Some remarks on the group of R-equivalence classes of semisimple groups over arithmetic fields

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Abstract

We compute the group of R-equivalence classes of some simple simply connected algebraic groups of types ${}^{3,6}D_4$ and E_6 defined over an algebraic number field. It is also proved that the variety of a semisimple algebraic group G defined over a p-adic field K is rational over K if all its absolutely simple components are defined over K and have a type different from A_n, ${}^2D_n$; in particular, the corresponding group of R-equivalence classes is always trivial.

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1 Introduction

Let G be a semisimple algebraic group defined over either a non-archimedean locally compact field of characteristic zero, i. e. a finite extension of the p-adic number field $\mathbb{Q}_p$, or an algebraic number field F. As one of the applications of his norm principle [G1] Ph. Gille obtained in [G2], [G3] the following elegant formula for the computation of the group of R-equivalence classes $G(F)/R$. Consider a semisimple simply connected covering of the group G

$$1 \longrightarrow \mu \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1 \tag{1}$$

and a flasque resolution of the module μ

$$1 \longrightarrow \mu \longrightarrow E \longrightarrow P \longrightarrow 1. \tag{2}$$

If F is a local field as above then one has

$$G(F)/R \simeq \mathrm{H}^1(F, E). \tag{3}$$

If F is an algebraic number field then the sequences (1), (2) induce an exact sequence

$$\widetilde{G}(F)/R \longrightarrow G(F)/R \longrightarrow \mathrm{H}^1(F, E) \longrightarrow 1. \tag{4}$$

It follows from (4) that in order to get the same nice and complete formula (3) for the computation of the group $G(F)/R$ in the case of an algebraic number field F it is necessary to establish additionally that for an arbitrary absolutely simple simply connected F-group $\widetilde{H}$ the equality $\widetilde{H}(F)/R = 1$ holds.

If $\widetilde{H}$ is isotropic over F then by the Kneser – Tits conjecture the group of F-points $\widetilde{H}(F)$ coincides with its subgroup generated by unipotent elements; in particular, $\widetilde{H}(F)/R = 1$. At present we know (cf. [PR]) that the Kneser – Tits conjecture is true for all $\widetilde{H}$ with the possible exception of some groups of type E_6 and of F-rank 1. We show in § 2 that for the remaining cases the equality $\widetilde{H}(F)/R = 1$ is, in fact, an easy consequence of results of the paper [CM] and the exact sequence (4).

Consider now the anisotropic case. If $\widetilde{H}$ has type A_n then the equality $\widetilde{H}(F)/R = 1$ is again a trivial consequence of results of the paper [CM] (outer type) and Voskresenskii's theorem (inner type; cf. Appendix in [V]).

For anisotropic groups of the other types the Margulis – Platonov conjecture states that the group of F-points $\widetilde{H}(F)$ is projectively simple, i. e. it has no non-central normal subgroups, and hence the equality $\widetilde{H}(F)/R = 1$ also holds.

At present the Margulis – Platonov conjecture is proved for all groups with the exception of groups of types ${}^2A_n$, ${}^{3,6}D_4$, E_6 (cf. [PR], [R], [RPot], [S]). In the first part of the paper we establish the triviality of the group of R-equivalence classes for an arbitrary group $\widetilde{H}$ of type ${}^{3,6}D_4$.

Thus summarizing we get that in order to show that for an arbitrary semisimple algebraic group G defined over an algebraic number field F the natural map

$$G(F)/R \to \mathrm{H}^1(F, E)$$

is an isomorphism, and in particular $G(F)/R$ is an abelian group, it remains to be proven that for an arbitrary simple simply connected F-anisotropic group $\widetilde{H}$ of type E_6 one has $\widetilde{H}(F)/R = 1$.

It follows from formula (3) that for a local field as above the structure of the group $G(F)/R$ depends only on the discrete module μ. Therefore any two semisimple algebraic groups over F have the same group of R-equivalence classes if the kernels of the corresponding simply connected coverings coincide; in particular, $G(F)/R = 1$ if all absolutely simple components of the group G are defined over F and have inner type.

If one assumes additionally that G has no simple components of type ${}^1A_n$ then it is easy to show that G is R-trivial, i. e. $G(K)/R = 1$ for an arbitrary extension K/F. We explain this phenomenon in the second part of the paper. We prove more precisely that the variety G is rational over F. Here our argument is based mainly on ideas and methods of the paper [CP]. It should be noted that by Merkurjev's result [M1] the property to be R-trivial can not be extended to groups containing simple components of type A_n.

Note also that it is still unknown in general whether the R-triviality of G implies the property to be F-rational or at least to be a direct factor of a stably F-rational group. According to [CS] this always holds for algebraic tori.

In the paper we use the following notation and conventions.

If G is a semisimple algebraic group over an arbitrary field F, we denote the corresponding simply connected covering by $\widetilde{G}$ and by μ its fundamental group. The centralizer $C_G(S)$ of a maximal F-split torus S is called the anisotropic kernel of G and its commutator subgroup

$$C_G(S)^{(1)} = [\, C_G(S), C_G(S)\,]$$

is called the semisimple anisotropic kernel.

$G_{m,F}$ is the 1-dimensional F-split torus.

μ_n denotes the module of n-th roots of unity and ξ_n denotes a primitive n-th root of unity.

For a finite separable extension L/F the symbol $\mathrm{R}_{L/F}$ stands for the Weil functor of the field restriction.

Br (F) denotes the Brauer group of a field F. By "algebra over F" we mean a central simple associative finite-dimensional algebra over F. If A is an algebra over F, then A^{op} denotes the opposite algebra and $SL(1, A)$ denotes the set of elements in A^* with reduced norm equal 1. Clearly, $SL(1, A)$ is the group of F-points of the corresponding absolutely simple simply connected algebraic F-group $\mathbf{SL}(1, \mathrm{A})$.

$\mathbb{A}_F^n$ is the n-dimensional affine space over the field F. A connected algebraic group G defined over F is called F-rational if the variety G is rational over F, i.e. G is birationally isomorphic over F to the affine space $\mathbb{A}_F^n$. We say that a group G is stably F-rational if the variety $G \times \mathbb{A}_F^n$ is rational over F for some natural number n. Furthermore, we say that two algebraic groups G_1, G_2 over F are stably birationally isomorphic and denote this by $G_1 \approx G_2$ if the varieties $G_1 \times \mathbb{A}_F^n$ and $G_2 \times \mathbb{A}_F^m$ are birationally isomorphic over F for some natural numbers n, m.

By a point in "generic" position in an algebraic variety X we mean a point in some non-empty Zariski open subset $U \subset X$ which always can be easily specified from the argument.

In the paper we assume familiarity of the reader with the classification of absolutely simple algebraic groups in terms of Tits indices (cf. [T]).

For the sake of brevity a non-archimedean locally compact field of characteristic 0 is called a local field. If F is an algebraic number field, V_∞^F or simply V_∞ denotes the set of all archimedean valuations.

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2 Isotropic groups of type E_6

Let X be an algebraic variety defined over a field F. Recall first the notion of *R-equivalence* introduced by Manin in [Ma]. Two points $x, y \in X(F)$ are called *strictly R-equivalent* if there exists a rational map $\varphi : \mathbb{A}_F^1 \to X$ over F defined at the points 0, 1 and such that $\varphi(0) = x$, $\varphi(1) = y$. Points $x, y \in X(F)$ are called *R-equivalent* if there exists a finite sequence

$$x = x_1,\, x_2, \ldots, x_n = y$$

of points of $X(F)$ such that all neighbors x_i, x_{i+1} are strictly R-equivalent.

It is obvious that R-equivalence is an equivalence relation and therefore the set $X(F)$ is partitioned into disjoint classes of R-equivalent points. The set of R-equivalence classes is usually denoted by $X(F)/R$.

An element $g \in G(F)$ of an F-defined connected algebraic group G is called *R-trivial* if it is R-equivalent to 1. We denote the set of all R-trivial elements by $RG(F)$.

Lemma 2.1 *$RG(F)$ is a normal subgroup in $G(F)$.*

Proof is trivial. □

The group structure in $G(F)$ induces a group structure on the set of R-equivalence classes $G(F)/R$. The set $G(F)/R$ equipped with the natural group structure is called *the group of R-equivalence classes.*

Lemma 2.2 *There exists a natural isomorphism*

$$G(F)/RG(F) \simeq G(F)/R.$$

Proof follows from the definitions. $\square$

An arbitrary homomorphism $\varphi : G \to H$ defined over F induces a homomorphism $G(F) \to H(F)$ which takes $RG(F)$ to $RH(F)$. Therefore we have a well-defined group homomorphism

$$\varphi/R : G(F)/R \to H(F)/R.$$

Lemma 2.3 *1) If G_1, G_2 are connected algebraic groups defined over F then*

$$(G_1 \times G_2)(F)/R \simeq G_1(F)/R \times G_2(F)/R.$$

2) If L/F is a finite separable extension and G is a connected algebraic group defined over L, then one has

$$R_{L/F}(G)(F)/R \simeq G(L)/R.$$

Proof follows immediately from the definitions. $\square$

Proposition 2.4 ([CS], Proposition 11). *Let F be an infinite field and let G be a connected algebraic group over F. Then for an arbitrary non-empty open set $U \subset G$ defined over F the natural map*

$$U(F)/R \longrightarrow G(F)/R$$

is a bijection.

Corollary 2.5 *If the group G is rational over F, then $G(F)/R = 1$.*

Corollary 2.6 *If the groups G_1, G_2 are birationally isomorphic over F, then there exists a bijection*

$$G_1(F)/R \simeq G_2(F)/R.$$

In particular, $G_1(F)/R = 1$ if and only if $G_2(F)/R = 1$.

Corollary 2.7 *If the groups G_1, G_2 are stably birationally isomorphic over F then there exists a bijection*

$$G_1(F)/R \simeq G_2(F)/R.$$

The above mentioned results allow to reduce the computation of groups of R-equivalence classes of arbitrary connected reductive groups to the case of anisotropic reductive groups. Namely, let G be a connected reductive group defined over F. Denote a maximal F-split torus in G by S. It follows from the Bruhat – Tits decomposition that there exists a birational F-isomorphism

$$G \simeq U \times C_G(S) \times U,$$

where U is the unipotent radical of a minimal parabolic F-subgroup P containing S. Since the variety U is rational over F, Corollaries 2.5, 2.6 and Lemma 2.3 show that

$$G(F)/R \simeq C_G(S)(F)/R.$$

Furthermore, the bundle $C_G(S) \to C_G(S)/S$ is locally trivial, for S is an F-split torus, implying

$$G \approx C_G(S) \approx C_G(S)/S,$$

hence by Corollary 2.7 we get

$$G(F)/R \simeq [\,C_G(S)/S\,](F)/R. \tag{5}$$

For an absolutely simple simply connected group of type A_n the computation of the group of R-equivalence classes is given in the following two theorems.

Theorem 2.8 (cf. Appendix in [V]). *Let A be an algebra over an arbitrary field F. Then $RSL(1,A) = [A^*, A^*]$ and therefore*

$$SL(1,A)/R \simeq SL(1,A)/[A^*, A^*].$$

The factor group $SL(1,A)/[A^*, A^*]$ is known as the reduced Whitehead group and it is denoted by $SK_1(A)$. It is well-known that over local and global fields $SK_1(A) = 1$ and so we get immediately from Theorem 2.8

Corollary 2.9 *If G is a simply connected inner form of type A_n defined over either a local or global field F, then $G(F)/R = 1$.*

We pass now to the computation of the group of R-equivalence classes of outer forms of type A_n. Let L/F be a separable quadratic extension and let A be an algebra over L with an involution σ of the second kind. Denote by Σ' the set of all elements in A with reduced norm in F and by Σ the subgroup of Σ' generated by the σ-invariant elements in A.

