REPRESENTATIONS IN FINITE WREATH PRODUCTS

Enumerative theory and applications

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1. Introduction

What makes groups like $GL(n, \mathbf{k})$, $\mathbf{k}$ a field, or the symmetric groups S_n important representation groups, and do there exist "good" representation sequences essentially different from these canonical choices? It is one of the purposes of the present paper to provide a possible answer to this question in the case of permutation representations. Another purpose is to survey some important applications of the enumerative theory described in this paper, and to illustrate the general theory by examining a number of examples. In general, a necessary but certainly not sufficient condition to be imposed on a representation sequence $\{R_n\}$ is that the structure of its members R_n is rather well understood, and that computations in connection with group operations may be performed in an effective way. In dealing with possible generalizations of permutation representations we will have to start from some class $\mathcal{R}$ of sequences $R_0, R_1, \ldots$ of finite groups, which is a fairly natural and technically sufficiently controlled "neighborhood" of the sequence $\{S_n\}$, and it is far from clear from the outset what such a class might be. Also, our answer will have to depend on the precise meaning attached in this context to the word "good", i.e., we will have to exhibit a criterion, applicable to sequences $\{R_n\} \in \mathcal{R}$, when such a sequence is to be termed a good representation sequence. The following paragraphs are intended to motivate and explain the choices forming the basis of this investigation. We first turn to permutation representations.

For a group Γ denote by $s_\Gamma(n)$ the number of subgroups of index n in Γ. If, for instance, Γ is finitely generated or of finite (subgroup) rank then $s_\Gamma(n)$ is finite for all n. There is a well–known connection between the enumeration of Γ– actions on finite sets (i.e. permutation represention of Γ) and the problem of counting finite index subgroups in Γ, manifesting itself in a variety of results scattered throughout the literature. One of the earliest results in this direction is M. Hall's recursion formula

$$s_{F_r}(n) = n(n!)^{r-1} - \sum_{k=1}^{n-1} \big((n-k)!\big)^{r-1} s_{F_r}(k) \tag{1}$$

for the number of index n subgroups in the free group of rank r; cf. [17, Theorem 5.2]. Here, the connection with permutation representations becomes visible if we interpret $(n!)^{r-1}$ and $((n-k)!)^{r-1}$ as values of the function $|\mathrm{Hom}(F_r, S_m)|/(m!)$. Hall's formula was generalized by Dey [10, Theorem 6.10] to (non–trivial) free products. In retrospective, it appears that Hall's formula together with its background on Schreier systems and their associated subgroups in free groups developed in [20] marks the beginning of what has in recent years become known as the theory of subgroup growth; cf. for example [25] and [26]. Denote by $\iota_m(H)$ the number of solutions of the equation $x^m = 1$ in a finite group H, i.e., $\iota_m(H) = |\mathrm{Hom}(C_m, H)|$, where C_m is the cyclic group of order m. Chowla, Herstein and Scott [9] obtain the exponential generating function of the sequence $\{\iota_m(S_n)\}_{n=0}^{\infty}$ as

$$\sum_{n=0}^{\infty} \frac{\iota_m(S_n)}{n!} z^n = \exp\left(\sum_{d|m} \frac{z^d}{d} \right).$$

The more general relation

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(G,S_n)|}{n!} z^n = \exp\left(\sum_{d|m} \frac{s_G(d)}{d} z^d\right), \quad m = |G|$$

for an arbitrary finite group G, which exhibits the exponential generating function of the sequence $\{|\mathrm{Hom}(G,S_n)|\}_0^\infty$ as a particular type of entire function, was a starting point in [41] for the asymptotic enumeration of finite group actions; cf. [41, Theorem 5]. The condition that G is finite can in fact be relaxed considerably: If Γ is a group such that $|\mathrm{Hom}(\Gamma,S_n)| < \infty$ for all $n \geq 0$, then we have the (formal) relation

$$\sum_{n\geq 0} \frac{|\mathrm{Hom}(\Gamma,S_n)|}{n!} z^n = \exp\left(\sum_{n\geq 1} \frac{s_\Gamma(n)}{n} z^n\right). \tag{2}$$

The most general expression of this correspondence between group actions and subgroups of finite index to be found in the literature appears to be [13, Prop. 1], which apart from incorporating the results mentioned so far, can also be used for example to count torsion–free subgroups of finite index. The latter aspect has served as a starting point for a detailed analysis of the growth behaviour and the asymptotics of the number of free subgroups of finite index in a finitely generated virtually free group; cf. [38] and [39]. Formula (2) may be interpreted as a property of the sequence $\{S_n\}$ of symmetric groups: If Γ is, say, finitely generated, then the condition on $\mathrm{Hom}(\Gamma,S_n)$ is satisfied, and (2) expresses the generating function $\sum_{n\geq 0} |\mathrm{Hom}(\Gamma,S_n)| z^n/n!$ in terms of the subgroup numbers $s_\Gamma(n)$ of Γ. From a more abstract point of view this means that enumerating permutation representations of a group Γ yields *decodable information* on structural invariants of Γ.

The subject gained an interesting new perspective with the appearance of Chigira's paper [7]. In this paper Chigira studies the equation $x^m = 1$ in finite wreath products of the form $H \wr S_n$ and $H \wr A_n$, and in the Weyl groups W_n of type D_n, obtaining generating functions for the sequences $\{\iota_m(H \wr S_n)\}_{n=0}^\infty$, $\{\iota_m(H \wr A_n)\}_{n=0}^\infty$, and $\{\iota_m(W_n)\}_{n=0}^\infty$; cf. [7, Theorems 2, 4, 5]. Furthermore, he indicates how to modify Wilf's argument in [61] leading to an asymptotic formula for $\iota_m(S_n)$ with m fixed and $n \to \infty$ to obtain a similar asymptotic formula for $\iota_m(H \wr S_n)$; cf. [7, Theorem 3]. Chigira's results, limited as they are, nevertheless were the first to point towards the possibility of developing an enumerative theory for wreath product representations generalizing both (2) and the results of [7] in a far reaching and uniform way. Such a theory will be described in this paper.

Let H be a finite group, $N \trianglelefteq H$ a normal subgroup with H/N abelian, and let $\{\Pi_n\}$ denote either the sequence $\{S_n\}$ of symmetric groups or the sequence $\{A_n\}$ of alternating groups. Define

$$(H \wr \Pi_n)_N := \left\{(f,\pi) \in H \wr \Pi_n : \ \prod_i f(i) \in N\right\}, \quad n \geq 0. \tag{3}$$

The process of passing from $H \wr \Pi_n$ to the subgroup $(H \wr \Pi_n)_N$ is referred to as *localization* with respect to N. If $(H : N) \leq 2$ we call such a localization *tame*. Note that for instance W_n, the Weyl group of type D_n, is obtained as $W_n = (C_2 \wr S_n)_{\mathbf{1}}$. Our class

$\mathcal{R}$ will consist of all sequences $\{R_n\}$ of the form (3), and a sequence $\{R_n\} \in \mathcal{R}$ will be termed "good" if the generating function

$$\sum_{n\geq 0} \frac{|\mathrm{Hom}(\Gamma, R_n)|}{n!}\, z^n \tag{4}$$

can be expressed in terms of the representation sequence $\{R_n\}$ and invariants of Γ for all finitely generated groups Γ. It turns out that a sequence $\{(H \wr \Pi_n)_N\}$ is a good representation sequence in this sense if and only if $(H : N) \leq 2$, and our basic counting result (Theorem 1), to be explained in the next section, computes the generating function (4) as well as certain refinements of this series whenever $\{R_n\}$ has tame localization and Γ satisfies some rather mild finiteness assumptions (always met for instance if Γ is finitely generated). As a first illustration of Theorem 1 Chigira's enumerative results [7, Theorems 2, 4, 5] are rederived in Section 3. One of the key ingredients in the description of the generating function (4) given in Theorem 1 is the exterior function Φ_Γ, a certain power series, whose variables $z_{(\Gamma_\Pi, \Gamma_N)}$ are indexed by pairs of subgroups (Γ_Π, Γ_N) of index at most 2 in Γ. With each such subgroup pair (Γ_Π, Γ_N) there is associated a power series $\Theta_{\Gamma_\Pi}^{\Gamma_N}(z)$, and the generating function (4) is obtained (according to Theorem 1) by replacing each variable $z_{(\Gamma_\Pi, \Gamma_N)}$ of Φ_Γ with the corresponding power series $\Theta_{\Gamma_\Pi}^{\Gamma_N}(z)$. In Section 4, Theorem 1 is supplemented by giving an explicit formula for the exterior function Φ_Γ as an arithmetic mean of truncations of certain exponential functions (Theorem 2). Theorems 1 and 2 together constitute our main result concerning the exact enumeration of wreath product representations $\Gamma \to R_n$ with R_n of the form (3). The rest of the paper is organized as follows: In Section 5 we derive explicit formulae for the generating functions $\sum_{n\geq 0} |\mathrm{Hom}(G, R_n)| z^n/n!$, where G is a finite abelian 2–group and the representation sequence $\{R_n\}$ is any of $\{H \wr S_n\}$ or $\{H \wr A_n\}$ with a fixed finite group H, or $\{W_n\}$. This is a much more formidable task than the derivation of Chigira's corresponding results for finite cyclic groups, and serves well to illustrate some of the power of Theorems 1 and 2. Section 6 deals with rather precise asymptotic estimates for functions of the form $|\mathrm{Hom}(G, R_n)|$ with a finite group G and $R_n = (H \wr \Pi_n)_N$ as in (3). These estimates are obtained by combining Theorems 1 and 2 with some complex analytic machinery developed in [41]. Finally, Sections 7 and 8 present two applications of the theory developed so far. In Section 7 we show how some of the explicit formulae derived in Section 5 may be used to produce an efficient and extremely fast algorithm for computing the Euler characteristic of certain Quillen complexes, while Section 8 deals with some of the consequences of an asymptotic formula for the function $|\mathrm{Hom}(G, H \wr S_n)|$, explained in Section 6, for the theory of subgroup growth. The reader is referred to the table of contents for a more detailed exposition of the structure of this paper.

The present paper constitutes a considerably extended version of my talk at the conference Groups Korea '98, and I would like to thank the organizer Prof. A. Kim for the invitation and for inviting this report. I have deliberately included more material here than could be presented in a single lecture, in the hope of giving the reader a reasonably complete overview of these exciting new developments involving a fruitful interplay between group theory, combinatorics, and analysis. Most of the results mentioned in this paper have so far not appeared in print. However, as their publication is under way, I

have for the most part omitted proofs. The main exception occurs in Section 5, whose results appear to be of interest in a broader algebraic context.

Large parts of the research reported here were done while visiting the Mathematical Sciences Research Institute, Berkeley, California in 1996/97 during the Special Year on Combinatorics. It is my pleasure to thank Richard Stanley for the invitation as well as for fruitful discussions. Also, I would like to express my thanks to John Stallings and the participants of his seminar for their interest which manifested itself in a number of stimulating questions and discussions during a series of lectures delivered in 1996/97 on part of this material.

2. The basic counting principle

Denote by $S(\Omega)$ the symmetric group on the set Ω and put $S([n]) = S_n$ for the standard set $[n] := \{1, \dots, n\}$. We choose algebraic multiplication in $S(\Omega)$, i.e., the product $\pi_1 \cdot \pi_2$ of permutations $\pi_1, \pi_2 \in S(\Omega)$ is defined via

$$(\pi_1 \cdot \pi_2)(\omega) := \pi_2(\pi_1(\omega)), \quad \omega \in \Omega.$$

Consequently, group actions on sets will always be right actions, and, for a finite group H and a permutation group $\Pi(\Omega)$ on the finite set Ω, multiplication in the wreath product

$$H \wr \Pi(\Omega) = \Big\{(f, \pi) : \quad f : \Omega \to H,\ \pi \in \Pi(\Omega)\Big\}$$

is given by the formulae

$$\begin{aligned}(f_1, \pi_1) \cdot (f_2, \pi_2) &:= (f, \pi_1 \cdot \pi_2)\\ f(\omega) &:= f_1(\omega) f_2(\pi_1(\omega)), \quad \omega \in \Omega.\end{aligned}$$

The canonical projection of $H \wr \Pi(\Omega)$ onto $\Pi(\Omega)$ will be denoted by ϵ.

2.1. **Admissibility.** Let Γ be a group (finite or infinite), H a finite group, and let $\Sigma \subseteq \Gamma - \{1\}$ be a normal subset of Γ. Moreover, fix a non–empty set $\Lambda \subseteq \mathbb{N}$ of positive integers and collect all these data into a quadruple $\hat{Q} = (\Gamma, H, \Sigma, \Lambda)$. Given $\hat{Q}$ and a finite set Ω we denote by $\mathrm{Hom}_\Sigma^\Lambda(\Gamma, H \wr S(\Omega))$ the set of all homomorphisms $\tau : \Gamma \to H \wr S(\Omega)$ such that

(i) τ induces a fixed–point–free action of (the elements of) Σ on Ω,
(ii) the lengths of the orbits into which Ω decomposes under $\epsilon\tau$ are contained in the set Λ.

The elements of $\mathrm{Hom}_\Sigma^\Lambda(\Gamma, H \wr S(\Omega))$ will be called (Σ, Λ)–*admissible* representations of Γ in $H \wr S(\Omega)$. A quadruple $\hat{Q}$ is termed *admissible*, if the sets $\mathrm{Hom}_\Sigma^\Lambda(\Gamma, H \wr S_n)$ are finite for all $n \geq 0$. If, for instance, the group Γ is finitely generated or of finite (subgroup) rank, then $\hat{Q} = (\Gamma, H, \Sigma, \Lambda)$ is admissible for each normal set $\Sigma \subseteq \Gamma - \{1\}$, every finite group H, and every set Λ of positive integers. We observe that if $\hat{Q}$ is admissible the construction $\mathrm{Hom}_\Sigma^\Lambda(\Gamma, H \wr S(-))$ provides a covariant functor on the category Ens of

finite sets and bijections.[1] Let $N \trianglelefteq H$ be a normal subgroup with H/N abelian and let $\{\Pi_n\}_0^\infty$ denote either the sequence $\{S_n\}_0^\infty$ of symmetric groups or the sequence $\{A_n\}_0^\infty$ of alternating groups. For a set $M \subseteq \mathbb{N}_0$ of non–negative integers and an integer $n \geq 0$ we denote by

$$\mathrm{Hom}_\Sigma^{\Lambda,M}(\Gamma, (H \wr \Pi_n)_N) \tag{5}$$

the set of those (Σ,Λ)–admissible representations $\tau : \Gamma \to (H \wr \Pi_n)_N$ such that the number of orbits into which $[n]$ decomposes under $\epsilon\tau$ is contained in the set M. If $\hat{Q}$ is admissible the set (5) is finite for each set M of non–negative integers and every integer $n \geq 0$ (for the significance of the combinatorial parameters Σ, Λ, M see for instance [42, Examples 2,3]). Our first main result computes the exponential generating function of the sequence $\{|\mathrm{Hom}_\Sigma^{\Lambda,M}(\Gamma, (H \wr \Pi_n)_N)|\}_{n=0}^\infty$ whenever the quadruple $\hat{Q} = (\Gamma, H, \Sigma, \Lambda)$ is admissible, the representation sequence $\{(H \wr \Pi_n)_N\}$ has tame localization, and Γ satisfies one further very mild finiteness assumption. Before describing this result we take a closer look at its key ingredients.

2.2. **The series $\Theta_{\Gamma_\Pi}^{\Gamma_N}(z)$.** Let $\hat{Q} = (\Gamma, H, \Sigma, \Lambda)$ be an admissible quadruple, and let $N \trianglelefteq H$ be a normal subgroup with H/N abelian. Given a subgroup Γ' of (finite) index n in Γ, a homomorphism $\chi : \Gamma' \to H$, and a right transversal $\{\gamma_1, \ldots, \gamma_n\}$ for Γ' in Γ, define

$$\Gamma_\chi := \Big\{\gamma \in \Gamma : \prod_j \chi(\gamma_j \gamma \overline{\gamma_j \gamma}^{-1}) \in N\Big\},$$

where $^- : \Gamma \to \{\gamma_1, \ldots, \gamma_n\}$ associates with each element γ the representative $\overline{\gamma}$ of the coset $\Gamma'\gamma$. Consider the map $\psi : \Gamma \to H/N$ sending γ to $\prod_j \chi(\gamma_j \gamma \overline{\gamma_j \gamma}^{-1})N$. Since H/N is abelian the composition of χ with the canonical projection $H \to H/N$ factors through $\Gamma'/[\Gamma',\Gamma']$ inducing a homomorphism $\chi' : \Gamma'/[\Gamma',\Gamma'] \to H/N$, and we have $\psi = \chi' \circ V_{\Gamma\to\Gamma'}$, where $V_{\Gamma\to\Gamma'} : \Gamma \to \Gamma'/[\Gamma',\Gamma']$ is the transfer of Γ into Γ'. Since the transfer is a homomorphism and independent of the choice of representatives the same is true for ψ; consequently, $\Gamma_\chi = \ker(\psi)$ is a normal subgroup of index at most $(H : N)$ in Γ, and is independent of the system of representatives used in its definition. Let $\tau_{\Gamma'} : \Gamma \to S(\Gamma'\backslash\Gamma)$ describe the action of Γ by right multiplication on the set $\Gamma'\backslash\Gamma$ of right cosets of Γ' in Γ, and let $\Pi(\Gamma'\backslash\Gamma)$ be the subgroup of $S(\Gamma'\backslash\Gamma)$ isomorphic to Π_n. Given a positive integer n and two normal subgroups $\Gamma_\Pi, \Gamma_N \trianglelefteq \Gamma$ define

$$s(n, \Gamma_\Pi, \Gamma_N) := \sum_{\substack{(\Gamma:\Gamma')=n \\ \Gamma'\cap\Sigma=\emptyset \\ \tau_{\Gamma'}^{-1}(\Pi(\Gamma'\backslash\Gamma))=\Gamma_\Pi}} \big|\big\{\chi \in \mathrm{Hom}\,(\Gamma', H) : \ \Gamma_\chi = \Gamma_N\big\}\big|. \tag{6}$$

As one can show, admissibility of $\hat{Q}$ implies in particular that the cardinal number (6) is finite for every $n \in \Lambda$ and all normal subgroups $\Gamma_\Pi, \Gamma_N \trianglelefteq \Gamma$. With each pair (Γ_Π, Γ_N) of normal subgroups in Γ we associate a formal power series $\Theta_{\Gamma_\Pi}^{\Gamma_N}(z)$ via

$$\Theta_{\Gamma_\Pi}^{\Gamma_N}(z) := \sum_{n\in\Lambda} \frac{|H|^{n-1}\, s(n, \Gamma_\Pi, \Gamma_N)}{n}\, z^n.$$

[1] Such functors are often called species; cf for example [23].

