On multiplicities of simple subquotients in Generalized Verma modules

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Abstract

We reduce the problem on multiplicities of simple subquotients in an α-stratified Generalized Verma module to the analogous problem for classical Verma modules.

1 Introduction

The study of α-stratified modules over a simple complex finite-dimensional Lie algebra was originated in [CF] where several basic properties of such modules were obtained. The class of α-stratified modules contains the so-called Generalized Verma modules (GVM). These modules are completely different from another family of GVMs introduced and studied in [R]. α-stratified GVM were investigated in [FM, KM, Ma] where the BGG-like criterion for the existence of a non-trivial homomorphism between two α-stratified GVMs was established.

One of the most important results about classical Verma modules is the so-called Kazhdan-Lusztig Theorem describing the multiplicities of simple subquotient in a Verma module (see for example [BK] and references therein). An analogous result for GVM in the sense of [R] was obtained in [CC]. It happened that the answer obtained in [CC] is different from the classical Kazhdan-Lusztig theorem. The last means that the multiplicities of simple subquotients in a GVM (in the sense of [R]) can not be directly obtained from the analogous multiplicities in the corresponding Verma module.

In the present paper we calculate the multiplicities of simple subquotients in an α-stratified GVM. In fact, with arbitrary α-stratified GVM we associate certain Verma module and prove that the required multiplicities coincide with the multiplicities of simple subquotients in this Verma module. This analogy with Verma modules provides one more difference between α-stratified GVMs and GVMs in the sense of [R].

We have to note, that one related question for α-stratified modules was solved in [M, Theorem 13.4] in a complete generality. In fact, for any simple complex finite-dimensional Lie algebra $\mathfrak{G}$ and its "well-embedded" subalgebra $\mathfrak{G}_1$ of type A_n or C_n there were calculated the character of the unique simple quotient of GVM induced from a homogeneous

$\mathfrak{G}_1$-module. Here homogeneous means that this module is weight, dense and has weight subspaces of the same dimension (see [M] for details). In the case when $\mathfrak{G}_1$ is of type A_1, simple homogeneous means the same as simple α-stratified. Thus, using the mentioned result one can calculate the character of the unique simple quotient of an α-stratified GVM.

The paper is organized as follows: in section 2 we collect all necessary preliminaries. In section 3 we formulate our main result — Theorem 1, which is proved in section 4.

2 Preliminaries

Let $\mathbb{C}$ denotes the complex numbers, $\mathbb{Z}$ denotes the set of integers and $\mathbb{N}$ denotes the set of all positive integers. For a Lie algebra $\mathfrak{A}$ we will denote by $U(\mathfrak{A})$ its universal enveloping algebra.

Let $\mathfrak{G}$ be a simple complex finite-dimensional Lie algebra and $\mathfrak{H}$ be its Cartan subalgebra. Denote by Δ the corresponding root system and choose some base π in Δ. This defines the partition of Δ into two sets of positive (Δ^+) and negative (Δ^-) roots. We will write P for the abelian subgroup in $\mathfrak{H}^*$ generated by elements from Δ. For $\beta \in \Delta$ let $\mathfrak{G}_\beta$ denote the corresponding root subspace in $\mathfrak{G}$. Fix a Weyl-Chevalley basis X_α, $\alpha \in \Delta$, H_α, $\alpha \in \pi$. Set

$$\rho = \frac{1}{2} \sum_{\beta \in \Delta^+} \beta.$$

Fix $\alpha \in \pi$. Let $\mathfrak{G}^\alpha$ denote the $sl(2)$-subalgebra of $\mathfrak{G}$ corresponding to the root α. Set $\mathfrak{N}^\alpha_\pm = \sum_{\beta \in \Delta^+ \setminus \{\alpha\}} \mathfrak{G}_{\pm\beta}$, $\mathfrak{H}^\alpha = \{h \in \mathfrak{H} \,|\, \alpha(h) = 0\}$, $\pi_\alpha = \pi \setminus \{\alpha\}$. Then we have the following decomposition: $\mathfrak{G} = \mathfrak{G}^\alpha \oplus \mathfrak{N}^\alpha_- \oplus \mathfrak{H}^\alpha \oplus \mathfrak{N}^\alpha_+$. For $\mathfrak{H}_\alpha = \mathfrak{G}^\alpha \cap \mathfrak{H}$ one obtains $\mathfrak{G}^\alpha = \mathfrak{G}_\alpha \oplus \mathfrak{H}_\alpha \oplus \mathfrak{G}_{-\alpha}$.

For a $\mathfrak{G}$-module V and $\lambda \in \mathfrak{H}^*$ let V_λ denotes the weight space with respect to a weight λ. A $\mathfrak{G}$-module V will be called weight module if it decomposes into a direct sum of it's weight spaces. A weight $\mathfrak{G}$-module V is called α-stratified ([CF]) if the actions of X_α and $X_{-\alpha}$ are injective on V. All the modules considered in this paper are supposed to be weight modules.

Consider the quadratic Casimir operator $c = (H_\alpha + 1)^2 + 4X_{-\alpha}X_\alpha$ of $\mathfrak{G}_\alpha$. Any pair $a, b \in \mathbb{C}$ defines the unique indecomposable $\mathfrak{G}^\alpha$-module $N(a, b)$ such that $X_{-\alpha}$ acts bijectively on $N(a, b)$, all non-trivial weight spaces of $N(a, b)$ are one-dimensional, a is an eigenvalue of H_α and b is the eigenvalue of c. One has $N(a, b) \simeq N(a + 2l, b)$ for any $l \in \mathbb{Z}$.

Since $\mathfrak{H} = \mathfrak{H}_\alpha \oplus \mathfrak{H}^\alpha$ we can rewrite arbitrary $\lambda \in \mathfrak{H}^*$ as $\lambda = \lambda_\alpha + \lambda^\alpha$ where $\lambda_\alpha \in \mathfrak{H}_\alpha$ and $\lambda^\alpha \in \mathfrak{H}^\alpha$. Let $a, b \in \mathbb{C}$ and $\lambda \in \mathfrak{H}^*$ be such that $(\lambda - \rho)(H_\alpha) = (\lambda_\alpha - \rho)(H_\alpha) = a$. We can define the structure of a $\mathfrak{H}$-module on $N(a, b)$ by setting $hv = (\lambda - \rho)^\alpha(h)v$ for all $h \in \mathfrak{H}^\alpha$ and all $v \in N(a, b)$. Further we can consider $N(a, b)$ as $D = \mathfrak{H} + \mathfrak{G}^\alpha \oplus \mathfrak{N}^\alpha_+$-module by setting $\mathfrak{N}^\alpha_+ N(a, b) = 0$.

