf{E}\, e^{yW_j/B_n}\Big)^{-1} = 1 - \frac{y^2\sigma_j^2}{2B_n^2} + +\theta b\,\frac{x^3\beta_j}{B_n^3}. \tag{3.40}$$

¿From (3.39) it follows

$$\prod_{j=1}^n \mathbf{E}\, e^{yW_j/B_n} = \exp\Big\{\frac{1}{2}y^2 + \theta b x^3 L_n\Big\}, \quad 1 \leqslant y \leqslant 11x/10. \tag{3.41}$$

The following two relations (3.42) and (3.43) we verify for $1 \leqslant y \leqslant 11x/10$, using (3.35), (3.37), (3.40), and the condition $x \in [1, x_1]$. As the first step we easily obtain

$$\widehat{a}_j = \frac{\mathbf{E}\,(W_j/B_n)e^{yW_j/B_n}}{\mathbf{E}\, e^{yW_j/B_n}} = \frac{y\sigma_j^2}{B_n^2} + 2\theta b\,\frac{x^2\beta_j}{B_n^3}. \tag{3.42}$$

Then, taking into account (3.42), we deduce that

$$\widehat{\sigma}_j^2 = \frac{\mathbf{E}\,(W_j/B_n)^2 e^{yW_j/B_n}}{\mathbf{E}\, e^{yW_j/B_n}} - \widehat{a}_j^2 = \frac{\sigma_j^2}{B_n^2} + 5\theta b\,\frac{x\beta_j}{B_n^3}. \tag{3.43}$$

At last, using the well-known inequality $\widehat{\beta}_j \leqslant 8\,\mathbf{E}\,|\widehat{W}_j|^3$ and (3.35), (3.38), we see that

$$\widehat{\beta}_j \leqslant 8\,\frac{\mathbf{E}\,|W_j/B_n|^3 e^{yW_j/B_n}}{\mathbf{E}\,e^{yW_j/B_n}} \leqslant 10b\,\frac{\beta_j}{B_n^3}\,. \tag{3.44}$$

By (3.43), we first note that, for $1 \leqslant x \leqslant x_1$,

$$(1/2)^{2/3} \leqslant \mathfrak{B}_n^2 = 1 + 5\theta bxL_n \leqslant 21/16 \tag{3.45}$$

and then with the help of (3.44) and (3.45)

$$\mathfrak{L}_n \leqslant 20bL_n. \tag{3.46}$$

Let $f_j(t) = \varphi(t;\widehat{W}_j)e^{-i\widehat{a}_j t}$, $j = 1,\dots,n$. Using the well-known inequality

$$\bigl|f_j(t) - 1\bigr| \leqslant \frac{1}{2}\,\widehat{\sigma}_j^2 t^2 \leqslant \frac{1}{2}\,\widehat{\beta}_j^{2/3} t^2 \leqslant \frac{1}{2}\,\mathfrak{L}_n^{2/3}\mathfrak{B}_n^2 t^2, \qquad t \in \mathbb{R}, \tag{3.47}$$

and (3.45), we get the expansion

$$\log f_j(t) = -\sum_{k=1}^{\infty} \frac{1}{k}\bigl(1 - f_j(t)\bigr)^k$$

for $|t| \leqslant \mathfrak{L}_n^{-1/3}$ and $j = 1,\dots,n$. This expansion and the bounds (3.45), (3.47) permit to show, for $|t| \leqslant \mathfrak{L}_n^{-1/3}$, that

$$\begin{aligned}\sum_{j=1}^{n}\bigl|\log f_j(t) + 1 - f_j(t)\bigr| &\leqslant \sum_{j=1}^{n}\sum_{k=2}^{\infty}\frac{1}{k}\Bigl(\frac{1}{2}\widehat{\sigma}_j^2 t^2\Bigr)^k \leqslant \frac{3}{8}\sum_{j=1}^{n}\widehat{\sigma}_j^4 t^4\\ &\leqslant \frac{3}{8}\,t^4\bigl(\max_m \widehat{\sigma}_m\bigr)\sum_{j=1}^{n}\widehat{\beta}_j \leqslant \frac{3}{8}\,\mathfrak{L}_n^{4/3}\mathfrak{B}_n^4 t^4.\end{aligned} \tag{3.48}$$