Remark. Observe that if two pairs (Γ_Π, Γ_N) and (Γ'_Π, Γ'_N) of normal subgroups are conjugate under an automorphism of Γ then $\Theta^{\Gamma_N}_{\Gamma_\Pi}(z) = \Theta^{\Gamma'_N}_{\Gamma'_\Pi}(z)$.

2.3. **The function Φ_Γ.** Put $\alpha_\Lambda(\Pi) := \max\limits_{n\in\Lambda}(S_n : \Pi_n)$, and assume that $(H : N) \le 2$ and that Γ contains only finitely many subgroups of index $k_0 := \max\big(\alpha_\Lambda(\Pi), (H : N)\big)$. Consider the system of subgroup pairs

$$\mathcal{U}_\Gamma := \Big\{(\Gamma_\Pi, \Gamma_N) : \ \Gamma_\Pi, \Gamma_N \le \Gamma,\ (\Gamma : \Gamma_\Pi) \le \alpha_\Lambda(\Pi),\ (\Gamma : \Gamma_N) \le (H : N)\Big\}$$

and fix a set $E \subseteq \Gamma$ which generates Γ as a normal subgroup. With each pair $(\Gamma_\Pi, \Gamma_N) \in \mathcal{U}_\Gamma$ we associate a formal variable $z_{(\Gamma_\Pi,\Gamma_N)}$ and a discrete variable $\nu(\Gamma_\Pi, \Gamma_N)$ taking non–negative integral values, and form the power series

$$\Phi_\Gamma\big(z_{(\Gamma_\Pi,\Gamma_N)};\ (\Gamma_\Pi, \Gamma_N) \in \mathcal{U}_\Gamma\big) := \sum_\nu \prod_{(\Gamma_\Pi,\Gamma_N)} z^{\nu(\Gamma_\Pi,\Gamma_N)}_{(\Gamma_\Pi,\Gamma_N)}/\nu(\Gamma_\Pi, \Gamma_N)!,$$

where the right–hand sum is extended over those maps $\nu : \mathcal{U}_\Gamma \to \mathbb{N}_0$ satisfying the following $2|E| + 1$ conditions:

$$\begin{cases} \displaystyle\sum_{(\Gamma_\Pi,\Gamma_N)} \nu(\Gamma_\Pi, \Gamma_N) \ \in M \\ \displaystyle\sum_{e\notin\Gamma_\Pi} \nu(\Gamma_\Pi, \Gamma_N) \equiv 0 \mod \alpha_\Lambda(\Pi), \quad e \in E \\ \displaystyle\sum_{e\notin\Gamma_N} \nu(\Gamma_\Pi, \Gamma_N) \equiv 0 \mod (H : N), \quad e \in E. \end{cases} \tag{7}$$

The function Φ_Γ is called the *exterior function* associated with the data Γ, H, N, $\{\Pi_n\}_0^\infty$, Λ, and M. Denote by $\mathcal{J}_\Gamma$ the intersection of all subgroups of index k_0 in Γ. The quotient $\Gamma/\mathcal{J}_\Gamma$ is an elementary abelian 2–group of rank $\log_2\big(\sum_{k\le k_0} s_\Gamma(k)\big)$. The following result contains some important observations concerning the exterior function Φ_Γ.

Proposition 1 ([42], Prop. 2)**.** (i) *The function Φ_Γ does not depend on the choice of the normal generating system E.*
(ii) *The functions Φ_Γ and $\Phi_{\Gamma/\mathcal{J}_\Gamma}$, formed with respect to the same data H, N, $\{\Pi_n\}_0^\infty$, Λ, and M, coincide up to a canonical identification of variables*

$$z_{(\Gamma_\Pi,\Gamma_N)} \mapsto z_{(\Gamma_\Pi/\mathcal{J}_\Gamma,\Gamma_N/\mathcal{J}_\Gamma)}, \quad (\Gamma_\Pi, \Gamma_N) \in \mathcal{U}_\Gamma.$$

We shall see later (Theorem 2 below) that the exterior function Φ_Γ can be expressed explicitly as an arithmetic mean of truncations of certain exponential functions.

2.4. **The result.** This is the following.

Theorem 1 ([42], Theorem 1)**.** *Let $\hat{Q} = (\Gamma, H, \Sigma, \Lambda)$ be an admissible quadruple, $N \le H$ a subgroup of index $(H : N) \le 2$, $M \subseteq \mathbb{N}_0$ a set of non–negative integers, and let $\{\Pi_n\}_0^\infty$ denote either the sequence $\{S_n\}_0^\infty$ of symmetric groups or the sequence*

$\{A_n\}_0^\infty$ of alternating groups. Assume that Γ contains only finitely many subgroups of index $\max\big(\alpha_\Lambda(\Pi), (H:N)\big)$. *Then*

$$\sum_{n\geq 0} \frac{|\mathrm{Hom}_\Sigma^{\Lambda,M}(\Gamma, (H\wr\Pi_n)_N)|}{n!}\, z^n = \Phi_\Gamma\big(\Theta_{\Gamma_\Pi}^{\Gamma_N}(z);\ (\Gamma_\Pi,\Gamma_N)\in\mathcal{U}_\Gamma\big), \tag{8}$$

i.e., the exponential generating function of the sequence $\big\{|\mathrm{Hom}_\Sigma^{\Lambda,M}(\Gamma, (H\wr\Pi_n)_N)|\big\}_0^\infty$ *is obtained from the series Φ_Γ by replacing each variable $z_{(\Gamma_\Pi,\Gamma_N)}$ with the corresponding power series $\Theta_{\Gamma_\Pi}^{\Gamma_N}(z)$.*

3. Some examples

As a first illustration of Theorem 1 we derive the enumerative results in Chigira's paper [7].

3.1. **Symmetric wreath products.** Putting $\Pi_n = S_n$ $(n\geq 0)$ and $N = H$ we have $(H\wr\Pi_n)_N = H\wr S_n$, $\alpha_\Lambda(\Pi) = 1$, and $\mathcal{U}_\Gamma = \{(\Gamma,\Gamma)\}$. Dropping the subscript (Γ,Γ) the exterior function Φ_Γ becomes

$$\Phi_\Gamma(z) = \sum_{\nu\in M} z^\nu/\nu! =: e_M(z),$$

and Theorem 1 yields the following.

Corollary 1. *Let $\hat{Q} = (\Gamma, H, \Sigma, \Lambda)$ be an admissible quadruple and $M\subseteq\mathbb{N}_0$ a set of non–negative integers. Then*

$$\sum_{n\geq 0} \frac{|\mathrm{Hom}_\Sigma^{\Lambda,M}(\Gamma, H\wr S_n)|}{n!}\, z^n = e_M\left(\sum_{n\in\Lambda} \frac{|H|^{n-1}\, s_{\Gamma,\Sigma}^H(n)}{n}\, z^n\right), \tag{9}$$

where $s_{\Gamma,\Sigma}^H(n) := \sum_{\substack{(\Gamma:\Gamma')=n\\ \Gamma'\cap\Sigma=\emptyset}} |\mathrm{Hom}(\Gamma', H)|$.

The special case of Corollary 1 where $H = 1$ is [13, Prop. 1], which in turn yields (2) by setting $\Sigma = \emptyset$, $\Lambda = \mathbb{N}$, and $M = \mathbb{N}_0$. If we let $\Gamma = G$ and H be arbitrary finite groups and put $\Sigma = \emptyset$, $\Lambda = \mathbb{N}$, and $M = \mathbb{N}_0$, then we find from Corollary 1 that

$$\sum_{n=0}^\infty \frac{|\mathrm{Hom}(G, H\wr S_n)|}{n!}\, z^n = \exp\left(\sum_{d|m} \frac{|H|^{d-1}\, s_G^H(d)}{d}\, z^d\right), \quad m = |G|, \tag{10}$$

where $s_G^H(d) := s_{G,\emptyset}^H(d) = \sum_{(G:U)=d} |\mathrm{Hom}(U, H)|$. For $G = C_m$ a cyclic group of order m formula (10) specializes to

$$\sum_{n=0}^\infty \frac{\iota_m(H\wr S_n)}{n!}\, z^n = \exp\left(\sum_{d|m} \frac{|H|^{d-1}\, \iota_{m/d}(H)}{d}\, z^d\right). \tag{11}$$

This is [7, Theorem 2]. For a more detailed discussion of Corollary 1 the reader is referred to [42, Sect. 3.1].

Remarks. 1. For $M = \mathbb{N}_0$ formula (9) is equivalent to the recursion formula

$$n\, h^H_{\Gamma,\Sigma}(n,\Lambda) = \sum_{\substack{1\le k\le n\\ k\in\Lambda}} |H|^{k-1}\, s^H_{\Gamma,\Sigma}(k)\, h^H_{\Gamma,\Sigma}(n-k,\Lambda), \quad n \ge 1, \tag{12}$$

where $h^H_{\Gamma,\Sigma}(n,\Lambda) := |\mathrm{Hom}^{\Lambda}_{\Sigma}(\Gamma, H \wr S_n)|/(n!)$. Hall's formula (1) for free groups is a special case of this.

2. For a permutation $\pi \in S(\Omega)$ on a finite set Ω we denote by $a_j(\pi)$ the number of j–cycles occurring in the cycle decomposition of π, and by $a(\pi) = \sum a_j(\pi)$ the total number of cycles of π. Given sets $\Lambda \subseteq \mathbb{N}$, $M \subseteq \mathbb{N}_0$, and a permutation group $\Pi \le S(\Omega)$ on Ω define the (generalized) cycle index $\mathrm{Cyc}^{\Lambda,M}_{\Pi}(x_1,\dots,x_{|\Omega|})$ of Π with respect to the sets Λ and M as

$$\mathrm{Cyc}^{\Lambda,M}_{\Pi}(x_1,\dots,x_{|\Omega|}) := \frac{1}{|\Pi|} \sum_{\substack{\pi\in\Pi\\ a(\pi)\in M\\ a_j(\pi)=0 (j\notin\Lambda)}} \prod_{j=1}^{|\Omega|} x_j^{a_j(\pi)} \quad \in \mathbb{Q}[x_1,\dots,x_{|\Omega|}]$$

(the ordinary cycle index $\mathrm{Cyc}_{\Pi}(x_1,\dots,x_{|\Omega|})$ of Π corresponds to the case where $\Lambda = \mathbb{N}$ and $M = \mathbb{N}_0$). Then we have for an admissible quadruple $\hat{Q} = (\Gamma, H, \Sigma, \Lambda)$ and every set M of non–negative integers that

$$|\mathrm{Hom}^{\Lambda,M}_{\Sigma}(\Gamma, H \wr S_n)| = n!\, \mathrm{Cyc}^{\Lambda,M}_{S_n}(0!\,|H|^0\, s^H_{\Gamma,\Sigma}(1),\dots,(n-1)!\,|H|^{n-1}\, s^H_{\Gamma,\Sigma}(n)),$$

i.e., the number $|\mathrm{Hom}^{\Lambda,M}_{\Sigma}(\Gamma, H\wr S_n)|$ of (Σ,Λ)–admissible representations $\tau : \Gamma \to H \wr S_n$ having the property that the number of orbits into which $[n]$ decomposes under the Γ–action induced by τ is contained in the set M is obtained from the integral polynomial $n!\,\mathrm{Cyc}^{\Lambda,M}_{S_n}(x_1,\dots,x_n)$ by the substitutions $x_j = (j-1)!\,|H|^{j-1}\, s^H_{\Gamma,\Sigma}(j)$, $1 \le j \le n$.

3.2. **Alternating wreath products.** In Theorem 1 put $\Pi_n = A_n$ $(n \ge 0)$, $N = H$, $\Sigma = \emptyset$, $\Lambda = \mathbb{N}$, and $M = \mathbb{N}_0$. Then $(H \wr \Pi_n)_N = H \wr A_n$, and the set $\mathcal{U}_\Gamma$ can be identified with the system of all subgroups of index at most 2 in Γ. Let $\Gamma = G = \langle\zeta\rangle$ be a cyclic group of order m with m even. Then there are two subgroups of index at most 2 in G, $G_1 = G$ and $G_2 = \langle\zeta^2\rangle$, and we denote by z_1, ν_1 respectively z_2, ν_2 their associated variables. Put $E = \{\zeta\}$. Our exterior function Φ_G becomes

$$\Phi_G(z_1,z_2) = \sum_{\substack{\nu_1,\nu_2\ge 0\\ \nu_2\equiv 0(2)}} \frac{z_1^{\nu_1} z_2^{\nu_2}}{\nu_1!\nu_2!} = \exp(z_1)\cosh(z_2) = \frac{1}{2}\left\{ \mathrm{e}^{z_1+z_2} \,+\, \mathrm{e}^{z_1-z_2}\right\}.$$

For $d \mid m$ consider the subgroup $U = \langle\zeta^d\rangle$ of index d in G. The generator ζ acts on the coset space $U\backslash G$ as the d-cycle $\tau_U(\zeta) = (U\cdot 1, U\cdot\zeta, \dots, U\cdot\zeta^{d-1})$. Consequently, $\tau_U(G) \subseteq A(U\backslash G)$ if and only if d is odd. From this we see that

$$s(d,G_1,G) = \begin{cases} \iota_{m/d}(H), & \text{d odd}\\ 0, & \text{d even}\end{cases} \quad \text{and} \quad s(d,G_2,G) = \begin{cases} \iota_{m/d}(H), & \text{d even}\\ 0, & \text{d odd.}\end{cases}$$

It follows that

$$\Theta_1(z) \;:=\; \Theta_{G_1}^{G}(z) = \sum_{\substack{d\mid m\\ d\equiv 1(2)}} \frac{|H|^{d-1}\,\iota_{m/d}(H)}{d} z^d,$$

$$\Theta_2(z) \;:=\; \Theta_{G_2}^{G}(z) = \sum_{\substack{d\mid m\\ d\equiv 0(2)}} \frac{|H|^{d-1}\,\iota_{m/d}(H)}{d} z^d,$$

and, by Theorem 1

$$\sum_{n=0}^{\infty} \frac{\iota_m(H\wr A_n)}{n!}\, z^n = \frac{1}{2}\left\{ \exp\left(\sum_{d\mid m} \frac{|H|^{d-1}\,\iota_{m/d}(H)}{d}\, z^d\right) + \exp\left(\sum_{d\mid m} \frac{(-|H|)^{d-1}\,\iota_{m/d}(H)}{d}\, z^d\right)\right\}. \tag{13}$$

For odd m this equation coincides with (11). Hence, we have obtained Chigira's result [7, Theorem 4]. The special case of (13) where $H = 1$ and $m = 2$ was already found in [35, Sect. 5].

3.3. **The Weyl groups of type D_n.** If we put $\Pi_n = S_n$ $(n \geq 2)$, $H = C_2$, and $N = \mathbf{1}$, then $(H \wr \Pi_n)_N = W_n$ is the Weyl group of the (crystallographic) root system of type D_n. We consider again the cyclic group $\Gamma = G = \langle\zeta\rangle$ of order m with m even, and put $\Sigma = \emptyset$, $\Lambda = \mathbb{N}$, and $M = \mathbb{N}_0$. Keeping the notation of the previous subsection, $\mathcal{U}_\Gamma$ can again be identified with the set $\{G_1, G_2\}$, and, as before

$$\Phi_G(z_1, z_2) = \frac{1}{2}\left\{ \mathrm{e}^{z_1+z_2} \;+\; \mathrm{e}^{z_1-z_2}\right\}.$$

For $d \mid m$ consider the subgroup $U = \langle\zeta^d\rangle$ of index d in G and a homomorphism $\chi : U \to C_2$. If m/d is odd then $\chi(u) = 1$ for all $u \in U$, and $G_\chi = G_1$. If on the other hand m/d is even then there are two homomorphisms $\chi : U \to C_2$: the trivial homomorphism $\chi_0 \equiv 1$ with $G_{\chi_0} = G_1$ and a non–trivial homomorphism χ_1. In the latter case

$$\prod_{i=0}^{d-1} \chi_1(\zeta^{i+1}\overline{\zeta^{i+1}}^{-1}) = \chi_1(\zeta^d) \neq 1,$$

i.e., $\zeta \notin G_{\chi_1}$, and hence $G_{\chi_1} = G_2$. We conclude that

$$s(d, G, G_1) = 1 \quad (d \mid m) \quad \text{and} \quad s(d, G, G_2) = \begin{cases} 1, & \text{m/d even} \\ 0, & \text{m/d odd.} \end{cases}$$

It follows that

$$\Theta_1(z) := \Theta_G^{G_1}(z) = \sum_{d\mid m} \frac{2^{d-1}}{d} z^d \quad \text{and} \quad \Theta_2(z) := \Theta_G^{G_2}(z) = \sum_{\substack{d\mid m\\ m/d\equiv 0(2)}} \frac{2^{d-1}}{d} z^d,$$

and hence

$$\sum_{n=0}^{\infty} \frac{\iota_m(W_n)}{n!}\, z^n = \frac{1}{2}\, \exp\Bigg(\sum_{\substack{d\mid m \\ m/d\equiv 1(2)}} \frac{2^{d-1}}{d}\, z^d \Bigg) \Bigg\{ 1 + \exp\Bigg(\sum_{\substack{d\mid m \\ m/d\equiv 0(2)}} \frac{2^d}{d}\, z^d \Bigg) \Bigg\}. \tag{14}$$

Again, this formula also holds for odd m in virtue of (11), and we have found Chigira's result [7, Theorem 5].