The $\mathfrak{G}$-module

$$M_\alpha(\lambda, b) = U(\mathfrak{G}) \bigotimes_{U(D)} N(a, b)$$

is called generalized Verma module associated with $\mathfrak{G}, \mathfrak{H}, \pi, \alpha, \lambda, b$. One can easily prove that $M_\alpha(\lambda, b)$ is α-stratified if and only if $b \neq (a+1+2l)^2$ for all $l \in \mathbb{Z}$ (see also [CF, Theorem 2.1]). We will denote by $L_\alpha(\lambda, b)$ the unique simple quotient of $M_\alpha(\lambda, b)$. It is well-known that $M_\alpha(\lambda, b)$ has a composition series ([CF, Theorem 2.8(i)]). For $\lambda \in \mathfrak{H}^*$ we will write $M(\lambda)$ for the Verma module with the highest weight $\lambda - \rho$ ([D, 7.1.4]) and $L(\lambda)$ for it's unique simple quotient.

3 Main Theorem

Fix an analytic branch of the square root function satisfying the condition $\sqrt{1} = 1$. For arbitrary $\lambda \in \mathfrak{H}^*$ and $b \in \mathbb{C}$ we set

$$f(\lambda, b) = \lambda - \frac{\lambda(H_\alpha) + \sqrt{b}}{\alpha(H_\alpha)}\alpha.$$

Theorem 1. *Suppose that $M_\alpha(\lambda, b)$ is α-stratified. Then the multiplicity of $L_\alpha(\mu, d)$ as a simple subquotient in a composition series of $M_\alpha(\lambda, b)$ equals to the multiplicity of $L(f(\mu, d))$ as a simple subquotient in a composition series of $M(f(\lambda, b))$.*

4 Proof of the Main Theorem

For $u \in \mathbb{C}$ consider the $\mathfrak{G}^\alpha$-module

$$T(u) = \bigoplus_{a \in \mathbb{C}/2\mathbb{Z}} N(a, u)$$

and the corresponding induced module

$$M_T(\lambda, u) = U(\mathfrak{G}) \bigotimes_{U(D)} T(u).$$

A weight $\mathfrak{G}$-module V will be called normal provided $X_{-\alpha}$ acts bijectively on V. It follows from the definition of $N(a, b)$ that $M_T(\lambda, b)$ is normal.

Lemma 1. *Let V be a normal weight $\mathfrak{G}$-module and W be a normal submodule of V. Then the module V/W is normal.*

Proof. Since V is normal it follows that $X_{-\alpha}$ acts surjectively on V/W. Moreover, since W is normal it follows that the pre-image of any element from W is contained in W and thus $X_{-\alpha}$ acts injectively on V/W. Combining these results we obtain that V/W is normal. □

Consider a normal $\mathfrak{G}$-module V. Let $U(\alpha)$ denotes the localization of $U(\mathfrak{G})$ with respect to the multiplicative set $\{X^n_{-a} \mid n \in \mathbb{N}\}$. $U(\alpha)$ is well-defined by [M, Lemma 4.2]. Since V is normal, we can define the $U(\alpha)$-module $V(\alpha) = U(\alpha) \otimes_{U(\mathfrak{G})} V$. By [M, Lemma 4.3] there exists the unique polynomial extension $\{\theta_x \mid x \in \mathbb{C}\}$ of the family of automorphisms $\theta_x : U(\alpha) \to U(\alpha)$, $x \in \mathbb{Z}$ such that $\theta_x(v) = X_{-\alpha}^x v X_{-\alpha}^{-x}$, $x \in \mathbb{Z}$. For a $U(\alpha)$-module W and $x \in \mathbb{C}$ we will denote by $\theta_x(W)$ the $U(\alpha)$-module which is equal to W as a vector space and $v \cdot w = \theta(v)w$ for all $v \in U(\alpha)$, $w \in W$. Clearly, one can consider any $U(\alpha)$-module as a $U(\mathfrak{G})$-module by restriction.

Set $P_\alpha = \{\sum_{\beta\in\pi_\alpha} z_\beta\beta \mid z_\beta \in \mathbb{C}\}$ and $P(\alpha) = P + P_\alpha$. Let V be a weight $\mathfrak{G}$-module and $\lambda \in \mathfrak{H}^*$. We will denote by $V(\lambda)$ the direct summand $\sum_{\mu\in\lambda+P(\alpha)} V_\mu$ of V. For $\lambda_1, \lambda_2 \in \mathfrak{H}^*$ let $x(\lambda_1, \lambda_2)$ denote the unique complex number such that $\lambda_2 - (\lambda_1 + x(\lambda_1, \lambda_2)\alpha)$ belongs to P_α. A weight $\mathfrak{G}$-module V will be called α-homogeneous provided $V(\lambda_2) \simeq \theta_{x(\lambda_1,\lambda_2)}V(\lambda_1)$ for all $\lambda_1, \lambda_2 \in \mathfrak{H}^*$. It follows immediately from the definition that $M_T(\lambda, b)$ is α-homogeneous. One can easily see that the quotient of an α-homogeneous module by an α-homogeneous submodule is again α-homogeneous.

Let V be an α-homogeneous $\mathfrak{G}$-module. We will say that V is solid if fixed are maps $\psi(y) = \theta^{-1}_{x(y\alpha,0)} \circ \varphi(y) : V(0) \to V(y\alpha)$, $y \in \mathbb{C}$ where $\varphi(y)$, $y \in \mathbb{C}$ are isomorphisms. We will say that an α-homogeneous submodule W of V is solid provided

$$W(\lambda_1) = \psi(x(\lambda_1, 0)) \circ \psi^{-1}(x(\lambda_2, 0))(W(\lambda_2)).$$

It follows immediately from the definition that $M_T(\lambda, b)$ can be considered as a solid α-homogeneous module. One can easily see that the quotient of a solid α-homogeneous module by a solid α-homogeneous submodule (whose solid structure is inherited from the big module) is again solid α-homogeneous.

