Coxeter Decompositions of Spherical Tetrahedra.

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Introduction

Let P be a polyhedron in a spherical space S^3.

Definition 1. *A polyhedron P is a* **Coxeter polyhedron** *if all dihedral angles of P are the integer parts of π.*

Definition 2. *A polyhedron P admits a* **Coxeter decomposition** *if P can be tiled by N Coxeter polyhedra F_i,...,F_N ($1 < N$), such that any two tiles F_i and F_j with a common face are symmetric with respect to this face.*

In this paper we classify all Coxeter decompositions of convex spherical tetrahedra. The same problem for hyperbolic tetrahedra was solved in [1], but the table of the decompositions of the first type was uncomplete. We present the revised list in appendix.

Definitions

All polyhedra F_i in a Coxeter decomposition are obviously congruent. We call these polyhedra **fundamental** and denote by F. A plane containing a face of a fundamental polyhedron will be called **mirror** if it contains no face of P.

Remark 1. *A Coxeter decomposition of any polyhedron can be prolonged to the decomposition of the whole sphere (the definition of a Coxeter decomposition of $\mathbf{S}^3$ coincides with definition 2).*

Remark 2. *In this paper we deal with polyhedra bounded by the mirrors of a Coxeter decomposition of some polyhedron or the space.*

Definition 3. *Fix a Coxeter decomposition of a polyhedron P.*

A **vertex** *A of P is* **fundamental** *if it does not belong to any mirror.*

A **dihedral angle** *of P made up by faces α and β is called* **fundamental** *if $\alpha \cap \beta$ does not belong to any mirror.*

Remark 3. *All the definitions above can be used for polytopes of arbitrary dimension, in particular, for polygons.*

Notation

$N > 1$ is a number of the decomposition tiles F in P.
$Vol(T)$ is a volume of a tetrahedron T.
k-edge is an edge of F with dihedral angle $\frac{\pi}{k}$.
k-l-m-vertex is a vertex of F incident to k-edge, l-edge and m-edge.
By tetrahedra we mean convex tetrahedra.
The Coxeter tetrahedra S_i are represented by their Coxeter diagrams in Table 3.

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1 Fundamental polyhedron

Lemma 1. *1) Let P be a polyhedron with a Coxeter decomposition. Then any polyhedron contained in P admits a Coxeter decomposition too.*

2) Let P be a polyhedron admitting a Coxeter decomposition and all dihedral angles of P are fundamental. Then P is a Coxeter polyhedron.

3) If P contains no pair of opposite points of $\mathbf{S}^3$, then F is a tetrahedron.

Proof. The statements 1) and 2) are obvious in view of remark 2; 3) is true, since there are no non-degenerate Coxeter polyhedra in $\mathbf{S}^3$ except the tetrahedra.

□

Lemma 2. *Let F be a spherical Coxeter tetrahedron. Then there are only finitely many tetrahedra P admitting a Coxeter decomposition with fundamental tetrahedron F.*

Proof. Any spherical tetrahedron is completely determined by its dihedral angles. The dihedral angles of P should be multiples of the dihedral angles of F. Therefore, there are finitely many admissible sets of dihedral angles of P.

□

1.1 Volumes

Clearly, a volume of P should be a multiple of a volume of F, i.e. $\frac{Vol(P)}{Vol(F)} \in \mathbf{Z}$

This condition gives a way to conclude that some tetrahedra cannot tessellate some other tetrahedra. For instance, the Coxeter diagrams of F and P should be different (since volumes of F and P are completely determined by the dihedral angles of F and P).

The volume of a spherical Coxeter tetrahedron S_i equals to $\frac{Vol(\mathbf{S}^3)}{Ord(S_i)}$, where $Vol(\mathbf{S}^3)$ is a volume of the sphere and $Ord(S_i)$ is an order of the group generated by reflections in the faces of S_i. Obviously, $\frac{Vol(S_i)}{Vol(S_j)} = \frac{Ord(S_j)}{Ord(S_j)}$. See the list of the orders $Ord(S_i)$ in Table 3.

2 Types of Coxeter decompositions

There are three types of Coxeter decompositions:

Definition 4. *If P and any non-fundamental tetrahedron in P has a non-fundamental dihedral angle, then the decomposition is of the* **first type**,
if all dihedral angles of P are fundamental, then the decomposition is of the **second type**,
if P has a non-fundamental dihedral angle, but there is a non-fundamental tetrahedron $T \neq P$, $T \in P$ with fundamental dihedral angles, then the decomposition is of the **third type**.

The decompositions of the first and the second types will be studied separately. The decompositions of the third type will be constructed from the decompositions of the first and the second types.

2.1 Decompositions of the first type

It is proved in [1] that the decompositions of the first type with a fixed fundamental tetrahedron F can be classified by the following **inductive procedure**:

Step 0. Take a fundamental tetrahedron F, let $P_0 = F$.

Step 1. Construct all tetrahedra $P_1, ..., P_l$ admitting a Coxeter decomposition which consist of two copies of P_0.

Step 2. Construct all tetrahedra $P_{l+1}, ..., P_k$ from P_1 and P_0 and all tetrahedra $P_{k+1}, ..., P_s$ from two copies of P_1.

Step m. Suppose that after $m-1$ steps we have a sequence of tetrahedra $F = P_0, P_1, ..., P_m, ..., P_n$. Let $T^{i,j}_1$,...,$T^{i,j}_{k_{i,j}}$ be all the tetrahedra which we can obtain glueing P_i and P_j together. Suppose that for any $i, j < m-1, 1 \leq k \leq k_{i,j}$ any of $T^{i,j}_k$ is in the list P_0,...,P_n. Note that these conditions are satisfied after the steps 0, 1 and 2. Then at the m-th step we construct all the tetrahedra from P_{m-1} and P_i, for all $i \leq m-1$ (we do not put a new tetrahedron P_r in the sequence if $P_r = P_t$, $t < r$, and P_r and P_t have the same decomposition).

Since there are finitely many Coxeter tetrahedra in $\mathbf{S}^3$, we can do this procedure for all Coxeter tetrahedra. Thus, we obtain *all* Coxeter decompositions of the first type.

Spherical Coxeter tetrahedra are listed in Table. 3. Each of them is fundamental for a finitely many Coxeter decompositions. Let us find these decompositions using the inductive procedure.

1–5. The tetrahedra S_1–S_5 have connected Coxeter diagrams. The decompositions of the first type with $F = S_1$,...,S_5 were classified by inductive procedure (see Table 3 for the list).

6–8. Consider a decomposition with $F = S_6$, S_7 or S_8. Then F has a face α orthogonal to any other face of F. Hence, all the fundamental tetrahedra in the decomposition have a common vertex A opposite to the face α (see the proof of Lemma 6). Thus, all such decompositions can be constructed as follows. Take a sphere centered in A and a triangle p on this sphere. Fix a Coxeter decomposition of p such that p has a non-fundamental vertex (if all vertices of p are fundamental, we obtain a decomposition of the third type). Connect the point A by lines with all the vertices of the fundamental triangles f. In this construction we used the stereographic projection from the point opposite to A in 3-dimensional sphere. The decompositions of the spherical triangles with non-fundamental angles are listed in Table 3. The list was obtained by the inductive procedure.