4. Theorem 1 revisited

Let Γ, H, N, $\{\Pi_n\}_0^\infty$, Σ, Λ, and M be as in Theorem 1. As stated in Section 2 the exterior function Φ_Γ does in fact not depend on the (normal) generating system of Γ used in its definition, and its calculation may be performed in the finite group $\Gamma/\mathcal{J}_\Gamma$, where $\mathcal{J}_\Gamma$ denotes the intersection of all subgroups of index k_0 in Γ. Our first task is to describe an explicit formula for Φ_Γ. Put

$$\mathcal{U}_\Gamma^+ := \Big\{ \Gamma' :\ \Gamma' \le \Gamma,\ (\Gamma : \Gamma') \le k_0 \Big\}$$

and denote by $\pi : \Gamma \to \Gamma/\mathcal{J}_\Gamma$ the canonical projection. The quotient $V = \Gamma/\mathcal{J}_\Gamma$ is an elementary abelian 2–group of rank $r = \log_2\big(\sum_{k\le k_0} s_\Gamma(k)\big)$, which we view as a vector space over the field $\mathbf{F}_2$ with two elements. Let $\langle\cdot,\cdot\rangle$ be the usual inner product on V with respect to some fixed basis, i.e., $\langle u, v\rangle = \sum u_i v_i$, and define a map $\hat{}: V \to \mathcal{U}_\Gamma^+$ via $\hat{v} := \pi^{-1}(v^\perp)$. Its inverse is the map $^- : \mathcal{U}_\Gamma^+ \to V$ given by $\overline{\Gamma} = \mathfrak{o}$ and $\overline{\Gamma'} = v$ with $v \ne \mathfrak{o}$ and $\langle v, x\rangle = 0$ for all $x \in \Gamma'/\mathcal{J}_\Gamma$ if $(\Gamma : \Gamma') = 2$. Under this bijection the set $\mathcal{U}_\Gamma$ is identified with the subspace

$$\mathcal{U}_V = V_{\alpha_\Lambda(\Pi)} \oplus V_{(H:N)}$$

of dimension $R = r\big(\log_2 \alpha_\Lambda(\Pi) + \log_2(H:N)\big)$ in $V \oplus V$, where $V_1 := \{\mathfrak{o}\}$ and $V_2 := V$. The inner product on V induces the inner product

$$\big\langle (u,v), (x,y)\big\rangle = \langle u, x\rangle + \langle v, y\rangle$$

on $V \oplus V$, and, hence, on $\mathcal{U}_V$. Denote by $\Phi_V = \Phi_V\big(z_{(u,v)};\ (u,v) \in \mathcal{U}_V\big)$ the power series obtained from Φ_Γ by replacing each variable $z_{(\Gamma_\Pi,\Gamma_N)}$ with the corresponding variable $z_{(u,v)}$, where $\hat{u} = \Gamma_\Pi$ and $\hat{v} = \Gamma_N$. Moreover, for a set $M \subseteq \mathbb{N}_0$ of non–negative integers and a (formal) power series $f(z_1, \dots, z_s) = \sum_{\nu_1,\dots,\nu_s\ge 0} a_{\nu_1,\dots,\nu_s} z_1^{\nu_1} \dots z_s^{\nu_s}$ denote by $[f(z_1,\dots,z_s)]_M$ the truncation

$$\big[f(z_1,\dots,z_s)\big]_M = \sum_{\substack{\nu_1,\dots,\nu_s\ge 0 \\ \nu_1+\dots+\nu_s\in M}} a_{\nu_1,\dots,\nu_s}\, z_1^{\nu_1} \dots z_s^{\nu_s}$$

of f with respect to M. The following result provides an explicit formula for the function Φ_V, and hence also for the function Φ_Γ.

Theorem 2. *We have*

$$\Phi_V = \frac{1}{|\mathcal{U}_V|} \sum_{(u,v)\in\mathcal{U}_V} \Bigg[\exp\Bigg(\sum_{\langle (x,y),(u,v)\rangle = 0} z_{(x,y)} \ - \sum_{\langle (x,y),(u,v)\rangle = 1} z_{(x,y)} \Bigg) \Bigg]_M .$$

This is essentially [47, Theorem 3.1]. Combining Theorem 1 with Theorem 2 and Proposition 1 we now obtain the following more explicit version of Theorem 1.

Theorem 3. *Under the assumptions and with the notation of Theorem* 1 *we have*

$$\sum_{n\geq 0} \frac{|\mathrm{Hom}_\Sigma^{\Lambda,M}(\Gamma,(H\wr\Pi_n)_N)|}{n!}\, z^n = \frac{1}{|\mathcal{U}_V|}\sum_{(u,v)\in\mathcal{U}_V}\left[\exp\left(\sum_{\langle(x,y),(u,v)\rangle=0}\Theta_{\hat{x}}^{\hat{y}}(z)-\sum_{\langle(x,y),(u,v)\rangle=1}\Theta_{\hat{x}}^{\hat{y}}(z)\right)\right]_M, \quad (15)$$

where the members of the right–hand sum are truncated with respect to the variables $\Theta_{\hat{x}}^{\hat{y}}(z)$.

Remark. In order to apply Theorem 2 one has to correctly assign the inner functions $\Theta_{\Gamma_\Pi}^{\Gamma_N}(z)$ to the variables $z_{(u,v)}$ of Φ_V. This is trivial if $\dim(\mathcal{U}_V)\leq 2$, since in this case any bijection between $\mathcal{U}_\Gamma$ and $\mathcal{U}_V$ such that (Γ,Γ) is mapped to $(\mathfrak{o},\mathfrak{o})$ leads to the same (correct) expression for Φ_Γ, and hence for the generating function $\sum|\mathrm{Hom}_\Sigma^{\Lambda,M}(\Gamma,(H\wr\Pi_n)_N)|z^n/n!$ in question. In Theorem 3 the correct assignment is achieved formally by introducing the function $\hat{}: V\to\mathcal{U}_\Gamma^+$, but this leaves us with the task to analyze the bijection $\hat{}$, which may be trivial or not; cf. for instance Section 5.2, where this problem arises, and is solved (in Lemma 7) by means of duality theory.

5. Finite abelian 2–groups

In this section, we derive explicit formulae for the generating functions $\sum_{n\geq 0}|\mathrm{Hom}(G,R_n)|z^n/n!$, where G is a finite abelian 2–group and the representation sequence $\{R_n\}$ is any of $\{H\wr S_n\}$ or $\{H\wr A_n\}$ with a fixed finite group H, or $\{W_n\}$. This is a much more formidable task than the derivation of Chigira's corresponding results on finite cyclic groups, and serves well to demonstrate some of the power of Theorem 3. The material of this section is taken from [47, Sect. 4]. We will use mostly standard notation from the theory of partitions, as explained in the first chapter of [30], and the reader is referred to this source for any concept or notation left undefined. For a prime p and a partition λ we denote by $G(p,\lambda)$ the finite abelian p–group defined by

$$G(p,\lambda) := \bigoplus_{j\geq 1} C_{p^{\lambda_j}} = \bigoplus_{i\geq 1} m_i(\lambda)C_{p^i}.$$

Observe that by the structure theorem for finite abelian groups $G(p,\lambda)\cong G(p,\mu)$ implies $\lambda=\mu$. Philip Hall [18] determined, for any three partitions λ,μ,ν, a polynomial $g_{\mu,\nu}^\lambda(t)$ such that, for every prime p, the number of subgroups $H\leq G\cong G(p,\lambda)$ with $H\cong G(p,\mu)$ and $G/H\cong G(p,\nu)$ is $g_{\mu,\nu}^\lambda(p)$. The reader may consult [30, Chap. II] for the main results on these Hall polynomials, which will be important in what follows. Throughout this section, with the sole exception of Subsection 5.3, where we consider finite abelian p–groups for an arbitrary prime p, $G=G(2,\lambda)$ will be a finite abelian 2–group of type $\lambda=(\lambda_1,\ldots,\lambda_r)$, and $\{e_1,\ldots,e_r\}$ will be a basis for G with e_j of order $|e_j|=2^{\lambda_j}$ for all $j\in[r]$. However, see [47, Sect. 4] for an extension of the subsequent

results to finite abelian p–groups respectively arbitrary finite abelian groups. For any group G we denote by $\mathcal{L}(G)$ the lattice of all subgroups of G, ordered by inclusion.

5.1. **Orbits of $Aut(G)$ on minimal and maximal subgroups.** An essential ingredient in constructing the explicit formulae promised above is the determination of the orbits of $\mathrm{Aut}(G)$ on subgroups of index 2 in G. We will use duality theory in order to make this determination by examining orbits of $\mathrm{Aut}(G)$ on involutions of G. With each element $g = \sum_{j=1}^{r} \alpha_j e_j \in G$ we associate a character $\chi_g : G \to \mathbb{C}^*$ via

$$e_j \mapsto \exp\Big(\frac{2\pi i \alpha_j}{|e_j|}\Big), \quad 1 \le j \le r.$$

It is well known that the map $\phi : G \to \mathrm{Hom}(G, \mathbb{C}^*)$ given by $g \mapsto \chi_g$ is a group isomorphism, where multiplication in $\mathrm{Hom}(G, \mathbb{C}^*)$ is pointwise; cf., for example, [57, Lemma 8.1.3]. For a subgroup $H \le G$ define

$$H^* := \Big\{ g \in G : \ H \le \ker(\chi_g) \Big\}.$$

By a result of Baer [3]

- the map on $\mathcal{L}(G)$ determined by $H \mapsto H^*$ is an antiautomorphism of $\mathcal{L}(G)$,
- we have $(H^*)^* = H$ for all $H \le G$.

An important consequence is the following.

Lemma 1. *Let H be a subgroup of G. Then $H^* \cong G/H$.*

Proof. By Baer's result the interval $[H, G]$ is isomorphic to the lattice $\mathcal{L}(H^*)^{\mathrm{op}}$, the dual of the lattice $\mathcal{L}(H^*)$, and $\mathcal{L}(H^*) \cong \mathcal{L}(H^*)^{\mathrm{op}}$. Hence, G/H and H^* are abelian 2–groups with isomorphic subgroup lattices. Our claim follows now for example from another classical result of Baer; cf. [4] or [57, Theorem 2.6.8]. □

Our next goal is to define, for an endomorphism α of G, an endomorphism α^*, which is an adjoint to α in the appropriate sense. By a matrix of α with respect to the basis $\{e_1, \ldots, e_r\}$ we mean any matrix $(\alpha_{jk}) \in M_r(\mathbb{Z})$ satisfying $\alpha(e_j) = \sum_{k=1}^{r} \alpha_{jk} e_k$ for $1 \le j \le r$. If (α_{jk}) is a matrix for α, then α^* is defined as the endomorphism of G such that the matrix (α^*_{jk}) given by

$$\alpha^*_{jk} := \alpha_{kj} \, \frac{|e_k|}{|e_j|}, \quad 1 \le j, k \le r$$

is a matrix for α^* with respect to the same basis $\{e_1, \ldots, e_r\}$. Observe that since $|\alpha(e_j)|$ divides $|e_j|$ for all j, α^*_{jk} is an integer for all $j, k \in [r]$. Also, if (α_{jk}) is a fixed matrix for α with respect to a given basis, the set of all matrices for α with respect to this basis consists of all $(\beta_{jk}) \in M_r(\mathbb{Z})$ such that $|e_k|$ divides $\alpha_{jk} - \beta_{jk}$ for all pairs j, k. Hence, α^* is well defined.

Lemma 2. *Let α be an automorphism of G. Then*

(i) $\alpha^* \in \mathrm{Aut}(G)$ *and* $(\alpha^*)^{-1} = (\alpha^{-1})^*$,

(ii) $\chi_{g^\alpha}(h) = \chi_g(h^{\alpha^*})$ *for all* $g, h \in G$.

Proof. A straightforward matrix calculation shows that for $\alpha, \beta \in \mathrm{End}(G)$ we have $(\alpha\beta)^* = \beta^*\alpha^*$. The first claim follows immediately from this and the fact that $\mathrm{id}^* = \mathrm{id}$. To prove the second claim, it suffices to show that the claimed equality holds when $g = e_j$ and $h = e_k$ for arbitrary $j, k \in [r]$, and the equality in these cases follows immediately from the definitions. □

Lemma 3. *Let H and K be subgroups of G. Then H and K are conjugate under the action of $\mathrm{Aut}(G)$ if and only if H^* and K^* are conjugate under this action. In particular, if $(G : H) = (G : K) = 2$, then H and K are conjugate under $\mathrm{Aut}(G)$ if and only if the involutions generating H^* and K^* are conjugate.*

Proof. By Lemma 2 we have for every $g \in G$ and $\alpha \in \mathrm{Aut}(G)$ that

$$\ker(\chi_{g^\alpha}) = \ker(\chi_g)^{(\alpha^*)^{-1}},$$

and it is easy to see that for a subgroup $H \leq G$

$$H = \bigcap_{g \in H^*} \ker(\chi_g).$$

Given these two facts, and the fact that $(\alpha^{-1})^* = (\alpha^*)^{-1}$, it is straightforward to show that if $H^\alpha = K$ then $(K^*)^{\alpha^*} = H^*$. The particular statement follows from Lemma 1. □

With the desired duality result in hand, we now investigate the action of $\mathrm{Aut}(G)$ on involutions in G. We will make use of the following well–known fact (see, for example, [31, Chap. XI, Prop. 3]).

Lemma 4. *Let X be a cyclic subgroup of G such that the order of X equals the exponent of G. Then X has a complement in G.*

For a subgroup X of order 2 in G define

$$m(X) := \max\Big\{ |U| : \ X \leq U \leq G, \ U \text{ cyclic} \Big\}.$$

Lemma 5. *Let $X \leq G$ with $|X| = 2$ and let $x = \sum_{j=1}^r \alpha_j e_j$ be the involution generating X. Then*

$$m(X) = \min\Big\{ 2^{\lambda_j} : \ \alpha_j \not\equiv 0 \ (2^{\lambda_j}) \Big\}.$$

We omit the proof; cf. [47, Lemma 4.12]. Our next result explains the significance of the invariant $m(X)$.

Lemma 6. *Let $X, Y \leq G$ with $|X| = |Y| = 2$. Then X and Y are conjugate under the action of $\mathrm{Aut}(G)$ if and only if $m(X) = m(Y)$.*

Proof. We only consider the non–trivial direction. Assume then that there exist cyclic subgroups $T, U \leq G$ such that $X \leq T$, $Y \leq U$, and $|T| = m(X) = m(Y) = |U|$. Put $K := \big\langle e_j : \ |e_j| > m(X) \big\rangle$. Note that every maximal cyclic subgroup of K has order greater than $m(X)$. Thus $T \cap K = \{0\}$, since otherwise we would have $X \leq K$ and $m(X) > |T|$. Let $\overline{G} = G/K$, and let $\overline{T}$ be the image of T under the canonical projection from G onto $\overline{G}$. Then $\overline{T} \cong T$, and $|\overline{T}|$ is the exponent of $\overline{G}$. By Lemma 4, there exists

a complement $\overline{S}$ to $\overline{T}$ in $\overline{G}$. Let S be the full preimage of $\overline{S}$ in G. Clearly, $G = S \oplus T$. By the same argument, there is a complement to U in G. It follows that there exists an automorphism of G mapping T to U. □

For a positive integer i define

$$\mathcal{M}_i := \Big\{ X \leq G : \ |X| = 2 \text{ and } m(X) = 2^i \Big\}.$$

Corollary 2. *For each $i \in \mathbb{N}$ we have $|\mathcal{M}_i| = 2^{\lambda'_i} - 2^{\lambda'_{i+1}}$, where λ' is the partition conjugate to λ.*

Proof. Let $x = \sum_{j=1}^r \alpha_j e_j \in G$ with $|x| = 2$, and let $X = \langle x \rangle$. Since x is of order 2, α_j must be divisible by $2^{\lambda_j - 1}$ for all j. By Lemma 5 we have $m(X) = 2^i$ if and only if $\alpha_j \equiv 0 \ (2^{\lambda_j})$ whenever $\lambda_j < i$ and there exists some j such that $\lambda_j = i$ and $\alpha_j \not\equiv 0 \ (2^{\lambda_j})$. There are $\big(2^{m_i(\lambda)} - 1\big) \prod_{k>i} 2^{m_k(\lambda)} = 2^{\lambda'_i} - 2^{\lambda'_{i+1}}$ ways to choose such an element x. □

5.2. **Reduction formulae.** Our next task is to give a more explicit description of the map $^- : \mathcal{U}_G^+ \to V$, defined in Section 4, in the case when $k_0 = 2$. This will enable us to correctly assign the functions $\Theta_{G_\Pi}^G(z)$ respectively $\Theta_G^{G_N}(z)$ to the variables of the formula given in Theorem 2, and hence to correctly evaluate (15), in the case when R_n is $H \wr A_n$ or W_n. As a result we obtain an identity expressing the generating function for $|\mathrm{Hom}(G, H \wr A_n)|$ in terms of that for $|\mathrm{Hom}(G, H \wr S_n)|$ and the inner functions $\Theta_{G_\Pi}^G(z)$, along with an analogous formula expressing the generating function for $|\mathrm{Hom}(G, W_n)|$ in terms of that for $|\mathrm{Hom}(G, C_2 \wr S_n)|$ and functions $\Theta_G^{G_N}(z)$.