Theorem 2.10 ([CM]). *Let $G = \mathbf{SU}(\mathrm{A}, \sigma)$. Then we get*

$$G(F)/R \simeq \Sigma'/\Sigma.$$

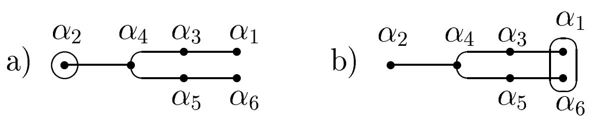
The factor group Σ'/Σ is known as the reduced unitary Whitehead group and it is denoted by $SUK_1(A,\sigma)$. It is well-known that the group $SUK_1(A,\sigma)$ is trivial over local and global fields and also over an arbitrary field if the index of A is square-free. Therefore we get from Theorem 2.10

Corollary 2.11 *1) If F is a local or global field, then $SU(A,\sigma)/R = 1$.*

2) If F is an arbitrary field and the index of A is square-free, then we have $SU(A,\sigma)/R = 1$.

Theorem 2.12 *Let F be an algebraic number field and let G be an absolutely simple simply connected algebraic group over F of type E_6 and of F-rank 1. Then $G(F)/R = 1$.*

Proof. According to the Tits classification [T] there are two possibilities for the Tits F-index of the group G:



Denote a maximal F-split torus of G by S and let T be an arbitrary torus over F containing S. Let $\Sigma = \Sigma(G,T)$ be the root system of the group G relative to T and let

$$\Pi = \{\alpha_1, \ldots, \alpha_6\} \subset \Sigma$$

be an arbitrary basis. In what follows we use without explanation the standard notation $x_\alpha(u)$, $h_\alpha(t)$, w_α from the book [St], where $\alpha \in \Sigma$ and u, t are parameters, and also use relations indicated on p. 32 of that book. If $X \subseteq \Sigma$, we denote by G_X the subgroup generated by the root subgroups $x_{\pm\alpha}(u)$ for all roots $\alpha \in X$.

Consider first diagram a). It follows from (5) that it suffices to prove

$$[\,C_G(S)/S\,](F)/R = 1$$

or equivalently $H(F)/R = 1$, where

$$H = C_G(S)^{(1)}/C_G(S)^{(1)} \cap S.$$

We get from diagram a) that the group $\widetilde{H} = C_G(S)^{(1)}$ is simply connected of type ${}^2A_5$ and is generated by the root subgroups

$$x_{\pm\alpha_1}(\ ),\ x_{\pm\alpha_3}(\ ), \ldots, x_{\pm\alpha_6}(\ ).$$

According to [T] the torus S is the identity component of the group given by the following system of equations for an element $t \in T$:

$$\begin{cases} \alpha_1(t) = 1 \\ \alpha_3(t) = 1 \\ \quad\cdots \\ \alpha_6(t) = 1 \end{cases}$$

Simple computations show that

$$S = \langle h_{\alpha_1}(t)h_{\alpha_3}(t^2)h_{\alpha_4}(t^3)h_{\alpha_5}(t^2)h_{\alpha_6}(t)h_{\alpha_2}(t^2) \mid t \in \overline{F}^* \rangle,$$

whence $C_G(S)^{(1)} \cap S = \mu_2$.

Thus we established that H is the right-hand term of an exact sequence

$$1 \longrightarrow \mu \longrightarrow \widetilde{H} \longrightarrow H \longrightarrow 1,$$

where $\mu = \mu_2$. Recall (cf. [CS]) that an exact sequence (2) is called a flasque resolution of a module μ if P is a permutation F-torus and E is an F-torus such that its cocharacter module $E_* = \mathrm{Hom}\,(G_m, E)$ is coflasque, i. e. $\mathrm{H}^1(\Gamma, E_*) = 1$ for any open subgroup $\Gamma \leq \mathrm{Gal}\,(\overline{\mathrm{F}}/\mathrm{F})$.

In our case $\mu = \mu_2$, so a flasque resolution is of the form

$$1 \longrightarrow \mu \longrightarrow G_{m,F} \xrightarrow{\phi} G_{m,F} \longrightarrow 1,$$

where ϕ is the second power map. It follows that $E \simeq G_{m,F}$ and therefore according to (4) and Corollary 2.11 we finally obtain

$$H(F)/R = \mathrm{H}^1(F, E) = \mathrm{H}^1(F, G_{m,F}) = 1.$$

Consider next diagram b). In contrast to the previous case the center Z of the anisotropic kernel $C_G(S)$ does not coincide with S, because it has dimension two.

We first describe the structure of the torus Z. It is clear that there are only two possibilities for Z: either

$$Z = R_{L/F}(G_{m,L}),$$

where L/F is the quadratic extension over which G has inner type, or

$$Z = G_{m,F} \times R^{(1)}_{L/F}(G_{m,L}).$$

In the first case the module $Z_2 \leq Z$ of elements of order 2 is of the form $Z_2 = R_{L/F}(\mu_2)$ and in the second case we have $Z_2 = \mu_2 \times \mu_2$.

From diagram b) it follows that Z is the identity component of the group given by the following system of equations

$$\begin{cases} \alpha_2(t) = 1 \\ \quad\dots \\ \alpha_5(t) = 1 \end{cases}$$

for $t \in T$. Solving this system we obtain

$$Z = \langle h_{\alpha_2}(t)h_{\alpha_4}(t^2)h_{\alpha_3}(u)h_{\alpha_1}(u^2/t^2)h_{\alpha_5}(t^3/u)h_{\alpha_6}(t^4/u^2) \mid t, u \in \overline{F}^* \rangle,$$

whence

$$Z_2 = \langle\, h_{\alpha_2}(-1)h_{\alpha_5}(-1),\ h_{\alpha_2}(-1)h_{\alpha_3}(-1)\,\rangle,$$

so that $Z_2 \simeq R_{L/F}(\mu_2)$, as the Galois group acts non-trivially on the vertices of the Dynkin diagram corresponding to the roots α_3 and α_5.

Thus we proved that

$$Z = R_{L/F}(G_{m,L})$$

implying $\mathrm{H}^1(K, Z) = 1$ for any extension K/F. To complete the computation of the group $G(F)/R$ we need the following two statements.

Lemma 2.13 *Let F be an arbitrary field and G_1 be a connected algebraic group over F. If G_2 is a connected F-subgroup of G_1 such that for any extension K/F the kernel of the natural morphism $\mathrm{H}^1(K, G_2) \to \mathrm{H}^1(K, G_1)$ is trivial then the variety G_1 is birationally isomorphic over F to the product $G_2 \times (G_1/G_2)$.*

Proof follows immediately from the surjectivity of the morphism

$$G_1(E) \to (G_1/G_2)(E),$$

where E is the field generated by the coordinates of a generic point ω over F of the variety G_1/G_2. □

Proposition 2.14 *Let G be an adjoint group of type ${}^{1,2}D_4$ defined over a global field F. Then $G(F)/R = 1$.*

Proof follows immediately from the Gille sequence (4). Namely, it is known (cf. [PR]) that the group of F-points of the simply connected covering $\widetilde{G}$ is projectively simple, and therefore $\widetilde{G}(F)/R = 1$. Furthermore, the center of $\widetilde{G}$ equals either $\mu_2 \times \mu_2$ (inner type) or $R_{L/F}(\mu_2)$ (outer type) and it is easy to check that in both cases the torus E a flasque resolution of the center of $\widetilde{G}$ has trivial cohomology group in dimension 1. □

It is easy now to complete the proof of Theorem 2.12. By Lemma 2.13, one has

$$C_G(S) \simeq Z \times C_G(S)/Z,$$

hence

$$C_G(S)/R \simeq [\,C_G(S)/Z\,](F)/R.$$

But $C_G(S)/Z$ is an adjoint group of type ${}^2D_4$, as

$$C_G(S)/Z \simeq C_G(S)^{(1)}/C_G(S) \cap Z,$$

and $|\,C_G(S) \cap Z\,| = 4$. Therefore in view of Proposition 2.14 we get

$$[\,C_G(S)/Z\,](F)/R = 1.$$

Theorem 2.12 is proved completely. □

Remark 2.15 It is proved in [CP] that the variety of an arbitrary simple group G of type E_6 defined over any field F and of F-rank ≥ 2 is rational over F; in particular $G(F)/R = 1$.

Remark 2.16 In the preprint [Th], Theorem 4.12, Nguyen Thang computed the group of R-equivalence classes of a reductive algebraic group G defined over a global or local field F without using (as he claimed) Gille's results [G1]-[G3] and, in particular, he also proved the triviality of $G(F)/R$ for an isotropic group G of type E_6. Unfortunately, there is a glaring mistake in the preprint. At the beginning of § 4 Nguyen Thang identified two R-structures on the group $\mathrm{H}^1(F, \mu)$, where μ is the kernel of the corresponding simply connected covering $\widetilde{G} \to G$, one of which is the natural one and the other one comes from the morphism $G(F) \to \mathrm{H}^1(F, \mu)$. At this point he did not realize that in the case of an arbitrary field F these two structures are different and that the main results of P. Gille mentioned above state exactly that they coincide in the case of global and local fields. If one assumes the validity of this key fact then there is nothing to prove and all the other results of [Th] related to the R-equivalence property are trivial consequences of it.

We conclude this section by proving two statements which we need in the second part of the paper.

Let A be an algebra over an arbitrary field F of prime degree p and let G be a semisimple algebraic F-group, the simply connected covering $\widetilde{G}$ of which has the form

$$\widetilde{G} = \mathbf{SL}(1, \mathrm{A}_1) \times \cdots \times \mathbf{SL}(1, \mathrm{A}_s),$$

where $A_1, \ldots, A_s$ are algebras over F which coincide with either A or A^{op}. Put $\mu = \operatorname{Ker} \varphi$, where $\varphi : \widetilde{G} \to G$ is the correponding covering map, and let

$$\widetilde{H} = \mathbf{SL}(1, \mathrm{A}_1) \times 1 \times \cdots \times 1$$

and $H = \varphi(\widetilde{H})$.

Proposition 2.17 *The variety G is birationally isomorphic over F to the direct product $H \times G/H$; in particular, if $p = 2,\ 3$ then G is rational over F.*

Proof. If the number s of components of $\widetilde{G}$ equals 1, the statement is trivial, since H coincides with G. In general consider the restriction

$$\varphi|_{\widetilde{H}} : \widetilde{H} \longrightarrow G.$$

Since p is a prime number we have that either the center $Z_1 = \mu_p$ of $\widetilde{H}$ is a subgroup of μ or $\varphi|_{\widetilde{H}}$ is injective.

