Irreducible tensor products of representations of finite quasi-simple groups of Lie type

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Abstract. Let G be a finite quasi-simple group of Lie type defined over a field of characteristic p. We determine whether G can possess irreducible Brauer characters α, β (of degree > 1) in characteristic other than p such that $\alpha \otimes \beta$ is irreducible.

1. Introduction

Let G be a finite group, r a fixed prime (or $r = 0$). Consider the following statement

$\mathcal{P}(r)$: $\alpha\otimes\beta$ is not irreducible for any nonlinear Brauer characters $\alpha, \beta \in \mathrm{IBr}_r(G)$

(by $\mathrm{IBr}_r(G)$ with $r = 0$ we mean $\mathrm{Irr}(G)$).

One motivation for studying the property $\mathcal{P}(r)$ comes from the desire to classify the overgroups of the irreducible subgroups of the finite classical groups $G(q)$. According to Aschbacher's Theorem [A], if M is a maximal subgroup of $G(q)$, then either M belongs to one of 8 collections $\mathcal{C}_i$, $1 \leq i \leq 8$, of "natural" subgroups of $G(q)$, or $M \in \mathcal{S}$, a collection of quasi-simple groups which act irreducibly on the natural module V of $G(q)$. For instance, $\mathcal{C}_4$ consists of stabilizers of tensor decompositions of V. Conversely, if $M \in \cup_{i=1}^{8}\mathcal{C}_i$, then the maximality of M is determined in [KL]. Now suppose $M \in \mathcal{S}$. Could it happen that $M < N < G$ for some $N \in \mathcal{C}_4$, in other words, could the (irreducible) representation of M on V be tensor-decomposable?

This problem of tensor-decomposable irreducible modules has been investigated by Isaacs [Is] for solvable groups G. In particular, he conjectures that if the solvable group G has two faithful complex characters α and β such that $\alpha\beta$ is irreducible,

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then G is abelian. He also proved this conjecture in the case where every minimal normal subgroup of G is cyclic.

The goal of this paper is to find all finite quasi-simple groups with property $\mathcal{P}(r)$. It has been shown by Bessenrodt and Kleshchev [BK1] that the symmetric group $\mathbb{S}_n$ ($n \geq 5$) satisfies $\mathcal{P}(0)$. They also classified all complex irreducible characters α, β of degree > 1 of the alternating group $\mathbb{A}_n$ ($n \geq 5$) such that $\alpha \otimes \beta$ is irreducible: it happens if and only if $n = a^2$ for some $a \geq 3$, one of the characters α, β is the nontrivial irreducible component of the natural permutation character, and the other one corresponds to the partition (a^a). More recently, Bessenrodt and Kleshchev have shown in [BK2] that $\mathbb{S}_n$ satisfies $\mathcal{P}(r)$ for any odd prime r, and that $\mathbb{S}_n$ satisfies $\mathcal{P}(2)$ if n is odd. The cases of complex characters of $2\mathbb{S}_n$ and $2\mathbb{A}_n$ are being studied. We note that there are examples of irreducible tensor products of complex spin representations of $\mathbb{A}_9$, $\mathbb{A}_{10}$. Also, some examples of irreducible tensor products of representations of $\mathbb{S}_n$ in characteristic 2 have been found by Gow and Kleshchev [GK].

We investigated the 26 sporadic finite simple groups (and their covers) using the character tables that we found in GAP. We found that M_{11}, J_1 and $2HS$ satisfy $\mathcal{P}(r)$ for all primes r, and that HN, Ly, He, Fi_{23} and J_4 satisfy $\mathcal{P}(0)$ whereas all other sporadic groups do not even satisfy $\mathcal{P}(0)$. The list of tensor-decomposable irreducible representations that we found can be reproduced in GAP with minimal effort and is hence not included in this paper. Instead, we list a couple of examples of irreducible tensor products $\Phi \otimes \Psi$ for $G = BM$, $2 \cdot BM$:

$$\begin{aligned} &\Phi(1) = 4371, && \Psi(1) = 53936390144,\\ &\Phi(1) = 96256, && \Psi(1) = 90807234375, \end{aligned}$$

and for $G = M$:

$$\begin{aligned} &\Phi(1) = 196883, && \Psi(1) = 8980616927734375,\\ &\Phi(1) = 19360062527, && \Psi(1) = 8980616927734375,\\ &\Phi(1) = 21296876, && \Psi(1) = 3503434660075044981. \end{aligned}$$

We focus our attention on the finite quasi-simple groups G of Lie type defined over a field $\mathbb{F}_q$ of characteristic p, $q = p^f$, and $p \neq r$.