We have $V = G/\Phi(G)$, where $\Phi(G)$ is the Frattini subgroup of G, and the elements $\bar{e}_j := e_j + \Phi(G)$ for $j \in [r]$ form a basis of V. Let $t = \sum_{j=1}^r \alpha_j e_j \in G$. Then t is an involution if and only if (i) $\alpha_j \equiv 0 \ (2^{\lambda_j})$ or $\alpha_j \equiv 2^{\lambda_j - 1} \ (2^{\lambda_j})$ for all j, and (ii) $\alpha_j \not\equiv 0 \ (2^{\lambda_j})$ for some $j \in [r]$. In order to describe the bijection $^-$, formed with respect to the basis $\{\bar{e}_1, \ldots, \bar{e}_r\}$, between the maximal subgroups of G and the non–zero elements of $G/\Phi(G)$, we define a correspondence between the minimal subgroups of these two groups and examine the relationship between duality in G and orthogonality in $G/\Phi(G)$. For an involution $t \in G$ as above define a vector $v(t) \in V$ via

$$v(t) = \sum_{\alpha_j \equiv 2^{\lambda_j - 1} (2^{\lambda_j})} \bar{e}_j.$$

If we think of t and $v(t)$ as vectors with respect to the bases $\{e_1, \ldots, e_r\}$ and $\{\bar{e}_1, \ldots, \bar{e}_r\}$, then $v(t)$ is the $\mathbf{F}_2$–vector which has zero entries exactly where t has trivial entries $\alpha_j \equiv 0 \ (2^{\lambda_j})$.

Lemma 7. *Let M be a maximal subgroup of G and let $t \in G$ be the involution generating M^*. Then $\bar{M} = v(t)$.*

Proof. Write $t = \sum_{j=1}^r \tau_j e_j$. By definition, for $g = \sum_{j=1}^r \alpha_j e_j$ we have

$$\chi_g(t) = \chi_t(g) = (-1)^{\sum_{j \in S(t)} \alpha_j} = (-1)^{|\{j \in S(t) : \, \alpha_j \equiv 1 (2)\}|},$$

where

$$S(t) := \Big\{ j \in [r] : \ \tau_j \equiv 2^{\lambda_j - 1} \, (2^{\lambda_j}) \Big\}.$$

Thus, $g \in M$ if and only if $|\{j \in S(t) : \ \alpha_j \equiv 1 \, (2)\}|$ is even. Noting that $\Phi(G) = \{2g : g \in G\}$, we find that

$$g \ + \ \Phi(G) \ = \ \sum_{\alpha_j \equiv 1 (2)} \overline{e}_j,$$

and, hence, that

$$\big\langle g + \Phi(G), v(t) \big\rangle \equiv \big| \big\{ j \in S(t) : \ \alpha_j \equiv 1 \, (2) \big\} \big| \ \bmod \ 2,$$

whence the lemma. □

Our next result provides the desired reduction formulae (compare the remark at the end of Section 2.2, and Lemmas 3 and 6).

Proposition 2. *For each i such that $m_i(\lambda) \neq 0$ let $T_i \leq G$ be a subgroup of order 2 with $m(T_i) = 2^i$. Let $M_i = T_i^*$, and put $\Theta_i(z) := \Theta^G_{M_i}(z)$ and $\Theta^{(i)}(z) := \Theta^{M_i}_G(z)$. For all $i \in \mathbb{N}$ with $m_i(\lambda) = 0$ put $\Theta_i(z) = \Theta^{(i)}(z) = 0$. For a given finite group H write $\mathcal{S}_{G,H}(z)$ for $\sum_{n \geq 0} \frac{|\mathrm{Hom}(G, H \wr S_n)|}{n!} z^n$ and $\mathcal{A}_{G,H}(z)$ for $\sum_{n \geq 0} \frac{|\mathrm{Hom}(G, H \wr A_n)|}{n!} z^n$. Moreover, write $\mathcal{W}_G(z)$ for $\sum_{n \geq 0} \frac{|\mathrm{Hom}(G, W_n)|}{n!} z^n$. For $a \in \mathbb{N}$ define*

$$C_a(z) := \sum_{i < a} \big(2^{\lambda'_i} - 2^{\lambda'_{i+1}} \big) \Theta_i(z) \ + \ 2^{\lambda'_a} \Theta_a(z),$$

and

$$C^{(a)}(z) := \sum_{i < a} \big(2^{\lambda'_i} - 2^{\lambda'_{i+1}} \big) \Theta^{(i)}(z) \ + \ 2^{\lambda'_a} \Theta^{(a)}(z).$$

Then for any finite group H we have

$$\mathcal{A}_{G,H}(z) = \frac{\mathcal{S}_{G,H}(z)}{2^r} \left[1 \ + \ \sum_{a \in \mathbb{N}} \big(2^{\lambda'_1 - \lambda'_{a+1}} - 2^{\lambda'_1 - \lambda'_a} \big) e^{-C_a(z)} \right], \tag{16}$$

where r is the rank of G. Also

$$\mathcal{W}_G(z) = \frac{\mathcal{S}_{G,C_2}(z)}{2^r} \left[1 \ + \ \sum_{a \in \mathbb{N}} \big(2^{\lambda'_1 - \lambda'_{a+1}} - 2^{\lambda'_1 - \lambda'_a} \big) e^{-C^{(a)}(z)} \right]. \tag{17}$$

Proof. We will prove the formula for $\mathcal{A}_{G,H}(z)$, the proof of formula (17) being essentially the same. For $i \in \mathbb{N}$ such that $m_i(\lambda) > 0$ define

$$V_i := \big\langle \bar{e}_j : \ \lambda_j \geq i \big\rangle,$$

and

$$i^+ := \begin{cases} \min \big\{ k > i : \, m_k(\lambda) > 0 \big\}, & i \neq \lambda_1 \\ \infty, & i = \lambda_1. \end{cases}$$

Note that $|V_i| = 2^{\lambda'_i}$. By Theorem 3 and Lemmas 5 and 7 we have

$$\mathcal{A}_{G,H}(z) = 2^{-r} \sum_{v \in V} \exp\Big(\sum_{x \in v^\perp} \Theta_x \; - \sum_{x \in V - v^\perp} \Theta_x \Big),$$

where $\Theta_\mathfrak{o} = \Theta_G^G(z)$, and, for $x \in V^\sharp := V - \{\mathfrak{o}\}$, $\Theta_x = \Theta_i(z)$ if and only if $x \in V_i - V_{i^+}$. Clearly, as x runs through V, the function Θ_x runs precisely through all $\Theta_M^G(z)$ such that $(G : M) \le 2$. Thus, by Corollary 1 and the definition of the $\Theta_M^G(z)$,

$$\mathcal{S}_{G,H}(z) = \exp\Big(\sum_{x \in V} \Theta_x \Big),$$

and, consequently

$$\mathcal{A}_{G,H}(z) = \frac{\mathcal{S}_{G,H}(z)}{2^r} \Big[1 \; + \sum_{v \in V^\sharp} \exp\Big(-2 \sum_{x \in V - v^\perp} \Theta_x \Big) \Big].$$

For fixed $v \in V^\sharp$, we have

$$2 \sum_{x \in V - v^\perp} \Theta_x = \sum_{m_i(\lambda) \neq 0} a_i(v) \Theta_i(z),$$

where

$$\begin{aligned} a_i(v) &= 2\big(|V_i - v^\perp| - |V_{i^+} - v^\perp| \big) \\ &= \begin{cases} 2^{\lambda'_i} - 2^{\lambda'_{i+1}}; & V_{i^+} \not\le v^\perp \\ 2^{\lambda'_i}; & V_{i^+} \le v^\perp, V_i \not\le v^\perp \\ 0; & V_i \le v^\perp. \end{cases} \end{aligned}$$

Note that $V_i \le v^\perp$ if and only if $v \in V_i^\perp = \langle \bar{e}_j : \lambda_j < i \rangle$. Write $v = \sum_{j=1}^r \alpha_j \bar{e}_j$ and put

$$a(v) := \max\big\{ \lambda_j : \alpha_j \neq 0 \big\}.$$

Then $v \notin V_i^\perp$ if and only if $i \le a(v)$, and we find that

$$2 \sum_{x \in V - v^\perp} \Theta_x = \sum_{i < a(v)} \big(2^{\lambda'_i} - 2^{\lambda'_{i+1}} \big) \Theta_i(z) \; + \; 2^{\lambda'_{a(v)}} \Theta_{a(v)}(z) = C_{a(v)}(z).$$

For $a \in \mathbb{N}$, the number of $v \in V^\sharp$ such that $a(v) = a$ is $2^{\lambda'_1 - \lambda'_{a+1}} - 2^{\lambda'_1 - \lambda'_a}$, whence (16). □

5.3. **The function $\mathcal{S}_{G(p,\lambda),H}(z)$.** Let $G = G(p,\lambda)$ be an abelian p–group of type λ. Then $|G| = p^{|\lambda|}$, and a subgroup of G of index p^ρ is isomorphic to $G(p,\mu)$ for some μ such that $|\lambda| - |\mu| = \rho$. By (10) we have

$$\mathcal{S}_{G(p,\lambda),H}(z) = \sum_{n=0}^\infty \frac{|\mathrm{Hom}(G(p,\lambda), H \wr S_n)|}{n!} z^n = \exp\Big(\sum_{\rho=0}^{|\lambda|} p^{-\rho} |H|^{p^\rho - 1} s_G^H(p^\rho) z^{p^\rho} \Big), \quad (18)$$

where $s_G^H(p^\rho) = \sum_{(G:U) = p^\rho} |\mathrm{Hom}(U,H)|$. We will make this generating function more explicit by giving a formula for $s_G^H(p^\rho)$. To this end we make use of a known formula which enumerates subgroups in $G(p,\lambda)$ of given isomorphism type. Recall that for a prime power q, the q–binomial coefficient $\binom{n}{k}_q$ is defined as the number of k–dimensional

subspaces in an n–dimensional vector space over the field with q elements. It is well known that

$$\binom{n}{k}_q = \prod_{j=0}^{k-1} \frac{q^{n-j}-1}{q^{k-j}-1}.$$

A partial ordering $\subseteq$ is defined on the set of all partitions by $\mu \subseteq \lambda :\Leftrightarrow \mu_i \leq \lambda_i$ for all i.

Lemma 8 (Miller, [32]). *Let λ, μ be partitions, and let p be a prime. Then $G(p,\lambda)$ contains a subgroup isomorphic to $G(p,\mu)$ if and only if $\mu \subseteq \lambda$. If $\mu \subseteq \lambda$, then the number of subgroups of $G(p,\lambda)$ isomorphic to $G(p,\mu)$ is*

$$\prod_{i\geq 1} p^{\mu'_{i+1}(\lambda'_i-\mu'_i)} \binom{\lambda'_i-\mu'_{i+1}}{\mu'_i-\mu'_{i+1}}_p.$$

We will use the notation $f_p(\mu,H) := |\mathrm{Hom}(G(p,\mu),H)|$, where p is a prime, μ is a partition, and H is a finite group.

Corollary 3. *Let $G = G(p,\lambda)$ be a finite abelian p–group, and let H be any finite group. Then*

$$s_G^H(p^\rho) = \sum_{\substack{\mu\subseteq\lambda \\ |\lambda|-|\mu|=\rho}} \prod_{i\geq 1} p^{\mu'_{i+1}(\lambda'_i-\mu'_i)} \binom{\lambda'_i-\mu'_{i+1}}{\mu'_i-\mu'_{i+1}}_p f_p(\mu,H), \quad 0\leq\rho\leq|\lambda|.$$

Combining formula (18) with Corollary 3 we obtain an explicit formula for $\mathcal{S}_{G(p,\lambda),H}(z)$.

Proposition 3. *Let $G = G(p,\lambda)$ be an abelian p–group of type λ, and let H be any finite group. Then*

$$\mathcal{S}_{G,H}(z) = \exp\left(\sum_{\rho=0}^{|\lambda|} \sum_{\substack{\mu\subseteq\lambda \\ |\lambda|-|\mu|=\rho}} p^{-\rho}|H|^{p^\rho-1} \prod_{i\geq 1} p^{\mu'_{i+1}(\lambda'_i-\mu'_i)} \binom{\lambda'_i-\mu'_{i+1}}{\mu'_i-\mu'_{i+1}}_p f_p(\mu,H) z^{p^\rho}\right). \tag{19}$$

As an example let $G = rC_{p^\alpha}$ be the homogeneous abelian p–group of rank r and exponent p^α, and let H be a finite group. The type λ of G is of the form $\lambda = (\alpha^r)$, $\lambda' = (r^\alpha)$, and (19) yields

$$\mathcal{S}_{rC_{p^\alpha},H}(z) = \exp\left(\frac{1}{|H|}\sum_{\rho=0}^{\alpha r} p^{-\rho}(|H|z)^{p^\rho} \sum_{\substack{\mu\subseteq(\alpha^r) \\ |\mu|=\alpha r-\rho}} f_p(\mu,H) \prod_{i=1}^{\alpha} p^{\mu'_{i+1}(r-\mu'_i)} \binom{r-\mu'_{i+1}}{\mu'_i-\mu'_{i+1}}_p\right). \tag{20}$$

For $\alpha = 1$, i.e., G elementary abelian, (20) simplifies further to give

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(C_p^r, H\wr S_n)|}{n!} z^n = \exp\left(\frac{1}{|H|}\sum_{\rho=0}^{r} p^{-\rho}\binom{r}{\rho}_p |\mathrm{Hom}(C_p^{r-\rho},H)|\,(|H|z)^{p^\rho}\right). \tag{21}$$

Here we have used multiplicative notation to be in consistency with the notation of Section 7, where this formula (as well as formulae (23) and (25) below) will play a crucial role.

5.4. **The functions $\Theta_i(z)$.** Let $G = G(2,\lambda)$ and fix any finite group H. Given the main results of the previous two subsections we can obtain an explicit formula for the exponential generating function of $|\mathrm{Hom}(G, H \wr A_n)|$ by finding explicit formulae for the functions $\Theta_i(z)$ defined in Proposition 2. By definition,

$$\Theta_i(z) = \sum_{\rho=0}^{|\lambda|} \frac{|H|^{2^\rho-1} s(2^\rho, M, G)}{2^\rho} z^{2^\rho},$$

where M is a maximal subgroup of G such that $m(M^*) = 2^i$ and

$$s(2^\rho, M, G) = \sum_{\substack{(G:U)=2^\rho \\ \tau_U^{-1}(A(U\backslash G))=M}} |\mathrm{Hom}(U, H)|.$$

Our first step is to fix M and determine which subgroups $U \leq G$ satisfy $\tau_U^{-1}(A(U\backslash G)) = M$.

Lemma 9. *Let $G = G(2,\lambda)$, and let $M < G$ with $(G : M) = 2$. Let $U \leq G$. Then $\tau_U^{-1}(A(U\backslash G)) = M$ if and only if $U \leq M$ and G/U is cyclic.*

Proof. Clearly, $\tau_U(G)$ coincides with the image of G/U under its regular representation $\rho_U : G/U \to S(U\backslash G)$. Let $(G : U) = 2^\rho$. If $\rho = 0$ then the claim of the lemma holds trivially, so we can assume that $\rho > 0$. If $\bar{x} \in G/U$ has order 2^σ, then $\rho_U(\bar{x})$ is the product of $2^{\rho-\sigma}$ 2^σ–cycles. Thus $\rho_U(\bar{x}) \notin \mathrm{Aut}(U\backslash G)$ if and only if $\rho = \sigma$, i.e., if and only if $\bar{x}$ generates G/U. It follows that $\tau_U^{-1}(A(U\backslash G)) \neq G$ if and only if G/U is cyclic. If this last condition holds, G/U has a unique subgroup of index 2, so there is a unique subgroup of index 2 in G which contains U. □

Corollary 4. *Let M be a maximal subgroup of G, and let $T = M^*$. Denote by $\mathcal{C}(T)$ the set of all cyclic subgroups of G containing T. Then we have*

$$s(2^\rho, M, G) = \sum_{\substack{C \in \mathcal{C}(T) \\ |C|=2^\rho}} |\mathrm{Hom}(G/C, H)|, \quad 0 \leq \rho \leq |\lambda|.$$

The remainder of this subsection will be devoted to finding a more explicit formula for the sum given in Corollary 4. Of course, if S and T are minimal subgroups of G which are conjugate under the action of $\mathrm{Aut}(G)$, then $s(2^\rho, S^*, G) = s(2^\rho, T^*, G)$ for all ρ. Thus, we can examine one representative for each $\mathrm{Aut}(G)$ conjugacy class of minimal subgroups. As we saw in Lemma 6 the class of a minimal subgroup T is determined by its invariant $m(T)$, and the size of each class was determined in Corollary 2. The main new point is that if $U \leq G$ is a non–trivial cyclic subgroup and T is the minimal subgroup of G contained in U, then $m(T)$ is explicitly determined by the isomorphism type of G/U. This reduces the problem of giving the desired explicit formula to that of determining, for a fixed partition $\mu \subseteq \lambda$, the number of cyclic subgroups $C \leq G$ of

order 2^ρ such that $G/C \cong G(2,\mu)$. In other words, we need to know $g^\lambda_{\mu,(\rho)}(2)$ for $\mu \subseteq \lambda$ and $1 \leq \rho \leq \lambda_1$. Although the known explicit formulae for the Hall polynomials $g^\lambda_{\mu,\nu}(t)$ are quite complicated, a relatively simple formula exists when one of the partitions μ, ν has only one part. For partitions λ and μ with $\mu \subset \lambda$ define

$$i(\mu,\lambda) := \max\Big\{j : m_j(\mu) < m_j(\lambda)\Big\}.$$

Lemma 10. *Let C be a non–trivial cyclic subgroup of G with $G/C \cong G(2,\mu)$. Let T be the minimal subgroup of G contained in C. Then $m(T) = 2^{i(\mu,\lambda)}$.*