In the first case the group H is adjoint, hence it is a direct factor of G and there is nothing to prove. Let $Z_1 \cap \mu = 1$. By Lemma 2.13, it suffices to check that for any extension K/F one has

$$\mathrm{Ker}\,\big[\,\tau : \mathrm{H}^1(K,H) \longrightarrow \mathrm{H}^1(K,G)\,\big] = 1.$$

Take an arbitrary element $\lambda \in \mathrm{Ker}\,\tau$. Since the restriction $\varphi|_{\widetilde{H}}$ is injective there exists the unique $\lambda' \in \mathrm{H}^1(K,\widetilde{H})$ such that $\lambda = \theta(\lambda')$, where θ is the natural bijection

$$\theta : \mathrm{H}^1(K,\widetilde{H}) \longrightarrow \mathrm{H}^1(K,H).$$

By construction, $\theta(\lambda')$ is trivial in $\mathrm{H}^1(K,G)$. Therefore we get

$$\lambda' \in \mathrm{Im}\,[\,\mathrm{H}^1(K,\mu) \longrightarrow \mathrm{H}^1(K,\mathbf{SL}(1,\mathrm{A}_1)\times\cdots\times\mathbf{SL}(1,\mathrm{A}_s))\,]. \tag{6}$$

We may write μ as the direct product of its submodules $\mu = \nu_1 \times \cdots \times \nu_t$ each of them isomorphic to μ_p. Without loss of generality one can assume that the direct factors ν_i are generated by elements of the form

$$a_i = (\underbrace{1,\ldots,1}_{n_i-1},\, \xi_p^{m_{n_i,i}},\ldots,\xi_p^{m_{s,i}}),$$

where $m_{n_i,i},\ldots,m_{s,i}$ are natural numbers such that

$$m_{n_i,i} \not\equiv 0 \pmod p$$

for all $i = 1,\ldots,t$ and

$$n_1 = 1 > n_2 > \ldots > n_t. \tag{7}$$

Identify $\mathrm{H}^1(K,\mathbf{SL}(1,\mathrm{A_i}))$ and $\mathrm{H}^1(K,\nu_i)$ with $K^*/\mathrm{Nrd}(\mathrm{A_K})$ and $K^*/(K^*)^p$ respectively, where $A_K = A\otimes_F K$. Since $\lambda' \in \mathrm{H}^1(K,\widetilde{H})$ we have

$$\lambda' = (a{\cdot}\mathrm{Nrd}(\mathrm{A_K}),1,\ldots,1)$$

for some $a \in K^*$. On the other hand the condition (6) implies that there are $b_1,\ldots,b_t \in K^*$ such that

$$\lambda' = \prod_{i=1}^{t}(1,\ldots,1,b_i^{m_{n_i,i}}\mathrm{Nrd}(\mathrm{A_K}),\ldots,\mathrm{b_i^{m_{s,i}}}\mathrm{Nrd}(\mathrm{A_K})). \tag{8}$$

Taking (if necessary) another direct product decomposition of μ one can additionally assume that the n_2-th$,\ldots,n_t-$th components of a_1 are trivial. Then taking into account (8) and (7) we conclude that

$$1\cdot\mathrm{Nrd}(\mathrm{A_K}) = \mathrm{b_2^{m_{n_2,2}}}\cdot\mathrm{Nrd}(\mathrm{A_K})$$

$$1\cdot\mathrm{Nrd}(\mathrm{A_K}) = \mathrm{b_2^{m_{n_3,2}}b_3^{m_{n_3,3}}}\cdot\mathrm{Nrd}(\mathrm{A_K})$$

$$\cdots$$

$$1\cdot\mathrm{Nrd}(\mathrm{A_K}) = \mathrm{b_2^{m_{n_t,2}}\ldots b_t^{m_{n_t,t}}}\cdot\mathrm{Nrd}(\mathrm{A_K}).$$

From these equalities it follows that $b_2, \ldots, b_t \in \mathrm{Nrd}(\mathrm{A_K})$, as by assumption

$$m_{n_2,2} \ldots m_{n_t,t} \not\equiv 0 \pmod{p}.$$

Therefore the condition (8) can been rewritten in the form

$$(a\mathrm{Nrd}(\mathrm{A_K}), 1, \ldots, 1) = (\mathrm{b_1^{m_{n_1,1}}}\mathrm{Nrd}(\mathrm{A_K}), \ldots, \mathrm{b_1^{m_{s,1}}}\mathrm{Nrd}(\mathrm{A_K})).$$

As $Z_1 \cap \mu = 1$, we have that p does not divide at least one of the powers $m_{2,1}, \ldots, m_{s,1}$. It follows that $b_1 \in \mathrm{Nrd}(\mathrm{A_K})$ and hence $a \in \mathrm{Nrd}(\mathrm{A_K})$, i. e. $\lambda' = \lambda = 1$.

An inductive argument shows that the variety G is birationally isomorphic over F to the direct product of varieties of simple F-groups where each of them is F-isomorphic to either $\mathbf{SL}(1, \mathrm{A})$, $\mathbf{SL}(1, \mathrm{A^{op}})$ or $\mathbf{PSL}(1, A)$, $\mathbf{PSL}(1, A^{op})$. It follows that with the additional assumption $p = 2, 3$ the variety G is rational over F, for by the Chevalley – Grothendieck theorem (cf. [Ch], [DG]) the field of rational functions of an arbitrary connected group G is isomorphic to the function field of a generic maximal torus of G and on the other hand by Voskresenskii's theorem (cf. [V]) all tori of dimension at most 2 are rational over their definition fields. Thus Proposition 2.17 is proved. □

Let D be a quaternion skew field over F, let h be a skew hermitian form over D relative to the standard involution and let

$$G' = \mathbf{SU}(\mathrm{D}, \mathrm{h}) \times \cdots \times \mathbf{SU}(\mathrm{D}, \mathrm{h}).$$

Consider an arbitrary F-defined central subgroup $\mu \leq G'$ and the canonical homomorphism

$$\varphi : G' \to G = G'/\mu.$$

Denote

$$H' = \mathbf{SU}(\mathrm{D}, \mathrm{h}) \times 1 \times \cdots \times 1$$

and let $H = \varphi(H')$.

Proposition 2.18 *The variety G is birationally isomorphic over F to the direct product $H \times G/H$.*

Proof is carried out analogously as in the previous case. □

3 Groups of type ${}^{3,6}D_4$

Theorem 3.1 *Let F be an algebraic number field and let G be an absolutely simple simply connected linear algebraic group over F of type ${}^{3,6}D_4$. Then the equality $G(F)/R = 1$ holds.*

Proof. The variety of any isotropic group of type ${}^{3,6}D_4$ defined over an arbitrary field F of characteristic $\neq 2$ is rational over F (cf. [CP], Proposition 12), and in particular the corresponding group of R-equivalence classes is always trivial. This means that without loss of generality we may assume that G is an F-anisotropic group.

Let L/F be the minimal Galois extension over which G becomes a group of inner type and let $\Gamma = \mathrm{Gal}\,(\mathrm{L/F})$. Recall that if G is a group of type ${}^3D_4$ then $[L:F] = 3$ and $\Gamma \simeq \mathbb{Z}/3\mathbb{Z}$; otherwise we have $[L:F] = 6$ and $\Gamma \simeq S_3$. It easily follows from the Hasse principle for Galois cohomology that there exists a quadratic extension K/F which quasi-splits G and $K \not\subseteq L$ (see the proof of Proposition 6.15 in [PR]).

Let τ be the non-trivial automorphism K/F and let $\sigma \in \Gamma$ be an arbitrary element of order 3. Fix a subfield $F \subset P \subseteq L$ of degree 3 over F. In the case $[L:F] = 3$ the field P clearly coincides with L. If $[L:F] = 6$, let λ be the non-trivial automorphism L/P. Obviously we have $\lambda\sigma\lambda^{-1} = \sigma^2$.

Since K and L are linearly disjoint over F, we can extend the automorphisms

$$\sigma,\ \lambda \in \mathrm{Gal}\,(\mathrm{L/F}), \quad \tau \in \mathrm{Gal}\,(\mathrm{K/F})$$

in a natural way to automorphisms of the extension E/F, where $E = K\cdot L$. For simplicity we denote these extensions by the same letters σ, λ, τ.

By construction, G has a K-defined Borel subgroup B. As G is F-anisotropic, the intersection $T = B \cap \tau(B)$ is an F-defined maximal torus of G which splits over E. Let $\Sigma = \Sigma(T,G)$ be the root system of G relative to T and let $\Pi \subset \Sigma$ be the basis corresponding to the Borel subgroup B.

Our proof will be based on a criterion for an element $g \in G(E)$ in "generic" position to be F-defined. To formulate it we need a special parametrization of E-points of a Chevalley group $G(E)$.

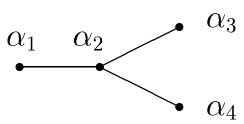
Let $W = W(T,G)$ be the Weyl group. It is well-known that the reflections w_α with respect to the simple roots $\alpha \in \Pi$ generate W. Consider an element $w_0 \in W$ having a maximal length with respect to the set of generators

$$S = \langle\, w_\alpha \mid \alpha \in \Pi \,\rangle$$

of W. It is known (cf. [B], chapter 6, §1) that such an element is unique and it is characterized by the property that it takes positive roots to negative ones. Recall also that the length r of its reduced decomposition

$$w_0 = w_{\alpha_1} \ldots w_{\alpha_{i_r}},$$

$\alpha_{i_j} \in \Pi$, coincides with the number of positive roots. We number elements of the basis $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ as follows:



(9)

Lemma 3.2 *In the notation above one has a decomposition*

$$w_0 = (w_{\alpha_2} w_{\alpha_1} w_{\alpha_3} w_{\alpha_4})\,(w_{\alpha_2} w_{\alpha_1} w_{\alpha_3} w_{\alpha_4})\,(w_{\alpha_2} w_{\alpha_1} w_{\alpha_3} w_{\alpha_4}). \qquad (10)$$

Proof. Easy computations show that the element $\nu \in W$ on the right-hand side of (10) takes the roots α_i to $-\alpha_i$, $i = 1, \ldots, 4$. This means that $\nu(\Sigma^+) = \Sigma^-$, where Σ^+ (resp. Σ^-) is the set of positive (resp. negative) roots with respect to Π, and thereby, by uniqueness, ν coincides with w_0. $\square$

Let us also denote

$$\beta_1 = \alpha_2,\ \beta_2 = \alpha_1,\ \beta_3 = \alpha_3,\ \beta_4 = \alpha_4$$

and further put $\beta_{i+4} = \beta_i$, $i = 1, \ldots, 8$. Consider the product morphism corresponding to the reduced decomposition (10)

$$\varphi : \mathbb{A}^{24} \times (G_{m,E})^4 \to G,$$

$$(u_1, \ldots, u_{12},\, v_1, \ldots, v_{12},\, t_1, \ldots, t_4) \longrightarrow \prod_{i=1}^{4} h_{\alpha_i}(t_i) \prod_{i=1}^{12} [\, x_{-\beta_i}(u_i) x_{\beta_i}(v_i)\,]. \qquad (11)$$

Proposition 3.3 *ϕ is a birational E-isomorphism.*

Proof: see [PR], ch. 9, §4, Lemma 20. $\square$

We now describe the action of Gal (E/F) on elements of $G(E)$. To this end recall preliminary the structure of quasi-split groups of type ${}^{3,6}D_4$ over an arbitrary field K. Let G_0 be a K-split group of type D_4. Let T_0 be a maximal K-split torus of G_0 and denote the corresponding root subgroups of G_0 by $\langle x_\alpha(u)\rangle$, $\alpha \in \Sigma$. Consider the group of algebraic automorphisms $\operatorname{Aut} G_0$ and the canonical homomorphism

$$\chi : \operatorname{Aut} G_0 \to \operatorname{Aut} \Delta,$$

where Δ is the Dynkin diagram (9).

According to [St], §10, p. 147, any automorphism $\eta \in \operatorname{Aut}\Delta$, i.e. any permutation of vertices $\{\alpha_1, \alpha_3, \alpha_4\}$ of Δ, can be lifted to an automorphism η_{G_0} of the group G_0 such that

$$\eta_{G_0}(x_{\pm\alpha_i}(u)) = x_{\pm\eta(\alpha_i)}(u),\ \ i = 1, \ldots, 4. \qquad (12)$$

It follows that χ is surjective and it admits a section, because the group $\operatorname{Aut} G_0$ is the semi-direct product of its subgroup of inner automorphisms $\overline{G_0}$ and its subgroup generated by the automorphisms (12) (which is obviously isomorphic to $\operatorname{Aut}\Delta$). In order to introduce no additional notation, we will denote the subgroup

$$\langle\, \eta_{G_0} \mid \eta \in S_3 = \operatorname{Aut}\Delta \,\rangle$$

of $\operatorname{Aut} G_0$ by the same symbol $\operatorname{Aut}\Delta$.

It is well-known that all K-quasi-split groups of type ${}^{3,6}D_4$ can be obtained from G_0 by twisting it using cocycles

$$\xi \in Z^1(K, \operatorname{Aut}\Delta).$$

We easily conclude from (12) that $\mathrm{Gal}\,(\mathrm{K_s}/\mathrm{K})$, where K_s is a separable closure of K, acts trivially on elements of the group $\operatorname{Aut}\Delta$, so that any cocycle ξ is the homomorphism

$$\xi : \mathrm{Gal}\,(\mathrm{K_s}/\mathrm{K}) \longrightarrow \operatorname{Aut}\Delta.$$

It follows then immediately from the definition of the twisting procedure and from (12) that the action of any element $\nu \in \mathrm{Gal}\,(\mathrm{K_s}/\mathrm{K})$ on the root subgroups $\langle x_{\alpha_i}(u)\rangle$ of G is given by

$$\nu\,[\,x_{\alpha_i}(u)\,] = x_{\xi(\nu)(\alpha_i)}(\nu(u)),$$

where $u \in K_s$ and $i = 1, \ldots, 4$.