Lemma 2. *Let V be a solid α-homogeneous module and W be a normal submodule in V. Then the submodule $\hat{W}$ of V defined by*

$$\hat{W}(\mu) = \sum_{\mu'\in\mathfrak{H}^*} \psi(x(\mu, 0)) \circ \psi^{-1}(x(\mu', 0))(W(\mu')),$$

$\mu \in \mathfrak{H}^*$ is the unique minimal solid normal α-homogeneous submodule containing W.

Proof. Clearly, $\hat{W}$ is solid, α-homogeneous and contains W. It is normal by the definition of θ_x. Its minimality follows directly from the construction. The uniqueness follows from the solidness. □

The submodule $\hat{W}$ constructed in Lemma 2 will be called the α-homogeneous hat of W. A $\mathfrak{G}$-module V is said to be simple normal if there are no non-trivial normal $\mathfrak{G}$-submodules in V.

Lemma 3. *Let V be a solid α-homogeneous $\mathfrak{G}$-module and W be its simple normal submodule. Let $\hat{W}$ be the α-homogeneous hat of W. Then $\hat{W}(\mu)$ is simple normal for any $\mu \in \mathfrak{H}^*$.*

Proof. Since W is simple normal it follows that $W = W(\mu')$ for some $\mu' \in \mathfrak{H}^*$. Thus $W(\mu') = \psi(x(\mu', 0)) \circ \psi^{-1}(x(\mu, 0))(\hat{W}(\mu))$. Suppose that $\hat{W}(\mu)$ is not simple normal and contains a non-trivial normal submodule N. Then $\psi(x(\mu', 0)) \circ \psi^{-1}(x(\mu, 0))(N)$ is a non-trivial normal submodule in $W(\mu')$ which contradicts our assumptions. □

Lemma 4. *Let V be a solid α-homogeneous and $W = W(\mu)$, $\mu \in \mathfrak{H}^*$ be a normal submodule in V. Suppose that W has a composition series*

$$W = W_0 \supset W_1 \supset \cdots \supset W_k = 0$$

such that all simple quotients $W^i = W_i/W_{i+1}$, $0 \leqslant i \leqslant k$ are normal. Let $\hat{W}$ be the α-homogeneous hat of W. Then $\hat{W}$ has the filtration

$$\hat{W} = \hat{W}_0 \supset \hat{W}_1 \supset \cdots \supset \hat{W}_k = 0$$

such that each $\hat{W}_i$ is the α-homogeneous hat of W_i for all $0 \leqslant i \leqslant k$. Moreover, $\hat{W}^i = \hat{W}_i/\hat{W}_{i+1}$ is the α-homogeneous hat of W^i in $V/\hat{W}_{i+1}$ and $\hat{W}^i(\xi)$ is simple normal for all $\xi \in \mathfrak{H}^*$.

Proof. Follows from Lemma 3 and Lemma 1 by trivial induction in k. □

Lemma 5. *Suppose that W is simple normal. Then W is of length at most 3 and contains the unique minimal submodule N such that X_{-a} acts injectively on N.*

Proof. Follows from the $sl(2)$-theory applied to the subalgebra $\mathfrak{G}^\alpha$ of $\mathfrak{G}$. □

Now we are ready to prove our main theorem.

Consider the module $M_T(\lambda, b)$. Clearly, it is normal and we can consider it as a solid α-homogeneous module with respect to an arbitrary solid structure. Consider its normal submodule $M_\alpha(\lambda, b)$. One can see that $M_T(\lambda, b)$ is the α-homogeneous hat of $M_\alpha(\lambda, b)$. Let $N = (M_T(\lambda, b))(f(\lambda, b))$. By Lemma 4 any composition series of $M_\alpha(\lambda, b)$ leads to a filtration of N with simple normal subquotients. By Lemma 5 each simple normal subquotient of N has the unique simple submodule on which $X_{-\alpha}$ acts injectively. Clearly, this correspondence is the bijection between the set of all simple subquotients of $M_\alpha(\lambda, b)$ and all simple subquotients of $M(f(\lambda, b))$ on which $X_{-\alpha}$ acts injectively. The rest follows from trivial observation that the module corresponding to $L_\alpha(\mu, d)$ is exactly $L(f(\mu, d))$. Theorem is proved.

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Schubert filtration for simple quotients of generalized Verma modules

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1 Introduction

Schubert filtration and Demazure character formula form a classical part in the theory of finite-dimensional representations of complex semi-simple finite-dimensional Lie algebras (and, more generally, simple algebraic groups, see [A, J]). Originally, this formula appears in [De] as an improvement of the Weyl character formula describing the character of a simple finite-dimensional module. Schubert filtration arises in a canonical way together with the Demazure formula.

Recently, an analogue of the Weyl character formula for the character of a simple quotient of the so-called α-stratified generalized Verma module (GVM) over a simply-laced Lie algebra was obtained ([FM]). The obtained formula differs from the Weyl formula by the multiplicand corresponding to a subalgebra of $U(\mathfrak{N}_-)$ acting freely on the simple module. This result stimulated an investigation to find an analogue for the Demazure formula and to construct the corresponding filtration of the simple module.

In the present paper we present an analogue of the Demazure formula and the corresponding filtration (Schubert filtration) for a simple quotient of an α-stratified GVM over a simple simply-laced finite-dimensional complex Lie algebra. The proof of the formula is very easy and based on the classical Demazure formula and the one obtained in [FM]. However, our construction of the corresponding filtration is quite non-trivial and, in fact, reduces to the construction of the analogues for the canonical generators of Demazure modules ([J]). Unlike the classical case, these generators can not be viewed as (semi)-primitive generators of the simple modules with respect to another choice of the basis in the root system of our algebra. A powerful tool in our proof is Mathieu's localization of $U(\mathfrak{G})$ ([Ma]) which allows us to reduce our problem (in part) to classical Verma modules.