9. Let F be a tetrahedron S_9 with arbitrary m and n. It is easy to prove by inductive procedure that there are lines a and b such that any tile F_i has an edges a_i and b_i on the lines a and b respectively (see Fig. 1a for an example). The number of decompositions of this kind is $(m-1)(n-1)-1$ (if $m \neq n$) or $\frac{m(m-1)}{2} - 1$ (if $m = n$).

10. Let F be a tetrahedron S_{10}. It is easy to prove by inductive procedure that all tiles of the decomposition have a common edge (see Fig. 1b). There are $m-2$ such a decompositions.

11. The tetrahedron S_{11} can not be fundamental for any Coxeter decompositions.

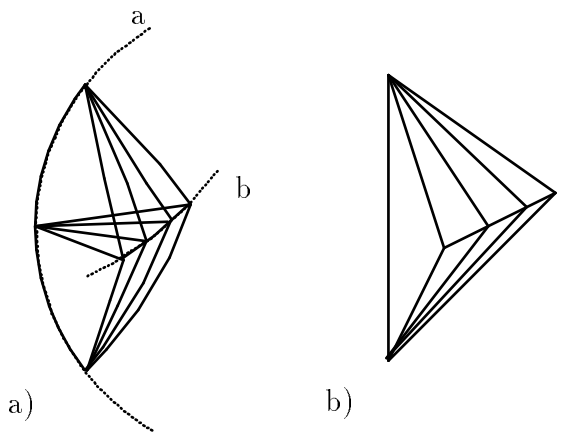


Figure 1.

2.2 Decompositions of the second type

Suppose that P admits a Coxeter decomposition and all dihedral angles of P are fundamental. It is evident that P is a Coxeter tetrahedron in this case.

Such decompositions can be of two types: either all vertices of P are fundamental or some vertices of P are non-fundamental.

The Lemmas 3, 4 and 5 are proved in [1].

Lemma 3. *(Subdiagram property).*

Let P be a tetrahedron admitting a Coxeter decomposition such that all vertices of P are fundamental, F be a fundamental tetrahedron, $\bar{P}$ and $\bar{F}$ be the Coxeter diagrams of P and F. Then for any node v in $\bar{P}$ there is a node w in $\bar{F}$ such that

$$\bar{P} \setminus v = \bar{F} \setminus w$$

(deleting the node from the Coxeter diagram we delete all the edges containing this node).

Lemma 4. *If all the vertices of P are fundamental, then $N \geq 8$.*

Lemma 5. *(Wide subdiagram property).*

Let P be a tetrahedron admitting a Coxeter decomposition of the second type such that some vertices of P are non-fundamental, F be a fundamental tetrahedron, $\bar{P}$ and $\bar{F}$ be the Coxeter diagrams of P and F. Then for any node v in $\bar{P}$ there exists a node w in F such that

either $\bar{P} \setminus v = \bar{F} \setminus w$,

or a triangle p corresponding to the subdiagram $\bar{P} \setminus v$ admits a Coxeter decomposition such that a fundamental triangle f corresponds to a subdiagram $\bar{F} \setminus w$ and all angles of p are fundamental.

Lemma 6. *Let P be a tetrahedron admitting a Coxeter decomposition without non-fundamental vertices, let F be a fundamental tetrahedron. Then the Coxeter diagram $\bar{F}$ contains no separate node.*

Proof. Suppose that $\bar{F}$ contains a separate node, that is F has a face α orthogonal to any other face of F. Since $N > 1$, there are at least two fundamental tetrahedra F_1 and F_2 with a common face. This common face cannot be α. Otherwise P contains two opposite points of sphere and has two vertices only (see Fig. 2a). Therefore all fundamental tetrahedra in P have a common vertex opposite to the face α (see Fig. 2b). This vertex is a non-fundamental vertex of P. Thus, if all vertices of P are fundamental then $\bar{F}$ has no separate node.

a) b)

Figure 2.

□

Remark 4. *Actually, it is easy to prove that if a simplex P admits a Coxeter decomposition without non-fundamental vertices, than a Coxeter diagram of a fundamental simplex is connected.*

Lemma 7. *Any spherical decomposition without non-fundamental vertices is one of two shown in the upper half of Table 4.*

Proof. Let P be a tetrahedron admitting a Coxeter decomposition of the second type, F be a fundamental tetrahedron. Then P and F satisfy the subdiagram and volume properties. By Lemma 4 $N \geq 8$. By Lemma 6 $\bar{F}$ contains no separate node. These conditions are satisfied only for two pairs of Coxeter tetrahedra (F,P). These pairs are (S_3,S_{11}) and (S_4,S_1). The numbers of tiles in these decompositions are $N = 12$ and $N = 120$ respectively. These decompositions are shown in the upper half of Table 4.

The first of these decompositions can be constructed by the following way. Take eight copies of F and construct a right octahedron (see Fig. 6a). Chose four faces of octahedron having no a common edge. Glue the rest four tetrahedra to these four faces.

For the construction of the second decomposition see Lemma 10.

□

In order to prove Lemma 8 we need a list of Coxeter decompositions of spherical triangles with fundamental angles. It is easy to check that such a decomposition is unique (see Fig. 3).

Lemma 8. *Any spherical decomposition of the second type with non-fundamental vertices is one of the four decompositions listed in the lower half of Table 4.*

Proof. Consider the Coxeter diagrams $\bar{P}$ and $\bar{F}$. By Lemma 5 $\bar{P}$ and $\bar{F}$ satisfy wide subdiagram property. A unique spherical decomposition of a triangle without non-fundamental angles is the decomposition shown in Fig 3. Hence, F has a dihedral angle $\frac{\pi}{5}$ and $\bar{P}$ has a subdiagram consisting of three separate nodes. Therefore F is either S_4 or S_8. P is one of the tetrahedra S_3, S_6, S_7, S_8, S_9, S_{10} and S_{11} (if $P = S_9$ then $m = n = 5$).

The decomposition shown in Fig. 3 consist of 15 fundamental triangles. Thus, $\frac{Vol(P)}{Vol(F)} \geq 15$. Hence, S_8 can be fundamental only for the tetrahedron S_{11}. There is a unique decomposition

with $(F,P)=(S_8,S_{11})$. To construct this decomposition draw the decomposition shown in Fig. 3 on a sphere, add the vertex A in a center of the sphere and join all the vertices of all triangles with A (see Fig. 6b).

Let F be the tetrahedron S_4. The tetrahedron S_7 has a 4-edge. Therefore S_4 cannot be fundamental for the decomposition of S_7. See Lemma 10 for proof that P cannot be a tetrahedron S_6, S_8 or S_{10} and for constructions of the decompositions of tetrahedra S_3, S_9 and S_{11} with fundamental tetrahedron F.