Coming back to our group G, which is, by construction, quasi-split over K, and to the notation introduced at the beginning of the section we obtain that up to numbering of the vertices $\{\alpha_1,\, \alpha_3,\, \alpha_4\}$ of the Dynkin diagram (9) one can always assume that σ and λ act on the root subgroups as follows:

$$\begin{aligned}
&\sigma[x_{\pm\alpha_1}(u)] = x_{\pm\alpha_3}[\sigma(u)], &\quad& \sigma[x_{\pm\alpha_3}(u)] = x_{\pm\alpha_4}[\sigma(u)],\\
&\sigma[x_{\pm\alpha_4}(u)] = x_{\pm\alpha_1}[\sigma(u)], && \sigma[x_{\pm\alpha_2}(u)] = x_{\pm\alpha_2}[\sigma(u)],\\
&\lambda[x_{\pm\alpha_1}(u)] = x_{\pm\alpha_1}[\lambda(u)], && \lambda[x_{\pm\alpha_2}(u)] = x_{\pm\alpha_2}[\lambda(u)],\\
&\lambda[x_{\pm\alpha_3}(u)] = x_{\pm\alpha_4}[\lambda(u)], && \lambda[x_{\pm\alpha_4}(u)] = x_{\pm\alpha_3}[\lambda(u)].
\end{aligned} \tag{13}$$

It remains also to be determined the action of τ on the root subgroups. As $T = B \cap \tau(B)$, the Borel subgroup $\tau(B)$ corresponds to negative roots Σ^- and hence τ takes positive roots to negative ones. The latter means that the natural embedding

$$\mathrm{Gal}\,(\mathrm{E}/\mathrm{L}) \to \mathrm{W}(\mathrm{T}, \mathrm{G})$$

takes τ to the element w_0 from Lemma 3.2.

Consider a Chevalley basis

$$\{H_{\alpha_i},\ i = 1, \ldots, 4,\ X_\alpha,\ \alpha \in \Sigma\}$$

of the Lie algebra of G. It follows from the equality $\tau(\alpha) = -\alpha$ that $\tau(X_\alpha) = c_\alpha\, X_{-\alpha}$, where $c_\alpha \in E$ is some constant.

Lemma 3.4 *$\tau[\,x_\alpha(u)\,] = x_{-\alpha}(c_\alpha\tau(u))$ for all $u \in E$.*

Proof follows from the equality $x_\alpha(u) = \exp(uX_\alpha)$. $\square$

The constants c_α, $\alpha \in \Sigma$, have the following properties.

Lemma 3.5 *1) $c_\alpha \in L$ for all $\alpha \in \Sigma$; moreover, $c_{\alpha_1} \in P$ and $c_{\alpha_2} \in F$;*
2) $c_{\alpha_3} = \sigma(c_{\alpha_1}),\ c_{\alpha_4} = \sigma^2(c_{\alpha_1})$;
3) $c_{-\alpha} = c_\alpha^{-1}$ for all $\alpha \in \Sigma$.

Proof. 1) The group G_α generated by elements $x_{\pm\alpha}(u)$ is, clearly, a simply connected group over L and has rank 1. Therefore G_α is isomorphic over L to $\mathbf{SL}(1, \mathrm{D}_\alpha)$, where D_α is a quaternion algebra over L. Since G_α splits over E, the algebra D_α contains a maximal subfield E. Let $E = L(\sqrt{d})$ and let $D_\alpha = (d, d_\alpha)$. Then a straightforward computation in the quaternion algebra D_α shows that

$$c_\alpha \equiv d_\alpha \pmod{\mathrm{N}_{E/L}(E^*)},$$

implying $c_\alpha \in L^*$.

By (13), the group G_{α_1} (resp. G_{α_2}) is defined over P (resp. over F) and splits over the quadratic extension $P(\sqrt{d})$ (resp. $K = F(\sqrt{d})$). Therefore applying the previous argument to G_{α_1} (resp. G_{α_2}) we obtain that $c_{\alpha_1} \in P$ (resp. $c_{\alpha_2} \in F$).

2) It follows immediately from the equalities

$$\tau\sigma[\,x_{\alpha_1}(u)\,] = \tau[\,x_{\alpha_3}(\sigma(u))\,] = x_{-\alpha_3}[\,c_{\alpha_3}\tau\sigma(u)\,]$$

$$\sigma\tau[\,x_{\alpha_1}(u)\,] = \sigma[\,x_{-\alpha_1}(c_{\alpha_1}\tau(u))\,] = x_{-\alpha_3}[\,\sigma(c_{\alpha_1})\sigma\tau(u)\,]$$

that $c_{\alpha_3} = \sigma(c_{\alpha_1})$, for $\tau\sigma = \sigma\tau$. One shows analogously that $c_{\alpha_4} = \sigma^2(c_{\alpha_1})$.

3) The equalities

$$X_\alpha = \tau^2(X_\alpha) = \tau(c_\alpha X_{-\alpha}) = c_\alpha c_{-\alpha}(X_\alpha)$$

show that $c_\alpha c_{-\alpha} = 1$. Lemma 3.5 is proved. $\square$

Corollary 3.6 *One has*

$$G_{\alpha_2} \overset{F}{\simeq} \mathbf{SL}(1, \mathrm{D}),\ \mathrm{G}_{\{\alpha_1,\alpha_3,\alpha_4\}} \overset{\mathrm{F}}{\simeq} \mathrm{R}_{\mathrm{P/F}}(\mathbf{SL}(1, \mathrm{T})),$$

where $D = (d, c_{\alpha_2})$ and $T = (d, c_{\alpha_1})$ are quaternion algebras over F and P respectively.

Proof. The statement follows immediately from (13) and Lemmas 3.4 and 3.5. $\square$

Lemma 3.7 *One has $\tau(h_{\alpha_i}(t)) = h_{\alpha_i}(\tau(t)^{-1}),\ i = 1, \ldots, 4,\ t \in E$.*

Proof follows from the fact that the one-dimensional E-split subgroups

$$T_{\alpha_i} = \langle h_{\alpha_i}(t) \mid t \in F_s \rangle$$

of T are defined and anisotropic over L, since τ acts on the character module of T by multiplication on -1. $\square$

Our next purpose is to establish a criterion for elements of $G(E)$ to be defined over F. Let $g \in G(E)$ be an element in "generic" position. Proposition 3.3 shows that g can be written uniquely in the form

$$g = \prod_{i=1}^{4} h_{\alpha_i}(t_i) \prod_{i=1}^{12} x_{-\beta_i}(u_i) x_{\beta_i}(v_i), \tag{14}$$

where $t_i,\, u_i,\, v_i \in E$. Recall that

$$\beta_1 = \alpha_2,\ \beta_2 = \alpha_1,\ \beta_3 = \alpha_3,\ \beta_4 = \alpha_4$$

and further $\beta_{i+4} = \beta_i$, $i = 1, \ldots, 8$. Put

$$g_1 = x_{-\alpha_2}(u_1)x_{\alpha_2}(v_1), \qquad g_2 = \prod_{i=2}^{4} x_{-\beta_i}(u_i)x_{\beta_i}(v_i),$$

$$g_3 = x_{-\alpha_2}(u_5)x_{\alpha_2}(v_5), \qquad g_4 = \prod_{i=6}^{8} x_{-\beta_i}(u_i)x_{\beta_i}(v_i),$$

$$g_5 = x_{-\alpha_2}(u_9)x_{\alpha_2}(v_9), \qquad g_6 = \prod_{i=10}^{12} x_{-\beta_i}(u_i)x_{\beta_i}(v_i),$$

so that

$$g = \prod_{i=1}^{4} h_{\alpha_i}(t_i)\ g_1 \ldots g_6. \tag{15}$$

By construction, we have $g_1,\, g_3,\, g_5 \in G_{\alpha_2}(E)$ and $g_2,\, g_4,\, g_6 \in G_{\{\alpha_1,\alpha_3,\alpha_4\}}(E)$.

Lemma 3.8 *The element g belongs to $G(K)$, i. e. $\sigma(g) = g$, $\lambda(g) = g$, if and only if the following conditions hold:*

1) $h_{\alpha_2}(t_2),\, g_1,\, g_3,\, g_5 \in G_{\alpha_2}(K)$;
2) $h_{\alpha_1}(t_1)h_{\alpha_3}(t_3)h_{\alpha_4}(t_4),\, g_2,\, g_4,\, g_6 \in G_{\{\alpha_1,\alpha_3,\alpha_4\}}(K)$.

Proof follows from (13) and from the injectivity of parameterization (11). $\square$

Lemma 3.9 *The following holds:*

1) $x = x_{-\alpha_2}(u)x_{\alpha_2}(v)$ lies in $G_{\alpha_2}(K)$ if and only if $u,\, v \in K$;

2) $h = h_{\alpha_2}(t)$ lies in $G_{\alpha_2}(K)$ if and only if $t \in K$;

3) The element

$$x = x_{-\alpha_1}(u_1)\, x_{\alpha_1}(v_1)\, x_{-\alpha_3}(u_3)\, x_{\alpha_3}(v_3)\, x_{-\alpha_4}(u_4)\, x_{\alpha_4}(v_4)$$

lies in $G_{\{\alpha_1,\alpha_3,\alpha_4\}}(K)$ *if and only if* $u_1,\, v_1 \in P{\cdot}K$ *and*

$$u_3 = \sigma(u_1),\; u_4 = \sigma^2(u_1),\; v_3 = \sigma(v_1),\; v_4 = \sigma^2(v_1);$$

4) The element

$$h = h_{\alpha_1}(t_1)\, h_{\alpha_3}(t_3)\, h_{\alpha_4}(t_4)$$

lies in $G_{\{\alpha_1,\alpha_3,\alpha_4\}}(K)$ *if and only if* $t_1 \in P{\cdot}K$ *and* $t_3 = \sigma(t_1),\, t_4 = \sigma^2(t_1)$.

Proof: see formulas (13). □

We now turn to the analysis of the equality $g = \tau(g)$. In the decomposition (14) we put

$$t = \prod_{i=1}^{4} h_{\alpha_i}(t_i), \quad r_i = x_{-\beta_i}(u_i)\, x_{\beta_i}(v_i),$$

$i = 1, \ldots, 12$. Then the equation $g = \tau(g)$ is rewritten in the form

$$t\, r_1 \ldots r_{12} = \tau(t)\, \tau(r_1) \ldots \tau(r_{12}). \tag{16}$$

It is clear that $\tau(r_i) \in G_{\beta_i}$ since by Lemma 3.4 we have

$$\tau(r_i) = \tau[\, x_{-\beta_i}(u_i)\, x_{\beta_i}(v_i)\,] = x_{\beta_i}(c_{-\beta_i}\tau(u_i))\, x_{-\beta_i}(c_{\beta_i}\tau(v_i)) =$$

$$= h_{\beta_i}(1+\tau(u_iv_i))\, x_{-\beta_i}[\, c_{\beta_i}\tau(v_i)/(1+\tau(u_iv_i))\,]\, x_{\beta_i}[\, \tau(u_i)/c_{\beta_i}(1+\tau(u_iv_i))\,].$$