We need as well the following well-known bound

$$\sum_{j=1}^{n}\bigl|f_j(t) - 1 + \frac{1}{2}\,\widehat{\sigma}_j^2 t^2\bigr| \leqslant \frac{1}{6}\,|t|^3\sum_{j=1}^{n}\widehat{\beta}_j = \frac{1}{6}\,\mathfrak{L}_n\mathfrak{B}_n^3|t|^3, \qquad t \in \mathbb{R}.$$

Applying this bound to (3.48) we easily get with the help of (3.45)

$$\Bigl|\sum_{j=1}^{n}\log f_j(t) + \frac{1}{2}\,\mathfrak{B}_n^2 t^2\Bigr| \leqslant \mathfrak{L}_n|t|^3, \qquad |t| \leqslant \mathfrak{L}_n^{-1/3}. \tag{3.49}$$

Let us return to the formula (3.36) and rewrite it in the form

$$1-\ \mathbf{P}\big(\mathbf{S}_n<y\big)=e^{-y^2}\prod_{j=1}^{n}\mathbf{E}\,e^{yW_j/B_n}\big(\widehat{I}_1+\widehat{I}_2+\widehat{I}_3\big),\tag{3.50}$$

where

$$\widehat{I}_1=\frac{1}{2\pi}\int\limits_{-(3bL_n^{1/3})^{-1}}^{(3bL_n^{1/3})^{-1}}\frac{G(t;x)}{y+it}\,dt,\qquad \widehat{I}_2=\frac{1}{2\pi}\int\limits_{(3bL_n^{1/3})^{-1}\leqslant|t|\leqslant(2^7bL_n)^{-1}}\frac{G(t;x)}{y+it}\,dt,$$

$$\widehat{I}_3=\lim_{T\to\infty}\frac{1}{2\pi}\int\limits_{(2^7bL_n)^{-1}<|t|\leqslant T}\frac{G(t;x)}{y+it}\,dt,\qquad G(t;x)=e^{-iyt}\prod_{j=1}^{n}\varphi(t;\widehat{W}_j).$$

Using (3.42), (3.45), (3.46), (3.49), and the inequality $\big|e^z-1\big|\leqslant|z|e^{|\operatorname{Re}z|}$ $(z\in\mathbb{C})$, we see that, for $|t|\leqslant L_n^{-1/3}/(3b)$ and $1\leqslant x\leqslant x_1$,

$$\begin{aligned}
&\Big|G(t;x)-\exp\Big\{-\frac{1}{2}t^2\Big\}\Big|\\
&=\Big|\exp\Big\{-\frac{1}{2}t^2+bL_n\big(i2\theta x^2t+5\theta xt^2+20\theta|t|^3\big)\Big\}-\exp\Big\{-\frac{1}{2}t^2\Big\}\Big|\\
&\leqslant cbL_n\big(x^2|t|+xt^2+|t|^3\big)\exp\Big\{-\frac{1}{2}t^2+bL_nt^2\big(5x+20|t|\big)\Big\}\\
&\leqslant cbx^2L_n(|t|+|t|^3)\exp\Big\{-\Big(\frac{1}{2}-\frac{5}{16}-\frac{1}{16}\Big)t^2\Big\}=cbx^2L_n(|t|+|t|^3)\exp\Big\{-\frac{1}{8}t^2\Big\}.
\end{aligned}$$

With the help of this bound we easily obtain, for $9x/10\leqslant y\leqslant 11x/10$,

$$\begin{aligned}
\widehat{I}_1&=\frac{1}{2\pi}\int\limits_{-(3bL_n^{1/3})^{-1}}^{(3bL_n^{1/3})^{-1}}\frac{e^{-t^2/2}}{y+it}\,dt+c\theta bx^2L_n\\
&=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{e^{-t^2/2}}{y+it}\,dt+c\theta bx^2L_n+\frac{c\theta}{x}\,e^{-(18b^2L_n^{2/3})^{-1}}\\
&=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{e^{-t^2/2}}{y+it}\,dt+cb\theta x^2\big(1+b^3L_n^{1/3}/x^3\big)L_n.
\end{aligned}\tag{3.51}$$