Observe that G cannot satisfy $\mathcal{P}(p)$. Indeed, let us view G as the fixed point subgroup $\mathcal{G}^F$, where $\mathcal{G}$ is a simple algebraic group defined over $\overline{\mathbb{F}}_p$ and F is the corresponding Frobenius endomorphism. Then the Steinberg tensor product theorem asserts that any irreducible $\overline{\mathbb{F}}_pG$-module can be written as the tensor product $V_1 \otimes V_2^{(p)} \otimes \ldots \otimes V_f^{(p^{f-1})}$ for some restricted irreducible $\overline{\mathbb{F}}_pG$-modules $V_1, \ldots, V_f$; and conversely, any such tensor product is irreducible. This remarkable theorem gives us many examples of modules whose tensor product is irreducible. Even more, in combination with a result of Seitz, this theorem allows us to describe all pairs (U, V) of $\overline{\mathbb{F}}_pG$-modules such that $U \otimes V$ is irreducible. For, let

$$V = V_1 \otimes V_2^{(p)} \otimes \ldots \otimes V_f^{(p^{f-1})}, \quad U = U_1 \otimes U_2^{(p)} \otimes \ldots \otimes U_f^{(p^{f-1})},$$

where each U_i, V_i is restricted and irreducible. Then

$$U \otimes V \simeq (U_1 \otimes V_1) \otimes (U_2 \otimes V_2)^{(p)} \otimes \ldots \otimes (U_f \otimes V_f)^{(p^{f-1})}.$$

Suppose $U \otimes V$ is irreducible. Then for each i, the G-module $W_i := U_i \otimes V_i$ is irreducible. In particular, the $\mathcal{G}$-module W_i is irreducible. Now [S1, (1.6)] states that W_i is restricted, and it also gives us the list of all pairs (U_i, V_i) of restricted modules such that the $\mathcal{G}$-module W_i is irreducible. Conversely, suppose that we are given irreducible restricted modules U_i, V_i, $1 \leq i \leq f$, such that each W_i is irreducible. Since W_i is restricted in this case, the G-module W_i is irreducible. Hence the G-module $U \otimes V$ is irreducible by the Steinberg tensor product theorem.

The main result of the paper is the following

Theorem 1.1. *Let G be a finite quasi-simple group of Lie type defined over a field $\mathbb{F}_q$. Then either G has property $\mathcal{P}(r)$ for any $r \not| q$, or one of the following holds.*
(i) $q \leq 3$.
(ii) $G = Sp_{2n}(q)$, *where q is odd.*
(iii) $G = Sp_{2n}(q)$, $2|q$.
(iv) $G = F_4(q)$, $2|q$.
(v) $G = {}^2F_4(q)$, $2|q$.
(vi) $G/Z(G) = L_3(4)$, $PSp_4(9)$, $\Omega_7(9)$, $P\Omega_8^+(9)$.

Throughout the paper, $r \not| q$ means that either $r = 0$, or r does not divide q. We prove more detailed results in the body of the paper. In particular we give examples where property $\mathcal{P}(r)$ does not hold. For example, it is shown in Proposition 5.5 that when G is $Sp_{2n}(3)$ or $Sp_{2n}(5)$, then $\mathcal{P}(r)$ does not hold. Using GAP one sees that $SL_3(4)$, $Sp_6(2)$, $Sp_8(2)$, and $Sp_{10}(2)$ do not have property $\mathcal{P}(0)$. Thus the cases (i), (ii) and (iii) in Theorem 1.1 actually occur. On the other hand, we show that when q is odd and ≥ 7, then $Sp_{2n}(q)$ has property $\mathcal{P}(0)$. Moreover we conjecture that $Sp_{2n}(q)$ has property $\mathcal{P}(r)$ when q is odd and ≥ 7. We also conjecture, but are not able to show, that none of the cases (iv) and (v) can happen. In particular, the cases (iv) and (v) cannot happen if $r = 0$, see Proposition 8.3. Furthermore we conjecture that $Sp_{2n}(q)$, with $2|q$ and $q \neq 2$ does have property $\mathcal{P}(r)$.

For reader's convenience, we formulate the main result in the complex case.

Theorem 1.2. *Let G be a finite quasi-simple group of Lie type defined over a field $\mathbb{F}_q$. Then either G has property $\mathcal{P}(0)$, or one of the following holds.*
(i) $q \leq 3$.
(ii) $G = Sp_{2n}(5)$.
(iii) $G = Sp_{2n}(q)$, $2|q$.
(iv) $G/Z(G) = L_3(4)$, $\Omega_7(9)$, $P\Omega_8^+(9)$.

The dual problem of determining whether the product of two conjugacy classes of a finite group could form a single conjugacy class has been investigated by Arad

[Ar]. Also, property $\mathcal{P}(0)$ is closely related to factorizations of quasi-primitive characters intensively investigated by Ferguson and Turull [FT].

An interesting by-product of the paper is a new lower bound $q^2(q^2+1)$ for the minimum degree $\ell(G)$ of nontrivial projective representations of $G = G_2(q)$ in cross characteristic, provided that $q = 3^f > 3$. (The Landázuri-Seitz-Zalesskii bound for this group is $q(q^2-1)$.)

2. Preliminaries

We are going to consider representations of a finite group G over an algebraically closed field $\mathbb{F}$ of characteristic r.