Recall that in Chevalley groups one has

$$h_\beta(t)\, x_\alpha(u)\, h_\beta(t)^{-1} = x_\alpha(t^{<\alpha,\beta>}u).$$

Using this relation we can rewrite the element $\tau(g)$ in the form

$$\begin{gathered} \tau(g) = \prod_{i=1}^{4} h_{\alpha_i}(\tau(t_i)^{-1}) \prod_{i=1}^{12} h_{\beta_i}(1+\tau(u_iv_i))\cdot \\ \prod_{i=1}^{12} x_{-\beta_i}[\, d_{\beta_i}\tau(v_i)(1+\tau(u_iv_i))\,]\, x_{\beta_i}[\, \tau(u_i)/d_{\beta_i}(1+\tau(u_iv_i))\,], \end{gathered} \tag{17}$$

where d_{β_i} are the "constants" (which depend on $u_{i+1}, \ldots, u_{12},\, v_{i+1}, \ldots, v_{12}$ but do not depend on $u_i,\, v_i$) given by

$$\begin{gathered} d_{\beta_{12}} = c_{\beta_{12}}, \quad d_{\beta_{11}} = c_{\beta_{11}}(1+\tau(u_{12}v_{12}))^{<\beta_{11},\beta_{12}>}, \\ d_{\beta_{10}} = c_{\beta_{10}}(1+\tau(u_{12}v_{12}))^{<\beta_{10},\beta_{12}>}\,(1+\tau(u_{11}v_{11}))^{<\beta_{10},\beta_{11}>}, \\ \ldots \\ d_{\beta_1} = c_{\beta_1}(1+\tau(u_{12}v_{12}))^{<\beta_1,\beta_{12}>} \ldots (1+\tau(u_2v_2))^{<\beta_1,\beta_2>}. \end{gathered} \tag{18}$$

Substituting (17) in the equation $g = \tau(g)$ and using the injectivity of parameterization (11) we obtain that the equality $g = \tau(g)$ holds iff

$$\begin{cases} u_{12} = d_{\beta_{12}}\tau(v_{12})(1+\tau(u_{12}v_{12})), \quad v_{12} = \tau(v_{12})/d_{\beta_{12}}(1+\tau(u_{12}v_{12})) \\ \qquad\qquad\qquad\qquad \dots \\ u_1 = d_{\beta_1}\tau(v_1)(1+\tau(u_1v_1)), \qquad v_1 = \tau(u_1)/d_{\beta_1}(1+\tau(u_1v_1)) \\ \prod\limits_{i=1}^{4} h_{\alpha_i}(t_i\tau(t_i)) = \prod\limits_{i=1}^{12} h_{\beta_i}(1+\tau(u_iv_i)) \end{cases}$$

or equivalently

$$\begin{aligned} u_{12} &= d_{\beta_{12}}\tau(v_{12})\,(1-d_{\beta_{12}}v_{12}\tau(v_{12}))^{-1}, \\ &\dots \\ u_1 &= d_{\beta_1}\tau(v_1)\,(1-d_{\beta_1}v_1\tau(v_1))^{-1}, \end{aligned} \tag{19}$$

$$\prod_{i=1}^{4} h_{\alpha_i}(t_i\tau(t_i)) = \prod_{i=1}^{12} h_{\beta_i}(1-d_{\beta_i}v_i\tau(v_i))^{-1}. \tag{20}$$

Thus we proved the following

Theorem 3.10 *Let $g \in G(E)$ be an element in "generic" position and let $u_1, \dots, u_{12}$, $v_1, \dots, v_{12}$, $t_1, \dots, t_4$ be its parameters in the decomposition (14). Then we have*

1) $g^{1-\tau}$ belongs to $T(E)$ iff relations (19) hold. Furthermore, for any integer $i = 1, \dots, 6$ we have

$$(g_i \dots g_6)^{1-\tau} = \prod_j h_{\beta_j}(1-d_{\beta_j}v_j\tau(v_j)),$$

where $g_1, \dots, g_6$ are the components of g in the decomposition (15), d_{β_j} are the constants given by (18), and the product is taken over all roots β_j occurring in $g_i, \dots, g_6$.

2) $g = \tau(g)$ iff relations (19), (20) hold.

Remark 3.11 *Substituting (19) in (18) it is easy to make sure that*

$$d_{\beta_i} = c_{\beta_i}(1-d_{\beta_{12}}v_{12}\,\tau(v_{12}))^{-<\beta_i,\beta_{12}>} \dots (1-d_{\beta_{i+1}}v_{i+1}\,\tau(v_{i+1}))^{-<\beta_i,\beta_{i+1}>}.$$

Let us come back to the proof of our main Theorem 3.1. Our strategy will consist in showing that for any element $g \in G(F)$ there are R-trivial elements $x_1, \dots, x_l \in G(F)$ such that

$$g = x_1 \dots x_l. \tag{21}$$

To this end, we show preliminary that it suffices to restrict ourselves by consideration of elements g from any subset $V \subset G(F)$ open in real topology.

Lemma 3.12 *Let $G_\infty = \prod_{v\in V_\infty} G(F_v)$ and let $U \subset G_\infty$ be an arbitrary open in real topology subset. Then the normal subgroup N of $G(F)$ generated by $G(F)\cap U$ coincides with $G(F)$.*

Proof. It is known (cf. [PR]) that the groups $G(F_v)$, $v \in V_\infty$, have no non-central normal subgroups. It follows that N is dense in G_∞ and hence $NU = G_\infty$. It follows that for any $g \in G(F)$ there exist $n \in N$ and $u \in U$ such that $g = nu$. But $u = n^{-1}g \in U \cap G(F) \subseteq N$, implying $g \in N$, as required. □

As U we take the following set. Let $v \in V_\infty^F$. If $F_v \simeq \mathbb{C}$, we put $U_v = G(F_v)$. If $F_v \simeq \mathbb{R}$, we denote by U_v the set consisting of all elements $g \in G(F_v)$ which lie in the image of the map (11) and such that their corresponding parameters $v_1, \dots, v_{12}$ satisfy the condition

$$1 - d_{\beta_i} v_i \tau(v_i) > 0,$$

if $\beta_i = \alpha_2$, and the condition

$$\mathrm{N}_{P\otimes F_v/F_v}(1 - d_{\beta_i} v_i \tau(v_i)) > 0,$$

if $\beta_i = \alpha_1, \alpha_3, \alpha_4$. Finally, let

$$U = \Pi_{v\in V_\infty} U_v$$

and let us show that an arbitrary element $g \in G(F)\cap U$ admits the decomposition (21).

We start with the decomposition of g in form (15). Recall that in this decomposition the factors g_1, g_3, $g_5 \in G_{\alpha_2}(K)$ and g_2, g_4, $g_6 \in G_{\{\alpha_1,\alpha_3,\alpha_4\}}$. To carry out a transition from (15) to the decomposition (21) we establish that multiplying g from the right-hand side by proper R-trivial elements $x_1, \dots, x_s$ from $G(F)$ one can “transpose” the factors $g_1, \dots, g_6$ in such a way to collect separately all g_i lying in G_{α_2} and all g_i lying in $G_{\{\alpha_1,\alpha_3,\alpha_4\}}$. As a result we will get an element of the form

$$g\, x_1 \dots x_s = x_{s+1}\, x_{s+2},$$

where $x_{s+1} \in G_{\alpha_2}$ and $x_{s+2} \in G_{\{\alpha_1,\alpha_3,\alpha_4\}}$. Since

$$x_{s+1}\, x_{s+2} \in G(F)$$

it follows from the injectivity of (11) that x_{s+1}, x_{s+2} are invariant under σ, λ, τ, i. e. they belong to $G_{\alpha_2}(F)$ and $G_{\{\alpha_1,\alpha_3,\alpha_4\}}(F)$ respectively, and it remains only to note that x_{s+1}, x_{s+2} are also R-trivial elements since by virtue of Corollary 3.6 the varieties G_{α_2}, $G_{\{\alpha_1,\alpha_3,\alpha_4\}}$ are rational over F.

The following two statements guarantee the possibility of rearrangement of components g_i and g_{i+1} in the decomposition (15). Let $g_1 \in G_{\alpha_2}(K)$ and

$g_2 \in G_{\{\alpha_1,\alpha_3,\alpha_4\}}(K)$ be two elements in "generic" position such that $g_2^{\tau-1} \in T$ and $(g_1g_2)^{\tau-1} \in T$. Put

$$(g_1g_2)^{\tau-1} = \prod_{i=1}^{4} h_{\alpha_i}(a_i).$$

Under proving Theorem 3.10 we showed that $a_3 = \sigma(a_1)$, $a_4 = \sigma^2(a_1)$ and the parameters a_1, a_2 are reduced norms in the quaternion algebras $D_1 = (d, c_{\beta_1})$ and $D_2 = (d, c_{\alpha_2}\mathrm{N}_{P/F}(a_1))$ over P and F respectively.

Lemma 3.13 *Suppose that for all valuations $v \in V_\infty^F$ such that $F_v \simeq \mathbb{R}$ the elements a_2 and $\mathrm{N}_{P/F}(a_1) = a_1a_3a_4$ are positive. Then there exists $x \in G(F)$ such that*

$$g_1\, g_2\, x = z_2\, z_1,$$

where $z_2 \in G_{\{\alpha_1,\alpha_3,\alpha_4\}}$ and $z_1 \in G_{\alpha_2}$.

Proof. We find the required element x in the form

$$x = g_2^{-1}g_1^{-1}\prod_{i=1}^{4} h_{\alpha_i}(t_i)\, z_2\, z_1, \tag{22}$$

where

$$z_1 = x_{-\alpha_2}(u_2)\, x_{\alpha_2}(v_2),$$

$$z_2 = x_{-\alpha_1}(u_1)\, x_{\alpha_1}(v_1)\, x_{-\alpha_3}(u_3)\, x_{\alpha_3}(v_3)\, x_{-\alpha_4}(u_4)\, x_{\alpha_4}(v_4)$$

and $u_1, \dots, u_4$, $v_1, \dots, v_4$, $t_1, \dots, t_4$ are some parameters. We have to pick up these parameters in such a way that x is defined over F.

Clearly, the elements g_1, g_2 are defined over K. So by virtue of Lemma 3.9 the condition for x to be defined over K is equivalent to the following relations:

$$u_3 = \sigma(u_1),\ u_4 = \sigma^2(u_1),\ v_3 = \sigma(u_1),$$

$$v_4 = \sigma^2(v_1),\ t_3 = \sigma(t_1),\ t_4 = \sigma^2(t_1)$$

and t_2, u_1, $v_1 \in K$, u_2, v_2, $t_1 \in PK$.

Assume in addition that the parameters u_1, u_2 satisfy relations

$$u_2 = d_{\beta_2}\tau(v_2)\,[\,1 - d_{\beta_2}v_2\tau(v_2)\,]^{-1},$$

$$u_1 = d_{\beta_1}\tau(v_1)\,[\,1 - d_{\beta_1}v_1\tau(v_1)\,]^{-1},$$

where

$$d_{\beta_2} = c_{\alpha_2},\ d_{\beta_1} = c_{\alpha_1}(1 + \tau(u_2v_2))^{-1} = c_{\alpha_1}(1 - c_{\alpha_2}v_2\tau(v_2)),$$

which is equivalent to

$$z_1^{\tau-1} \in T,\ (z_2\, z_1)^{\tau-1} \in T$$

(cf. formulas (19) and Remark 3.11).

It is easy to make sure that under these conditions we have

$$(z_2\, z_1)^{\tau-1} = h_{\alpha_2}(1 - d_{\beta_2} v_2 \tau(v_2))\, h_{\alpha_1}(1 - d_{\beta_1} v_1 \tau(v_1))\cdot$$

$$h_{\alpha_3}[\, \sigma(1 - d_{\beta_1} v_1 \tau(v_1))\,]\, h_{\alpha_4}[\, \sigma^2(1 - d_{\beta_1} v_1 \tau(v_1))\,],$$

and hence the element x is defined over F if and only if

$$\begin{cases} a_2 = (1 - d_{\beta_2} v_2 \tau(v_2))\, t_2 \tau(t_2) \\ a_1 = (1 - d_{\beta_1} v_1 \tau(v_1))\, t_1 \tau(t_1) \end{cases}$$

or equivalently

$$\begin{cases} a_2 = (1 - c_{\alpha_2} v_2 \tau(v_2))\, t_2 \tau(t_2) \\ a_1 = [\, 1 - c_{\alpha_1} a_2 v_1 \tau(v_1)/t_2 \tau(t_2)\,]\, t_1 \tau(t_1) \end{cases}$$

Thus we reduce our problem to checking conditions $a_2 \in \mathrm{Nrd}\, D_3$ and $a_1 \in \mathrm{Nrd}\, D_4$, where D_3, D_4 are quaternion algebras over F and P respectively of the form $D_3 = (d, c_{\alpha_2})$, $D_4 = (d, c_{\alpha_1} a_2)$.