By Lemma 2.2, (3.45), (3.46), and the condition $1\leqslant x\leqslant x_1$,

$$\begin{aligned}
\Big|\prod_{j=1}^{n}\varphi(t;\widehat{W}_j)\Big|&\leqslant\big(cbxL_nt^3+1\big)\exp\big\{-t^2\big(1-5bxL_n\big)/3\big\}\\
&\leqslant c\big(|t|^3+1)\exp\big\{-(1-5/16)t^2/3\big\}\leqslant c\big(|t|^3+1\big)\exp\big\{-t^2/5\big\}
\end{aligned}$$

for $|t| \leqslant \left(2^7 b L_n\right)^{-1}$, and we have

$$\left|\widehat{I}_2\right| \leqslant \frac{c}{x} \int\limits_{|t|>(3bL_n^{1/3})^{-1}} |t|^3 e^{-t^2/5}\, dt \leqslant \frac{c}{x}\, e^{-(50b^2 L_n^{2/3})^{-1}} \leqslant \frac{c}{x}\, b^4 L_n^{4/3}. \tag{3.52}$$

It remains to estimate $\widehat{I}_3$. We see that

$$\begin{aligned}\widehat{I}_3 = \lim_{T\to\infty} \frac{1}{2\pi} &\int\limits_{(2^7 bL_n)^{-1}<|t|\leqslant T} \frac{e^{-iyt}}{it} \prod_{j=1}^n \varphi(t;\widehat{W}_n)\, dt \\ - \frac{y}{2\pi} &\int\limits_{|t|>(2^7 bL_n)^{-1}} \frac{e^{-iyt}}{it(y+it)} \prod_{j=1}^n \varphi(t;\widehat{W}_n)\, dt := \widehat{I}_{3,1} + \widehat{I}_{3,2}.\end{aligned}$$

Note that

$$\left|\widehat{I}_{3,2}\right| \leqslant cbxL_n, \qquad y \leqslant 11x/10. \tag{3.53}$$

Let us represent $\widehat{I}_{3,1}$ in the form

$$\begin{aligned}\widehat{I}_{3,1} &= \lim_{T\to\infty} \frac{1}{2\pi} \int\limits_{-T}^{T} \frac{e^{-iyt}}{it} Q(t)\, dt - \frac{1}{2\pi} \int\limits_{|t|\leqslant(2^7 bL_n)^{-1}} \frac{e^{-iyt}}{it} Q(t)\, dt \\ &\quad + \frac{1}{2\pi} \int\limits_{|t|>(2^7 bL_n)^{-1}} \frac{e^{-iyt}}{it} e^{-t^2/2}\, dt \\ &:= \widehat{I}_{3,1,1}(y) - \widehat{I}_{3,1,2}(y) + \widehat{I}_{3,1,3}(y),\end{aligned} \tag{3.54}$$

where $Q(t) = \prod_{j=1}^n \varphi(t;\widehat{W}_n) - e^{-t^2/2}$. Since $\widehat{I}_{3,1,1} = \Phi(y) - \mathbf{P}\left(\widehat{S}_n < y\right)$, where $\widehat{S}_n = \widehat{W}_1 + \cdots + \widehat{W}_n$, we obtain, by the Berry–Esseen inequality and (3.45), (3.46),

$$\left|\widehat{I}_{3,1,1}\right| \leqslant cL_n + \left|\Phi\big(y/\mathfrak{B}_n\big) - \Phi(y)\right| \leqslant cbL_n, \qquad 9x/10 \leqslant y \leqslant 11x/10.$$

By Lemma 2.2 and (3.45), (3.46), we arrive at the inequality $\left|\widehat{I}_{3,1,2}\right| \leqslant cbxL_n$. In addition it is easy to see that $\left|\widehat{I}_{3,1,3}\right| \leqslant cbL_n$. Applying the last three bounds to (3.54) we obtain $\left|\widehat{I}_{3,1}(y)\right| \leqslant cbxL_n$ for $9x/10 \leqslant y \leqslant 11x/10$. Comparing this inequality with (3.53), we deduce

$$\left|\widehat{I}_3(y)\right| \leqslant cbxL_n \quad \text{for} \quad 9x/10 \leqslant y \leqslant 11x/10. \tag{3.55}$$

Let us return to (3.50). By Corollary 2.1 and (3.51), we see that

$$\widehat{I}_1 = \big(1-\Phi(y)\big)e^{y^2/2} + c\theta bx^2\big(1 + b^3 L_n^{1/3}/x^3\big)L_n.$$

Applying this relation and (3.52), (3.55), (3.41) to (3.50), we arrive at the assertion of the lemma.