□

To prove Lemma 10 we need the following

Lemma 9. *Let P be a tetrahedron admitting a Coxeter decomposition of the second type. Let A be a vertex of P. Suppose that A is not a 2-2-2-vertex. Then A is a fundamental vertex of P.*

Proof. Suppose that A is non-fundamental. Consider a small sphere centered in A. The section of P by this sphere is a triangle p admitting a Coxeter decomposition. All angles of p are fundamental, since all dihedral angles of P are fundamental. Then the angles of the triangle p are $(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$. This contradicts to the assumption that A is not a 2-2-2-vertex.

□

Lemma 10. *Let F be S_4. Then any decomposition of the second type is one of the decompositions shown in Table 4.*

Proof. By Lemmas 7 and 8 P is one of the tetrahedra S_1, S_3, S_6, S_8, S_9, S_{10} and S_{11} ($m=n=5$ if $P=S_9$). First we construct the decompositions of the tetrahedra S_1, S_3, S_9 and S_{11}. Then we prove that P cannot be S_6, S_8 and S_{10}.

Representation of a decomposition by the tessellations of the faces. Let A be a k-l-m-vertex of F. Then a decomposition of a neighborhood of A corresponds to the decomposition of $\mathbf{S}^2$ into triangles $(\frac{\pi}{k},\frac{\pi}{l},\frac{\pi}{m})$. Therefore we know how the faces of F tessellate any spherical plane (see Fig. 3d for this tessellation). Any face of P is obviously one of the triangles of Fig. 3d. Note that the tessellation shown in Fig. 3d is not a Coxeter decomposition of the sphere, since it contains triangles of different form.

Let $\angle\alpha\beta$ be a dihedral angle formed by two faces of F. Suppose that we know the tessellation of the plane α by the faces of F. Then we can construct a tessellation of the plane β (use Table 3). Take a tetrahedron P. Suppose that the faces of P are tessellated by faces of F. Suppose that any face of P is a triangle from Fig. 3d. Suppose also that the dihedral angles of P are good, i.e. the faces are attached correctly. Obviously, the decomposition of P is determined by the tessellations of the faces. Let us prove that it is determined correctly.

Consider a decomposition of $\mathbf{S}^3$ with fundamental tetrahedron S_4. Take any mirror α. Find on the plane α a triangle decomposed as a first face of P . Find the planes β, γ and δ making with α the required angles. The tetrahedron T bounded by the planes α, β, γ and δ is decomposed with the fundamental tetrahedron S_4. On the other hand, T is equal to P, since there are equal faces α and α' in P and T, and the dihedral angles adjacent to α and α' are equal. Since we glued the faces of T correctly, the faces of T and P are tessellated by the same way. Thus, a good tessellation of the faces of P determines a decomposition of P.

Representation of a decomposition by 5-edges. Since any fundamental tetrahedron has a unique 5-edge, we can represent the decomposition by the set of 5-edges. For example

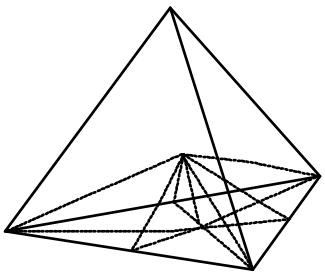


Figure 3.

the 5-edges in the decomposition of the tetrahedron 3.17 in Table 3 form a regular tetrahedron (see Fig. 3).

Decompositions of S_1, S_3, S_9 and S_{11} are represented in Fig. 8, 9, 10 and 11 respectively (we use the representation by tessellations of the faces of P). The inner 5-edges, i.e. 5-edges not lying on the faces, are also denoted there. These decompositions contains 75, 120, 144 and 900 fundamental tetrahedra respectively.

The decomposition of S_{11} can be also represented by the following way. Consider a decomposition of S_{11} with the fundamental tetrahedron F_3 (see Fig. 6a) and subdivide all the fundamental tetrahedra F_3 as shown in Fig. 8.

Impossibility of the decompositions of tetrahedra S_6, S_8 and S_{10}. To prove the impossibility of these decompositions we find a face α of P such that α is not a triangle from Fig. 3d.

Let $P = S_6$. By symmetry the 3-edges of S_6 have the same length. Let α be a face of P containing both 3-edges. Then any vertex of α is either 2-2-3-vertex or 2-3-3-vertex. By Lemma 9 any vertex of α is a fundamental vertex of $P = S_6$. Then the face α cannot be a triangle from Fig. 3d.

Let $P = S_8$. Let α be a face containing 3-edge and 5-edge. By Lemma 9 any vertex of α is a fundamental vertex of S_8. But Fig. 3d contains no triangle whose edges are 2-edge, 3-edge and 5-edge and all vertices are fundamental.

Let $P = S_{10}$. Then the number m in Table 3 equals to 3 or 5. Two vertices of P belongs to 2-edges only. Hence, the neighborhoods of these vertices are decomposed as shown in Fig. 4. The rest vertices are fundamental 2-2-m-vertices. Let α be one of the faces containing both non-fundamental vertices. Then α cannot be a triangle of Fig. 3d.

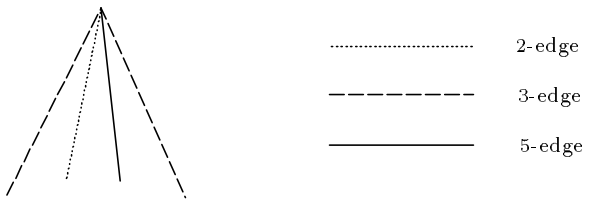


Figure 4.

□

2.3 Decompositions of the third type

The decompositions of the first and the second types are classified. Therefore, it is possible to classify the tetrahedra of the third type using a **wide inductive procedure**.

Let P be a tetrahedron admitting a decomposition of the third type, F be a fundamental tetrahedron and T_1,...,T_k be all distinct tetrahedra of the second type in P. The decompositions of the first type are constructed by the inductive procedure from the fundamental tetrahedron F. The same inductive procedure can be used for construction of the decompositions of the third type from F, T_1,...,T_k. The only change we need is to add the tetrahedra $P_1 = T_1$,...,$P_k = T_k$ at the Step 0.

Since there are finitely many possibilities for the Coxeter tetrahedron F and for the tetrahedra of the second type T_1,...,T_k, we can find all the decompositions of the third type using this procedure finitely many times.

Lemma 11. *Let P be a tetrahedron admitting a Coxeter decomposition D of the third type, T be a tetrahedron in P admitting a Coxeter decomposition D_1 with all fundamental vertices.*

Then P admits a Coxeter decomposition D_2 of the first type with the fundamental tetrahedron T and $D = D_1(D_2)$.

Proof. Let F be a tetrahedron fundamental for the decompositions of T. By Lemmas 7 and 8 the pair (F,T) is one of six pairs shown in Table 4. Denote these tetrahedra F_i, T_i, $i = 1$,...,6 correspondingly to the number of the pair in Table 4. Consider these pairs one after another. It is sufficient to prove that any decomposition of the third type is a product of the decompositions of the first and the second types. To classify the decompositions of the third type we use the inductive procedure.