Extending the proof of [S2, (1.4)] and of [Hup, (V.17.5)], we can prove the following lemma

Lemma 2.1. *Let R be a finite group with a normal subgroup Q.*

(i) *Suppose that R/Q is a group with trivial Schur multiplier. Let φ be an R-invariant irreducible $\mathbb{F}$-representation of Q. Then φ extends to an irreducible representation ψ of R.*

(ii) *Let χ be any $\mathbb{F}$-representation of R. Suppose in addition that $\chi|_Q$ is a direct sum of copies of a single irreducible representation φ of R, which is extendible to R. Let ψ be a fixed extension of φ to R. Then $\chi = \psi \otimes \omega$ for some $\mathbb{F}$-representation ω of R with $Q \leq \mathrm{Ker}(\omega)$. If R/Q is perfect, then any extension of φ to R coincides with ψ.*

Proof. 1) Suppose φ is R-invariant. Let ρ be a simple R-submodule of φ^R. Then $0 \neq \mathrm{Hom}_R(\rho, \varphi^R) \simeq \mathrm{Hom}_Q(\rho|_Q, \varphi)$, i.e. φ is a simple quotient of the Q-module $\rho|_Q$. By Clifford's Theorem, $\rho|_Q$ is a direct sum of, say, s copies of φ. By [Hup, (V.17.5)], $\rho = \psi \otimes \tau$, where ψ is a projective representation of R, and τ is a projective representation of degree s of R/Q. Since R/Q has trivial Schur multiplier, τ is actually a representation. The factor sets in ψ and τ are inverse sets, so ψ is also a representation. We have shown that φ extends to a representation ψ of R.

2) Now consider any χ as in (ii). We can again use [Hup, (V.17.5)]. It implies that $\chi = \mu \otimes \omega$, where μ is a projective representation of R, and ω is a projective representation of R/Q. Moreover, the proof shows that μ can be taken as any extension of φ. In particular, we can take $\mu = \psi$, and so μ is actually a representation. The factor sets in μ and ω are inverse sets, so ω is also a representation.

Finally, suppose that ψ' is another extension of φ to R, and R/Q is perfect. Due to (ii), $\psi' = \psi \otimes \gamma$, where γ is a representation of R/Q of degree 1. Since R/Q is perfect, $\gamma = 1$ and $\psi' = \psi$. $\square$

Usually, we will use Lemma 2.1 in the following set-up. Q is the set

$$\{[X, Y, z] | X, Y \in \mathbb{F}_q^m, z \in \mathbb{F}_q\}$$

(for a natural integer m and an odd prime power q) endowed with the following group operation

$$[X, Y, z] \circ [X', Y', z'] = [X + X', Y + Y', z + z' + \sum_{i=1}^{m} (x_i y_i' - x_i' y_i)],$$

if

$$X = (x_1, \dots, x_m), \quad Y = (y_1, \dots, y_m),$$
$$X' = (x_1', \dots, x_m'), \quad Y' = (y_1', \dots, y_m').$$

Clearly, $[Q, Q] = Z(Q) = \Phi(Q)$ coincides with the subgroup $\{t_c := [0, 0, c] \mid c \in \mathbb{F}_q\}$. Moreover, $[x, Q] = Z(Q)$ for any $x \in Q \setminus Z(Q)$, hence Q is of special type.

Lemma 2.2. *Let Q be the above group with $(m, q) \neq (1, 3)$, $(1, 9)$. Suppose that P' is a semi-direct product of Q by $A \times B$, where*

(a) *$A \times B$ acts trivially on $Z(Q)$;*

(b) *A acts faithfully on Q and A is perfect;*

(c) $\mathrm{Mult}(B) = 1$.

Let φ be a non-linear irreducible $\mathbb{F}$-representation of Q. Then

(i) *φ extends to an irreducible representation of P'. Fix any extension ψ of φ.*

(ii) *If χ is any $\mathbb{F}$-representation of P' and all irreducible constituents of $\chi|_Q$ are isomorphic to φ, then $\chi = \psi \otimes \omega$ for some $\mathbb{F}$-representation ω of P' with $Q \leq \mathrm{Ker}(\omega)$.*

Proof. 1) Let L be any maximal subgroup of $Z(Q)$. We show that Q/L is extraspecial. Indeed, $[Q/L, Q/L] = [Q, Q]L/L = Z(Q)/L$. If $xL \in Z(Q/L)$, then $[x, Q] \leq L$, whence $x \in Z(Q)$, i.e. $Z(Q/L) = Z(Q)/L$. Thus $[Q/L, Q/L] = Z(Q/L)$ has order p. If M is any maximal subgroup of Q/L, then clearly $M \geq Z(Q/L)$, whence $\Phi(Q/L) \geq Z(Q/L)$. But $Q/Z(Q) = Q/\Phi(Q)$ is elementary abelian, therefore $\Phi(Q/L) = Z(Q/L)$, i.e. Q/L is extraspecial, as desired.

Let S be the group of all automorphisms of Q that fix $Z(Q)$ pointwise. It is known that $S \simeq Sp_{2m}(q)$. The assumption on (m, q) guarantees that $\mathrm{Mult}(S) = 1$. We make the set $R := \{(x, f) \mid x \in Q, f \in S\}$ into a group by defining $(x, f) \cdot (y, g) = (x \cdot f(y), fg)$. Then R is a semi-direct product of Q by S, and $Z(R) = Z(Q)$. The proof of [S2, (1.4)] shows that φ is R-invariant.