Lemma 3.7. *Let $1 \leqslant x \leqslant x_2 = B_n/(8\widehat{\beta}^{1/3})$. Under the assumptions of Theorem 4 on $X_1, \dots, X_n$ the random variables*

$$\Big\{X_j - \frac{ax}{B_n}\big(X_j^2 - \sigma_j^2\big)\Big\}_{j=1}^n, \ \Big\{X_j\mathbf{I}\big\{|X_j| \leqslant B_n/x\big\}\Big\}_{j=1}^n, \ \Big\{Y_j - \frac{x}{2B_n}\big(Z_j^2 - \mathbf{E}\,Z_j^2\big)\Big\}_{j=1}^n,$$

where $a = 1/4, 1/2, 3/4$, satisfy the assumptions of Lemma 3.6 with $b = 4^3$.

Proof. We shall prove prove the lemma for the random variables

$$W_{a,j} = X_j - \frac{ax}{B_n}\big(X_j^2 - \sigma_j^2\big), \qquad j = 1, \dots, n,$$

only. For the remaining random variables it can be proved in a similar way. Rewrite $yW_{a,j}/B_n$ in the form

$$\frac{y}{B_n}W_{a,j} = -\frac{ayx}{B_n^2}\Big(X_j - \frac{B_n}{2ax}\Big)^2 + \frac{ayx\sigma_j^2}{B_n^2} + \frac{y}{4ax}, \tag{3.56}$$

where $9x/10 \leqslant y \leqslant 11x/10$. In view of (3.37) and the condition $1 \leqslant x \leqslant x_2$, we conclude from (3.56) that

$$yW_{a,j}/B_n \leqslant 6/5 \qquad \text{for } 0 \leqslant y \leqslant 11x/10. \tag{3.57}$$

Therefore we get the bound

$$\big(y/B_n\big)^m|W_{a,j}|^m e^{yW_{a,j}/B_n} \leqslant (6/5)^m e^{6/5}, \qquad m = 0, 1, \dots, 3 \tag{3.58}$$

for the same y. Let us verify the condition (3.34). In view of (3.57), we obtain with the help of Taylor's formula

$$\begin{aligned}\mathbf{E}\,e^{yW_{a,j}/B_n} &= \mathbf{E}\,e^{yW_{a,j}/B_n}\mathbf{I}\big\{|X_j| \leqslant B_n/x\big\} + \mathbf{E}\,e^{yW_{a,j}/B_n}\mathbf{I}\big\{|X_j| > B_n/x\big\}\\ &= 1 + \frac{y}{B_n}\,\mathbf{E}\,W_{a,j}\mathbf{I}\big\{|X_j| \leqslant B_n/x\big\} + \frac{y^2}{2B_n^2}\,\mathbf{E}\,W_{a,j}^2\mathbf{I}\big\{|X_j| \leqslant B_n/x\big\}\\ &+ \frac{\theta y^3}{6B_n^3}e^{6/5}\,\mathbf{E}\,|W_{a,j}|^3\mathbf{I}\big\{|X_j| \leqslant B_n/x\big\} + e^{6/5}\theta\,\mathbf{P}\,\big(|X_j| > B_n/x\big).\end{aligned} \tag{3.59}$$

With the help of (3.3) and integrating by parts, we easily deduce, as in the proof of Lemma 3.2, that

$$\begin{gathered}\mathbf{P}\,\big(|X_j| > B_n/x\big) \leqslant x^3\beta_j/B_n^3,\\ \Big|\mathbf{E}\,X_j\mathbf{I}\big\{|X_j| \leqslant B_n/x\big\}\Big| \leqslant 3x^2\beta_j/(2B_n^2), \quad \sigma_j^2 - \mathbf{E}\,X_j^2\mathbf{I}\big\{|X_j| \leqslant B_n/x\big\} \leqslant 3\beta_j x/B_n.\end{gathered} \tag{3.60}$$

In view of (3.60), we have

$$
\begin{aligned}
&\big|\,\mathbf{E}\,W_{a,j}\mathbf{I}\big\{|X_j|\leqslant B_n/x\big\}\big|\leqslant\big|\,\mathbf{E}\,X_j\mathbf{I}\big\{|X_j|\leqslant B_n/x\big\}\big|\\
&+\max\big\{\,\mathbf{P}\,\big(|X_j|>B_n/x\big),\ \sigma_j^2-\,\mathbf{E}\,X_j^2\mathbf{I}\big\{|X_j|\leqslant B_n/x\big\}\big\}\\
&\leqslant\big(\tfrac{3}{2}+\tfrac{9}{4}\big)x^2\beta_j/B_n^2=\frac{15}{4}\,x^2\beta_j/B_n^2.
\end{aligned}\tag{3.61}
$$