Let us briefly describe the structure of the paper. In Section 2 we collect all necessary preliminaries. In Section 3 we obtain an analogue for the Demazure formula. In Section 4 we define the Mathieu's functor and describe its basic properties. In Section 5 we prove an auxiliary lemma describing some elements from $U(sl(3,\mathbb{C}))$ having very nice properties. Finally, in Section 6 we construct an analogue for the Schubert filtration.

2 Notations and preliminary results

Let $\mathbb{C}$ denote the complex field; $\mathbb{Z}$ the set of integers; $\mathbb{N}$ the set of positive integers and $\mathbb{Z}_+$ the set of non-negative integers. All the notations that will be used without preliminary definition can be found in [D]. For a Lie algebra $\mathfrak{A}$ by $U(\mathfrak{A})$ we will denote the universal enveloping algebra of $\mathfrak{A}$.

Consider a simple simply-laced complex finite-dimensional Lie algebra $\mathfrak{G}$. Let $\mathfrak{H}$ be a Cartan subalgebra of $\mathfrak{G}$ and Δ be the corresponding root system. Fix a basis π in Δ and an element $\alpha \in \pi$. This lead to the decomposition $\Delta = \Delta_+ \cup \Delta_-$. We will denote by ρ the half-sum of all positive roots. Let W denote the Weyl group of Δ (acting in $\mathfrak{H}^*$) generated by the reflections s_β, $\beta \in \Delta$. Fix a Weyl-Chevalley basis in $\mathfrak{G}$ consisting of the elements $X_{\pm\beta}$, $\beta \in \Delta_+$ and H_β, $\beta \in \pi$. By $\mathfrak{N}_\pm$ we will denote the standard components of the triangular decomposition of $\mathfrak{G}$ with respect to the basis π. Let $\mathfrak{G}(\alpha)$ be a Lie subalgebra of $\mathfrak{G}$ generated by $X_{\pm\alpha}$ and D be a subalgebra generated by $\mathfrak{G}(\alpha)$, $\mathfrak{H}$ and $\mathfrak{N}_+$. Let $c = (H_\alpha + 1)^2 + 4X_{-\alpha}X_\alpha$ be a Casimir operator of $U(\mathfrak{G}(\alpha))$.

Consider a vector space $\Omega = \mathfrak{H}^* \oplus \mathbb{C}$. For each $(\lambda, p) \in \Omega$ there is a canonical way to put into correspondence the so-called generalized Verma module (GVM) as it was done, for example, in [FM1]. Indeed, consider a unique indecomposable $\mathfrak{G}(\alpha)$-module $N(\lambda, p)$ defined by the following conditions:

- $\lambda(H_\alpha) - 1$ is an eigenvalue of H_α on $N(\lambda, p)$;
- p^2 is the eigenvalue of c on $N(\lambda, p)$;
- all weight subspaces of $N(\lambda, p)$ are one-dimensional;
- $X_{-\alpha}$ acts bijectivly on $N(\lambda, p)$.

Clearly, $N(\lambda, p)$ can be considered as a D-module by making use of $\lambda - \rho$. The induced module

$$M(\lambda, p) = U(\mathfrak{G}) \bigotimes_{U(D)} N(\lambda, p).$$

is the GVM associated with $\mathfrak{G}$, $\mathfrak{H}$, π, α, λ and p. Clearly, $M(\lambda, p)$ has the unique simple quotient which will be denoted by $L(\lambda, p)$. There is an action of W on Ω defined as follows ([FM]): Consider a partition of π: $\pi = \pi_1 \cup \pi_2$ where $\pi_1 = \{\gamma \in \pi \,|\, \alpha + \gamma \in \Delta\}$, $\pi_2 = \{\gamma \in \pi \,|\, \alpha + \gamma \notin \Delta\}$. For $(\lambda, p) \in \Omega$ and $\beta \in \pi_1$ denote

$$n^\pm_\beta(\lambda, p) = \frac{1}{2}(\lambda(H_\alpha + 2H_\beta) \pm p)$$

and define $(\lambda_\beta, p_\beta) \in \Omega$, where $\lambda_\beta = \lambda - n^-_\beta(\lambda, p)\beta$, $p_\beta = n^+_\beta(\lambda, p)$.

For each $\beta \in \pi$ set

$$s_\beta(\lambda, p) = \begin{cases} (\lambda, -p), & \beta = \alpha \\ (s_\beta \lambda, p), & \beta \in \pi_2 \setminus \{\alpha\} \\ (\lambda_\beta, p_\beta), & \beta \in \pi_1. \end{cases}$$

Let $r \in \mathbb{C}$. Consider a linear space $B_r = \sum_{\beta \in \pi \setminus \{\alpha\}} \mathbb{C}\beta + r\alpha$ with a fixed point $r\alpha$ and let $\Omega_r = B_r \times \mathbb{C}$. There exists a root system $\Delta_{\alpha,r}$ in Ω_r with respect to which W (with respect to the action defined above) is the Weyl group. Let $(\cdot,\cdot)_r$ denote the corresponding non-degenerated form on Ω_r and $\zeta = \zeta_{\alpha,r} : \Delta \to \Delta_{\alpha,r}$ be a natural bijection.

Let (λ, p), $(\mu, q) \in \Omega_r$ and $\beta \in \Delta_{\alpha,r}$. We will write $(\lambda, p) \stackrel{\beta}{\to} (\mu, q)$ if $(\mu, q) = s_\beta(\lambda, p)$ and $(\beta, (\lambda, p))_r \in \mathbb{N}$ for $\beta \neq \zeta(\alpha)$. Then $(\mu, q) \ll (\lambda, p)$ will describe the fact that there exists a sequence β_1, $\beta_2, \ldots, \beta_k$ in $\Delta_{\alpha,r}$ such that $(\mu, q) \stackrel{\beta_1}{\to} s_{\beta_1}(\mu, q) \stackrel{\beta_2}{\to} \ldots\ s_{\beta_{k-1}} \ldots s_{\beta_1}(\mu, q) \stackrel{\beta_k}{\to} (\lambda, p)$. Denote by P^{++} the set of all $(\lambda, p) \in \Omega$ such that $w(\lambda, p) \ll (\lambda, p)$ for all $w \in W$.