The first condition automatically holds, as, by assumption, a_2 is everywhere locally positive. It is obvious that the second condition also holds since on one hand, $a_1 \in \mathrm{Nrd}\, D_1$ and on the other, the quaternion algebras D_1, D_4 locally coincide over F_v for all $v \in V_\infty^F$. Thus Lemma 3.13 is proved. $\square$

Conversely, let $g_1 \in G_{\alpha_2}(K)$ and $g_2 \in G_{\{\alpha_1,\alpha_3,\alpha_4\}}(K)$ be two elements in "generic" position such that $g_1^{\tau-1} \in T$ and $(g_2\, g_1)^{\tau-1} \in T$. As above, put

$$(g_2\, g_1)^{\tau-1} = \prod_{i=1}^{4} h_{\alpha_i}(a_i).$$

Lemma 3.14 *Suppose that for all valuations $v \in V_\infty^F$ such that $F_v \simeq \mathbb{R}$ the elements a_2 and $\mathrm{N}_{P/F}(a_1) = a_1 a_3 a_4$ are positive. Then there exists $x \in G(F)$ such that*

$$g_2\, g_1\, x = z_1\, z_2,$$

where $z_2 \in G_{\{\alpha_1,\alpha_3,\alpha_4\}}$ and $z_1 \in G_{\alpha_2}$.

Proof is carried out analogously as in the previous one and so we omit it. $\square$

The following statement completes the proof of our main Theorem 3.1.

Proposition 3.15 *The elements x from the previous two lemmas are R-trivial in the group $G(F)$.*

Proof. Since two cases are equivalent, we consider the element x from Lemma 3.13 only. According to (22), we can write x in the form

$$x = g_2^{-1} g_1^{-1}\, t\, z_2\, z_1,$$

where

$$g_1,\, z_1 \in G_{\alpha_2}(K), \quad g_2,\, z_2 \in G_{\{\alpha_1,\alpha_3,\alpha_4\}}(K), \quad t \in T(K), \quad z_1^{\tau-1} \in T,$$

and moreover the corresponding parameter a in the equality $z_1^{\tau-1} = h_{\alpha_2}(a)$ is positive as an element of F_v for all $v \in V_\infty^F$ such that $F_v \simeq \mathbb{R}$.

Let us consider the set

$$Y = \{(s_1, s_2, s_4) \in G_{\alpha_2}(K) \times G_{\{\alpha_1,\alpha_3,\alpha_4\}}(K) \times G_{\{\alpha_1,\alpha_3,\alpha_4\}}(K) \mid \\ s_2^{-1} s_1^{-1} s_4\, z_1 \in G(F)\}$$

and the morphism

$$\lambda : Y \longrightarrow G(F),\ (s_1,\, s_2,\, s_4) \longrightarrow s_2^{-1} s_1^{-1} s_4\, z_1.$$

It is clear that the image of λ contains 1 and x and so it suffices to show that

i) *Y coincides with the set of F-points of some algebraic affine F-variety X;*

ii) *X is birationally isomorphic over F to an algebraic group H defined over F which has the trivial group of R-equivalence classes.*

Let

$$s_1 = h_{\alpha_2}(t_1)\, x_{-\alpha_2}(u_1)\, x_{\alpha_2}(v_1)$$
$$s_2 = h_{\alpha_1}(t_2)\, h_{\alpha_3}(t_3)\, h_{\alpha_4}(t_4)\, x_{-\alpha_1}(u_2)\, x_{\alpha_1}(v_2)\, x_{-\alpha_3}(u_3)\, x_{\alpha_3}(v_3)\, x_{-\alpha_4}(u_4)\, x_{\alpha_4}(v_4)$$
$$s_4 = h_{\alpha_1}(t_5)\, h_{\alpha_3}(t_6)\, h_{\alpha_4}(t_7)\, x_{-\alpha_1}(u_5)\, x_{\alpha_1}(v_5)\, x_{-\alpha_3}(u_6)\, x_{\alpha_3}(v_6)\, x_{-\alpha_4}(u_7)\, x_{\alpha_4}(v_7).$$

Clearly, we have

$$s_1 \in G_{\alpha_2}(K), \quad s_2,\, s_4 \in G_{\{\alpha_1,\alpha_3,\alpha_4\}}(K).$$

Then Lemmas 3.8 and 3.9 show that

$$u_1,\, v_1 \in K, \quad t_2,\, t_5,\, u_2,\, v_2,\, u_5,\, v_5 \in P{\cdot}K,$$

and the parameters

$$t_3,\, t_4,\, t_6,\, t_7,\, u_3,\, u_4,\, u_6,\, u_7,\, v_3,\, v_4,\, v_6,\, v_7$$

are uniquely determined by t_2, t_5, u_2, u_5, v_2, v_5.

Furthermore, we have already known that the condition

$$s_2^{-1} s_1^{-1} s_4\, z_1 \in G(F)$$

is equivalent to

$$(s_1\, s_2)^{\tau-1} \in T, \quad (s_4\, z_1)^{\tau-1} \in T, \quad (s_1\, s_2)^{\tau-1} = (s_4\, z_1)^{\tau-1}.$$

Then according to formulas (19), Theorem 3.10, item 1), Remark 3.11 the parameters u_1, u_2, u_5 are uniquely determined by v_1, v_2, v_5 and the remainder parameters t_1, t_2, t_5, v_1, v_2, v_5 satisfy the system of equations

$$\left\{ \begin{array}{l} t_2\, \tau(t_2)\, (1 - c_{\alpha_1} v_2\, \tau(v_2)) = t_5\, \tau(t_5)\, (1 - d_{\beta_1} v_5\, \tau(v_5)) \\ t_1\, \tau(t_1)\, (1 - d_{\beta_2} v_1\, \tau(v_1)) = a \end{array} \right. \tag{23}$$

where

$$d_{\beta_1} = c_{\alpha_1} a^{-<\alpha_1,\alpha_2>} = c_{\alpha_1} a$$

$$d_{\beta_2} = c_{\alpha_2}\, N_{P/F}\, [\, t_2\, \tau(t_2)\, (1 - c_{\alpha_1} v_2\, \tau(v_2))\,]^{-<\alpha_2,\alpha_1>} =$$

$$= c_{\alpha_2} N_{P/F}\, [\, t_2\, \tau(t_2)\, (1 - c_{\alpha_1} v_2\, \tau(v_2))\,].$$

Using a proper biregular substitution we can rewrite the second equation of (23) in the form

$$t_1\, \tau(t_1) - a\, v_1\, \tau(v_1) = d_{\beta_2},$$

hence the system (23) is equivalent to

$$\left\{ \begin{array}{l} \mathrm{Nrd}_{\mathrm{T}_1}(\mathrm{w}_2) = \mathrm{Nrd}_{\mathrm{T}_2}(\mathrm{w}_5) \\ \mathrm{Nrd}_{\mathrm{T}_3}(\mathrm{w}_1) = \mathrm{c}_{\alpha_2} \mathrm{N}_{\mathrm{P/F}}[\mathrm{Nrd}_{\mathrm{T}_1}(\mathrm{w}_2)] \end{array} \right. \tag{24}$$

where T_1, T_2 and T_3 are the quaternion algebras over P and F respectively of the form

$$T_1 = (d, c_{\alpha_1}), \quad T_2 = (d, c_{\alpha_1} a), \quad T_3 = (d, a),$$

and $w_2 \in T_1$, $w_5 \in T_2$, $w_1 \in T_3$.

Note that according to our assumption a is positive in all real completions F_v; hence the quaternion algebra T_3 splits over all F_v, $v \in V_\infty^F$. This implies that c_{α_2} is a reduced norm in T_3 and thus the system (24) is equivalent to

$$\left\{ \begin{array}{l} \mathrm{Nrd}_{\mathrm{T}_1}(\mathrm{w}_2) = \mathrm{Nrd}_{\mathrm{T}_2}(\mathrm{w}_5) \\ \mathrm{Nrd}_{\mathrm{T}_3}(\mathrm{w}_1) = \mathrm{N}_{\mathrm{P/F}}\, [\, \mathrm{Nrd}_{\mathrm{T}_1}(\mathrm{w}_2)\,] \end{array} \right. \tag{25}$$

which determines an F-subgroup H of the reductive group

$$R_{P/F}(T_1^*) \times R_{P/F}(T_2^*) \times T_3^*.$$

Lemma 3.16 $H(F)/R = 1$.

Proof. It is obvious that H is a reductive group. Its semisimple part $H^{(1)}$ coincides with

$$R_{P/F}(\mathbf{SL}(1,\mathrm{T}_1)) \times \mathrm{R}_{\mathrm{P/F}}(\mathbf{SL}(1,\mathrm{T}_2)) \times \mathbf{SL}(1,\mathrm{T}_3)$$

and its central torus is isomorphic over F to

$$Z \simeq \{(p,\, p,\, N_{P/F}(p)) \in R_{P/F}(G_{m,P}) \times R_{P/F}(G_{m,P}) \times G_{m,F} \mid \\ p \in R_{P/F}(G_{m,P})\},$$

in particular, Z has trivial Galois cohomology group in dimension 1.

It follows then from Lemma 2.13 that

$$H(F)/R = [\, H/Z\,](F)/R = [\, H^{(1)}/H^{(1)} \cap Z\,]\,(F)/R,$$

and therefore by virtue of the sequence (4) the group $H(F)/R$ is a term of an exact sequence

$$H^{(1)}(F)/R = 1 \longrightarrow H(F)/R \longrightarrow \mathrm{H}^1(F,E) \to 1,$$

where E is the torus from a flasque resolution of $H^{(1)} \cap Z$. It remains to note that $\mathrm{H}^1(F,E) = 1$, as $H^{(1)} \cap Z = \mu_2$. Lemma 3.16 and hence Proposition 3.15 are proved. □

4 Semisimple p-adic groups of inner type

Throughout this section F denotes a local field whose residue field $\overline{F}$ has characteristic $\neq 2$. Let $\widetilde{G}_1, \ldots, \widetilde{G}_s$ be absolutely simple simply connected algebraic groups of inner type defined over F. Put

$$\widetilde{G} = \widetilde{G}_1 \times \cdots \times \widetilde{G}_s.$$

The main result of this section is the following theorem.

Theorem 4.1 *Let $\widetilde{G}$ be a group without components of types ${}^1A_n$. Then for an arbitrary F-defined central subgroup $\mu \leq \widetilde{G}$ the factor variety $G = \widetilde{G}/\mu$ is rational over F.*

In fact we prove a slightly more general statement. Namely, in the notation above suppose that all simple components of $\widetilde{G}$ are groups of one of the following types:

$$B_n\ (n \geq 2),\ C_n\ (n \geq 2),\ {}^1D_n\ (n \geq 4),\ G_2,\ F_4,\ {}^{1,2}E_6,\ E_7,\ E_8,\ {}^{3,6}D_4.$$

Theorem 4.2 *For an arbitrary F-defined central subgroup $\mu \leq \widetilde{G}$ the variety of the group $G = \widetilde{G}/\mu$ is rational over F.*

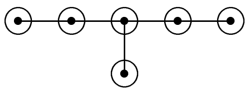
Proof is based on the classification of p-adic groups and Proposition 2.17. Recall first that by the Kneser theorem (cf. [PR], ch. VI) groups of types G_2, F_4, E_8 split over F and they have no centers. Therefore they are direct factors in G and without loss of generality we may assume that $\widetilde{G}$ consists of simple groups of types $B_n,\ C_n,\ {}^1D_n,\ {}^{3,6}D_4,\ E_7,\ {}^{1,2}E_6$ only.