Using (3.60) and (3.37), we obtain, by elementary calculations,

$$
\big|\,\mathbf{E}\,W_{a,j}^2\mathbf{I}\big\{|X_j|\leqslant B_n/x\big\}-\sigma_j^2\big|\leqslant 6x\beta_j/B_n,\quad \mathbf{E}\,|W_{a,j}|^3\mathbf{I}\big\{|X_j|\leqslant B_n/x\big\}\leqslant 6x^3\beta_j.\tag{3.62}
$$

Applying (3.60)–(3.62) to (3.59), we see, bearing in mind the condition $1\leqslant x\leqslant x_2$, that $W_{a,1},\dots,W_{a,n}$ satisfy the condition (3.34) of Lemma 3.6 with $b=4^3$.

Now let us verify the first condition in (3.35). Write, using (3.57) and (3.58),

$$
\begin{aligned}
&\left|\,\mathbf{E}\,W_{a,j}e^{yW_{a,j}/B_n}-\frac{y\sigma_j^2}{B_n}\right|\leqslant\left|\,\mathbf{E}\,W_{a,j}e^{yW_{a,j}/B_n}\mathbf{I}\big\{|X_j|\leqslant B_n/x\big\}-\frac{y\sigma_j^2}{B_n}\right|\\
&+\left|\,\mathbf{E}\,W_{a,j}e^{yW_{a,j}/B_n}\mathbf{I}\big\{|X_j|>B_n/x\big\}\big)\right|\leqslant\big|\,\mathbf{E}\,W_{a,j}\mathbf{I}\big\{|X_j|\leqslant B_n/x\big\}\big|\\
&+\left|\,\frac{y}{B_n}\,\mathbf{E}\,W_{a,j}^2\mathbf{I}\big\{|X_j|\leqslant B_n/x\big\}-\frac{y\sigma_j^2}{B_n}\right|+\frac{y^2}{2B_n^2}\,e^{6/5}\,\mathbf{E}\,|W_{a,j}|^3\mathbf{I}\big\{|X_j|\leqslant B_n/x\big\}\\
&+e^{6/5}\,\frac{6B_n}{5y}\,\mathbf{P}\,\big(|X_j|>B_n/x\big).
\end{aligned}
$$

Again applying again (3.60)–(3.62) to the right-hand side of this inequality we get that $W_{a,1},\dots,W_{a,n}$ satisfy the first condition in (3.35) with $b=4^3$. The inequalities

$$
\left|\,\mathbf{E}\,W_{a,j}^2e^{yW_{a,j}/B_n}-\sigma_j^2\right|\leqslant 4^3x\beta_j/B_n,\qquad \mathbf{E}\,|W_{a,j}|^3e^{yW_{a,j}/B_n}\leqslant 4^3\beta_j
$$

can be verified in the same way. Thus, the random variables $W_{a,1},\dots,W_{a,n}$ satisfy the assumptions of Lemma 3.6 with $b=4^3$ which proves the lemma.

Lemma 3.8. *For $0\leqslant x\leqslant 2R_{1n}$, we have*

$$
\mathbf{P}\,\big(|S_n|/V_n>x\big)\leqslant c\big(1-\Phi(x)\big)e^{65L_nx^3}.
$$

Proof. Obviously, it is sufficient to prove the assertion of the lemma for $x\geqslant 1$. Write

$$
\begin{aligned}
\mathbf{P}\,\big(S_n>xV_n\big)=\;&\mathbf{P}\,\big(S_n>xV_n,V_n^2<B_n^2/2\big)+\mathbf{P}\,\big(S_n>xV_n,\big|V_n^2-B_n^2\big|\leqslant B_n^2/2\big)\\
&+\mathbf{P}\,\big(S_n>xV_n,V_n^2>3B_n^2/2\big).
\end{aligned}\tag{3.63}
$$

By Lemma 3.2, we have

$$P\big(S_n > xV_n, V_n^2 < B_n^2/2\big) \leqslant \mathbf{P}\left(V_n^2 < B_n^2/2\right) \leqslant \exp\big\{-B_n^4/(3\cdot 2^7\widehat{A}_n)\big\}. \tag{3.64}$$