For a weight $\mathfrak{G}$-module V by $\operatorname{ch} V$ we will denote its (formal) character, see [D, Section 7.5]. The main result of [FM] is the following:

Theorem 1. *Suppose that $(\lambda, p) \in P^{++}$. Let $\Delta(\alpha)$ be a root subsystem of Δ generated by $\pi \setminus \{\alpha\}$ and set $K = \Delta_+ \setminus \Delta(\alpha)$. Let $W(\alpha)$ denote the Weyl group of $\Delta(\alpha)$ and ρ_α denote the half-sum of all positive roots in $\Delta(\alpha)$. Then there exists an element $a(\lambda, p) \in \mathfrak{H}^*$ such that*

$$\operatorname{ch}(L(\lambda, p)) = \left(\sum_{i=-\infty}^{+\infty} e^{i\alpha} \right) \left(\prod_{\beta \in -K \setminus \{\alpha\}} (1 - e^{-\beta})^{-1} \right) \times$$
$$\times \left(\sum_{w \in W(\alpha)} (-1)^{l(w)} e^{w(\lambda - \rho + a(\lambda,p) + \rho_\alpha) - a(\lambda,p)} \right) \left(\sum_{w \in W(\alpha)} (-1)^{l(w)} e^{w(\rho_\alpha)} \right)^{-1}.$$

3 Demazure formula for α-stratified modules

In this section we obtain an analogue for the Demazure character formula in the case of simple modules without highest weights. Suppose that $(\lambda, p) \in P^{++}$. For $\beta \in \Delta(\alpha)$ set

$$d_\beta = (1 - e^\beta)^{-1}(1 - e^\beta s_\beta).$$

Let

$$T = \left(\sum_{i=-\infty}^{+\infty} e^{i\alpha} \right) \left(\prod_{\beta \in -K \setminus \{\alpha\}} (1 - e^{-\beta})^{-1} \right).$$

We recall that the standard length function on W ($W(\alpha)$) with respect to π ($\pi \setminus \{\alpha\}$) is defined ([H]).

Theorem 2. *There exists an element $a(\lambda, p) \in \mathfrak{H}^*$ such that for any reduced decomposition $w = s_{\beta_1} s_{\beta_2} \dots s_{\beta_k}$ of the longest element $w \in W(\alpha)$ the following equality holds:*

$$\operatorname{ch} L(\lambda, p) = Te^{a(\lambda,p)} \left(d_{\beta_1} d_{\beta_2} \dots d_{\beta_k} e^{\lambda-\rho-a(\lambda,p)} \right).$$

Proof. By the virtue of Theorem 1, there exists $a(\lambda, p) \in \mathfrak{H}^*$ such that $\operatorname{ch} L(\lambda, p) = Te^{a(\lambda,p)} Q$, where

$$Q = \left(\sum_{w \in W(\alpha)} (-1)^{l(w)} e^{w(\lambda-\rho-a(\lambda,p)+\rho_\alpha)} \right) \left(\sum_{w \in W(\alpha)} (-1)^{l(w)} e^{w(\rho_\alpha)} \right)^{-1}.$$

From the other hand, by the Weyl formula ([D, Theorem 7.5.9]) Q can be considered as the character of a simple finite-dimensional modules over a Lie algebra associated with $\Delta(\alpha)$. From the Demazure character formula [Z, Theorem 2.5.3] we obtain that

$$Q = d_{\beta_1} d_{\beta_2} \dots d_{\beta_k} e^{\lambda-\rho-a(\lambda,p)}$$

for any reduced decomposition $w = s_{\beta_1} s_{\beta_2} \dots s_{\beta_k}$ of the longest element $w \in W(\alpha)$. This completes the proof. $\square$

To construct an analogue of the Schubert filtration associated with the formula obtained in Theorem 2 we need some auxiliary notations and definitions.

4 Mathieu's localization of $U(\mathfrak{G})$

In this section we define a localization of $U(\mathfrak{G})$ ([Ma]). It allows us to reduce our problem of construction of the Schubert filtration to the problem of construction of a special filtration in the Verma modules with an integral dominant highest weight. We have to note that originally the localization of $U(\mathfrak{G})$ in [Ma] is defined in a very general situation. Since we need it only for our case of modules induced from $sl(2)$, we will define it only in this restricted case.

Consider a multiplicative subset $\{X_{-a}^n \mid n \in \mathbb{N}\}$ in $U(\mathfrak{G})$. Let $U(\alpha)$ denote the localization of $U(\mathfrak{G})$ by this set, which is well-defined by [Ma, Lemma 4.2]. Let V be a weight $\mathfrak{G}$-module such that $X_{-\alpha}$ acts bijectivly on V (we will call such module normal). Then we can define the $U(\alpha)$-module $V(\alpha) = U(\alpha) \otimes_{U(\mathfrak{G})} V$.

By [Ma, Lemma 4.3] there exists the unique polynomial extension $\{\theta_x \mid x \in \mathbb{C}\}$ of the family of automorphisms $\theta_x : U(\alpha) \to U(\alpha)$, $x \in \mathbb{Z}$ such that $\theta_x(v) = X_{-\alpha}^x v X_{-\alpha}^{-x}$, $x \in \mathbb{Z}$. For a $U(\alpha)$-module W and $x \in \mathbb{C}$ we will denote by $\theta_x(W)$ the shift of W by θ_x, i.e. an $U(\alpha)$-module which is equal to W as a vector space and $u \cdot w = \theta(u)w$ for all $u \in U(\alpha)$ and $w \in W$. Clearly, one can consider any $U(\alpha)$-module as an $U(\mathfrak{G})$-module by restriction.