Applying Lemma 2.1(i) to the group R, we see that φ extends to an $\mathbb{F}$-representation, say ρ of R.

2) The faithful action (by conjugation) of A on Q yields an embedding $\Phi : A \hookrightarrow S$. Now we can embed $T := Q : A$ in R by sending any element xa, $x \in Q$ and $a \in A$, to $(x, \Phi(a))$. Then clearly φ extends to an $\mathbb{F}$-representation τ of T, namely we can take $\tau = \rho|_T$. By Lemma 2.1(ii), the perfectness of A implies that τ is the unique extension of φ to T.

3) Next we observe that τ is P'-invariant. Indeed, if $y \in P'$, then τ^y is an extension of $\varphi^y = \varphi$ (since P' acts trivially on $Z(Q)$) from $Q^y = Q$ to $T^y = T$. But τ is the unique extension of τ, so $\tau^y = \tau$. Now Lemma 2.1 (i) applied to the group P' with the normal subgroup T implies that τ extends to an $\mathbb{F}$-representation, say ψ of P'. We have just established assertion (i).

4) Let χ be as in (ii). Since any $\mathbb{F}Q$-representation is semisimple, $\chi|_Q$ is a direct sum of some copies of φ. Due to (i), φ extends to ψ. By Lemma 2.1 (ii), $\chi = \psi \otimes \omega$ for some $\mathbb{F}P'$-representation ω with $Q \leq \mathrm{Ker}(\omega)$. $\square$

In most cases, our finite Lie-type group G has a subgroup Q of *special type*, i.e. a p-subgroup Q ($p \neq r$ a prime) such that $[Q,Q] = Z(Q) = \Phi(Q)$ has exponent p, and $[x,Q] = Z(Q)$ for any $x \in Q \setminus Z(Q)$. Moreover, we can select Q such that $P := N_G(Q)$ is a "big" parabolic subgroup of G. Furthermore, P has a normal subgroup $P' := Q : S$, which is a split extension of Q by a perfect subgroup S, and the quotient P/P' is solvable. Also, usually we can check that $Z(P') = Z(Q)$ and $\mathrm{Mult}(S) = 1$.

Suppose that V and W are irreducible $\mathbb{F}$-representations of G. Let $\Omega(V)$, resp. $\Omega(W)$, be the set of nontrivial linear characters of $Z(Q)$ occuring in V, resp. in W. Then by Lemma 2.2,

$$V|_{P'} = V^0 \oplus \sum_{\lambda \in \Omega(V)} E^\lambda \otimes V^\lambda, \quad W|_{P'} = W^0 \oplus \sum_{\lambda \in \Omega(W)} E^\lambda \otimes W^\lambda. \tag{1}$$

Here, E^λ is the unique extension to P' of the irreducible representation of Q whose restriction to $Z(Q)$ contains λ, V^λ and W^λ are certain S-modules, viewed as P'-modules, and $V^0 = C_V(Z(Q))$, $W^0 = C_W(Z(Q))$.

A key role in our proof is played by the following

Lemma 2.3. *In the above notation, suppose there is $\lambda \in \Omega(V)$ such that $\lambda^{-1} \in \Omega(W)$. Let a subgroup C of P normalize Q and centralize $Z(Q)$. Then $(V \otimes W)|_C$ has a submodule T, on which Q acts trivially and which is isomorphic to $V^\lambda \otimes W^{\lambda^{-1}}$ as P'-module. In particular, if $V \otimes W$ is irreducible, then*

$$\dim(V) \cdot \dim(W) \leq (G : C) \cdot \dim(V^\lambda) \cdot \dim(W^{\lambda^{-1}}).$$

Proof. Since $E^\lambda \otimes V^\lambda$ is the λ-weight space of $Z(Q)$ on V, C fixes $E^\lambda \otimes V^\lambda$. Similarly, C fixes $E^{\lambda^{-1}} \otimes W^{\lambda^{-1}}$. Thus C fixes $U := (E^\lambda \otimes V^\lambda) \otimes (E^{\lambda^{-1}} \otimes W^{\lambda^{-1}})$. Since Q acts trivially on V^λ and on $W^{\lambda^{-1}}$, the Q-module U is the sum of $\dim(V^\lambda) \cdot \dim(W^{\lambda^{-1}})$ copies of $E^\lambda \otimes E^{\lambda^{-1}}$. Clearly, the Q-character afforded by $E^\lambda \otimes E^{\lambda^{-1}}$ is the sum of all linear Q-characters. Now if T is the Q-fixed point subspace of U, then $\dim(T) = \dim(V^\lambda) \cdot \dim(W^{\lambda^{-1}})$. Clearly T is a C-module. As P'-module, $U \simeq (E^\lambda \otimes E^{\lambda^{-1}}) \otimes (V^\lambda \otimes W^{\lambda^{-1}})$. Since $E^{\lambda^{-1}}$ is dual to E^λ, $E^\lambda \otimes E^{\lambda^{-1}}$ containes the trivial P'-submodule. This shows that $T \simeq V^\lambda \otimes W^{\lambda^{-1}}$ as P'-submodule. $\square$

Lemma 2.4. *Let G be a finite group with property $\mathcal{P}(r)$. Let H be a normal subgroup of G of prime index t, and let $\gamma, \delta \in \mathrm{IBr}_r(H)$ be characters of degree > 1 such that*

(i) *γ extends to G;*

(ii) *For a fixed $h \in G \setminus H$, if $\delta^h \neq \delta$, then $\gamma \otimes \delta^h \neq \gamma \otimes \delta$.*

Then the H-character $\gamma \otimes \delta$ is not irreducible.