Now let us estimate the second summand on the right-hand side of (3.63). Since, for $|y| \leqslant 1/2$,

$$\sqrt{1+y} \geqslant 1 + y/2 - y^2/4,$$

it is easy to see that

$$\begin{aligned}
&\mathbf{P}\left(S_n > xV_n, \left|V_n^2 - B_n^2\right| \leqslant B_n^2/2\right) \leqslant P\Big(S_n > xB_n\Big\{1+\frac{1}{4B_n^2}\left(V_n^2 - B_n^2\right)\Big\}, V_n^2 \geqslant B_n^2\Big)\\
&+ \mathbf{P}\left(S_n > xB_n\Big\{1 + \frac{3}{4B_n^2}\left(V_n^2 - B_n^2\right)\Big\}, V_n^2 < B_n^2\right)\\
&\leqslant \mathbf{P}\left(S_n - \frac{x}{4B_n}\left(V_n^2 - B_n^2\right) > xB_n\right) + \mathbf{P}\left(S_n - \frac{3x}{4B_n}\left(V_n^2 - B_n^2\right) > xB_n\right) := p_1 + p_2.
\end{aligned} \tag{3.65}$$

By Lemma 3.7, the random variables $\left\{W_{1/4,j}\right\}_{j=1}^n$ and $\left\{W_{3/4,j}\right\}_{j=1}^n$ satisfy the assumptions of Lemma 3.6 with $b = 4^3$. Therefore we have

$$p_k \leqslant c\big(1-\Phi(x)\big)e^{65L_n x^3}, \qquad k = 1, 2, \tag{3.66}$$

for $x \leqslant \min\big\{B_n/(8\widehat{\beta}^{1/3}), 2^{-10}L_n^{-1}\big\}$.

It remains to estimate the third summand on the right-hand side of (3.63). We see that

$$\begin{aligned}
\mathbf{P}\left(S_n > xV_n, V_n^2 > 3B_n^2/2\right) &\leqslant \mathbf{P}\Big(\sum_{j=1}^n X_j\mathbf{I}\big\{|X_j| \leqslant B_n/x\big\} > xB_n\Big)\\
&\quad + \mathbf{P}\Big(\sum_{j=1}^n X_j\mathbf{I}\big\{|X_j| > B_n/x\big\} > xB_n/5\Big) := p_3 + p_4.
\end{aligned} \tag{3.67}$$

By Lemma 3.7, we note that the random variables $X_j\mathbf{I}\big\{|X_j| \leqslant B_n/x\big\}$, $j = 1, \dots, n$ satisfy the assumptions of Lemma 3.6. By this lemma, we conclude that (3.66) holds for $k = 3$. On the other hand, we have the simple bound

$$p_4 \leqslant \mathbf{P}\left(\sum_{j=1}^n \mathbf{I}\big\{|X_j| > B_n/x\big\} > x^2/5\right) \leqslant \sum_{m=[x^2/5]}^n \sum_{\{l_1,\dots,l_m\}} \prod_{j=1}^m \mathbf{P}\left(|X_{l_j}| > B_n/x\right), \tag{3.68}$$

where the summation in the second sum on the right-hand side 0f (3.68) is taken over all subpopulations $\{l_1,\dots,l_m\}$ of size m of the population $\{1,\dots,n\}$. In view of (3.3) and Stirling's formula, we obtain from (3.68), for $x \leqslant B_n^3/(5e^2 n\widehat{\beta})$,

$$p_4 \leqslant \sum_{m=[x^2/5]}^{n} \binom{n}{m} \Big(\frac{\widehat{\beta} x^3}{B_n^3}\Big)^m \leqslant \frac{c}{\Gamma([x^2/5]+1)} \Big(\frac{n\widehat{\beta} x^3}{B_n^3}\Big)^{[x^2/5]} \leqslant ce^{-ex^2/5}. \tag{3.69}$$

Applying (3.64), (3.66) for $k=1,2,3$, and (3.69) to (3.63), we arrive at the bound, for $0 \leqslant x \leqslant 2R_{1n}$,

$$\mathbf{P}\left(S_n > xV_n\right) \leqslant c\big(1-\Phi(x)\big)e^{65L_n x^3}.$$

Similarly we obtain, for $-2R_{1n} \leqslant x \leqslant 0$,

$$\mathbf{P}\left(S_n < xV_n\right) \leqslant c\Phi(x)e^{65L_n x^3}.$$

¿From the last two bounds the assertion of the lemma follows.

Proof of Theorem 4. To prove Theorem 4 we use some ideas of the papers by Wang and Jing (1999) and Shao (1997). Without loss of generality we assume that $x \geqslant 6^4$, the parameter $R_{1n} > 2$, and $L_n \leqslant c$, where c is sufficiently small.