5 Some $sl(3, \mathbb{C})$ theory

Let $\mathfrak{G} = sl(3, \mathbb{C})$, $\pi = \{\alpha, \beta\}$, $\Delta(\alpha) = \{\pm\beta\}$. Denote by $\mathfrak{B}$ the subalgebra in $\mathfrak{G}$ generated by $\mathfrak{H}$ and $\mathfrak{G}_\gamma$, $\gamma \in (\Delta_+ \cup \{\alpha\})$. For a weight element v (of weight μ) in a weight $\mathfrak{G}$-module V such that $cv = t^2 v$ we set

$$X_+^v = \frac{1}{2}(\mu(H_\alpha) + 1 - t)X_\beta - X_{-\alpha}X_{\alpha+\beta},$$
$$X_-^v = \frac{1}{2}(\mu(H_\alpha) + 1 + t)X_\beta - X_{-\alpha}X_{\alpha+\beta}.$$

Proposition 1. *Let V be a simple weight $\mathfrak{G}$-module and $v \in V$ be a non-zero element of weight μ such that $cv = t^2 v$ for some $t \in \mathbb{C}$. Then $c(X_+ v) = (t+1)^2 v$ and $c(X_- v) = (t-1)^2 v$.*

Proof. Consider the elements $v_1 = X_\beta v$ and $v_2 = X_{-\alpha}X_{\alpha+\beta}v$ in V and write down the results of the action of c on them. By the direct calculation we obtain that the vector space generated by v_1, v_2 is invariant under c and the matrix of the transformation c written in the basis v_1, v_2 has the following form:

$$\begin{pmatrix} t^2 - 2(\mu(H_\alpha)+1) + 1 & t^2 - (\mu(H_\alpha)+1)^2 \\ 4 & t^2 + 2(\mu(H_\alpha)+1) + 1 \end{pmatrix}.$$

One can check that the eigenvalues of this matrix are $(t+1)^2$, $(t-1)^2$ and the corresponding eigenvectors are

$$v_+ = \left(\frac{1}{2}(\mu(H_\alpha) + 1 - t)X_\beta - X_{-\alpha}X_{\alpha+\beta}\right)v,$$
$$v_- = \left(\frac{1}{2}(\mu(H_\alpha) + 1 + t)X_\beta - X_{-\alpha}X_{\alpha+\beta}\right)v.$$

This completes our proof. □

6 Schubert filtration for α-stratified modules

The aim of this section is to prove the following theorem, which presents a natural generalization of the Schubert filtration of a simple finite-dimensional module.

Theorem 3. *Let (λ, p) be an element in P^{++}. There is an element $a(\lambda, p) \in \mathfrak{H}^*$ such that for any reduced decomposition $w = s_{\beta_1}s_{\beta_2}\dots s_{\beta_k}$ of the longest element $w \in W(\alpha)$ there exists a (canonical) filtration*

$$L(\lambda, p) = L_1 \supset L_2 \supset \dots \supset L_k \supset L_{k+1} = 0$$

of $L(\lambda, p)$ (viewed as a D-module) by the D-modules L_j, $j = 1, 2, \dots, k$ such that

$$\mathrm{ch}\, L_j = Te^{a(\lambda,p)}\left(d_{\beta_j}\dots d_{\beta_k}e^{\lambda-\rho-a(\lambda,p)}\right).$$

Proof. To simplify our notations we set $L = L(\lambda, p)$. According to [Ma, Proposition 5.7], there exists $x \in \mathbb{C}$ such that the $\mathfrak{G}$-module $L' = \theta_x(L)$ is not α-stratified and hence is a (non-split) extension of a highest weight module $L(1)$ and a module $L(2)$ which is a highest weight module with respect to the base $s_\alpha(\pi)$. Moreover, it might happened that $L(1)$ is not uniquely defined. In this case from $sl(2,\mathbb{C})$-theory it follows that $L(1)$ can be chosen in two different ways (as $L(1)^1$ or $L(1)^2$) and $L(1)^1 \subset L(1)^2$. We choose $L(1) = L(1)^2$. Since $X_{-\alpha}$ is invertible in $U(\alpha)$ we obtain that $L(1)$ is a submodule in L' and $L(2) \simeq L'/L(1)$.

Let μ be the highest weight of $L(1)$. From the definition of Mathieu's localization it follows that $\mu = \lambda - \rho + y\alpha$ for some $y \in \mathbb{C}$. Since $(\lambda, p) \in P^{++}$ we obtain that $p \in \mathbb{Z}$ and $p \neq 0$, hence $\mu(H_\alpha) = |p| - 1$. From this it follows that the unique simple quotient of $L(1)$ is finite-dimensional (or, equivalently, μ is integral dominant).

Let $\mathfrak{G}^\alpha$ denote the Lie algebra corresponding to $\Delta(\alpha)$. Clearly, $\mathfrak{G}^\alpha$ is a subalgebra of $\mathfrak{G}$. Consider a $\mathfrak{G}^\alpha$-module $M = U(\mathfrak{G}^\alpha)L_\mu$. One can see that M is a simple finite-dimensional $\mathfrak{G}^\alpha$-module.

Consider a reduced decomposition $w = s_{\beta_1}s_{\beta_2}\dots s_{\beta_k}$ of the longest element $w \in W(\alpha)$. According to the classical Demazure character formula and the classical Schubert filtration one can consider a Schubert filtration

$$M = M_1 \supset M_2 \supset \cdots \supset M_k \supset M_{k+1} = 0$$

of M corresponding to the above decomposition.

Let v_j denotes a (canonical) generator of M_j for $j \in \{1, 2, \dots, k\}$. First of all we have to look at the module $L(1)$ a bit closer. Clearly, $L(1)$ is obtained from the Verma module $M(\mu + \rho)$ by factoring out all non-trivial submodules of the form $M(g(\mu+\rho))$, $g \in W(\alpha)$. Consider a unital subalgebra $\hat{K}$ of $U(\mathfrak{N}_-)$ generated by all elements $X_{-\gamma}$, $\gamma \in K$. The original module $M(\mu+\rho)$ is $\mathfrak{N}_-$-free. Any canonical generator of $M(g(\mu+\rho))$, $g \in W(\alpha)$ is obtained from a canonical generator of $M(\mu+\rho)$ as a linear combination of elements from $U(\mathfrak{G}^\alpha)$. We note that $\alpha \notin \Delta(\alpha)$. Since any γ from K contains α in a decomposition with respect to π it follows that for any non-zero $v \in M$ and any non-zero $u \in \hat{K}$, $uv \neq 0$.