1) Suppose $(F,T)=(F_1,T_1)$ or $(F,T)=(F_3,T_3)$. Then all dihedral angles of T are right and fundamental. Therefore, T cannot be glued together with another tetrahedron (otherwise the result of the glueing cannot be non-degenerate tetrahedron).

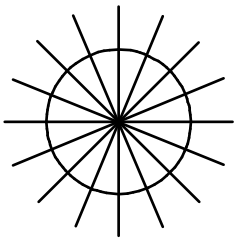
2) The rest four pairs (F,T) have a common fundamental tetrahedron F. Use the inductive procedure to prove that if P contains T_i $(i = 2, 4, 5, 6)$, then P is composed of several copies of T_i. At Step 0 we have $P_0 = F$, $P_1 = T_2$, $P_2 = T_4$, $P_3 = T_5$, $P_4 = T_6$. It is sufficient to check that T_i can be glued together with no tetrahedron X_i (where X_i are tetrahedra listed in Table 3.4). A simple test using computer program shows that non of X_i has a face decomposed like a face of T_i.

   No face of the tetrahedra T_i $(i = 2, 4, 5, 6)$ admits a reflection symmetry. Therefore, any two adjacent copies of T_i are symmetric with respect to the common face.

□

3 Tables

$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{k}\right)$



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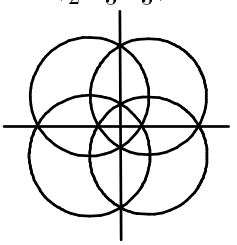

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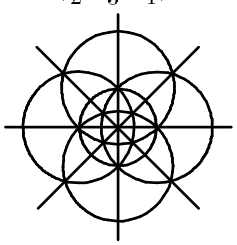

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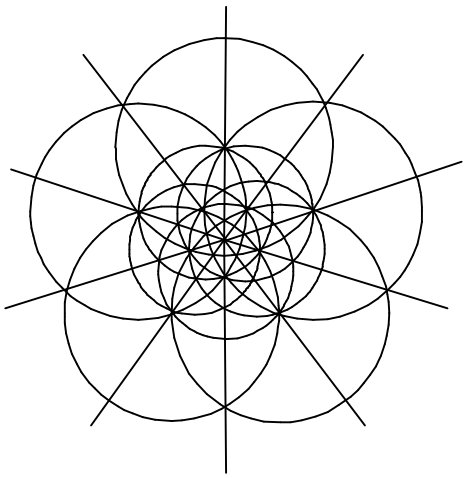

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Table 1. Decompositions of spherical triangles with non-fundamental vertex

The decompositions of the sphere are given in stereographic projection. The triangles are represented by the values of their angles. The multiple π is omitted. In the most part of cases the group G generated by reflections of p coincide with the group generated by reflections of fundamental triangle f. In the rest cases the triangles fundamental for G are presented above the tables.

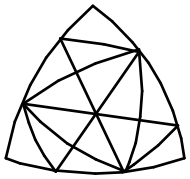


Figure 5. Decomposition of a spherical triangle without non-fundamental vertices
The decomposition contains 15 fundamental triangles.

Notation

The tetrahedra with the dihedral angles $\frac{k_i \pi}{q_i}$ i = 1,...,6 are represented by the following diagrams: take a Coxeter diagram of the tetrahedron whose dihedral angles equals to $\frac{\pi}{q_i}$ $i = 1,...,6$ and subdivide the edges corresponding to the dihedral angles of the size $\frac{k_i \pi}{q_i}$ into k_i parts.

The list of the decompositions of the first type was obtained by an inductive procedure. The non-trivial decompositions of the first type are listed here for each of the Coxeter tetrahedra. The Coxeter diagram of the fundamental tetrahedron is presented before the list of the decompositions with this fundamental tetrahedron.
The numbers (k,l ; m,n,p,q) under the decompositions describe the decompositions.
k is the number of fundamental tetrahedra in the decomposition,
l is a number of glueing necessary for construction of the decomposition (glueing the tetrahedra with l_1 and l_2 we get a tetrahedron with $l = 1 + \max\{l_1, l_2\}$),
m and n are numbers of the decomposed tetrahedra which should be glued together to obtain the decomposition,
p and q are numbers of the glued faces of the tetrahedra m and n respectively (the nodes of the diagrams are numbered from left to right by the numbers 0,1,2,3).

Throughout Fig. 3-Fig. 11 we denote the 2-edges, 3-edges and 5-edges as shown below:

2-edge 3-edge 5-edge

The inner 5-edges, i.e. 5-edges does not lying on a face of tetrahedra, are denoted by a following way: the ends of these edges are marked by a small circles, triangles or squares, the ends of the same edge are marked by a same sign.
Let the length of the 5-edge of the fundamental tetrahedron be equal to unit. Then each of the tetrahedra shown in Fig. 9 and Fig. 10 contains unique inner 5-edges of the length two. The tetrahedron shown in Fig. 11 contains 6 inner edges of the length four intersecting in the center of the tetrahedron.

The tessellation of the sphere shown in Fig. 3d can be constructed by the following way. Take an icosahedron and truncate all its vertices by the planes going through the middle points of the edges. Project this truncated icosahedron on the sphere, then use the stereographic projection to obtain Fig. 3a. Subdivide all the triangles and pentagons of Fig. 3a as shown in Fig. 3b and 3c respectively.

Table 2. Spherical Coxeter tetrahedra

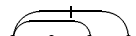
n	diagram	order	n	diagram	order	n	diagram	order
S_1		$2^3 \cdot 3 \cdot 5$	S_4		$2^6 \cdot 3^2 \cdot 5^2$	S_8		$2^4 \cdot 3 \cdot 5$
S_2		$2^7 \cdot 3$	S_5		$2^7 \cdot 3^2$	S_9		$2^2 \cdot m \cdot n$
S_3		$2^6 \cdot 3$	S_6		$2^4 \cdot 3$	S_{10}		$2^3 \cdot m$
			S_7		$2^5 \cdot 3$	S_{11}		2^4

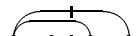
Table 3. Spherical tetrahedra of the first type

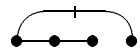
1

0

1
(2,3 ; 0,0,0,0)

2 
(3,4 ; 1,0,0,1)

3 
(4,4 ; 1,1,2,2)

4 
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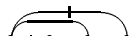
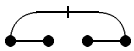
5 
(6,5 ; 2,2,1,1)

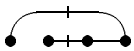
6 
(6,5 ; 2,2,3,3)

7
(7,5 ; 3,2,0,3)

8
(8,6 ; 4,4,2,2)

9
(9,6 ; 6,2,3,2)

10
(12,6 ; 6,6,1,1)

11 
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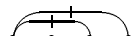
12 
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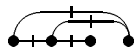
13 
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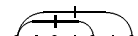
14 
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2