Proof. Consider the inertia group I of δ. Then $H \leq I \leq G$. If $I = G$, then δ extends to G by [F, 3.2.14], and so $\gamma \otimes \delta$ is reducible by the condition $\mathcal{P}(r)$ for

G. Suppose $I < G$. Then $I = H$, and δ has exactly t distinct conjugates δ_i, $i = 1, \dots, t$, and $\delta^G|_H = \sum_{i=1}^t \delta_i$. Let γ' denote an extension of γ to G. Suppose $\gamma \otimes \delta$ is irreducible. Then $\gamma \otimes \delta_i$, $i = 1, \dots, t$, are all irreducible and distinct, hence the G-character $\gamma' \otimes \delta^G$ is irreducible, again contradicting the condition $\mathcal{P}(r)$ for G. □

We will frequently use the following trivial statements.

Lemma 2.5. *Let $G = H * C$ be a central product of a subgroup H and an abelian subgroup C. Then any irreducible $\mathbb{F}H$-module V extends to G.*

Proof. Let W be a simple submodule of V^G. Then $W = A \otimes B$, where A is an irreducible H-module and B an irreducible C-module. Since C is abelian, $\dim(B) = 1$. Next, $0 \neq \mathrm{Hom}(W, V^G) \simeq \mathrm{Hom}(W|_H, V)$, hence $W|_H = V$. □

Lemma 2.6. *Let V be a G-module. Let $H \lhd G$ and let U be a simple submodule, resp. a simple quotient, of $V|_H$. Suppose U extends to G and G/H is abelian. Then the G-module V has a submodule, resp. a quotient, $\tilde{U}$, which is an extension of U.*

Proof. 1) We claim that, if W is an arbitrary irreducible constituent of the G-module U^G, then $W|_H \simeq U$. Indeed, since U is G-invariant, $U^G|_H$ is the sum of $|G/H|$ copies of U. Hence, $W|_H$ is a direct sum of copies of U. By [Hup, (V.17.5)], $W = W_1 \otimes W_2$, where W_1 is a projective representation of G and W_2 a projective representation of G/H. We can choose W_1 to be an extension $\hat{U}$ of U to G. Since the factor sets in W_1 and W_2 are inverse sets, W_2 is also a linear representation of G/H. But G/H is abelian, hence $\dim W_2 = 1$ and W is an extension of U.

2) Consider the case where U is a submodule of $V|_H$. Then $0 \neq \mathrm{Hom}(U, V|_H) \simeq \mathrm{Hom}(U^G, V)$. Thus V has a submodule V_1 which is a nonzero quotient of U^G. By 1), V_1 has a submodule which is an extension of U.

3) Consider the case where U is a quotient of $V|_H$. Then $0 \neq \mathrm{Hom}(V|_H, U) \simeq \mathrm{Hom}(V, U^G)$. Thus V has a quotient V_1 which is a nonzero submodule of U^G. By 1), V_1 has a quotient which is an extension of U. □

Observe that one cannot drop the assumption G/H is abelian in Lemma 2.6. For example, one can have examples where V is the tensor product of an extension of U and an irreducible G/H-module of dimension > 1, and yet V is irreducible.

3. Linear Groups

Let $G = GL_n(q)$, $S = SL_n(q)$ with $q = p^f$. It is easy to check that all covers of $L_2(q)$ with $q > 3$ and $q \neq 9$ have property $\mathcal{P}(r)$ for all r. On the other hand, the universal cover of $L_2(9) \simeq Sp_4(2)'$ does not have property $\mathcal{P}(r)$ for any r. In what follows we assume that $n \geq 3$, $(n, q) \neq (3, 2)$, $(3, 4)$, $(4, 2)$, $(4, 3)$, $(4, 9)$. Let $\mathbb{F}_q^n$

be the natural module for G, with a basis $(e_1, \dots, e_n)$. We consider the parabolic subgroup $P = St_S(\langle e_2\rangle, \langle e_2, \dots, e_n\rangle_{\mathbb{F}_q})$ of S. Then $Q = O_p(P)$ is a group of order q^{2n-3} of special type. The center $Z(Q)$ has order q and contains $q-1$ transvections t_c, where $c \in \mathbb{F}_q^\bullet$ and t_c has matrix $\mathrm{diag}\left(\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, I_{n-2}\right)$ in the given basis of V. It is easy to see that Q has a complement

$$\left\{\mathrm{diag}(a, b, X) \mid a, b \in \mathbb{F}_q^\bullet, X \in GL_{n-2}(q), ab \cdot \det(X) = 1\right\},$$

and all the transvections t_c are conjugate in P. Let P' be the (unique) normal subgroup of type $Q : SL_{n-2}(q)$ in P. Then $Z(P') = Z(Q)$. Our assumption on (n, q) guarantees that S is the universal cover of $PSL_n(q)$, and that the group $SL_{n-2}(q)$ is perfect and acts faithfully on Q.