By the elementary inequality $2B_nV_n \leqslant B_n^2 + V_n^2$, we have

$$\mathbf{P}\left(S_n \geqslant xV_n\right) \geqslant \mathbf{P}\Big(S_n - \frac{x}{2B_n}\left(V_n^2 - B_n^2\right) \geqslant xB_n\Big). \tag{3.70}$$

In view of Lemma 3.7, the random variables $W_{1/2,j}$, $j=1,\dots,n$, satisfy the assumptions of Lemma 3.6 with $b = 4^3$. Applying Lemma 3.6 with $y=x$, we obtain

$$\mathbf{P}\Big(S_n - \frac{x}{2B_n}\left(V_n^2 - B_n^2\right) \geqslant xB_n\Big) = \big(1-\Phi(x)\big)e^{4^3\theta L_n x^3}\big(1+c\theta L_n x^3\big). \tag{3.71}$$

Using Lemma 3.4 and (3.3), we note that

$$\begin{aligned}
\mathbf{P}\left(S_n \geqslant xV_n\right) &\leqslant \mathbf{P}\left(S_n^* \geqslant xV_n^*\right) + \sum_{k=1}^{n} \mathbf{P}\Big(S_n \geqslant xV_n,\, |X_k| \geqslant \frac{B_n}{x}\Big) \\
&\leqslant \mathbf{P}\left(S_n^* \geqslant xV_n^*\right) + \max_k \mathbf{P}\Big(S_n - X_k \geqslant \sqrt{x^2-1}(V_n^2 - X_k^2)^{1/2}\Big) \sum_{k=1}^{n} \mathbf{P}\Big(|X_k| \geqslant \frac{B_n}{x}\Big) \\
&\leqslant \mathbf{P}\left(S_n^* \geqslant xV_n^*\right) + cL_n x^3 \max_k \mathbf{P}\Big(S_n - X_k \geqslant x\sqrt{1-x^{-2}}(V_n^2 - X_k^2)^{1/2}\Big).
\end{aligned} \tag{3.72}$$

In view of (3.37), $\min_k \left(B_n^2 - \sigma_k^2\right) \geqslant B_n^2\big(1 - L_n^{2/3}\big)$ and we deduce with the help of Lemma 3.8 that, for $x \leqslant 2\big(1 - 4L_n^{2/3}\big)R_{1n}$,

$$\max_k \mathbf{P}\Big(S_n - X_k \geqslant x\sqrt{1-x^{-2}}(V_n^2 - X_k^2)^{1/2}\Big) \leqslant c\big(1-\Phi(x)\big)e^{65L_n x^3}. \tag{3.73}$$

Now we also assume that $x \leqslant \widehat{R}_{1n}$. It follows from (3.70)–(3.73), that in order to prove the theorem for $x \leqslant \widehat{R}_{1n}$ it is sufficient to show that

$$\mathbf{P}\left(S_n^*/V_n^* \geqslant x\right) \leqslant \big(1-\Phi(x)\big)\big(1+c\min\{1, M_{2n}(x)\}\big). \tag{3.74}$$

Using the well-known estimate $\sqrt{1+y} \geqslant 1+y/2-y^2$, $y \geqslant -1$, we see that

$$\begin{aligned}\mathbf{P}\left(S_n^* \geqslant xV_n^*\right) &\leqslant \mathbf{P}\left(S_n^* \geqslant x\widehat{V}_n\right) = \mathbf{P}\left(S_n^* \geqslant xB_n\left(1+B_n^{-2}\big(\widehat{V}_n^2-B_n^2\big)\right)^{1/2}\right)\\ &\leqslant \mathbf{P}\left(S_n^* \geqslant xB_n\left(1+(2B_n^2)^{-1}\big(\widehat{V}_n^2-B_n^2\big)-B_n^{-4}\big(\widehat{V}_n^2-B_n^2\big)^2\right)\right).\end{aligned} \tag{3.75}$$

With the help of Lemma 3.5 we deduce from (3.75) that

$$\mathbf{P}\left(S_n^* \geqslant xV_n^*\right) \leqslant \mathbf{P}\left(S_n^* \geqslant xB_n\left(1-\frac{1}{x^2}M_{2n}(x,\gamma)+\frac{1}{2B_n^2}\big(\widehat{V}_n^2-\mathbf{E}\,\widehat{V}_n^2\big)\right)\right)+cL_n^2e^{-x^2/2} \tag{3.76}$$

for all $\gamma \in [0,1/2]$, where

$$M_{2n}(x,\gamma) = x^3L_n^{1-\gamma}+x^2B_n^{-2}\sum_{j=1}^n \mathbf{E}\,X_j^2\mathbf{I}\big\{|X_j| > L_n^{(1-\gamma)/4}B_n/(6x^{3/4})\big\}.$$