Proof. For $j \in \mathbb{N}$ define

$$G_{\geq j} := \big\langle e_k :\ \lambda_k \geq j \big\rangle \quad \text{and} \quad G_{<j} := \big\langle e_k :\ \lambda_k < j \big\rangle.$$

Note that for any j we have $T \leq G_{\geq j}$ if and only if $C \cap G_{\geq j} \neq 0$. By Lemma 5 we have $m(T) \geq 2^j$ if and only if $T \leq G_{\geq j}$. Thus, $m(T) < 2^j$ if and only if $C \cap G_{\geq j} = 0$. We can now finish the proof by showing that $C \cap G_{\geq j} = 0$ if and only if $i(\mu,\lambda) < j$. Let $\bar{G} = G/C$, and for $K \leq G$ let $\bar{K}$ be the image of K under the natural projection from G onto $\bar{G}$. If $C \cap G_{\geq j} = 0$, then $\bar{G}_{\geq j} \cong G_{\geq j}$. This implies that $(\lambda_1, \ldots, \lambda_{\lambda'_j}) \subseteq \mu$, and, since $\mu \subseteq \lambda$, we must have $m_k(\mu) = m_k(\lambda)$ for all $k \geq j$. Thus, $i(\mu,\lambda) < j$. If, on the other hand, $C \cap G_{\geq j} \neq 0$, then there is some $\nu \subset (\lambda_1, \ldots, \lambda_{\lambda'_j})$ (note the proper containment) such that $\bar{G}_{\geq j} \cong G(2,\nu)$. Since G is abelian and $\bar{G} = \bar{G}_{\geq j} + \bar{G}_{<j}$, $\bar{G}$ is a homomorphic image of $H = \bar{G}_{\geq j} \oplus \bar{G}_{<j}$. Now $H \cong G(2,\rho)$ for some partition ρ, and for $k \geq j$ we have $m_k(\mu) \leq m_k(\rho) \leq m_k(\nu)$. There is some $k \geq j$ such that $m_k(\nu) < m_k(\lambda)$, so $i(\mu,\lambda) \geq j$. □

If λ and μ are partitions with $\mu \subseteq \lambda$, then $\lambda - \mu$ is a sequence of non–negative integers, as is $\lambda' - \mu'$. The sequence $\lambda - \mu$ is called a horizontal ρ–strip if $\lambda'_i - \mu'_i \leq 1$ for all i and $|\lambda| - |\mu| = \rho$. If $\lambda - \mu$ is a horizontal ρ–strip, define

$$I(\lambda,\mu) := \Big\{i \in \mathbb{N} :\ \lambda'_i - \mu'_i = 1,\ \lambda'_{i+1} - \mu'_{i+1} = 0\Big\}.$$

For any partition λ put

$$n(\lambda) := \sum_i (i-1)\,\lambda_i = \sum_i \binom{\lambda'_i}{2}.$$

Proposition 4 ([30], Chap. II, 4.13)**.** *Let λ and μ be partitions with $\mu \subseteq \lambda$, and let $\rho \in \mathbb{N}$. If $\lambda - \mu$ is not a horizontal ρ–strip, then $g^\lambda_{\mu,(\rho)}(t) = 0$. If $\lambda - \mu$ is a horizontal ρ–strip, then*

$$g^\lambda_{\mu,(\rho)}(t) = \frac{t^{n(\lambda)-n(\mu)}}{1-t^{-1}} \prod_{i \in I(\lambda,\mu)} \big(1 - t^{-m_i(\lambda)}\big).$$

For $\rho, i \in \mathbb{N}$ define

$$\mathcal{P}(\lambda,i,\rho) := \Big\{\mu \subseteq \lambda :\ \lambda - \mu \text{ is a horizontal } \rho\text{–strip, and } i(\mu,\lambda) = i\Big\}.$$

Combining the results obtained so far we can now give the desired explicit formula for the function $\Theta_i(z)$.

Proposition 5. *For each $i \in \mathbb{N}$ such that $m_i(\lambda) > 0$ we have*

$$\Theta_i(z) = \frac{2^{1+n(\lambda)}}{|H|(2^{\lambda'_i} - 2^{\lambda'_{i+1}})} \sum_{0<\rho\le\lambda_1} \frac{1}{2^\rho} \sum_{\mu\in\mathcal{P}(\lambda,i,\rho)} \frac{1}{2^{n(\mu)}} \prod_{j\in I(\lambda,\mu)} \frac{2^{m_j(\lambda)}-1}{2^{m_j(\lambda)}} f_2(\mu,H)(|H|z)^{2^\rho}.$$

Proof. Let M be a maximal subgroup of $G = G(2,\lambda)$ such that $m(M^*) = 2^i$, and let $C(M^*)$ denote the number of conjugates of M^* under the action of $\mathrm{Aut}(G)$. By Corollary 4 and Lemma 10 we have

$$s(2^\rho, M, G) = \frac{1}{C(M^*)} \sum_{\mu\in\mathcal{P}(\lambda,i,\rho)} g^\lambda_{\mu,(\rho)}(2) f_2(\mu,H), \quad 0 \le \mu \le |\lambda|.$$

Applying Corollary 2 and Proposition 4 we find that for $0 \le \rho \le |\lambda|$

$$s(2^\rho, M, G) = \frac{2^{1+n(\lambda)}}{2^{\lambda'_i} - 2^{\lambda'_{i+1}}} \sum_{\mu\in\mathcal{P}(\lambda,i,\rho)} 2^{-n(\mu)} \prod_{j\in I(\lambda,\mu)} \big(1 - 2^{-m_j(\lambda)}\big) f_2(\mu,H),$$

whence the result. □

An explicit formula for the exponential generating function $\mathcal{A}_{G(2,\lambda),H}(z)$ of the sequence $\{|\mathrm{Hom}(G(2,\lambda), H \wr A_n)|\}_{n=0}^\infty$ can now be obtained by combining the results of Propositions 2, 3, and 5. If, for instance, we take for G the homogeneous abelian 2–group of rank r and exponent 2^α, then $m(M^*) = 2^\alpha$ for all $M < G$ with $(G : M) = 2$, i.e., all maximal subgroups of G are conjugate under the action of $\mathrm{Aut}(G)$. In the context of Proposition 5 we have $\mathcal{P}(\lambda,\alpha,\rho) = \{\mu\}$, where μ is the partition consisting of $r-1$ parts α, and a last part $\alpha - \rho$. Also, for this μ, $I(\lambda,\mu) = \{\alpha\}$. Hence,

$$\Theta_\alpha(z) = \frac{2^{1-r}}{|H|} \sum_{\rho=1}^{\alpha} 2^{\rho(r-2)} \left|\mathrm{Hom}\big((r-1)C_{2^\alpha} \oplus C_{2^{\alpha-\rho}}, H\big)\right| \big(|H|z\big)^{2^\rho},$$

and, by Proposition 2

$$\sum_{n=0}^\infty \frac{|\mathrm{Hom}(G, H\wr A_n)|}{n!} z^n = 2^{-r} \bigg(\sum_{n=0}^\infty \frac{|\mathrm{Hom}(G, H\wr S_n)|}{n!} z^n \bigg)$$
$$\times \bigg\{ 1 + (2^r - 1) \exp\bigg(- \frac{2}{|H|} \sum_{\rho=1}^{\alpha} 2^{\rho(r-2)} \left|\mathrm{Hom}\big((r-1)C_{2^\alpha} \oplus C_{2^{\alpha-\rho}}, H\big)\right| \big(|H|z\big)^{2^\rho} \bigg) \bigg\}. \tag{22}$$

For $\alpha = 1$, that is, G elementary abelian, formula (22) simplifies to

$$\sum_{n=0}^\infty \frac{|\mathrm{Hom}(C_2^r, H\wr A_n)|}{n!} z^n = 2^{-r} \bigg(\sum_{n=0}^\infty \frac{|\mathrm{Hom}(C_2^r, H\wr S_n)|}{n!} z^n \bigg)$$
$$\times \bigg\{ 1 + (2^r - 1) \exp\bigg(- \frac{2^{r-1}}{|H|} \left|\mathrm{Hom}(C_2^{r-1}, H)\right| \big(|H|z\big)^2 \bigg) \bigg\} \tag{23}$$

in multiplicative notation.

5.5. **The functions $\Theta^{(i)}(z)$.** By definition,

$$\Theta^{(i)}(z) = \sum_{\rho=0}^{|\lambda|} 2^{2^\rho-\rho-1} s(2^\rho, G, M) z^{2^\rho},$$

where M is a maximal subgroup of G with $m(M^*) = 2^i$ and

$$s(2^\rho, G, M) = \sum_{(G:U)=2^\rho} \big|\{\chi \in \mathrm{Hom}(U, C_2) : \ G_\chi = M\}\big|.$$

Our next result identifies the invariant G_χ attached to a homomorphism $\chi : U \to C_2$, where U is a subgroup of G.

Lemma 11. *Let $G = G(2, \lambda)$, and let $U \leq G$. Let $\chi : U \to C_2$ be a homomorphism with kernel K. Then*

(i) *If $K = U$ or if G/K is not cyclic, then $G_\chi = G$.*

(ii) *If $(U : K) = 2$ and G/K is cyclic, then G_χ is the unique subgroup of index 2 in G which contains K.*

Proof. We will use multiplicative notation. If $K = U$ it follows immediately that $G_\chi = G$, so we may assume that $(U : K) = 2$. Let $T = \{t_1, \ldots, t_{2^\rho}\}$ be a right transversal for U in G. For any $g \in G$, the map $t_i \mapsto \overline{t_i g}$ is a permutation of T, and, since G is abelian and χ is a homomorphism, this gives

$$\prod_{i=1}^{2^\rho} \chi(t_i g \overline{t_i g}^{-1}) = \chi(g^{2^\rho}).$$

Therefore, $G_\chi = \{g \in G : \ g^{2^\rho} \in K\}$. Thus $G_\chi < G$ if and only if there exists some $\bar{g} \in G/K$ such that $\bar{g}^{2^\rho} \neq 1$. Since $|G/K| = 2^{\rho+1}$ the latter condition is equivalent to G/K being cyclic. If G/K is cyclic then it contains a unique subgroup of index 2, and hence there is a unique subgroup of index 2 in G which contains K, which therefore must coincide with G_χ. □

For a partition λ and a non–negative integer ρ we define a partition $\lambda - \rho$ via $(\lambda - \rho)_i := \max\{\lambda_i - \rho, 0\}$.

Proposition 6. *For each $i \in \mathbb{N}$ with $m_i(\lambda) > 0$ we have*

$$\Theta^{(i)}(z) = |G| \sum_{\rho=0}^{i-1} \frac{(2z)^{2^\rho}}{2^{1+2\rho+|\lambda-\rho|}}.$$

Proof. Let $M = \langle t \rangle^*$ with $t = 2^{\lambda_j - 1} e_j$, where $\lambda_j = i$. By Lemma 5, $m(M^*) = 2^i$. Note that if $K \leq U \leq G$ with $(U : K) = 2$ then there is a unique homomorphism from U to C_2 with kernel K. By Lemmas 1 and 11

$$s(2^\rho, G, M) = \big|\{K \leq M : \ G/K \cong C_{2^{\rho+1}}\}\big| = \big|\{C \leq G : \ M^* \leq C, \ C \cong C_{2^{\rho+1}}\}\big|.$$

If $\rho \geq i$ then $s(2^\rho, G, M) = 0$ by the definition of $m(M^*)$. Now assume $\rho < i$, and let $M^* \leq C \leq G$ with $C \cong C_{2^{\rho+1}}$. Then there is a unique generator $c = \sum_{k=1}^r \gamma_k e_k$, $0 \leq \gamma_k < |e_k|$, of C such that $\gamma_j = 2^{i-\rho-1}$, and for $k \neq j$ we must have $2^{\lambda_k} \mid 2^\rho \gamma_k$.

Conversely, any c satisfying these conditions generates a group of order $2^{\rho+1}$ containing t. We now choose such a c. Fix $k \neq j$. If $\lambda_k \geq \rho$, then there are 2^ρ ways to choose γ_k, while if $\lambda_k < \rho$, then there are 2^{λ_k} ways to choose γ_k. These choices can be made independently, so there are 2^σ ways to choose c, where

$$\sigma = \rho\big(\sum_{k\geq\rho} m_k(\lambda) - 1\big) + \sum_{k<\rho} k m_k(\lambda) = |\lambda| - (\rho + |\lambda - \rho|).$$

This gives

$$s(2^\rho, G, M) = \begin{cases} \frac{|G|}{2^{\rho+|\lambda-\rho|}}, & \rho < i \\ 0, & \rho \geq i, \end{cases}$$

whence the proposition. □

Proposition 6, when combined with Propositions 2 and 3 yields an explicit formula for the exponential generating function $\mathcal{W}_{G(2,\lambda)}(z)$ of the sequence $\{|\mathrm{Hom}(G(2,\lambda), W_n)|\}_{n=0}^\infty$. For our standard example, the homogeneous abelian 2–group G of rank r and exponent 2^α, we find from Proposition 6 that

$$\Theta^{(\alpha)}(z) = \frac{1}{2} \sum_{0\leq\rho<\alpha} 2^{\rho(r-2)} (2z)^{2^\rho},$$

and Proposition 2 yields

$$\sum_{n=0}^\infty \frac{|\mathrm{Hom}(G, W_n)|}{n!} z^n = 2^{-r} \bigg(\sum_{n=0}^\infty \frac{|\mathrm{Hom}(G, C_2 \wr S_n)|}{n!} z^n \bigg) \times \bigg\{ 1 + (2^r - 1) \exp\bigg(- 2^{r-1} \sum_{0\leq\rho<\alpha} 2^{\rho(r-2)} (2z)^{2^\rho} \bigg) \bigg\}. \quad (24)$$

For $\alpha = 1$ this gives

$$\sum_{n=0}^\infty \frac{|\mathrm{Hom}(C_2^r, W_n)|}{n!} z^n = 2^{-r} \bigg(\sum_{n=0}^\infty \frac{|\mathrm{Hom}(C_2^r, C_2 \wr S_n)|}{n!} z^n \bigg) \Big\{ 1 + (2^r - 1)\, \mathrm{e}^{-2^r z} \Big\}. \quad (25)$$

6. Asymptotic estimates

In this section we will be concerned with the asymptotic behaviour of arithmetic functions of the form $|\mathrm{Hom}(G, R_n)|$, where G is a finite group and $\{R_n\}$ is a representation sequence of the type considered in Theorem 1. We begin with the case of symmetric representations.

6.1. **The function $|\mathrm{Hom}(G, H \wr S_n)|$.** In view of formula (10) the problem of obtaining asymptotic estimates for the function $\mathrm{Hom}(G, H \wr S_n)|$ is a special case of the following.

(A) *Given a polynomial $P(z) \in \mathbb{R}[z]$, derive asymptotic information on the coefficients α_n of the entire function $\exp(P(z)) = \sum_{n=0}^\infty \alpha_n z^n$, which is explicit in $P(z)$ and n.*

Problem (A) was studied already around 1920 by Pólya in connection with his investigation concerning the zeros of the derivatives of certain analytic functions; cf. [53]. His result is however not sufficiently explicit for our purposes. Also, Pólya's method does not give complete asymptotic expansions but only information on the first term. In the case $P(z) = z$, a solution of (A) is, of course, given by Stirling's formula

$$n! \sim (2\pi)^{1/2}\, n^{n+1/2}\, \mathrm{e}^{-n}, \quad (n \to \infty), \tag{26}$$

or, on a different level of precision, by the expansion

$$n! \approx (2\pi)^{1/2}\, n^{n+1/2}\, \mathrm{e}^{-n} \left\{ 1 + \sum_{\nu=1}^{\infty} \mathfrak{c}_\nu\, n^{-\nu} \right\} \quad (n \to \infty), \tag{27}$$

of factorials derived from Stirling's asymptotic expansion of $\log \Gamma(z)$. As is well known, the coefficients $\mathfrak{c}_\nu$ in (27) can be expressed in terms of Bernoulli numbers via the (formal) identity

$$\exp\left(\sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)}\, z^{-(2k-1)} \right) = 1 + \sum_{\nu=1}^{\infty} \mathfrak{c}_\nu\, z^{-\nu}; \tag{28}$$

cf. for example [62, Sect. 12.33] or [11, Sect. 3.10].[2] In dealing with the asymptotics of the function $|\mathrm{Hom}(G, H \wr S_n)|$ the crux is to obtain results for (A) analogous to (26) or (27) for a sufficiently large class of polynomials including in particular all polynomials

$$P_G^H(z) = \sum_{d|m} \frac{|H|^{d-1} s_G^H(d)}{d}\, z^d, \quad m = |G|$$

associated with a pair (G, H) of finite groups, while maintaining this high level of explicitness in $P(z)$ and n. Indeed, interpreted in this rather strict sense, (A) is not a well–posed problem, since the class of functions $\{\mathrm{e}^{P(z)} : \ P(z) \in \mathbb{R}[z]\}$ turns out to be too large to allow for a uniform asymptotic behaviour of the coefficients α_n. In [41] problem (A) is investigated under certain restrictions on $P(z)$. Let $P(z) = \sum_{\mu=1}^{m} c_\mu z^\mu$ be a polynomial of exact degree $m \geq 1$ with real coefficients c_μ, and put $\exp(P(z)) = \sum_{n=0}^{\infty} \alpha_n z^n$. The following result, which provides a generalization of Stirling's formula, conveys some of the flavour of [41].