Let V be any irreducible $\mathbb{F}S$-module of dimension > 1. Then by Lemma 2.2,

$$V|_{P'} = V^0 \oplus \sum_{\lambda \in \Omega(V)} E^\lambda \otimes V^\lambda$$

in the notation of (1), where $\dim(E^\lambda) = q^{n-2}$ and $\Omega(V) = \mathrm{IBr}_r(Z(Q)) \setminus \{1_Q\}$. The subgroup $\left\{\mathrm{diag}(a^{-1}, 1, a, I_{n-3}) \mid a \in \mathbb{F}_q^\bullet\right\}$ permutes the summands $E^\lambda \otimes V^\lambda$, $\lambda \in \Omega(V)$, transitively, hence all V^λ are of the same dimension (and $\Omega(V) \neq \emptyset$). We begin with the following estimate

Lemma 3.1. *In the above notation,*

$$\dim(V^\lambda) \leq \frac{\dim(V)}{q^{n-1} - 1}.$$

If $\dim(V^\lambda) = 1$ *and* $n \geq 5$, *then* V *is one of Weil modules.*

Proof. Let $R = St_S(e_2)$ and let $A = O_p(R)$. Then A is an elementary abelian group of order q^{n-1}. Let $V(\mu)$ be the homogeneous component of $V|_A$ corresponding to $\mu \in \mathrm{IBr}_r(A)$. Fix a nontrivial $\mu \in \mathrm{IBr}_r(A)$. Then $\dim(V) = \dim(V(1)) + (q^{n-1} - 1)\dim(V(\mu))$. In particular, $\dim(V(\mu)) \leq \dim(V)/(q^{n-1} - 1)$. Let χ be the Brauer character of V. Then $\chi(t_1) = \dim(V(1)) - \dim(V(\mu))$. In the meantime,

$$\dim(V) = \dim(V^0) + q^{n-2}(q-1)\dim(V^\lambda), \quad \chi(t_1) = \dim(V^0) - q^{n-2}\dim(V^\lambda),$$

whence

$$\dim(V^\lambda) = \frac{\dim(V) - \chi(t_1)}{q^{n-1}} = \dim(V(\mu)) \leq \frac{\dim(V)}{q^{n-1} - 1}.$$

Next suppose that $\dim(V^\lambda) = 1$. Then $\dim V(\mu) = 1$ and $\dim(V) = \dim(V(1)) + (q^{n-1} - 1)$. Observe that $V(1) = C_V(A)$. Now we can look at the opposite parabolic subgroup R' and its radical A' of order q^{n-1}. Since $S = \langle A, A'\rangle$ (cf. [GM]) and V is irreducible and non-linear, $C_V(A) \cap C_V(A') = 0$. The above computation shows that $\dim(C_V(A)) = (\dim(V) + (q^{n-1} - 1)\chi(t_1))/q^{n-1}$. Similarly, if $t' \in A'$ is a non-identity element, then $\dim(C_V(A')) = (\dim(V) + (q^{n-1} - 1)\chi(t'))/q^{n-1}$. Observe that t_1 and t' are S-conjugate, as $n > 2$. Hence, $\dim(C_V(A)) = \dim(C_V(A'))$ and so $\dim(V) \geq 2\dim(C_V(A))$. We conclude that $\dim(V) \leq 2(q^{n-1}-1)$. It remains to apply the main result of [GT1]. □

Observe that Lemma 3.1 remains true for an $\mathbb{F}G$-module V.

Proposition 3.2. *Suppose $n \geq 2$, and $(n,q) \neq (3,4)$. Then the groups $SL_n(q)$ with $q \geq 4$, and $GL_n(q)$ with $q \geq 3$, have property $\mathcal{P}(r)$ for any $r \not| q$.*

Proof. The case $(4,3)$ can be checked using GAP whereas the case $(4,9)$ can be handled using the methods of [MM]. Thus we may assume that $n \geq 3$, and $(n,q) \neq (3,4)$, $(4,3)$, $(4,9)$. First we assume that $G = GL_n(q)$, $q \geq 3$, has irreducible $\mathbb{F}$-representations V, W of dimension > 1 such that $V \otimes W$ is irreducible. Consider the parabolic subgroup $\hat{P} = St_G(\langle e_2\rangle, \langle e_2, \dots, e_n\rangle_{\mathbb{F}_p})$ of G. Then $Q = O_p(\hat{P})$, and Q has a complement

$$D := \left\{\mathrm{diag}(a,b,X) \mid a,b \in \mathbb{F}_q^\bullet, X \in GL_{n-2}(q)\right\}$$

in $\hat{P}$. Consider the following subgroups

$$A := \left\{\mathrm{diag}(a,a,X) \mid a \in \mathbb{F}_q^\bullet, X \in GL_{n-2}(q)\right\}, \quad B := \left\{\mathrm{diag}(1,a,I_{n-2}) \mid a \in \mathbb{F}_q^\bullet\right\}$$