Let $\widetilde{G}' \leq \widetilde{G}$ be a subgroup consisting of all simple components of the first five types mentioned above and analogously let $\widetilde{G}'' \leq \widetilde{G}$ be a subgroup consisting of all simple components of type E_6. It is clear that $\widetilde{G} \simeq \widetilde{G}' \times \widetilde{G}''$ and that the center of $\widetilde{G}'$ (resp. $\widetilde{G}''$) is a 2-primary (resp. 3-primary) group. It follows

$$\mu \simeq \mu' \times \mu'',$$

where $\mu' = \mu \cap \widetilde{G}'$, $\mu'' = \mu \cap \widetilde{G}''$, implying that it suffices to consider separately the cases when $\widetilde{G}$ either consists of components of type E_6 only or it has no components of such a type.

We begin with the first case, i. e. with simple components of type ${}^{1,2}E_6$. By [T], there are three possibilities for the Tits F-indices of $\widetilde{G}_1, \ldots, \widetilde{G}_s$:



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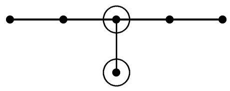

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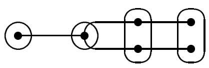


(28)

For a group $\widetilde{G}_i$ with the diagram (27) a semisimple F-anisotropic kernel is an almost direct product of two simple simply connected F-groups H_1, H_2 of inner type A_2. Therefore one has

$$H_1 \simeq \mathbf{SL}(1, D_1), \quad H_2 \simeq \mathbf{SL}(1, D_2),$$

where D_1, D_2 are skew fields over F of degree 3.

Lemma 4.3 *D_2 coincides with either D_1 or ${D_1}^{op}$; in particular for an arbitrary extension E/F algebras $D_1 \otimes_F E$ and $D_2 \otimes_F E$ have the same reduced norms.*

Proof. From the tables in [T] it follows that any extension E/F splitting D_1 also splits D_2 and vice versa. Therefore the algebras D_1, D_2 generate the same cyclic subgroup in $\mathrm{Br}\,(\mathrm{F})$ and it remains to note that $[D_i]$ has order 3 in $\mathrm{Br}\,(\mathrm{F})$. $\square$

As the first step of the proof of Theorem 4.2 we reduce the general case to the one where all simple components of $\widetilde{G}$ are either groups of inner type or groups of outer type splitting over the same quadratic extension L/F.

As $\mathrm{char}\,\overline{\mathrm{F}} \neq 2$, there exist only three quadratic extensions of F. Let L_1, L_2, L_3 be such extensions and we number the components $\widetilde{G}_1, \dots, \widetilde{G}_s$ of $\widetilde{G}$ in such a way that first we have groups $\widetilde{G}_1, \dots, \widetilde{G}_{i_1}$ of inner type and then groups

$$\{\widetilde{G}_{i_1+1}, \dots, \widetilde{G}_{i_2}\},\ \{\widetilde{G}_{i_2+1}, \dots, \widetilde{G}_{i_3}\},\ \{\widetilde{G}_{i_3+1}, \dots, \widetilde{G}_{s}\}$$

of outer type splitting over L_1, L_2, L_3 respectively. Put

$$\widetilde{H}_0 = \widetilde{G}_1 \times \cdots \times \widetilde{G}_{i_1}, \quad \widetilde{H}_1 = \widetilde{G}_{i_1+1} \times \cdots \times \widetilde{G}_{i_2},$$

$$\widetilde{H}_2 = \widetilde{G}_{i_2+1} \times \cdots \times \widetilde{G}_{i_3}, \quad \widetilde{H}_3 = \widetilde{G}_{i_3+1} \times \cdots \times \widetilde{G}_{s}$$

and denote

$$\nu_0 = \widetilde{H}_0 \cap \mu, \quad \nu_1 = \widetilde{H}_1 \cap \mu, \quad \nu_2 = \widetilde{H}_2 \cap \mu, \quad \nu_3 = \widetilde{H}_3 \cap \mu.$$

Lemma 4.4 $\mu \simeq \nu_0 \times \nu_1 \times \nu_2 \times \nu_3$.

Proof. It suffices to show that ν_0, ν_1, ν_2, ν_3 generate μ. Recall that the center Z_i of $\widetilde{G}_i$ is a group of order 3 and as $\mathrm{Gal}\,(\overline{\mathrm{F}}/\mathrm{F})$-module it has the following structure (see [PR]):

a) $Z_i \simeq \mu_3$, *if $\widetilde{G}_i$ is a group of inner type;*

b) $Z_i \simeq R^{(1)}_{L/F}(\mu_3)$, *if $\widetilde{G}_i$ is a group of outer type splitting over L.*

Case 1. Suppose that $\xi_3 \in F$. Then for any $\sigma \in \mathrm{Gal}\,(\overline{\mathrm{F}}/\mathrm{F})$ and any non-trivial element $x \in Z_i$ in case a) we have $\sigma(x) = x$ and in case b) we have $\sigma(x) = x$, if $\sigma|_L = 1$, and $\sigma(x) = x^2$ otherwise.

Let

$$z = (z_0,\ z_1,\ z_2,\ z_3) \in \mu$$

be an arbitrary non-trivial element. We have to prove that $z_i \in \nu_i$, $i = 0, \dots, 3$. Take three elements σ_1, σ_2, $\sigma_3 \in \mathrm{Gal}\,(\overline{\mathrm{F}}/\mathrm{F})$ such that $\sigma_i|_{L_i} \neq 1$ and $\sigma_i|_{L_j} = 1$ if $i \neq j$. Since μ is defined over F, we get $\sigma_1(z) \in \mu$. On the other hand according to our choice of σ_1 we have $\sigma_1(z) = (z_0,\ z_1^2,\ z_2,\ z_3)$. It follows that

$$\sigma_1(z)\, z^{-1} = (1,\ z_1,\ 1,\ 1) \in \mu,$$

whence $z_1 \in \nu_1$. One can check analogously that

$$\sigma_2(z)\, z^{-1} = (1,\ 1,\ z_2,\ 1), \quad \sigma_3(z)\, z^{-1} = (1,\ 1,\ 1,\ z_3),$$

hence $z_2 \in \nu_2$, $z_3 \in \nu_3$ implying $z_0 \in \nu_0$ as well.

Case 2 : $\xi_3 \notin F$. Then $F(\xi_3)$ coincides with one of the fields L_1, L_2, L_3. Let $F(\xi_3) = L_1$. As above, we indicate first how $\sigma \in \mathrm{Gal}\,(\overline{\mathrm{F}}/\mathrm{F})$ acts on elements of the center Z_i.

Let $x \in Z_i$. Clearly, in case a) one has $\sigma(x) = x$, if $\sigma|_{L_1=F(\xi_3)} = 1$, and $\sigma(x) = x^2$ otherwise. It is easy also to make sure that in case b) we have

if $L \neq L_1$, then

$$\sigma(x) = \begin{cases} x, & \text{if } \sigma|_{L_1} = \sigma|_L = 1 \ \text{ or } \ \sigma|_L \neq 1,\ \sigma|_{L_1} \neq 1; \\ x^2, & \text{if } \sigma|_{L_1} \neq 1,\ \sigma|_L = 1 \ \text{ or } \ \sigma|_{L_1} = 1,\ \sigma|_L \neq 1; \end{cases} \tag{29}$$

and if $L = L_1$, then $\sigma(x) = x$.

Let

$$z = (z_0,\, z_1,\, z_2,\, z_3) \in \mu.$$

It follows immediately from (29) that for any $\sigma \in \mathrm{Gal}\,(\overline{\mathrm{F}}/\mathrm{F})$ such that $\sigma|_{L_1} = \sigma|_{L_2} = 1$ and $\sigma|_{L_3} \neq 1$ one has

$$\sigma(z) = (z_0,\, z_1,\, z_2,\, z_3^2),$$

whence $z_3 \in \nu_3$. One shows analogously that $z_2 \in \nu_2$ and so the element $z^{'} = (z_0,\, z_1,\, 1,\, 1)$ also lies in μ. Finally, picking up $\sigma \in \mathrm{Gal}\,(\overline{\mathrm{F}}/\mathrm{F})$ such that $\sigma|_{L_1} \neq 1$ we conclude from the equality

$$\sigma(z^{'}) = (z_0^2,\, z_1,\, 1,\, 1)$$

that $z_0 \in \nu_0$, $z_1 \in \nu_1$. Thus Lemma 4.4 is proved completely. □

Let us continue the proof of Theorem 4.2 in the case where all components of $\widetilde{G}$ have type E_6. By Lemma 4.4, we have an isomorphism

$$G \simeq H_0 \times H_1 \times H_2 \times H_3,$$

where $H_i = \widetilde{H}_i/\nu_i$, $i = 0, \ldots, 3$. Therefore one can assume that all simple components of G are groups of either inner type or outer type E_6 splitting over the same quadratic extension L/F.

Case 1: $\widetilde{G}_1, \ldots, \widetilde{G}_s$ are groups of inner type. Let S be a maximal F-split torus of G. We have to show the rationality of the variety of the group $C_G(S)/S$. From the diagrams (26), (27) it follows that an F-anisotropic semisimple group $C_G(S)/S$ is an almost direct product of simple F-groups of inner type A_2 and moreover, by virtue of Lemma 4.3, each of them is related to either D or D^{op} where D is a fixed skew field over F of degree 3 (recall that over a local field the 3-component of $\mathrm{Br}\,(\mathrm{F})$ has order 3). So our statement follows immediately from Proposition 2.17.

Case 2: $\widetilde{G}_1, \dots, \widetilde{G}_s$ are groups of outer type splitting over a quadratic extension L/F. Let $\widetilde{S}_i$ be a maximal F-split torus of $\widetilde{G}_i$. It follows from the diagram (28) that the centralizer $C_{\widetilde{G}_i}(\widetilde{S}_i)$ is isomorphic over F to the torus T_i of the form

$$T_i \simeq G_m \times G_m \times R_{L/F}(G_{m,L}) \times R_{L/F}(G_{m,L}).$$

As the center Z_i of $\widetilde{G}_i$ lies in the last two components of T_i, the following lemma completes the proof.

Lemma 4.5 *Let $\nu \leq R_{L/F}(G_{m,L})$ be an arbitrary subgroup over F of order 3 and let*

$$\widetilde{P} = R_{L/F}(G_{m,L}) \times \cdots \times R_{L/F}(G_{m,L}).$$

Then for any subgroup $\mu \leq \nu \times \cdots \times \nu$ defined over F the variety $P = \widetilde{P}/\mu$ is rational over F.

Proof is trivial. Namely, if

$$\nu \times 1 \times \cdots \times 1 \leq \mu,$$

then $P \simeq P_1 \times P_2$, where

$$P_1 = 1 \times R_{L/F}(G_{m,L}) \times \cdots \times R_{L/F}(G_{m,L})/\mu_1, \quad P_2 = R_{L/F}(G_{m,L})/\nu$$

and

$$\mu_1 = \mu \cap \left[1 \times R_{L/F}(G_{m,L}) \times \cdots \times R_{L/F}(G_{m,L})\right].$$

As $\dim P_2 = 2$, the variety P_2 is rational over F and by induction so is P.