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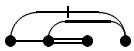
1 
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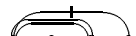
3 
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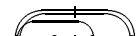
4 
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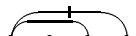
5 
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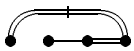
6 
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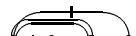
8 
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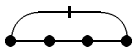
10 
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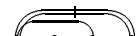
14 
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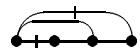
19 
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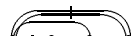
22 
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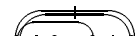
24 
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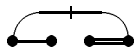
27 
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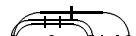
28 
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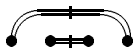
30 
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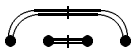
31 
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34
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35
(24,8 ; 20,11,0,1)

36
(26,8 ; 20,15,0,2)

37
(28,8 ; 20,18,0,1)

38
(14,8 ; 22,2,1,2)

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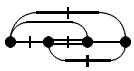
41 
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(31,8 ; 27,25,3,3)

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45
(34,8 ; 27,27,2,2)

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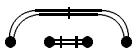
47 
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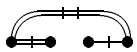
50 
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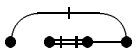
52 
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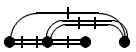
54 
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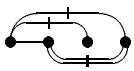
56 
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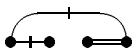
57 
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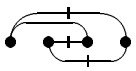
58 
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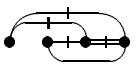
59 
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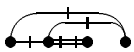
61 
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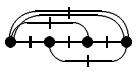
62 
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(46,9 ; 45,22,1,2)

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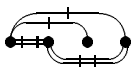
(54,9 ; 46,29,0,2)

65 
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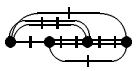
66 
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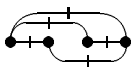
67 
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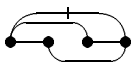
3

0

1 
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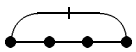
2 
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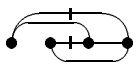
5 
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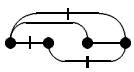
6 
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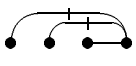
7 
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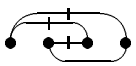
8 
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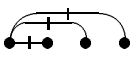
9 
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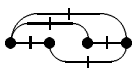
10 
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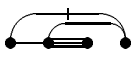
1 
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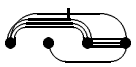
3 
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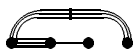
5 
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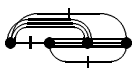
7 
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9
(8,6 ; 5,5,2,2)

10
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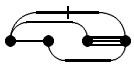
11 
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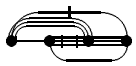
13 
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(11,7 ; 10,3,0,2)

17
(14,7 ; 10,8,0,0)

18
(17,7 ; 12,9,0,0)

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20
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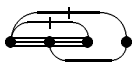
22 
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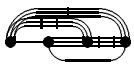
26 
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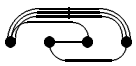
27 
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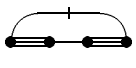
32 
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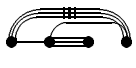
33 
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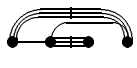
34 
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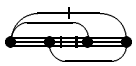
37 
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41
(76,10 ; 29,28,3,3)

42
(48,10 ; 30,13,0,2)

43
(55,10 ; 30,18,0,0)

44
(70,10 ; 30,21,3,1)

45
(66,10 ; 30,26,0,2)

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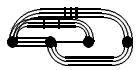
49 
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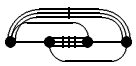
52 
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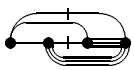
56 
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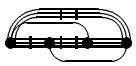
57 
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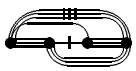
60 
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(103,11 ; 43,42,3,3)

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(126,11 ; 44,37,0,1)

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(80,11 ; 45,17,3,3)

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(136,11 ; 45,44,0,2)

68
(81,11 ; 48,12,0,3)

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(127,11 ; 48,43,2,2)

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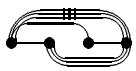
73 
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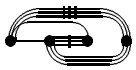
75 
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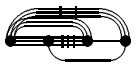
81 
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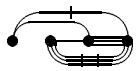
84 
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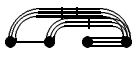
85 
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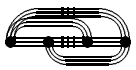
86 
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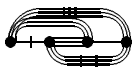
90 
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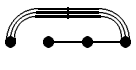
91 
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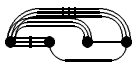
92 
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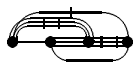
94 
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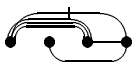
97 
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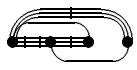
99 
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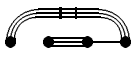
101 
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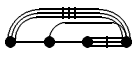
102 
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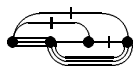
(272,12 ; 67,67,2,2)

104 
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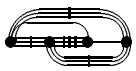
110 
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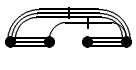
112 
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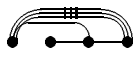
114 
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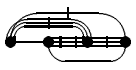
115 
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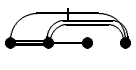
1 
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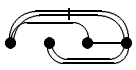
2 
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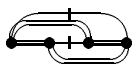
3 
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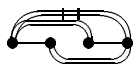
4 
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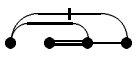
5 
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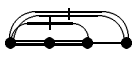
6
(6,5 ; 2,2,1,1)

7 
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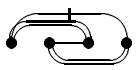
8 
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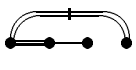
9
(12,6 ; 5,5,1,1)

10 
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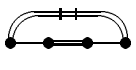
11 
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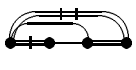
12 
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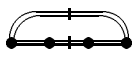
13 
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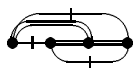
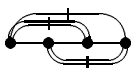
14 
(18,7 ; 7,7,3,3)

15
(19,7 ; 8,7,0,3)

16 
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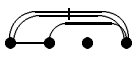
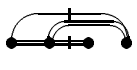
17 
(24,8 ; 10,10,0,0)

18 
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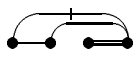
19 
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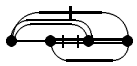
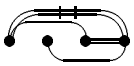
20 
(30,8 ; 12,12,1,1)

21
(33,8 ; 13,12,0,3)

22
(36,8 ; 13,13,0,0)

23
(21,8 ; 15,1,0,1)

24 
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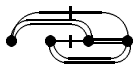
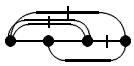
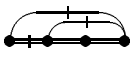
25 
(29,8 ; 15,8,3,1)

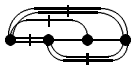
26 
(36,8 ; 16,9,0,0)

27
(48,9 ; 17,17,0,0)

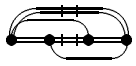
28
(48,9 ; 18,17,0,0)

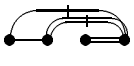
29 
(39,9 ; 20,7,1,2)

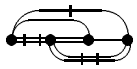
30
(60,9 ; 20,20,2,2)

31 
(42,9 ; 21,7,3,1)

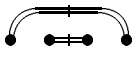
32
(45,9 ; 21,10,3,0)

33 
(52,9 ; 21,15,1,2)