Set $\widehat{M(1)} = \hat{K}v_1$ and $\widehat{L(1)} = U(D)\widehat{M(1)}$. This module can be extended to a normal D-submodule $\widehat{L_1}$ in L' such that $\widehat{L_1} \cap L(1) = \widehat{L(1)}$. Let $L_1 = \theta_{-x}(\widehat{L_1})$. From the construction of L_1 it follows that

$$\operatorname{ch} L_1 = Te^{a(\lambda,p)}\left(d_{\beta_1}\dots d_{\beta_k}e^{\lambda-\rho-a(\lambda,p)}\right)$$

and thus $L_1 = L$.

Let $l \in \mathbb{N}$ be such that $X^l_{\beta_1}v_1 = av_2$ for $a \in \mathbb{C}$, $a \neq 0$. If $\{\beta_1 \pm \alpha\} \cap \Delta = \varnothing$ then we can define $\widehat{M(2)} = X^l_{\beta_1}\widehat{M(1)}$ and $\widehat{L(2)} = U(D)\widehat{M(2)}$. At the same way as above the last module can be extended to a normal D-submodule $\widehat{L_2}$ in L' such that $\widehat{L_2} \cap L(1) = \widehat{L(2)}$. We set $L_2 = \theta_{-x}(\widehat{L_2})$. One can see that $\widehat{L(2)} \cap M = M_2$. Since $\hat{K}M$ is free over $\hat{K}$, it follows immediately that

$$\operatorname{ch} L_2 = Te^{a(\lambda,p)}\left(d_{\beta_2}\dots d_{\beta_k}e^{\lambda-\rho-a(\lambda,p)}\right)$$

and the module L_2 is constructed.

Finally, suppose that $\{\beta_1 \pm \alpha\} \cap \Delta$ is non-empty. Then the Lie subalgebra in $\mathfrak{G}$ generated by $X_{\pm\alpha}$ and $X_{\pm\beta_1}$ is isomorphic to $sl(3,\mathbb{C})$. Set $\beta = \beta_1$. Using the notations from the previous section consider the elements $(X_+^{v_1})^l v_1$ and $(X_-^{v_1})^l v_1$. Clearly, exactly one of these elements is non zero and thus coincides with v_2 up to a non-zero complex constant. Set $\varepsilon = +\ (-)$ if $X_+^l v_1 \neq 0$ ($X_-^l v_1 \neq 0$). Consider a vector-space $\widehat{M(2)}$ generated by $(X_\varepsilon^v)^l v$, where v runs through the set of all weight elements in $\widehat{M(1)}$ which are eigenelements with respect to c. Set $\widehat{L(2)} = U(D)\widehat{M(2)}$. Using Proposition 1 one can show that the character of this vector-space coincides with the character of the vector-space $\hat{K}M_2$. At the same way as above, module $\widehat{L(2)}$ can be extended to a normal D-submodule $\widehat{L_2}$ in L' such that $\widehat{L_2} \cap L(1) = \widehat{L(2)}$ and we set $L_2 = \theta_{-x}(\widehat{L_2})$. One can see that $\widehat{L(2)} \cap M = M_2$. Since $\hat{K}M$ is free over $\hat{K}$, it follows immediately that

$$\operatorname{ch} L_2 = Te^{a(\lambda,p)} \left(d_{\beta_2} \dots d_{\beta_k} e^{\lambda-\rho-a(\lambda,p)} \right)$$

and the module L_2 is constructed.

Now one can easily construct the modules L_i, $i \geqslant 3$ at the same way as above. Clearly, they are defined in a canonical way. □

We have to note, that in the case of the algebra $sl(n,\mathbb{C})$ the proof can be simplified by considering L as a Gelfand-Zetlin module. In this case the statement of the last theorem follows immediately from the geometric description of the GZ-support of L (see [MO] for details on GZ-modules).

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A note on simplicity of Generalized Verma modules

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Abstract

With each generalized Verma module over a complex simple finite-dimensional Lie algebra we associate a Verma module over the same algebra. We conjecture that simplicity of this Verma module implies the simplicity of the generalized Verma module. We prove this conjecture in a generic case.

1 Introduction

Generalized Verma modules (GVM) are the natural generalization of the famous Verma modules ([D, BGG]). They were studied intensively by several authors, see [RC, CF, FM, MO] and references therein. However, their structure is understood only in a few number of particular cases, see [FM, KM1, KM2, M, MO]. In this note we present one general conjecture on the structure of arbitrary GVM and prove it in a generic case. In fact, with each GVM we associate a Verma module and state that the simplicity of this Verma module implies the simplicity of the GVM.

2 Notations

For a Lie algebra $\mathfrak{A}$ we will denote by $U(\mathfrak{A})$ its universal enveloping algebra and by $Z(\mathfrak{A})$ the center of $U(\mathfrak{A})$.

Let $\mathfrak{G}$ be a simple complex finite-diminsional Lie algebra with a fixed Cartan subalgebra $\mathfrak{H}$ and the corresponding root system Δ with the chosen base π. This data lead to the standard triangular decomposition of $\mathfrak{G}$ and to the decomposition of Δ into positive and negative roots. For $\alpha \in \Delta$ let $\mathfrak{G}_\alpha$ denote the root space of $\mathfrak{G}$ corresponding to the root α. Let W be the Weyl group of Δ and let $\cdot$ denote its action on $\mathfrak{H}^*$. Let ρ denote the half-sum of all positive roots. For $\lambda \in \mathfrak{H}^*$ let $M(\lambda)$ denote the Verma module associated with λ and π ([D, Chapter 7]).

Fix a subset $S \subset \pi$. Let $\mathfrak{G}^S$ be a (semi-simple) subalgebra of $\mathfrak{G}$ generated by all $\mathfrak{G}_{\pm\alpha}$, $\alpha \in S$. Let Δ^S be the root subsystem of Δ with the base S and W^S be the corresponding Weyl group. Let $\mathfrak{H}_S$ denote the subspace of all $h \in \mathfrak{H}$ such that $\alpha(h) = 0$ for all $\alpha \in S$. Consider a subalgebra $\mathfrak{N}_\pm(S)$ generated by all $\mathfrak{G}_\alpha$, where α is a positive root in Δ which does not belong to Δ^S, and set $\mathfrak{B}(S) = \mathfrak{G}^S \oplus \mathfrak{H}_S \oplus \mathfrak{N}_+(S)$. Let ρ^S be the half-sum of all positive roots from Δ^S and $\rho_S = \rho - \rho^S$.