Theorem 4 ([41], Theorem 1)**.** *Suppose that the polynomial $P(z)$ meets Hayman's condition*

(H) $\qquad \alpha_n > 0$ for all sufficiently large n

and the gap condition

(G) $\qquad c_\mu = 0$ for $m/2 < \mu < m$.

Then the coefficients α_n of the entire function $\exp(P(z))$ satisfy the asymptotic formula

$$\alpha_n \sim \frac{K}{\sqrt{2\pi n}} \left(\frac{n}{m c_m}\right)^{-n/m} \exp\left(P\Big(\Big(\frac{n}{m c_m}\Big)^{1/m} \Big) \right) \quad (n \to \infty), \tag{29}$$

[2]Note that the k^{th} Bernoulli number in the sense of Whittaker and Watson is $(-1)^{k-1} B_{2k}$ in de Bruijn's and our notation.

where

$$K = K(P) := \begin{cases} m^{-1/2}, & m \text{ odd} \\ m^{-1/2} \exp\big(-\frac{c_{m/2}^2}{8c_m}\big), & m \text{ even.} \end{cases}$$

The proof is based on Hayman's method [22]. Moreover, under more stringent conditions on the polynomial $P(z)$ (still met by the polynomials $P_G^H(z)$), the asymptotic formula (29) can be extended to a full asymptotic expansion of α_n; cf. [41, Theorem 2]. The latter result can be seen as an analogue of Stirling's expansion (27) and the identity (28) for polynomials of higher degree. Its proof, which is based on the work of Harris and Schoenfeld [21], is several orders of magnitude harder than that of Theorem 4, and the result itself is too technical to be stated here. However, as a consequence of these developments, a complete asymptotic expansion of the function $|\mathrm{Hom}(G, H \wr S_n)|$ is obtained for arbitrary finite groups G and H in terms of $m = |G|$, $|H|$, and the numbers $s_G^H(d) = \sum_{(G:U)=d} |\mathrm{Hom}(U, H)|$ for $d \mid m$. Its main term yields the asymptotic formula

$$|\mathrm{Hom}(G, H \wr S_n)| \sim$$
$$K_G^H \, (n\,|H|)^{(1-1/m)n} \exp\left(-\frac{m-1}{m} n + \frac{1}{|H|} \sum_{\substack{d \mid m \\ d<m}} \frac{s_G^H(d)}{d} \, (n\,|H|)^{d/m} \right) \quad (n \to \infty) \quad (30)$$

with

$$K_G^H = \begin{cases} m^{-1/2}, & m \text{ odd} \\ m^{-1/2} \exp\big(-\frac{(s_G(m/2)\iota_2(H))^2}{2m|H|}\big), & m \text{ even,} \end{cases}$$

and the quotient

$$|\mathrm{Hom}(G, H \wr S_n)| \Big/ \left(K_G^H \, \big(n|H|\big)^{\frac{(m-1)n}{m}} \exp\left(-\frac{m-1}{m} n + \frac{1}{|H|} \sum_{\substack{d \mid m \\ d<m}} \frac{s_G^H(d)}{d} \, \big(n|H|\big)^{d/m} \right) \right)$$

is expanded as a Poincaré series in $(n|H|)^{-1/m}$ with coefficients given explicitly in terms of m, $|H|$, and the $s_G^H(d)$; cf. [42, Theorem 4] for more details. As a consequence of (30) we have in particular that

$$|\mathrm{Hom}(G, H \wr S_n)| / |\mathrm{Hom}(G, H \wr S_{n-1})| \sim (n|H|)^{1-1/m} \quad (n \to \infty), \quad (31)$$

and, using [42, Theorem 4] instead of (30), (31) also extends to a full asymptotic expansion of the quotient $|\mathrm{Hom}(G, H \wr S_n)| / |\mathrm{Hom}(G, H \wr S_{n-1})|$; cf. [42, Theorem 5]. These results, which incorporate the asymptotic enumeration of finite group actions[3] as a special case, are of considerable independent interest. They also have important consequences for the theory of subgroup growth; cf. Section 8 and the literature cited therein. We mention in passing that the analytic machinery developed in [41] has found other applications, for instance in connection with the asymptotic value distribution of orthogonal polynomials; cf. [46].

[3]Comments on the history of this problem can be found in the introduction of [41]; cf. also [41, Sect. 5.2 (c)].

6.2. **Further estimates.** Having obtained such a precise asymptotic estimate for the function $|\mathrm{Hom}(G, H \wr S_n)|$ it appears natural to ask about the asymptotic behaviour of closely related functions such as $|\mathrm{Hom}(G, H \wr A_n)|$ or $|\mathrm{Hom}(G, W_n)|$. Concerning the latter functions the following conjecture is made in [42].

Conjecture 1. (i) *For each pair (G, H) of finite groups there exists a constant $C = C(G, H) > 1$ such that*

$$|\mathrm{Hom}(G, H \wr A_n)| = \frac{|\mathrm{Hom}(G, H \wr S_n)|}{1 + s_G(2)} \left\{ 1 + \mathcal{O}\big(C^{-n^{1/|G|}}\big) \right\}.$$

(ii) *For each finite group G there exists a constant $\tilde{C} = \tilde{C}(G) > 1$ such that*

$$|\mathrm{Hom}(G, W_n)| = \frac{|\mathrm{Hom}(G, C_2 \wr S_n)|}{1 + s_G(2)} \left\{ 1 + \mathcal{O}\big(\tilde{C}^{-n^{1/|G|}}\big) \right\}.$$

More generally, one may ask about the asymptotic behaviour of the function $|\mathrm{Hom}(G, R_n)|$, where G is a finite group and the representation sequence $R_n = (H \wr \Pi_n)_N$ is of the type considered in Theorem 1 with $k_0 = 2$; in particular, one may ask under what conditions the estimate

$$|\mathrm{Hom}(G, (H \wr \Pi_n)_N)| = \frac{|\mathrm{Hom}(G, H \wr S_n)|}{(1 + s_G(2))^{\log_2(\alpha_\Pi(H:N)}} \left\{ 1 + \mathcal{O}\big(C^{-n^{1/|G|}}\big) \right\} \tag{32}$$

holds with some constant $C > 1$. Here,

$$\alpha_\Pi := \alpha_{\mathbb{N}}(\Pi) = \begin{cases} 1, & \{\Pi_n\} = \{S_n\} \\ 2, & \{\Pi_n\} = \{A_n\}. \end{cases}$$

Validity of (32) for some group G and some representation sequence $\{R_n\}$ as above implies in particular that the asymptotic expansion provided by [42, Theorem 4] for $|\mathrm{Hom}(G, H \wr S_n)|$ carries over without change to the function

$$\big(1 + s_G(2)\big)^{\log_2(\alpha_\Pi(H:N))} |\mathrm{Hom}(G, (H \wr \Pi_n)_N)|.$$

The partial results known at present in this direction, which we review below, treat the special case of (32) where G is arbitrary and R_n is of the form $R_n = (H \wr S_n)_N$ under certain restrictions on H and N, and the case where $G = C_2$ and the representation sequence $\{R_n\}$ is any of the sequences considered in Theorem 1. Theorem 5 below in particular establishes the second part of Conjecture 1. The results of this subsection are taken from [43].

6.2.1. *The Weyl group case.* Assume that $k_0 = 2$ and that $s_G(2) \neq 0$. By Theorem 3, formula (10), and the fact that $\sum_{(x,y)\in\mathcal{U}_V} \Theta_{\hat{x}}^{\hat{y}}(z) = P_G^H(z)$, we have

$$|\mathrm{Hom}(G, (H \wr \Pi_n)_N)| = \frac{|\mathrm{Hom}(G, H \wr S_n)|}{(1 + s_G(2))^{\log_2(\alpha_\Pi(H:N))}} \left\{ 1 + \sum_{(u,v)\in\mathcal{U}_V^\sharp} Q_u^v(n) \right\}, \tag{33}$$

where[4]

$$Q_u^v(n) := \big\langle z^n, \exp\big(P_u^v(z)\big)\big\rangle \big/ \big\langle z^n, \exp\big(P_G^H(z)\big)\big\rangle,$$

[4] $\langle z^\nu, f(z)\rangle$ denotes the coefficient of z^ν in the power series $f(z)$.

$$P_u^v(z) := P_G^H(z) \;-\; 2 \sum_{\substack{(x,y)\in\mathcal{U}_V \\ \langle (x,y),(u,v)\rangle = 1}} \Theta_{\hat{x}}^{\hat{y}}(z),$$

and $\mathcal{U}_V^\sharp := \mathcal{U}_V - \{(\mathfrak{o},\mathfrak{o})\}$. For each subgroup $U \leq G$ define a map $\psi_U^H : \mathrm{Hom}(U,H) \to V$ via $\chi \mapsto \bar{G}_\chi$, and let $X_U^H := \mathrm{Hom}(U,H)$. By a straightforward calculation

$$\big\langle z^d,\, P_u^v(z)\big\rangle = \frac{|H|^{d-1}}{d} \sum_{(G:U)=d} (-1)^{\overline{\langle \tau_U^{-1}(\Pi(U\setminus G)),u\rangle}} \Big[2\big|(\psi_U^H)^{-1}(v^\perp)\big| - \big|X_U^H\big|\Big]$$

$$(d \mid m,\ (u,v) \in \mathcal{U}_V^\sharp). \quad (34)$$

$P_u^v(z)$ is of degree $m = |G|$ with leading coefficient $(-1)^{\overline{\langle \tau_{\mathbf{1}}^{-1}(\Pi(\mathbf{1}\setminus G)),u\rangle}}|H|^{m-1}/m$, but apart from that not much can be said about the polynomials $P_u^v(z)$ in general. The situation improves somewhat if H is abelian, since in this case the sets X_U^H carry a natural group structure, and the maps ψ_U^H turn out to be group homomorphisms, a fact which via formula (34) allows us to obtain some rudimentary control over the polynomials $P_u^v(z)$.

Lemma 12 ([43], Prop. 2.1)**.** (i) *If H is abelian then for every subgroup $U \leq G$ the map ψ_U^H is a homomorphism.*
(ii) *If $H - N$ contains an involution then the map ψ_G^H is surjective.*

Combining Lemma 12 with formula (34) now yields the following.

Corollary 5. *Suppose that H is abelian and that $H - N$ contains an involution. Then*
(i) *the polynomials $P_u^v(z)$ with $(u,v) \in \mathcal{U}_V^\sharp \cap (\{\mathfrak{o}\} \times V)$ have non–negative coefficients,*
(ii) *for every $(u,v) \in \mathcal{U}_V \cap (V \times V^\sharp)$ we have $\langle z, P_u^v(z)\rangle = 0$.*

Corollary 5 contains the key observations concerning the polynomials $P_u^v(z)$ needed for our investigation of the sequence $\{|\mathrm{Hom}(G,(H \wr S_n)_N)|\}_0^\infty$. In conjunction with Theorem 4 it allows us to estimate the quantities $Q_u^v(n)$ occurring in (33), and to obtain the following.

Theorem 5 ([43], Cor. 4.2)**.** *Let G and H be finite groups, $s_G(2) \neq 0$, and let $N \leq H$ be a subgroup of index 2. Suppose that H is abelian and that $H - N$ contains an involution. Then we have for every real number ε satisfying $0 < \varepsilon < |X_G^H|$ that*

$$|\mathrm{Hom}(G,(H \wr S_n)_N)| = \frac{|\mathrm{Hom}(G, H \wr S_n)|}{1 + s_G(2)} \left\{1 \,+\, \mathcal{O}\big(C^{-n^{1/m}}\big)\right\}, \quad (35)$$

where m is the order of G and

$$C = C(G,H,\varepsilon) := \exp\Big(\frac{|X_G^H| - \varepsilon}{|H|^{1-1/m}}\Big).$$

Corollary 6. *For each finite group G and every real number ε satisfying $0 < \varepsilon < 1 + s_G(2)$ we have*

$$|\mathrm{Hom}(G,W_n)| = \frac{|\mathrm{Hom}(G, C_2 \wr S_n)|}{1 + s_G(2)} \left\{1 \,+\, \mathcal{O}\big(\tilde{C}^{-n^{1/m}}\big)\right\},$$

where $m = |G|$ *and*

$$\tilde{C} = C(G, C_2, \varepsilon) = \exp\Big(\frac{1 + s_G(2) - \varepsilon}{2^{1-1/m}}\Big).$$

Corollary 6 establishes part (ii) of Conjecture 1 concerning the W_n–representations of finite groups.

6.2.2. *The case* $G = C_2$. For $G = C_2$ the polynomials $P_u^v(z)$ occurring in (33) all have degree 2, as has $P_G^H(z)$. Moreover, in the special case of polynomials $P(z) \in \mathbb{R}[z]$ of degree 2 a rather sharp estimate, coming from the theory of orthogonal polynomials, is known for the coefficients of $\mathrm{e}^{P(z)}$, which does not require Hayman's condition (H) or any other further hypothesis.

Lemma 13 ([46], Prop. 3.4 (i))**.** *Let* $a, b \in \mathbb{R}$ *and let* $\exp(az + bz^2) = \sum_{n=0}^{\infty} \alpha_n z^n$. *If* $b \neq 0$ *then*

$$\alpha_n \ll n^{-1/2} \Big(\frac{2|b|}{n}\Big)^{n/2} \exp\Big(\frac{n}{2} + \big|\Im\big(\frac{a}{\sqrt{-2b}}\big)\big|\, n^{1/2}\Big), \quad n \geq 1. \tag{36}$$

Analyzing (33) for $G = C_2$ by means of formulae (34) and (36) gives the following.

Theorem 6. *Let* H *be a finite group,* $N \leq H$ *a subgroup of index* $(H : N) \leq 2$, *and let* $\{\Pi_n\}_0^\infty$ *denote either the sequence* $\{S_n\}_0^\infty$ *of symmetric groups or the sequence* $\{A_n\}_0^\infty$ *of alternating groups. Assume that* $\max(\alpha_\Pi, (H : N)) = 2$.
(i) *If* $N = H$ *or if* $H - N$ *contains an involution then we have*

$$\iota_2((H \wr \Pi_n)_N) = (\alpha_\Pi\,(H : N))^{-1}\, \iota_2(H \wr S_n)\, \Big\{1 + \mathcal{O}\big(C^{-\sqrt{n}}\big)\Big\},$$

where

$$C := \begin{cases} \mathrm{e}^{\iota_2(H)/\sqrt{|H|}}, & N = H \\ \mathrm{e}^{(\iota_2(H) - |2\iota_2(N) - \iota_2(H)|)/\sqrt{|H|}}, & \iota_2(N) \neq \iota_2(H). \end{cases}$$

(ii) *If* $H - N$ *contains no involution then* $\iota_2((H \wr \Pi_n)_N) = \iota_2(H \wr \Pi_n)$ *for all* $n \geq 0$.

This is [43, Theorem 6.1].

7. Computing the Euler characteristic of Quillen complexes

The purpose of this section is to show how some of the explicit formulae derived in Section 5 may be used to produce an efficient and extremely fast algorithm for the calculation of the Euler characteristic of certain Quillen complexes. We begin by recalling some definitions and outlining the basic strategy underlying our approach. The results of this section are taken from [48].

7.1. **The basic strategy.** Let P be a finite partially ordered set, and let $\mathfrak{A}(P)$ be its associated incidence algebra, i.e., the set of all functions $f : P \times P \to \mathbb{R}$ such that $f(x,y) = 0$ if $x \not\leq y$, with pointwise addition and scalar multiplication, and multiplication $f * g$ defined by

$$(f * g)(x,y) = \sum_{x \leq z \leq y} f(x,z)\, g(z,y).$$

$\mathfrak{A}(P)$ is an associative $\mathbb{R}$–algebra with (two–sided) identity δ given by

$$\delta(x,y) = \begin{cases} 1, & x = y \\ 0, & x \neq y. \end{cases}$$

The Möbius function μ_P is defined as the inverse of the zeta function ζ_P of P, the latter being given by

$$\zeta_P(x,y) = \begin{cases} 1, & x \leq y \\ 0, & \text{otherwise.} \end{cases}$$

The order complex $\Delta(P)$ of P is by definition the abstract simplicial complex whose k–simplices are the chains $x_0 < \ldots < x_k$ from P. The reduced Euler characteristic $\tilde{\chi}(\Delta(P))$ of $\Delta(P)$ is defined as $\tilde{\chi}(\Delta(P)) = \sum_i (-1)^i f_i$, where f_i denotes the number of i–dimensional simplices of $\Delta(P)$. Since $\Delta(P)$ contains precisely one simplex of dimension -1 (the empty chain), $\tilde{\chi}(\Delta(P))$ is related to the ordinary Euler characteristic $\chi(\Delta(P))$ by $\tilde{\chi}(\Delta(P)) = \chi(\Delta(P)) - 1$. For a finite group G and a prime p define $\mathcal{S}_p(G)$ to be the poset of all non–trivial p–subgroups of G, ordered by inclusion. Similarly, define $\mathcal{A}_p(G)$ as the subposet of $\mathcal{S}_p(G)$ whose elements are the non–trivial p–tori of G, i.e., the non–trivial elementary abelian p–subgroups of G. In [54], Quillen studied the complexes $\Delta(\mathcal{S}_p(G))$ and $\Delta(\mathcal{A}_p(G))$. He showed, among other things, that geometric realizations of these two complexes are homotopy equivalent, and that they are contractible if G has a normal p–subgroup. Moreover, he conjectured that the converse of the last result holds. This conjecture is still open, although significant progress has been made (see [1]). Here, we focus on the computation of the (reduced) Euler characteristic of these complexes for certain families of groups G.