34 
(51,9 ; 21,19,0,3)

35 
(66,9 ; 21,21,1,1)

36
(72,9 ; 22,22,1,1)

37 
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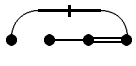
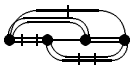
38 
(23,9 ; 24,0,0,1)

39 
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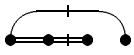
(72,10 ; 27,17,0,1)

41 
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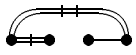
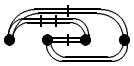
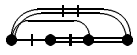
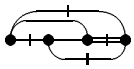
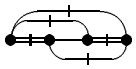
42
(78,10 ; 28,20,0,2)

43 
(84,10 ; 28,22,0,2)

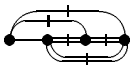
44 
(54,10 ; 29,12,3,2)

45 
(63,10 ; 29,16,3,1)

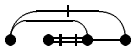
46 
(84,10 ; 30,16,0,2)

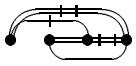
47 
(120,10 ; 30,30,2,2)

48
(81,10 ; 31,29,1,1)

49 
(102,10 ; 31,30,2,1)

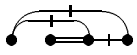
50
(74,10 ; 32,25,3,2)

51 
(90,10 ; 32,32,1,1)

52 
(91,10 ; 33,29,0,1)

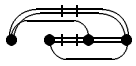
53
(73,10 ; 34,24,3,3)

54
(93,10 ; 34,31,3,3)

55 
(102,10 ; 34,34,1,1)

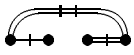
56
(108,10 ; 36,22,0,2)

57
(46,10 ; 38,38,1,1)

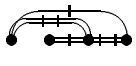
58 
(77,10 ; 39,21,0,2)

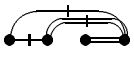
59
(120,11 ; 40,27,3,1)

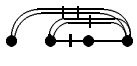
60
(144,11 ; 40,40,1,1)

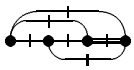
61 
(132,11 ; 41,40,0,1)

62
(135,11 ; 43,34,1,2)

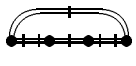
63 
(168,11 ; 43,43,2,2)

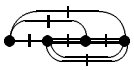
64 
(69,11 ; 45,5,0,2)

65 
(123,11 ; 45,41,3,1)

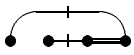
66 
(204,11 ; 47,46,1,2)

67
(201,11 ; 48,47,2,1)

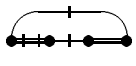
68 
(162,11 ; 48,48,2,2)

69 
(138,11 ; 49,26,0,1)

70
(95,11 ; 50,23,0,3)

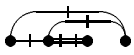
71 
(97,11 ; 50,38,3,2)

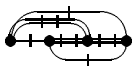
72
(171,11 ; 51,48,2,2)

73 
(110,11 ; 52,15,3,1)

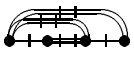
74
(154,11 ; 52,45,2,2)

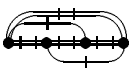
75
(146,11 ; 53,53,1,1)

76 
(186,11 ; 54,54,1,1)

77 
(270,12 ; 62,62,1,1)

78
(169,12 ; 65,57,3,2)

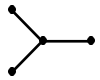
79 
(245,12 ; 67,39,2,2)

80 
(292,12 ; 67,52,1,1)

81
(190,12 ; 70,70,2,2)

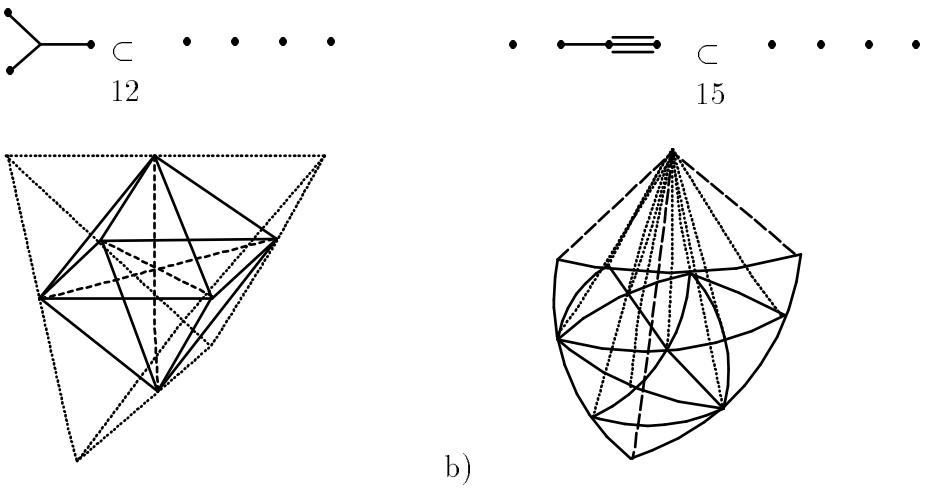
Table 4. Spherical tetrahedra of the second type

Decompositions with all fundamental vertices

	F	P	N	construction
1			12	see Fig. 6a
2			120	see Fig. 9

Decompositions with some non-fundamental vertices

	F	P	N	construction
3			15	see Fig. 6b
4			75	see Fig. 8
5			144	see Fig. 10
6			900	see Fig. 11



a) b)

Figure 6.