of D, and let $C = Q : A$. Clearly, C normalizes Q and centralizes $Z(Q)$. By Lemma 2.3, C has a simple submodule T inside $V \otimes W$ of dimension at most $\dim(V^\lambda) \cdot \dim(W^{\lambda^{-1}})$, which is acted on trivially by Q. Observe that $\hat{P} = CB$, and B centralizes C/Q. Hence, according to Lemmas 2.5 and 2.6, T extends to a $\hat{P}$-submodule inside $V \otimes W$, which we denote by the same symbol T. Thus $0 \neq \mathrm{Hom}(T, (V \otimes W)|_{\hat{P}}) \simeq \mathrm{Hom}(T^G, V \otimes W)$. But $V \otimes W$ is irreducible, hence $\dim(V) \cdot \dim(W) \leq (G : \hat{P}) \cdot \dim(V^\lambda) \cdot \dim(W^{\lambda^{-1}})$. Combined with Lemma 3.1, this inequality yields

$$1 \leq \frac{(q^n-1)(q^{n-1}-1)}{(q-1)^2 \cdot (q^{n-1}-1)^2},$$

a contradiction since $q \geq 3$.

Next we assume that $S = SL_n(q)$, $q \geq 4$, has irreducible $\mathbb{F}$-representations V, W of dimension > 1 such that $V \otimes W$ is irreducible. Consider the complement

$$D := \left\{\mathrm{diag}(a,b,X) \mid a,b \in \mathbb{F}_q^\bullet, X \in GL_{n-2}(q), ab \cdot \det(X) = 1\right\}$$

of Q in P. Consider the following subgroups

$$A := \left\{\mathrm{diag}(a,a,X) \mid a \in \mathbb{F}_q^\bullet, \det(X) = a^{-2}\right\}, \quad B := \left\{\mathrm{diag}(a,a^{-1},I_{n-2}) \mid a \in \mathbb{F}_q^\bullet\right\}$$

of D, and let $C = Q : A$. Clearly, C normalizes Q and centralizes $Z(Q)$. By Lemma 2.3, C has a simple submodule T inside $V \otimes W$ of dimension at most $\dim(V^\lambda) \cdot \dim(W^{\lambda^{-1}})$, which is acted on trivially by Q. Since B centralizes C/Q, T extends to a BC-submodule inside $V \otimes W$, which we denote by the same symbol T. Thus $0 \neq \mathrm{Hom}(T, (V \otimes W)|_{BC}) \simeq \mathrm{Hom}(T^S, V \otimes W)$. But $V \otimes W$ is irreducible, hence $\dim(V) \cdot \dim(W) \leq (S : BC) \cdot \dim(V^\lambda) \cdot \dim(W^{\lambda^{-1}})$. Combined with Lemma

3.1, this inequality yields

$$1 \leq \frac{(q^n-1)(q^{n-1}-1)\cdot \gcd(2,q-1)}{(q-1)^2\cdot(q^{n-1}-1)^2},$$

a contradiction since $q \geq 4$. $\square$

Proposition 3.3. *The groups $SL_n(2)$, $n \geq 5$, has property $\mathcal{P}(r)$ for any $r \neq 2$.*

Proof. We proceed by induction on $n \geq 5$. The cases $n = 5, 6$ can be checked directly using GAP.

Consider the case $n \geq 7$ and assume the contrary: $S = SL_n(2)$ has irreducible $\mathbb{F}$-representations V, W of dimension > 1 such that $V \otimes W$ is irreducible. We distinguish two subcases.

1) $\dim(V^{\lambda})$, $\dim(W^{\lambda^{-1}}) > 1$.

By Lemma 2.3, P has a submodule T inside $V \otimes W$, which is acted on trivially by Q and which is isomorphic to $V^{\lambda} \otimes W^{\lambda^{-1}}$ as P/Q-module. Since $P/Q \simeq SL_{n-2}(2)$, by induction hypothesis T is not irreducible. Hence we can choose a composition factor U of the P-module T such that $\dim(U) \leq \dim(T)/2$. Moreover, we can choose U as a submodule or a quotient of T. In the former case,

$$0 \neq \mathrm{Hom}(U, (V \otimes W)|_P) \simeq \mathrm{Hom}(U^S, V \otimes W),$$

hence $V \otimes W$ is a quotient of U^S. In the latter case,

$$0 \neq \mathrm{Hom}((V \otimes W)|_P, U) \simeq \mathrm{Hom}(V \otimes W, U^S),$$

hence $V \otimes W$ is a submodule of U^S.