By a result of P. Hall (see [19, (2.21)] or [55, Prop. 3.6]) the reduced Euler characteristic of $\Delta(P)$ and the Möbius function on P are related by

$$\tilde{\chi}(\Delta(P)) = \mu_{\hat{P}}(\hat{0}, \hat{1}),$$

where $\hat{P}$ is the poset obtained by adding a minimum element $\hat{0}$ and a maximum element $\hat{1}$ to P. By the definition of the Möbius function,

$$\tilde{\chi}(\Delta(P)) = -\sum_{x \in P \cup \{\hat{0}\}} \mu_{\hat{P}}(\hat{0}, x).$$

In particular, if $P = \mathcal{A}_p(G)$, we can think of $\hat{0}$ as the trivial subgroup $\mathbf{1} \leq G$. Then, if $T \leq G$ is a p–torus of rank r, we have (see [19, (2.7)] or [55, Ex. 5.2])

$$\mu_{\widehat{\mathcal{A}_p(G)}}(\mathbf{1}, T) = \mu_{\mathcal{L}(T)}(\mathbf{1}, T) = (-1)^r\, p^{\binom{r}{2}}.$$

Hence, the problem of determining $\tilde{\chi}(\Delta(\mathcal{A}_p(G)))$ is reduced to the problem of enumerating the p–tori in G by rank. For an arbitrary group G this is difficult. However, if

we know $|\mathrm{Hom}(C_p^r, G)|$ for each positive integer r then we can use Möbius inversion on the lattice of subgroups of C_p^r to determine the number $|\mathrm{Inj}(C_p^r, G)|$ of embeddings of C_p^r in G. The number of p–tori in G of rank r is then obtained by dividing $|\mathrm{Inj}(C_p^r, G)|$ by the order of $\mathrm{Aut}(C_p^r) = \mathrm{GL}_r(p)$. The inversion process just described together with certain analytic considerations indicated in the next subsection produces an expression for $\tilde{\chi}(\Delta(\mathcal{A}_p(G)))$ as an infinite series (formula (38)), but we can use this series representation together with a result of K. Brown to produce a finite algorithm which finds the desired Euler characteristic. Of course, the efficiency of this algorithm depends on being able to quickly compute the values $|\mathrm{Hom}(C_p^r, G)|$. If G is a member of one of the representation sequences $\{H \wr S_n\}$ or $\{H \wr A_n\}$ with some finite group H whose associated function $|\mathrm{Hom}(C_p^r, H)|$ is known, or if G is one of the Weyl groups W_n, formulae (21), (23), and (25) provide exactly this information.

7.2. **The algorithm (outline).** Define a function $F(z)$ by

$$F(z) := \sum_{s=0}^{\infty} \frac{z^{\binom{s}{2}}}{\prod_{j=1}^{s}(z^j - 1)}. \tag{37}$$

Then it is easy to show that the right–hand side of (37) is absolutely convergent for all $z \in \mathbb{C}$ such that $|z| \neq 1$, and that for all $z \in \mathbb{C}$ such that $|z| > 1$ we have $F(z) = P(1/z)$, where $P(z) = \sum_{n \geq 0} p(n) z^n$ is the partition generating function. Moreover, by establishing the estimate

$$\frac{|\mathrm{Hom}(C_p^{k+1}, G)| / \prod_{j=1}^{k+1}(p^j - 1)}{|\mathrm{Hom}(C_p^{k}, G)| / \prod_{j=1}^{k}(p^j - 1)} \;\leq\; \frac{1}{p^{k - r_p(G) + 1} - 1}$$

one shows that the series

$$\sum_{k=0}^{\infty} (-1)^k \, \frac{|\mathrm{Hom}(C_p^k, G)|}{\prod_{j=1}^{k}(p^j - 1)}$$

is absolutely convergent for every finite group G and each prime p. Here, $r_p(G)$ denotes the p–rank of G, i.e., the maximum over the ranks of the p–tori in G. From these facts and the identity

$$|\mathrm{Inj}(C_p^r, G)| = \sum_{k=0}^{r} (-1)^{r-k} \, p^{\binom{r-k}{2}} \begin{pmatrix} r \\ k \end{pmatrix}_p |\mathrm{Hom}(C_p^k, G)|,$$

which follows from the relation

$$|\mathrm{Hom}(C_p^r, G)| = \sum_{W \leq C_p^r} |\mathrm{Inj}(W, G)|$$

by Möbius inversion, one then deduces that

$$-\tilde{\chi}(\Delta(\mathcal{A}_p(G))) = P(1/p) \sum_{k=0}^{\infty} (-1)^k \, \frac{|\mathrm{Hom}(C_p^k, G)|}{\prod_{j=1}^{k}(p^j - 1)}. \tag{38}$$

This is an interesting representation of $\tilde{\chi}(p, G) := \tilde{\chi}(\Delta(\mathcal{A}_p(G)))$ as an (absolutely convergent) infinite sum, but, as it stands, it is of course not sufficient to produce a finite algorithm. However, by truncating the series on the right–hand side of (38), estimating

the remainder, and applying a result of Brown [6] to the effect that $\tilde{\chi}(p,G) \in p\mathbb{Z}$, we obtain the following.

Theorem 7. *Let G be a finite group and let p be a prime. Define*

$$\kappa(p,G) := \min\left\{k \geq r_p(G) : \ \frac{|\mathrm{Hom}(C_p^k, G)|}{\prod_{j=1}^k (p^j-1)} \ < \ \frac{p}{2P(1/p)}\right\}.$$

Then

$$-\tilde{\chi}(p,G) = \left[P(1/p) \sum_{k=0}^{\kappa(p,G)-1} (-1)^k \frac{|\mathrm{Hom}(C_p^k, G)|}{\prod_{j=1}^k (p^j-1)}\right]_p, \tag{39}$$

where for $x \in \mathbb{R}\backslash\frac{p}{2}\mathbb{Z}$, $[x]_p$ denotes the integer multiple of p which is closest to x.

Theorem 7 when combined with formulae (21), (23), and (25) provides a finite and fast algorithm for computing $\tilde{\chi}(p,G)$ when G is part of one of the representation sequences considered in Section 5, after the following two details are handled:

(i) Formula (39) involves $P(1/p)$, so we must be able to compute this value efficiently for each given prime p. Define

$$G(z) := \sum_{k=0}^{\infty} \prod_{j=1}^{k} \frac{1}{1-z^j}.$$

Then one can show that $G(z)$ converges absolutely if $|z| > 1$ and that $F(z)\,G(z) = 1$ for all such z. Since $F(z) = P(1/z)$ for $|z| > 1$ this gives an alternating series which converges to $1/P(1/p)$.

(ii) In order to determine $\kappa(p,G)$ one must know $r_p(G)$. The following lemma gives $r_p(G)$ in terms of $r_p(H)$ when G is one of the groups $H \wr S_n$ or $H \wr A_n$ with some finite group H, and also gives $r_p(W_n)$. We believe the results of this lemma to be well known, but have been unable to find an explicit reference other than [48].

Lemma 14. *Let $n \in \mathbb{N}$ and let p be a prime. Then*

(a) $r_p(S_n) = \lfloor n/p \rfloor$.

(b) $r_p(A_n) = \begin{cases} r_p(S_n) - 1, & p = 2 \text{ and } n \equiv 2,3\,(4) \\ r_p(S_n), & \text{otherwise.} \end{cases}$

(c) *Let H be any finite group and let Π_n be one of A_n or S_n. Then*

$$r_p(H \wr \Pi_n) = \begin{cases} r_p(\Pi_n), & r_p(H) = 0 \\ r_p(H)n, & r_p(H) > 0. \end{cases}$$

(d) $r_p(W_n) = \begin{cases} r_p(S_n), & p > 2 \\ n, & p = 2 \text{ and } n \equiv 0\,(2) \\ n-1, & p = 2 \text{ and } n \equiv 1\,(2). \end{cases}$

8. Subgroup growth of free products

The concept of *subgroup growth*, which has evolved during the last decade, provides a framework for investigations concerning arithmetic properties of the sequence $\{s_\Gamma(n)\}_1^\infty$ or related subgroup counting functions and their connection with the algebraic structure of the group Γ. Like Milnor's concept of word growth it can, to a large extent, be thought of as a part of "asymptotic group theory", where one studies groups from the point of view of the asymptotic behaviour of some of their invariants. Some of the major developments concerning the function $s_\Gamma(n)$ are described in Lubotzky's Galway notes [25] and [26]; cf. also [15], [27], [28], [29], [14], and the references therein.

Denote by $\mathcal{F}$ the smallest class of groups containing the class **Fin** of all finite groups and the infinite–cyclic group, which is closed under the operation of forming the free product of two (and hence of finitely many) of its members. Any group $\Gamma \in \mathcal{F}$ has the form

$$\Gamma = G_1 * \ldots * G_s * F_r, \quad 0 \le r, s < \infty, \;\; 1 < m_\sigma := |G_\sigma| < \infty, \tag{40}$$

with finite groups G_σ and a free group F_r of finite rank r for some $r \ge 0$. Note that by results of Kurosh and Baer/Levi the numbers r, s and the groups G_σ (up to order and isomorphism) are isomorphism invariants of Γ; cf. for example [24, § 35]. If Γ is as in (40) then its Euler characteristic (in the sense of Wall) is given by

$$\chi(\Gamma) = \sum_{\sigma=1}^{s} \frac{1}{m_\sigma} \; - \; r \; - \; s \; + \; 1.$$

This section deals with consequences of the asymptotic formula (30) and its refinement [42, Theorem 4] for the subgroup growth of $\mathcal{F}$–groups.

8.1. **The triangle Δ_Γ^H.** With each pair $(\Gamma, H) \in \mathcal{F} \times \mathbf{Fin}$ we associate an infinite triangular array $\Delta_\Gamma^H = \big(H_k^n(\Gamma, H)\big)_{0 \le k \le n}$ with rational entries defined by

$$H_k^n(\Gamma, H) := \binom{n}{k}^{-1} \frac{|\mathrm{Hom}(\Gamma, H \wr S_n)|}{|\mathrm{Hom}(\Gamma, H \wr S_k)|\,|\mathrm{Hom}(\Gamma, H \wr S_{n-k})|}, \quad 0 \le k \le n.$$

By definition, Δ_Γ^H is symmetric, $H_k^n(\Gamma, H) = H_{n-k}^n(\Gamma, H)$, and satisfies the boundary condition $H_0^n(\Gamma, H) = 1$, $n \ge 0$. In fact, the similarity of Δ_Γ^H with Pascal's triangle goes somewhat beyond these observations, and, for reasons to be explained in the next subsection, the investigation of the growth behaviour and the asymptotics of the function $s_\Gamma^H(n) = \sum_{(\Gamma:\Gamma')=n} |\mathrm{Hom}(\Gamma', H)|$ requires the discussion of a certain combinatorial property of the triangle Δ_Γ^H.

For Pascal's triangle the sum of the reciprocal values of the entries in the interior of a row converges to zero as the row number goes to infinity. More precisely, we have for each fixed integer $K \ge 1$ that

$$\sum_{k=K}^{n-K} \binom{n}{k}^{-1} = \mathcal{O}(n^{-K}) \quad (n \to \infty).$$

For $K = 1, 2, \ldots$ put

$$A^{(K)}_{\Gamma,H}(n) := \sum_{k=K}^{n-K} \big(H^n_k(\Gamma, H)\big)^{-1}.$$

Then the sum $A^{(K)}_{\Gamma,H}(n)$ also tends to zero for fixed K as $n \to \infty$ in an appropriately rapid fashion depending on K, provided that $\chi(\Gamma) < 0$, i.e., if Γ contains a non–abelian free subgroup. Indeed, suppose that Γ is as in (40) and put $h^H_\Gamma(n) := |\mathrm{Hom}(\Gamma, H \wr S_n)|/(n!)$. Then we find from (30) and Stirling's formula that

$$h^H_\Gamma(n) \sim L^H_\Gamma \, |H|^{n-1} \, n^{-1} \, \Phi^H_\Gamma(n) \quad (n \to \infty), \tag{41}$$

where

$$L^H_\Gamma := (2\pi)^{\frac{r-1}{2}} \, (m_1 \ldots m_s)^{-1/2} \, |H| \, \exp\left(- \frac{\big(\iota_2(H)\big)^2}{|H|} \sum_{\substack{\sigma \\ 2|m_\sigma}} \frac{\big(s_{G_\sigma}(m_\sigma/2)\big)^2}{2m_\sigma} \right) \tag{42}$$

and

$$\Phi^H_\Gamma(n) := \big(n|H|\big)^{-\chi(\Gamma)n} \exp\left(\chi(\Gamma)\, n \, + \, \frac{1}{|H|} \sum_{\sigma=1}^{s} \sum_{\substack{d_\sigma | m_\sigma \\ d_\sigma < m_\sigma}} \frac{s^H_{G_\sigma}(d_\sigma)}{d_\sigma} \big(n|H|\big)^{d_\sigma/m_\sigma} \, + \, \frac{r+1}{2} \log n \right). \tag{43}$$

Combining the estimate (41) with the strategy used in the proof of [40, Prop. 1] one obtains the following.

Proposition 7. *Suppose that $\chi(\Gamma) < 0$, and let $K \geq 1$ be any fixed natural number. Then we have for each finite group H that*

$$A^{(K)}_{\Gamma,H}(n) = \mathcal{O}\big(n^{K\chi(\Gamma)}\big) \quad (n \to \infty). \tag{44}$$

Proposition 7 tells us in particular that $A^H_\Gamma(n) := A^{(1)}_{\Gamma,H}(n) \to 0$ as $n \to \infty$, provided that $\chi(\Gamma) < 0$. On the other hand, $A^H_{C_\infty}(n) = n - 1$, and one infers from (41) that $A^H_\Gamma(n)$ is at least of order $\mathcal{O}(\sqrt{n})$ if $\Gamma = C_2 * C_2$ and grows exponentially if $|\Gamma| < \infty$. Hence, the following statements on a pair $(\Gamma, H) \in \mathcal{F} \times \mathbf{Fin}$ are equivalent:

(i) $\chi(\Gamma) < 0$.

(ii) $\lim\limits_{n\to\infty} A^H_\Gamma(n) = 0$.

(iii) $A^H_\Gamma(n)$ remains bounded.

This is the analogue of [37, Theorem 3] for the triangle Δ^H_Γ. Proposition 7 itself is analogous to [39, Theorem 2].

In other respects the triangle Δ^H_Γ tends to behave less satisfactory, from a combinatorial point of view, than its counterpart $\Delta(\underline{X}) = \big(V^\lambda_\mu(\underline{X})\big)_{0 \leq \mu \leq \lambda}$, the triangle associated with a finite order graph $\underline{X}$, which was introduced in [36] in connection with the problem

of counting free subgroups of finite index. For example, while "most" of the triangles $\Delta(\underline{X})$ have the unimodal property

$$V^{\lambda}_{\mu-1}(\underline{X}) < V^{\lambda}_{\mu}(\underline{X}) \quad \text{for} \quad 0 < \mu < \frac{\lambda+1}{2} \quad \text{and} \quad \lambda \geq 2,$$

this is not true for the triangles Δ^H_Γ.[5] Here the situation is more complicated. It follows in particular from (31) that there exists a constant $\mathcal{C}(\Gamma, H) > 0$ such that

$$\frac{H^n_k(\Gamma, H)}{H^n_{k-1}(\Gamma, H)} \geq \mathcal{C}(\Gamma, H) \Big(\frac{n-k+1}{k}\Big)^{-\chi(\Gamma)}, \quad 0 < k \leq n.$$

In fact one can take $\mathcal{C}(\Gamma, H) := \prod_\sigma \big(C_1(G_\sigma, H)/C_2(G_\sigma, H)\big)$, where $C_1(G_\sigma, H)$ and $C_2(G_\sigma, H)$ are chosen according to (31) in such a way that the inequalities

$$C_1(G_\sigma, H) n^{1-1/m_\sigma} \leq |\mathrm{Hom}(G_\sigma, H \wr S_n)|/|\mathrm{Hom}(G_\sigma, H \wr S_{n-1})| \leq C_2(G_\sigma, H) n^{1-1/m_\sigma}$$

hold for all $n \geq 1$. In the case $\chi(\Gamma) < 0$ one concludes from this for example that the quotients

$$\frac{H^n_k(\Gamma, H)}{H^n_{k-1}(\Gamma, H)} \quad \text{with} \quad 0 < k < \frac{n+1}{2} \quad \text{and} \quad n \geq 2$$

are at least bounded away from zero and that Δ^H_Γ contains arbitrary long unimodal sequences. Indeed, given any positive integer k_1, we have $H^n_{k-1}(\Gamma, H) < H^n_k(\Gamma, H)$ for $0 < k \leq k_1$ and all $n \geq n_1 := \lfloor k_1(1 + \mathcal{C}(\Gamma, H)^{1/\chi(\Gamma)})\rfloor$. Hence, for every $n \geq n_1$ the sequence

$$1 = H^n_0(\Gamma, H) < \ldots < H^n_{k_1}(\Gamma, H) > H^n_{n-k_1+1}(\Gamma, H) > \ldots > H^n_n(\Gamma, H) = 1$$

is unimodal of length $2k_1 + 1$.