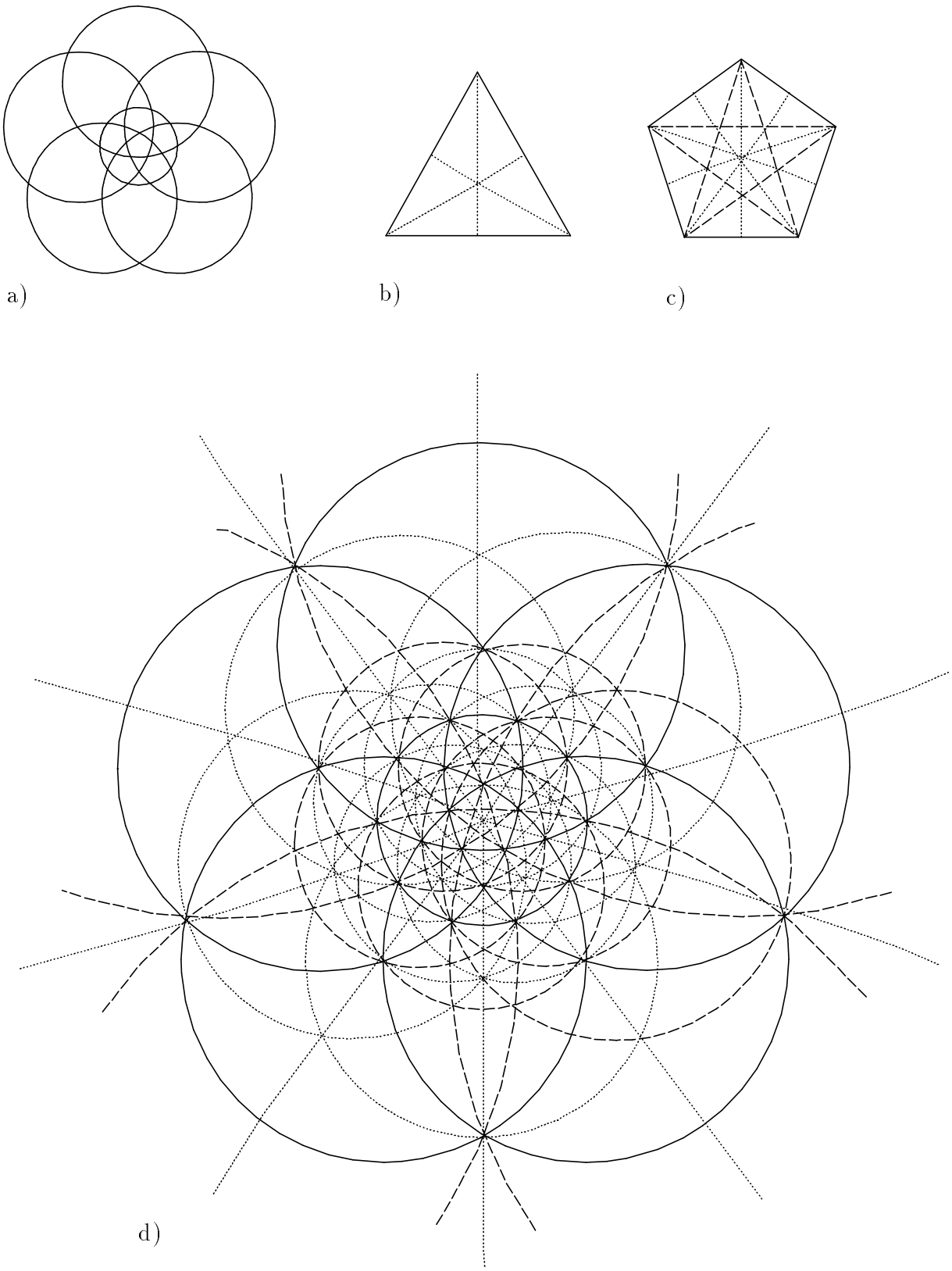


Figure 7. The tessellation of any spherical mirror in the decomposition of $\mathbf{S}^3$ with fundamental tetrahedron S_4

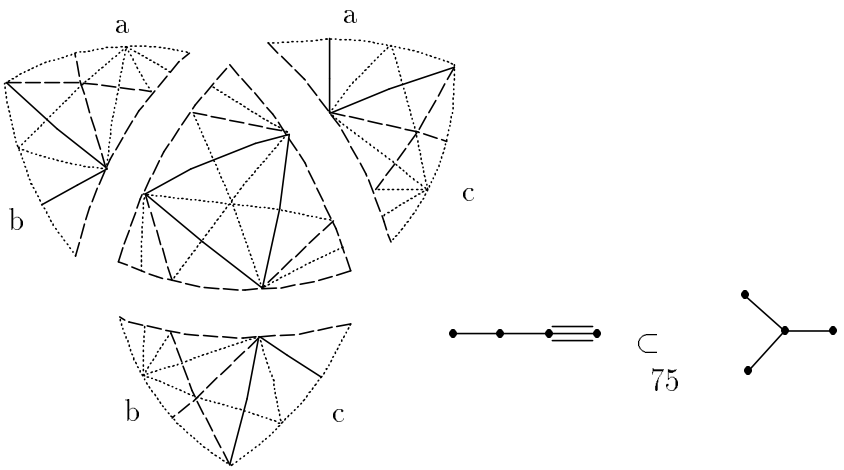


Figure 8.

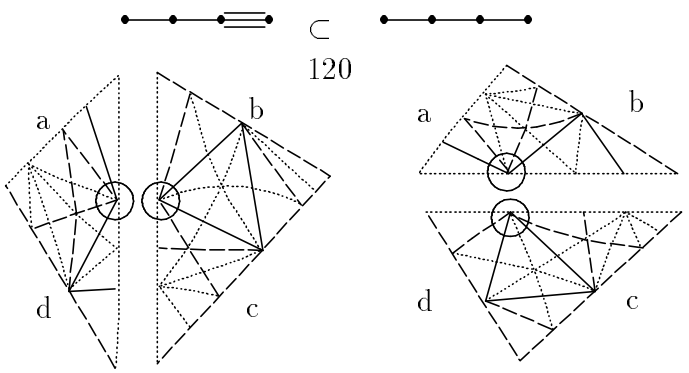


Figure 9.

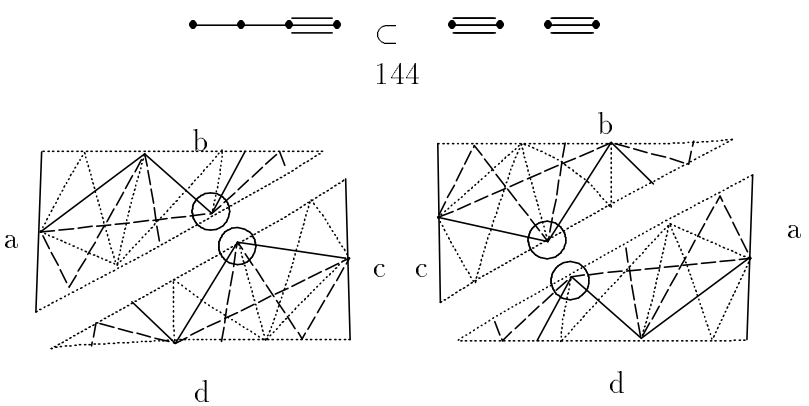


Figure 10.