Using (3.29) we obtain that, for $\gamma \in [0,1/2]$ and $x \leqslant \widehat{R}_{1n}$,

$$x^{-2}M_{2n}(x,\gamma) \leqslant xL_n^{1-\gamma}+18x^{3/4}L_n^{(3+\gamma)/4} \leqslant 10^{-1}.$$

By Lemma 3.7, the random variables $Y_j - x\big(Z_j^2-\mathbf{E}\,Z_j^2\big)/(2B_n)$, $j=1,\dots,n$, satisfy the assumptions of Lemma 3.6. Applying this lemma with

$$y = x\big(1-x^{-2}M_{2n}(x,\gamma)\big)$$

to (3.76), we obtain, for all $\gamma \in [0,1/2]$,

$$\begin{aligned}&\mathbf{P}\left(S_n^* \geqslant xV_n^*\right) \leqslant \big(1-\Phi(x\big(1-x^{-2}M_{2n}(x,\gamma)\big)\big(1+cL_nx^3\big)\\ &\leqslant 1-\Phi(x)+c\big(1-\Phi(x)\big)M_{2n}(x,\gamma)\exp\big\{M_{2n}(x,\gamma)+x^{-2}M_{2n}^2(x,\gamma)\big\}.\end{aligned} \tag{3.77}$$

Consider those x such that $x \leqslant \widehat{R}_{1n}$ and $M_{2n}(x) \leqslant 1$. Now (3.77) implies, for such x,

$$\mathbf{P}\left(S_n^* \geqslant xV_n^*\right) \leqslant \big(1-\Phi(x)\big)\big(1+cM_{2n}(x)\big). \tag{3.78}$$

Finally notice that (3.74) follows from (3.78) and Lemma 3.8.

To complete the proof of Theorem 4 note that for $\widehat{R}_{1n} \leqslant x \leqslant R_{1n}$ it follows from Lemma 3.8. Whereas for $x > R_{1n}$ it follows from Lemma 3.3. Thus, Theorem 4 is proved.

Proof of Corollary 4. Using (3.29) straightforward calculations yield

$$M_{2n}(x) \leqslant cx^2 \big(|x| L_n\big)^{4/5}. \tag{3.79}$$

The relations (1.14) and (1.15) now follow directly from Theorem 4 and (3.79).

Proof of Theorem 2. For $|x| > n^{1/6}/\beta^{1/3}$ the proof is similar as the proof of Theorem 4, using Lemma 3.8 and Lemma 3.3 for i.i.d. random variables. To obtain the statement of the theorem for $|x| \leqslant n^{1/6}/\beta^{1/3}$ we again use the bounds (3.70)–(3.74). In addition we need the following bound, due to Shao (1999),

$$\mathbf{P}\,\big(S_n \geqslant xV_n\big) \leqslant \big(1 + 1/x\big)\,\frac{1}{\sqrt{2\pi}x}\,\exp\big\{-x^2/2 + c\beta x^3/\sqrt{n}\big\} \tag{3.80}$$

for any $2 \leqslant x \leqslant n^{1/2}/2$. Taking into account Lemma 2.4 we see that the statement of the theorem for $1 \leqslant x \leqslant n^{1/6}/\beta^{1/3}$ immediately follows from (3.70)–(3.74), and (3.80). Similarly one can prove the statement of the theorem for $-n^{1/6}/\beta^{1/3} \leqslant x \leqslant -1$. Theorem 2 is proved.

Proof of Corollary 2. Note that the bound (1.9) is an obvious consequence of (3.79) and (3.80). To obtain (1.10) and (1.11) we consider two cases. First let $|x| \leqslant l^{-1/4}$. From Theorem 2 and (1.9) we have $r_n(x) \leqslant cx^2(|x|l)^{4/5} \leqslant l^{1/10}$.

Now let $l^{-1/4} < |x| \leqslant \big(l\rho(l)\big)^{-1/3}$. Then we have, by Theorem 2 and (1.9), $r_n(x) \leqslant cl|x|^3 \leqslant \big(\rho(l)\big)^{-1}$. The assertion of Corollary 2 follows from these estimates.

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