8.2. **Growth behaviour and asymptotics of the function $s^H_\Gamma(n)$.** Putting $\Sigma = \emptyset$ and $\Lambda = \mathbb{N}$ in formula (12) we find that the functions $h^H_\Gamma(n)$ and $\tilde{s}^H_\Gamma(n) := |H|^{n-1} s^H_\Gamma(n)$ are related by the transformation formula

$$n\, h^H_\Gamma(n) = \sum_{k=1}^{n} \tilde{s}^H_\Gamma(k)\, h^H_\Gamma(n-k), \quad n \geq 1. \tag{45}$$

In order to discuss the asymptotic behaviour of the function $s^H_\Gamma(n)$ adopt, for a moment, the following more formal point of view. Consider two sequences $1 = h_0, h_1, h_2, \ldots$ and $s_1, s_2, s_3, \ldots$ of real numbers satisfying a relation

$$\sum_{k=1}^{n} s_k\, h_{n-k} = c\, n\, h_n, \tag{46}$$

[5]For the definition of the triangles $\Delta(\underline{X})$ and an account of their connection with the growth behaviour and the asymptotics of the number of torsion–free subgroups of finite index in finitely generated virtually free groups we refer to [38]; cf. also [39, Sect. 4]. A more systematic investigation of finite order graphs and their associated triangles can be found in [37]; compare in particular [37, Theorem 1] for a precise statement concerning the unimodal property of the triangle $\Delta(\underline{X})$.

with some constant $c > 0$. We require that $s_n \geq 0$ and $h_n > 0$ for $n \geq 1$. Define the *triangle* $\Delta = (H_k^n)_{0 \leq k \leq n}$ *associated with the transformation* (46) by

$$H_k^n := \frac{h_n}{h_k\, h_{n-k}}, \quad 0 \leq k \leq n,$$

and for each fixed integer $K \geq 1$ put

$$A_n^{(K)} := \sum_{k=K}^{n-K} (H_k^n)^{-1}.$$

If the sequence $\{h_n\}_0^\infty$ is growing in an appropriately rapid fashion then information concerning the asymptotic behaviour of the h_n's can be transferred to the sequence $\{s_n\}_1^\infty$. The simplest result in this direction is the following observation of Wright, which allows us to derive an asymptotic formula for the sequence s_n once we are given an asymptotic formula for h_n.

Lemma 15 ([63], Theorem 3)**.** *If in the context of* (46) *we have* $A_n^{(1)} \to 0$ *as* $n \to \infty$ *then* $s_n \sim c\, n\, h_n$.

Formulae (41) and (45), in conjunction with Proposition 7 and Lemma 15, now yield the following.

Theorem 8. *Let* Γ *be an* $\mathcal{F}$*–group of the form* (40) *with* $\chi(\Gamma) < 0$*, and let* H *be any finite group. Then*

$$s_\Gamma^H(n) \sim L_\Gamma^H\, \Phi_\Gamma^H(n) \quad (n \to \infty), \tag{47}$$

where L_Γ^H *and* $\Phi_\Gamma^H(n)$ *are given by* (42) *and* (43).

As an example consider the modular group $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$. Here, $r = 0$, $s = 2$, $m_1 = 2$, $m_2 = 3$, and $\chi(\Gamma) = -1/6$, and Theorem 8 gives

$$s^H_{\mathrm{PSL}(2,\mathbb{Z})}(n) \sim$$
$$\frac{|H|\exp\big(-\frac{(\iota_2(H))^2}{4|H|}\big)}{2\sqrt{3\pi}}\,\big(n|H|\big)^{n/6}\,\exp\Big(-\frac{n}{6} + \big(\frac{n}{|H|}\big)^{1/2} + \big(\frac{n}{|H|^2}\big)^{1/3} + \frac{1}{2}\log n\Big). \tag{48}$$

From Theorem 8, the fact that $\Phi_\Gamma^H(n-1) = o\big(\Phi_\Gamma^H(n)\big)$, and Stirling's formula we also obtain an explicit estimate on the growth of the difference function $s_\Gamma^H(n) - s_\Gamma^H(n-1)$.

Theorem 9. *Let* Γ *and* H *be as in Theorem* 8*, and choose a constant* ρ *with* $0 < \rho < 1$*. Then there exists an integer* $n_1 = n_1(\Gamma, H, \rho) > 0$ *such that for* $n > n_1$

$$s_\Gamma^H(n) - s_\Gamma^H(n-1) >$$
$$\rho\,(2\pi)^{\chi(\Gamma)/2}\, L_\Gamma^H\,\big(|H|\,n!\big)^{-\chi(\Gamma)}\,\exp\Bigg(\frac{1}{|H|}\sum_{\sigma=1}^{s}\sum_{\substack{d_\sigma | m_\sigma \\ d_\sigma < m_\sigma}} \frac{s^H_{G_\sigma}(d_\sigma)}{d_\sigma}\,\big(n|H|\big)^{d_\sigma/m_\sigma} + \frac{1-\chi(\Gamma)+r}{2}\,\log n\Bigg). \tag{49}$$

In particular, the function $s_\Gamma^H(n)$ is strictly increasing for sufficiently large n, provided that $\chi(\Gamma) < 0$.

Note that for small values of n the function $s_\Gamma^H(n)$ need indeed not be increasing; cf. for example Newman's table of the number of subgroups of index n in $\mathrm{PSL}(2,\mathbb{Z})$ for $1 \le n \le 100$ in [49].

There exists a more elaborate asymptotic method for divergent power series due to E. M. Wright [64] and – in greater generality – to E. A. Bender [5], which involves the sums $A_n^{(K)}$ for all $K \ge 1$, and allows us, in the context of (46), to derive a full asymptotic expansion for the function s_n from a known expansion of the sequence h_n. If, instead of using the asymptotic formula (30) plus Stirling's formula, we start from [42, Theorems 4 and 5] and formulae (27) and (28), establish asymptotic expansions of the function $h_\Gamma^H(n)$ and the quotients $h_\Gamma^H(n-k)/h_\Gamma^H(n)$ for each fixed $k \ge 1$, and combine these expansions with Proposition 7 and, say, Bender's method, then we obtain a full asymptotic expansion for the function $s_\Gamma^H(n)$ of the form $L_\Gamma^H\, \Phi_\Gamma^H(n)$ times a Poincaré series in $(n|H|)^{-1/m_\Gamma}$, where $m_\Gamma := lcm(m_1, \ldots, m_s)$, whose coefficients are given explicitly in terms of Γ and H. This result, which represents an important refinement of the main result in [40], is described and proved (among other things) in [44].

8.3. **Combinatorial classification of $\mathcal{F}$–groups.** Let $(\Gamma, H) \in \mathcal{F} \times \mathbf{Fin}$. From Theorem 9 we infer in particular that $s_\Gamma^H(n) \neq 0$ for all sufficiently large n, provided that $\chi(\Gamma) < 0$. Hence, we can define a relation $\sim_a$ between two pairs $(\Gamma_1, H_1), (\Gamma_2, H_2) \in \mathcal{F} \times \mathbf{Fin}$ by

$$(\Gamma_1, H_1) \sim_a (\Gamma_2, H_2) :\Leftrightarrow s_{\Gamma_1}^{H_1}(n) \sim s_{\Gamma_2}^{H_2}(n) \quad (n \to \infty),$$

provided that both Γ_1 and Γ_2 contain a non–abelian free subgroup. We extend this definition to the whole class $\mathcal{F} \times \mathbf{Fin}$ by setting

$$(\Gamma_1, H_1) \sim_a (\Gamma_2, H_2) :\Leftrightarrow |\Gamma_1| = |\Gamma_2|, |H_1| = |H_2|, \text{ and } s_{\Gamma_1}^{H_1} = s_{\Gamma_2}^{H_2},$$

if one of the groups Γ_i involved is finite or virtually infinite–cyclic. Clearly, this defines an equivalence relation on $\mathcal{F} \times \mathbf{Fin}$ with the property that

$$(\Gamma_1, H_1) \sim (\Gamma_2, H_2) \Rightarrow (\Gamma_1, H_1) \sim_a (\Gamma_2, H_2),$$

where $\sim$ is the equivalence relation on $\mathcal{F} \times \mathbf{Fin}$ given by

$$(\Gamma_1, H_1) \sim (\Gamma_2, H_2) :\Leftrightarrow |H_1| = |H_2| \text{ and } s_{\Gamma_1}^{H_1} = s_{\Gamma_2}^{H_2}.$$

No useful characterization in terms of structural invariants is known for the relation $\sim$; however, Theorem 8 in conjunction with Schneider's transcendence theorem [58] implies such a characterization for the coarser equivalence relation $\sim_a$. From this characterization we also obtain a finiteness result on $\sim_a$–equivalence classes and hence a fortiori for $\sim$–classes.

Theorem 10. (a) *Let Γ_1 and Γ_2 be $\mathcal{F}$–groups with decompositions $\Gamma_i = \underset{\sigma_i=1}{\overset{s(\Gamma_i)}{*}} G_{\sigma_i}^{(i)} * F_{r(\Gamma_i)}$ $(i = 1, 2)$, let $m_{\sigma_i}^{(i)}$ denote the order of $G_{\sigma_i}^{(i)}$, and let H_1, H_2 be two finite groups. Then we have $(\Gamma_1, H_1) \sim_a (\Gamma_2, H_2)$ if and only if the following conditions are satisfied:*

(i) $\chi(\Gamma_1) = \chi(\Gamma_2)$.

(ii) $m_{\Gamma_1} = m_{\Gamma_2} (=: m)$.

(iii) $\prod_{\sigma_1=1}^{s(\Gamma_1)} m_{\sigma_1}^{(1)} = \prod_{\sigma_2=1}^{s(\Gamma_2)} m_{\sigma_2}^{(2)}$.

(iv) $r(\Gamma_1) = r(\Gamma_2)$.

(v) $|H_1| = |H_2|$.

(vi) $$\sum_{\substack{\sigma_1 \\ d \mid m_{\sigma_1}^{(1)}}} \frac{s_{G_{\sigma_1}^{(1)}}^{H_1}(m_{\sigma_1}^{(1)}/d)}{m_{\sigma_1}^{(1)}} = \sum_{\substack{\sigma_2 \\ d \mid m_{\sigma_2}^{(2)}}} \frac{s_{G_{\sigma_2}^{(2)}}^{H_2}(m_{\sigma_2}^{(2)}/d)}{m_{\sigma_2}^{(2)}}$$

for all integers d with $d > 1$ and $d \mid m$.

(vii) $$\sum_{\substack{\sigma_1 \\ 2 \mid m_{\sigma_1}^{(1)}}} \frac{\left(s_{G_{\sigma_1}^{(1)}}^{H_1}(m_{\sigma_1}^{(1)}/2)\right)^2}{m_{\sigma_1}^{(1)}} = \sum_{\substack{\sigma_2 \\ 2 \mid m_{\sigma_2}^{(2)}}} \frac{\left(s_{G_{\sigma_2}^{(2)}}^{H_2}(m_{\sigma_2}^{(2)}/2)\right)^2}{m_{\sigma_2}^{(2)}}.$$

In other words, the collection

$$\chi(\Gamma),\ m_\Gamma,\ \prod_\sigma m_\sigma,\ r,\ |H|,\ \sum_{\substack{\sigma \\ d \mid m_\sigma}} \frac{s_{G_\sigma}^H(m_\sigma/d)}{m_\sigma}\ (1 < d \mid m_\Gamma),\ \sum_{\substack{\sigma \\ 2 \mid m_\sigma}} \frac{\left(s_{G_\sigma}^H(m_\sigma/2)\right)^2}{m_\sigma}$$

is a complete system of invariants for the equivalence relation $\sim_a$ on $\mathcal{F} \times \mathbf{Fin}$.

(b) *A $\sim_a$–equivalence class of $\mathcal{F} \times \mathbf{Fin}$ contains only a finite number of non–isomorphic pairs (Γ, H).* [6]

If we identify $\mathcal{F}$ with the subclass $\mathcal{F} \times \{\mathbf{1}\}$ of $\mathcal{F} \times \mathbf{Fin}$ then the relations discussed above induce equivalence relations on $\mathcal{F}$, which we denote by the same symbols $\sim$ and $\sim_a$, and Theorem 10 yields the following.

Corollary 7. (a) *The collection*

$$\chi(\Gamma),\ m_\Gamma,\ \prod_\sigma m_\sigma,\ r,\ \sum_{\substack{\sigma \\ d \mid m_\sigma}} \frac{s_{G_\sigma}(m_\sigma/d)}{m_\sigma}\ (1 < d \mid m_\Gamma),\ \sum_{\substack{\sigma \\ 2 \mid m_\sigma}} \frac{\left(s_{G_\sigma}(m_\sigma/2)\right)^2}{m_\sigma}$$

is a complete set of invariants for the equivalence relation $\sim_a$ on the class $\mathcal{F}$.

(b) *A $\sim_a$–equivalence class of $\mathcal{F}$–groups contains only a finite number of non–isomorphic groups.*

[6] Two pairs (Γ_1, H_1) and (Γ_2, H_2) are called isomorphic if $\Gamma_1 \cong \Gamma_2$ and $H_1 \cong H_2$.

Using for instance the existence of non–isomorphic finite p–groups with isomorphic subgroup lattices or Rottlaender's work [56] it is easy to exhibit non–isomorphic $\mathcal{F}$–groups Γ_1 and Γ_2 such that $\chi(\Gamma_1), \chi(\Gamma_2) < 0$ and $\Gamma_1 \sim \Gamma_2$; and, using Corollary 7, one can also construct examples of free products $\Gamma_1, \Gamma_2 \in \mathcal{F}$ such that $s_{\Gamma_1} \neq s_{\Gamma_2}$ and $s_{\Gamma_1}(n) \sim s_{\Gamma_2}(n)$ as $n \to \infty$; cf. [40]. It follows that the three equivalence relations on the class $\mathcal{F}$ given by (i) isomorphism, (ii) the property that $s_{\Gamma_1} = s_{\Gamma_2}$, and (iii) the relation $\sim_a$ are all distinct.

8.4. **Counting finite index subgroups by isomorphism type.** Consider again the (inhomogeneous) modular group $\Gamma = \mathrm{PSL}(2,\mathbb{Z}) \cong C_2 * C_3$. By Kurosh's subgroup theorem, any subgroup $\Gamma' \leq \Gamma$ is of the form

$$\Gamma' \cong \underbrace{C_2 * \ldots * C_2}_{\lambda} * \underbrace{C_3 * \ldots * C_3}_{\mu} * F_\nu$$

with non–negative integers λ, μ, and ν. Hence, the isomorphism type of a subgroup $\Gamma' \leq \Gamma$ is conveniently described by means of the triple $t(\Gamma') = (\lambda, \mu, \nu) \in \mathbb{N}_0^3$. Moreover, if Γ' has finite index in Γ then an Euler characteristic calculation shows that $t(\Gamma')$ and $(\Gamma : \Gamma')$ are connected via the relation

$$3\,\lambda \,+\, 4\,\mu \,+\, 6\,(\nu - 1) = (\Gamma : \Gamma').$$

The problem to construct, classify, and enumerate finite index subgroups in the modular group with given restrictions on the isomorphism type goes back to Klein and Poincaré, and has received a large amount of attention since the 1880's, both as an intrinsic problem for the modular group, and in connection with the construction of modular forms; cf. for example [33], [34], [50], [51], [52], [59], [16], [2], and the literature cited therein. Despite these enormous efforts, not too much appears to be known concerning the asymptotic behaviour of the arithmetic functions associated with these counting problems. Denote by $a_\Gamma(\lambda, \mu, \nu)$ the number of finite index subgroups $\Gamma' \leq \Gamma$ such that $t(\Gamma') = (\lambda, \mu, \nu)$, let $t_i = (\lambda_i, \mu_i, \nu_i) \in \mathbb{N}_0^3$ be a sequence of isomorphism types such that $n_i := 3\lambda_i + 4\mu_i + 6(\nu_i - 1) \to \infty$ as $i \to \infty$, and consider the function $a_\Gamma(t_i)$. Then what can we say about the sequence $\{a_\Gamma(t_i)\}_1^\infty$, in particular what is its growth behaviour and asymptotics? If $t_i = (0, 0, \nu_i)$ with $\nu_i \geq 2$ and $\nu_i \to \infty$ as $i \to \infty$ then the general results obtained in [36], [38], and [39] concerning torsion–free subgroups of finite index in finitely generated virtually free groups apply; for example we find from [38, Theorem 5] that

$$a_\Gamma(t_i) \sim \frac{3}{\pi}\, 6^{\nu_i - 1}\, (\nu_i \,-\, 1)! \quad (i \to \infty).$$

To conclude this survey, we briefly want to indicate how the introduction of the parameter $H \in \mathbf{Fin}$ into the theory of subgroup growth of $\mathcal{F}$–groups combined with the results concerning the functions $s_\Gamma^H(n)$ outlined in the previous three subsections may help in the solution of the problem mentioned above. Indeed, replacing the function $s_\Gamma(n)$ by $s_\Gamma^H(n)$ for some non–trivial finite group H has the effect of making perceptable the distribution of isomorphism types within the layers of finite index subgroups in Γ, which before had only been measured as a whole through the sequence $s_\Gamma(n)$. To be

more precise, consider the equation

$$s_\Gamma^H(n) = \sum_{\substack{\lambda,\mu,\nu\geq 0 \\ 3\lambda+4\mu+6(\nu-1)=n}} \big(\iota_2(H)\big)^\lambda \big(\iota_3(H)\big)^\mu |H|^\nu \, a_\Gamma(\lambda,\mu,\nu) \tag{50}$$

with Γ and n fixed, and varying H. Then the set of triples (λ,μ,ν) (and thus also the values of the function a_Γ) involved in the right–hand sum will remain the same, while the coefficient $(\iota_2(H))^\lambda(\iota_3(H))^\mu|H|^\nu$ varies with H, as does the left–hand side $s_\Gamma^H(n)$. Hence, by varying H we are able to produce sufficiently many independent equations of the form (50) to solve for the a_Γ's in terms of the functions $s_\Gamma^H(n)$. This in turn allows us to bring the asymptotic formula (47) respectively its refinement [44, Theorem 1] to bear on the above problem, which, in conjunction with techniques for the asymptotic approximation of sums, leads to asymptotic estimates for the function $a_\Gamma(t_i)$ for a broad variety of sequences $\{t_i\}$ of isomorphism types. Moreover, this strategy is, of course, not confined to the modular group, but can be used to systematically investigate the corresponding more general problem for arbitrary $\mathcal{F}$–groups and some of their lifts; cf. [44] and [45].

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