Let

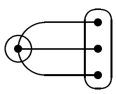
$$\nu \times 1 \times \cdots \times 1 \cap \mu = 1.$$

Denote the image of

$$R_{L/F}(G_{m,L}) \times 1 \times \cdots \times 1$$

in P by P_1. As the torus P_1 has trivial Galois cohomology group in dimension 1, we have, by Lemma 2.13, that $P \approx P_1 \times [\,P/P_1\,]$ and hence one can again use the induction argument. Lemma 4.5 is proved. □

Now we pass to the case where the center of $\widetilde{G}$ is a 2-primary group, i. e. there are no groups of type E_6 among the components $\widetilde{G}_1, \dots, \widetilde{G}_s$. For the reader's convenience recall the classification of the Tits F-indices of non-split groups of types B_n, C_n, ${}^1D_n$, ${}^{3,6}D_4$, E_7 over local fields (see details in [T]):



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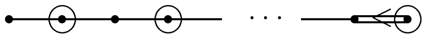
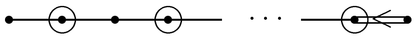
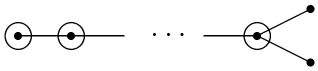
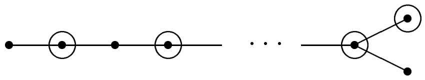
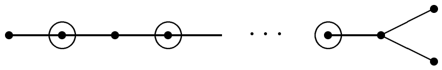
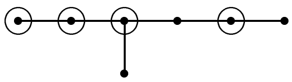
 (32)

 (33)

 (34)

 (35)

 (36)

 (37)

The center Z_i of $\widetilde{G}_i$ has the following structure (see [PR], ch. 6):

$$Z_i = \begin{cases} \mu_2, & \text{if } \widetilde{G}_i \text{ is a group of type } B_n, C_n, E_7; \\ \mu_4, & \text{if } \widetilde{G}_i \text{ is a group of type } {}^1D_{2n+1}; \\ \mu_2 \times \mu_2, & \text{if } \widetilde{G}_i \text{ is a group of type } {}^1D_{2n}; \\ R^{(1)}_{L/F}(\mu_2), & \text{if } \widetilde{G}_i \text{ is a group of type } {}^{3,6}D_4, \end{cases} \tag{38}$$

where in the latter case L denotes a cubic extension of F over which $\widetilde{G}_i$ has type ${}^{1,2}D_4$. In what follows we denote by T (resp. D) a skew field over F of degree 4 (resp. degree 2). We divide the further proof of the rationality of the variety $G = \widetilde{G}/\mu$ into three steps.

First step. We number the simple components of $\widetilde{G}$ in such a way that first we have the groups $\widetilde{G}_1, \ldots, \widetilde{G}_i$ of type ${}^{3,6}D_4$ and then the groups of the remainder types. Let

$$H_1 = \widetilde{G}_1 \times \cdots \times \widetilde{G}_i, \quad H_2 = \widetilde{G}_{i+1} \times \cdots \times \widetilde{G}_n.$$

Clearly, $\widetilde{G} = H_1 \times H_2$ and using (38) it is easy to show that $\mu = \mu_1 \times \mu_2$, where $\mu_1 = H_1 \cap \mu$, $\mu_2 = H_2 \cap \mu$. It follows that

$$G \simeq H_1/\mu_1 \times H_2/\mu_2$$

and hence one can assume that all simple components of G have types either B_n, C_n, ${}^1D_n$, E_7 or ${}^{3,6}D_4$.

According to (30) in the second case all simple components of $\widetilde{G}$ are quasi-split groups over F. So passing to the centralizer of a maximal F-split torus of $\widetilde{G}$ and repeating verbatim the argument from Lemma 4.5, it is easy to show that the variety G is birationally isomorphic over F to the direct product of the varieties of unipotent groups and F-rational tori of the form

$$R_{L/F}(G_{m,L}), \quad R_{L/F}(G_{m,L})/R^{(1)}_{L/F}(\mu_2).$$

This means that without loss of generality we may assume that $\widetilde{G}$ contains no simple components of exception type ${}^{3,6}D_4$.

Second step. We reduce the general case to the one where $\widetilde{G}$ contains no components of types B_n, C_n, E_7 and components with the Tits diagrams (34), (35). The distinctive feature of all such components is the fact that their semisimple anisotropic kernels are almost direct products of groups of type A_1, i. e. groups of the form $\mathbf{SL}(1,\mathrm{D})$. Therefore, replacing all components of $\widetilde{G}$ having the above mentioned property with the centralizers of their maximal F-split tori and then factorizing them by the maximal F-split tori it is easy to see that we are in conditions of the following general theorem.

Let

$$\widetilde{G} = \widetilde{G}_1 \times \cdots \times \widetilde{G}_s,$$

where all simple components $\widetilde{G}_i$ are either $\mathbf{SL}(1,\mathrm{D})$ or have the Tits diagram (36). Consider an arbitrary central subgroup $\mu \leq \widetilde{G}$ defined over F and put $G = \widetilde{G}/\mu$. Suppose also that $\widetilde{G}_1 = \mathbf{SL}(1,\mathrm{D})$ and denote its image in G by H.

Theorem 4.6 *The variety G is birationally isomorphic over F to the direct product $H \times G/H$.*

Proof is carried out in exactly the same fashion as Proposition 2.17. Namely, consider the restriction

$$\phi|_{\widetilde{G}_1} : \widetilde{G}_1 \longrightarrow G.$$

Since the center Z_1 of $\widetilde{G}_1$ consists of two elements there are two possibilities: either $Z_1 \leq \mu$ or $\phi|_{\widetilde{G}_1}$ is injective.

In the first case H is a direct factor in G and there is nothing to prove. Let $Z_1 \cap H = 1$. Then by Lemma 2.13, it suffices to show that for any extension K/F one has

$$\operatorname{Ker}[\,\tau : \mathrm{H}^1(\mathrm{K},\mathrm{H}) \longrightarrow \mathrm{H}^1(\mathrm{K},\mathrm{G})\,] = 1. \tag{39}$$

Recall that an F-group of type ${}^1D_{2n+1}$ having the Tits diagram (36) is isomorphic over F to a spinor group $\mathbf{Spin}(D,h)$, where h is a non-degenerate skew-hermitian form over the quaternion algebra D having trivial discriminant and odd dimension and moreover h is not hyperbolic.

By our assumptions, the centers Z_i of the components $\widetilde{G}_i$ coincide either with μ_2 or with μ_4. It follows that μ is a direct product of its submodules of the form μ_2, μ_4. Therefore repeating verbatim the same argument as in Proposition 2.17 it is easy to see that the equality (39) follows from the following two lemmas.

Lemma 4.7 *Let K/F be an arbitrary extension. Identifying* $\mathrm{H}^1(K,\mu_2)$ *with* $K^*/(K^*)^2$ *we get*

$$\mathrm{Ker}\,\left[\mathrm{H}^1(K,\mu_2) \longrightarrow \mathrm{H}^1(K,\mathrm{Spin}\,(D,h))\right] = \mathrm{Nrd}\,(D\otimes_F K)^*/(K^*)^2.$$

Proof. The exact sequence

$$\mathbf{SU}\,(D,h)(K) \longrightarrow \mathrm{H}^1(K,\mu_2) \longrightarrow \mathrm{H}^1(K,\mathbf{Spin}\,(D,h))$$

shows that our kernel consists of spinor norms. It is well-known (see. [Sch]) that over a local field any skew-hermitian form of dimension ≥ 4 is isotropic. Therefore a subgroup of $K/(K^*)^2$ consisting of spinor norms coincides exactly with $\mathrm{Nrd}\,(D\otimes_F K)/(K^*)^2$. $\square$

Lemma 4.8 *Let K/F be an arbitrary extension. Identifying* $\mathrm{H}^1(K,\mu_4)$ *with* $K^*/(K^*)^4$ *we get*

$$\mathrm{Ker}\,\left[\,\mathrm{H}^1(K,\mu_4) \longrightarrow \mathrm{H}^1(K,\mathbf{Spin}\,(D,h))\,\right] \subset \mathrm{Nrd}\,(D\otimes_F K)/(K^*)^4.$$

Proof. If

$$x(K^*)^4 \in \mathrm{Ker}\,\left[\,\mathrm{H}^1(K,\mu_4) \longrightarrow \mathrm{H}^1(K,\mathbf{Spin}\,(D,h))\,\right],$$

then, clearly, we have

$$x(K^*)^2 \in \mathrm{Ker}\,\left[\,\mathrm{H}^1(E,\mu_2) \longrightarrow \mathrm{H}^1(K,\mathbf{SU}\,(D,h))\,\right],$$

forcing x to be a multiplier of the skew-hermitian form h over K and hence a multiplier of its 3-dimensional anisotropic part h_{an} which has trivial discriminant.

The triviality of the discriminant of h_{an} implies that $\mathbf{PSU}(D,h_{an})$ is an adjoint group of inner type ${}^1A_3$, hence its variety is rational over F; in particular, the group of R-equivalence classes of $\mathbf{PSU}(D,h_{an})$ is R-trivial. Then according to [M2] the group of multipliers of the form h over K coincides with the subgroup of K^* generated by the norms of all finite extensions L/K over which h becomes hyperbolic. By [M2] Lemma 4, the latter group is contained in

$$N_{E\cdot K/K}\,[\,\mathrm{Nrd}\,(C(h_{an})\otimes_F K)\,]\,,$$

where $C(h_{an})$ is the Clifford algebra of h_{an} and E/F is its center. It remains to note that the components of the Clifford algebra $C(h_{an})$ coincide with either T or T^{op} and that if x is a reduced norm in $T\otimes_F K$ then it is also a reduced norm in $(T \otimes_F T) \otimes_F K = D \otimes_F K$. Thus Lemma 4.8 and hence Theorem 4.6 are proved. $\square$

Third step. Let all simple components $\widetilde{G}_1, \dots, \widetilde{G}_s$ of $\widetilde{G}$ have type ${}^1D_{2n+1}$ and the Tits diagram (36). As above, we can replace the components $\widetilde{G}_i$ with $C_{\widetilde{G}_i}(\widetilde{S}_i)/\widetilde{S}_i$, where $\widetilde{S}_i$ is a maximal F-split torus of $\widetilde{G}_i$. From the arguments in [CP] item 6.3, it follows that $C_{\widetilde{G}_i}(\widetilde{S}_i)/\widetilde{S}_i$ is glued by means of several copies of the groups $\mathbf{SL}\,(1,D)$ and $\mathbf{SL}\,(1,T)$, the latter one being not a direct factor in $C_{\widetilde{G}_i}(\widetilde{S}_i)$.

Lemma 4.9 *Let K/F be an arbitrary extension. Identifying $\mathrm{H}^1(K,\mu_2)$ with $K^*/(K^*)^2$ we get*

$$\mathrm{Ker}\left[\,\mathrm{H}^1(K,\mu_2) \longrightarrow \mathrm{H}^1(K,\mathbf{SL}\,(1,T))\,\right] \subseteq \mathrm{Nrd}\,(D\otimes_F K)/(K^*)^2.$$

Proof. It suffices to note that

$$\mathbf{SL}\,(1,T) \;\simeq\; \mathbf{Spin}\,(D,h_2),$$

where h_2 is a 2-dimensional F-anisotropic skew-hermitian form over D, and that the described kernel consists of spinor norms. $\square$

Lemma 4.10 *Let K/F be an arbitrary extension. Identifying $\mathrm{H}^1(K,\mu_4)$ with $K^*/(K^*)^4$ we get*

$$\mathrm{Ker}\left[\,\mathrm{H}^1(K,\mu_4) \longrightarrow \mathrm{H}^1(K,\mathbf{SL}\,(1,T))\,\right] \subseteq \mathrm{Nrd}\,(D\otimes_F K)/(K^*)^4.$$

Proof. If x is a reduced norm in $T \otimes_F K$ then x is also a reduced norm in $(T \otimes_F T) \otimes_F K = D \otimes_F K$. $\square$

Using Lemmas 4.9, 4.10 it is easy to complete the proof of Theorem 4.2. Namely, repeating verbatim the same argument as in Theorem 4.6 or in Proposition 2.17 we see that the group

$$H = \left[\,\prod_{i=1}^{s} C_{\widetilde{G}_i}(\widetilde{S}_i)/\widetilde{S}_i\,\right]/\mu.$$

is, in fact, birationally isomorphic over F to the direct product of its subgroup H' consisting of all its simple components of type A_1 and the corresponding factor group H/H'. Therefore the variety of our initial group G is birationally isomorphic over F to the direct product of varieties of unipotent groups, rational varieties $\mathbf{SL}\,(1,D)$, $\mathbf{PSL}\,(1,D)$ and the group G' which is glued by means of several copies of the rational group $\mathbf{SL}(1,T)/\mu_2 = \mathbf{SU}\,(D,h_2)$. In turn, by Proposition 2.18, the variety G' is birationally isomorphic over F to the direct product of several copies of the groups $\mathbf{SU}\,(D,h_2)$ and $\mathbf{PSU}(D,h_2)$. Theorem 4.2 is completely proved. $\square$

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