A $\mathfrak{G}$-module V is said to be an S-weight module if the action of $\mathfrak{H}_S$ is diagonalizable on V. For S-weight modules we retain all standard notations. A non-zero S-weight element v in an S-weight $\mathfrak{G}$-module will be called S-primitive if $\mathfrak{N}_+(S)v = 0$.

Consider a simple $\mathfrak{G}^S$ module V (not necessarily weight) and $\lambda \in \mathfrak{H}_S^*$. Setting $hv = (\lambda - \rho_S)(h)v$ and $av = 0$ for $h \in \mathfrak{H}_S$, $a \in \mathfrak{N}_+(S)$ and $v \in V$ we turn V into a $\mathfrak{B}(S)$-module. The $\mathfrak{G}$-module

$$M(\lambda, V) = U(\mathfrak{G}) \bigotimes_{U(\mathfrak{B}(S))} V$$

is called generalized Verma module (GVM) associated with V and λ. Basic properties of GVM can be found, for example, in [CF]. We note only that $M(\lambda, V)$ is an S-weight $\mathfrak{G}$-module and all S-primitive generators of $M(\lambda, V)$ are $v \in M(\lambda, V)_{\lambda-\rho_S} \simeq V$.

Unlike the classical case of Verma modules it is not clear if for any GVM $M(\lambda, V)$ and any $\mu \in \mathfrak{H}_S^*$ the corresponding $\mathfrak{G}^S$-module $M(\lambda, V)_\mu$ is of finite length or not. Using [K, Theorem 3.5] we can state that $M(\lambda, V)_\mu$ is of finite length in the case when V is a Harish-Chandra module (i.e. one coming from a group representation).

3 Basic construction

Let $M(\lambda, V)$ be a GVM. Then V has a central character, say χ_V, as a $\mathfrak{G}^S$-module by Quillen's lemma. By [D, Proposition 7.4.8] there exists a Verma module $M(\mu)$ over $\mathfrak{G}^S$ for some $\mu \in (\mathfrak{H}^S)^*$ having the same central character (χ_V). By [D, Proposition 7.4.7], a Verma module $M(\nu)$ over $\mathfrak{G}^S$, $\nu \in (\mathfrak{H}^S)^*$ has central character χ_V if and only if $\nu \in W^S(\mu)$. Among these Verma modules we choose $M(\nu)$ with ν lying in the anti-dominant Weyl chamber (with respect to S). Under this choice $M(\nu)$ is always simple. Consider a GVM $M(\lambda, M(\nu))$. One can easily show that $M(\lambda, M(\nu)) \simeq M(\lambda + \nu)$. Set $M(\lambda + \nu) = \mathfrak{f}(M(\lambda, V))$.

4 Main result

Conjecture 1. *If $\mathfrak{f}(M(\lambda, V))$ is simple then $M(\lambda, V)$ is simple.*

Remark 1. *We believe that in this conjecture one can replace $\mathfrak{G}$ by a symmetrizable Kac-Moody Lie algebra (but $\mathfrak{G}^S$ should remain finite dimensional one).*

Unfortunately we can not prove this conjecture in the complete generality. In what follows we will prove it in the so-called *generic* situation. In fact we will prove the following:

Theorem 1. *Suppose that the support of $M(\lambda + \nu) = \mathfrak{f}(M(\lambda, V))$ intersects the set $(W \cdot (\lambda + \nu)) - \rho$ only in the highest weight. Then $M(\lambda, V)$ is simple.*

Remark 2. *From the BGG criterion ([BGG, Theorem 2,3]) it follows that $M(\lambda+\nu)$ is simple under the conditions of Theorem 1.*

We call this situation generic since $(W \cdot \xi) - \rho$ intersect the support of $M(\xi)$ only in the highest weight for almost all Verma modules. Thus the above theorem can be applied in almost all cases and states that almost all GVMs are simple.

Proof. Consider $M(\lambda+\nu)$ as a GVM $M(\lambda, M(\nu))$. In this proof all weight spaces are assumed to be S-weight spaces and all weights are assumed to be S-weights. Suppose that $M(\lambda, V)$ is not simple and have a non-trivial submodule N. Clearly, N contains a non-zero S-primitive element $v \in M(\lambda, V)_\mu$, $\lambda \neq \mu$. Obviously, the $\mathfrak{G}^S$ module $M(\lambda, V)_\mu$ is obtained from V via tensor product with a with a finite dimensional module, say F. Thus by [K, Theorem 5.1] we can assume that $Z(\mathfrak{G}^S)$ acts on v via some character, say θ.

Consider a $\mathfrak{G}^S$ module $M(\lambda, M(\nu))_\mu$. Clearly, it is obtained from $M(\nu)$ via tensor product with F. Moreover, since $M(\nu)$ is a Verma module, then $M(\nu) \otimes F$ has a Verma flag ([D, Lemma 7.6.14]). Comparing the filtration in [D, Lemma 7.6.14] with [K, Theorem 5.1] we conclude that necessarily one of Verma sub-quotients of $M(\lambda, M(\nu))_\mu$, say $M(\xi)$, has the central character θ.

Consider the Verma module $M(\xi+\mu)$ as $M(\mu, M(\xi))$ and let w be its canonical generator. Since w is an S-primitive element we can calculate the action of $Z(\mathfrak{G})$ on w in terms of θ and μ using the S-Harish-Chandra homomorphism ([DFO, Proposition 2]). Moreover, the same can be done for v. This implies that the central characters of $M(\xi+\mu)$ and $M(\lambda, V)$ and thus the central characters of $M(\xi+\mu)$ and $M(\lambda+\nu)$ coincides. From Harish-Chandra theorem ([D, Theorem 7.4.7]) we obtain immediately that $\xi + \mu \in W \cdot (\lambda+\nu)$, which contradicts conditions of our theorem. This completes the proof. $\square$

Remark 3. *Using the S-Harish-Chandra homomorphism ([DFO, Proposition 2]) and [K, Theorem 3.5] one can also prove that $M(\lambda, V)$ has a (finite) composition series if V is a Harish-Chandra module.*

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