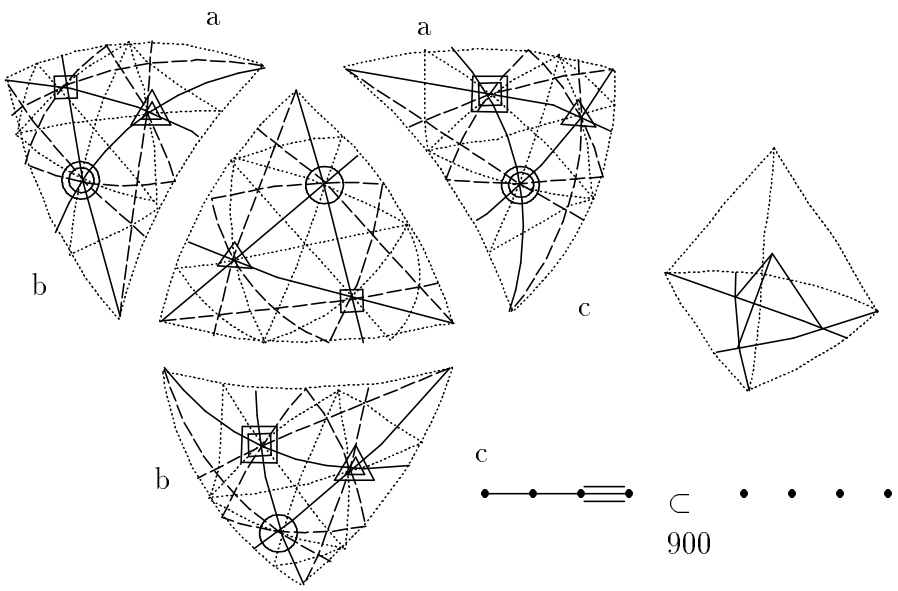
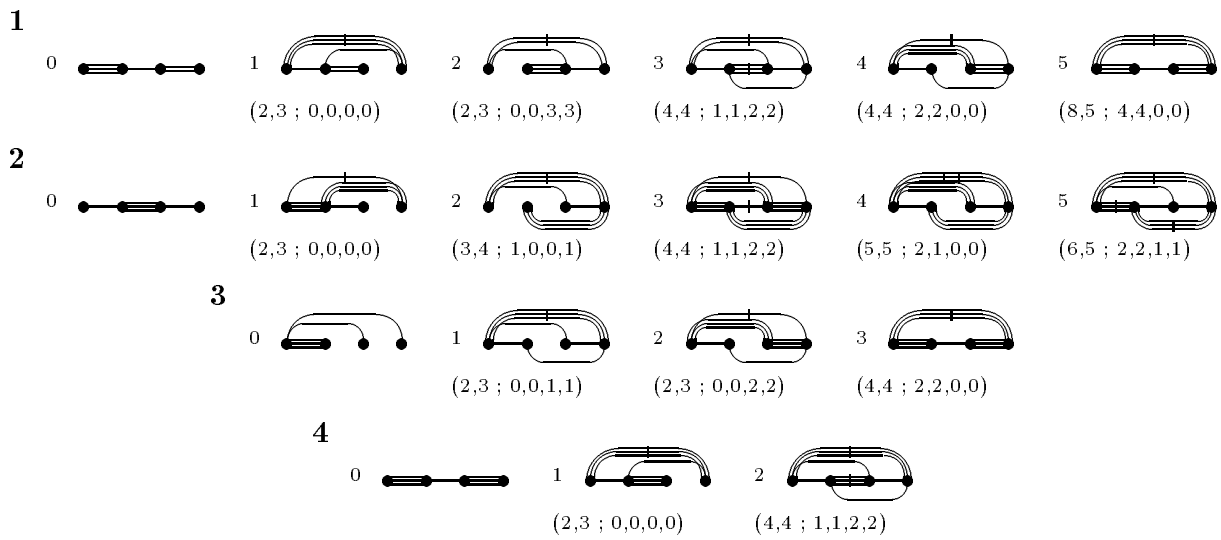


Figure 11.

Appendix

The table of hyperbolic decompositions of the first type contained in [1] was uncomplete. The revised version of the table is presented below.

Table 5. Hyperbolic tetrahedra of the first type



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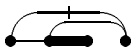
1 
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2
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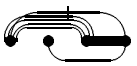
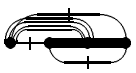
3 
(3,4 ; 1,0,0,1)

4
(4,4 ; 1,1,2,2)

5
(4,5 ; 3,0,0,2)

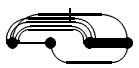
6
(5,5 ; 3,2,0,0)

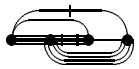
7 
(6,5 ; 3,3,1,1)

8
(6,5 ; 3,3,3,3)

9
(8,6 ; 5,5,2,2)

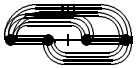
10
(8,6 ; 5,5,3,3)

11 
(10,6 ; 6,6,0,0)

12 
(10,6 ; 6,6,1,1)

13
(10,6 ; 6,6,3,3)

14
(12,6 ; 8,8,1,1)

15 
(16,7 ; 9,9,2,2)

16
(12,7 ; 10,5,0,0)

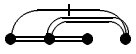
17
(20,7 ; 12,11,0,0)

18
(24,7 ; 14,14,1,1)

19
(24,8 ; 16,16,1,1)

6

0

1 
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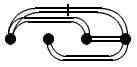
3 
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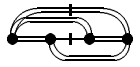
4 
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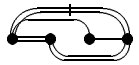
5 
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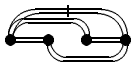
6
(6,5 ; 3,3,0,0)

7
(6,5 ; 3,3,1,1)

8 
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9
(12,6 ; 6,6,1,1)

7

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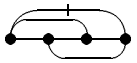
1 
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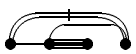
1 
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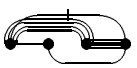
3 
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(4,4 ; 1,1,2,2)

5
(8,5 ; 3,3,0,0)

9

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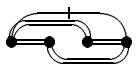
1 
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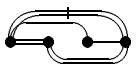
2 
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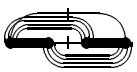
3 
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4
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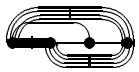
5 
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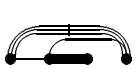
1 
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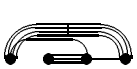
2 
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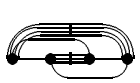
3 
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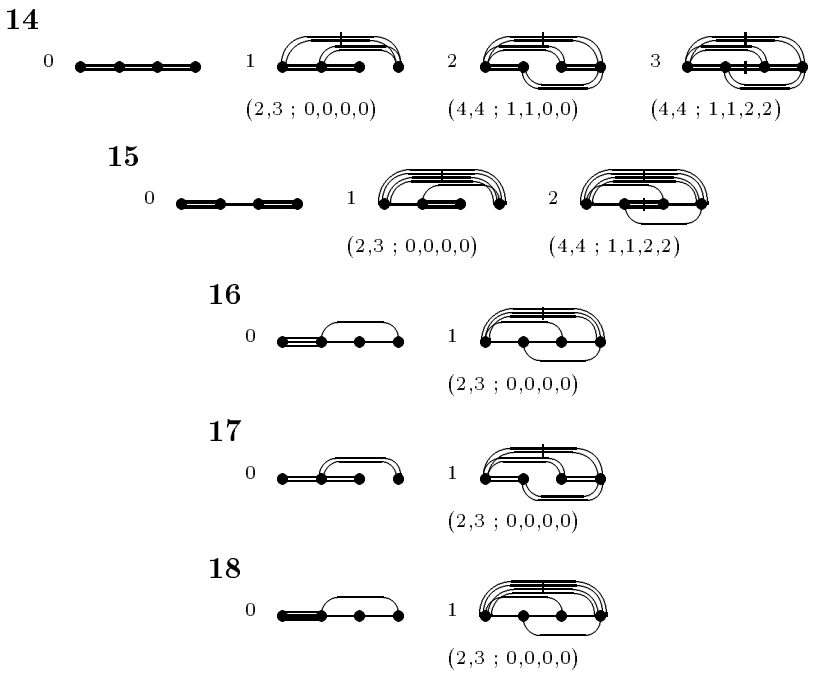

References

[1] Felikson A.A. Coxeter decompositions of hyperbolic tetrahedra, preprint, Bielefeld, 1998.

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