Let L be a Levi subgroup of P. Then $L \simeq SL_{n-2}(2)$, and U is actually an L-module. Observe that U^S is the Harish-Chandra induction $R_L^S(U)$ (where we use the parabolic subgroup P containing L). We introduce two more parabolic subgroups. The first one, $P_1 = St_S(\langle e_1 \rangle, \langle e_1, e_2, \ldots, e_{n-1} \rangle)$ is obviously conjugate to P. We can view U as a module U_1 for a Levi subgroup L_1 of P_1, and then $U^S = R_L^S(U) \simeq R_{L_1}^S(U_1)$. The second one is $P_2 = St_S(\langle e_1 \rangle, \langle e_1, e_2 \rangle)$ having $L_2 := L$ as a Levi subgroup. Let $\mathcal{S} = \{(1,2), (2,3), \ldots, (n-1,n)\}$ be the standard set of generators for the Weyl group of $S = SL_n(2)$. Since

$$P_1 = \left\{ \left[\begin{array}{c|c|c} 1 & * & * \\ \hline 0 & X & * \\ \hline 0 & 0 & 1 \end{array}\right] \mid X \in SL_{n-2}(2) \right\}, \quad P_2 = \left\{ \left[\begin{array}{cc|c} 1 & * & * \\ 0 & 1 & * \\ \hline 0 & 0 & X \end{array}\right] \mid X \in SL_{n-2}(2) \right\},$$

P_1 is the standard parabolic subgroup corresponding to the subset

$$\mathcal{I}_1 = \{(2,3), (3,4), \ldots, (n-2, n-1)\},$$

and P_2 is the standard parabolic subgroup corresponding to the subset

$$\mathcal{I}_2 = \{(3,4), (4,5), \ldots, (n-1, n)\}$$

of $\mathcal{S}$. Clearly, $\mathcal{I}_1$ and $\mathcal{I}_2$ are conjugate via the cycle $w = (2, 3, 4, \ldots, n-1, n)$ of the Weyl group of S. Now the independence property of the Harish-Chandra induction (cf. [HL], [DD]) implies that $R_{L_1}^S(U_1) \simeq R_{L_2}^S(U_2)$, where $U_2 = {}^{w}U_1$

(the w-conjugate of U_1). Thus U^S is isomorphic to the Harish-Chandra induction $(1 \circ 1 \circ U_2) \uparrow S$ from P_2 to S of the $P_2/O_p(P_2)$-module U_2 in the notation of [GT1].

Next we can write

$$(1 \circ 1 \circ U_2) \uparrow S \simeq (((1 \circ 1) \uparrow SL_2(2)) \circ U_2) \uparrow S.$$

But $(1 \circ 1) \uparrow SL_2(2)$ is a reducible 3-dimensional module. Therefore, any composition factor of U^S has dimension at most $(2/3)\dim(U^S)$.

The above arguments show that $\dim(V \otimes W) \leq (1/3)(S : P)\dim(T)$. Combined with Lemma 3.1, this yields

$$1 \leq \frac{1}{3} \cdot \frac{(2^n - 1)(2^{n-1} - 1)}{(2^{n-1} - 1)^2},$$

a contradiction.

2) Either $\dim(V^\lambda)$ or $\dim(W^{\lambda^{-1}}) = 1$.

For definiteness let us suppose $\dim(V^\lambda) = 1$. The above argument with Harish-Chandra induction shows that $\dim(V \otimes W) \leq (2/3) \cdot (S : P)\dim(V^\lambda)\dim(W^{\lambda^{-1}})$. Recall that $\dim(V^\lambda) = 1$, and $\dim(W^{\lambda^{-1}}) \leq \dim(W)/(2^{n-1} - 1)$ by Lemma 3.1. By [GT1], $\dim(V) \geq 2^n - 3$. We come to the inequality

$$2^n - 3 \leq \frac{2}{3} \cdot \frac{(2^n - 1)(2^{n-1} - 1)}{2^{n-1} - 1},$$

again a contradiction. □

Proposition 3.4. *Let $n \geq 3$ and let V, W be two nontrivial irreducible representations of $SL_n(3)$ in characteristic $r \neq 3$ such that $V \otimes W$ is absolutely irreducible. Then $r > 0$, n is even, and at least one of the modules V, W is a Weil module.*

Proof. We proceed by induction on $n \geq 3$. The cases $n = 3, 4$ can be checked directly using GAP. So we assume that $n \geq 5$ and distinguish two cases.

1) $\dim(V^\lambda)$, $\dim(W^{\lambda^{-1}}) > 1$.

By Lemma 2.3, P' has a submodule T inside $V \otimes W$, which is acted on trivially by Q and which is isomorphic to $V^\lambda \otimes W^{\lambda^{-1}}$ as P/Q-module. Since $P'/Q \simeq SL_{n-2}(3)$, by induction hypothesis T is not irreducible. Hence we can choose a composition factor L of the P-module T such that $\dim(L) \leq \dim(T)/2$. Moreover, we can choose L as a submodule or a quotient of T. Arguing as in the proof of Proposition 3.3, we see that $V \otimes W$ is either a quotient or a submodule of L^S.

Consider the subgroup $D := \{\mathrm{diag}(a, a, I_{n-2}) \mid a \in \mathbb{F}_q^\bullet\}$ in P. Since D centralizes P'/Q, L extends to a $P'D$-module of the same dimension, whence $L^{P'D}$ consists of two constituents of the same dimension. Hence we arrive at the bound $\dim(V \otimes W) \leq (1/2)(S : P'D)\dim(T)$. Combined with Lemma 3.1, this yields

$$1 \leq \frac{1}{2} \cdot \frac{(3^n - 1)(3^{n-1} - 1)}{2(3^{n-1} - 1)^2},$$

a contradiction.

2) Either $\dim(V^\lambda)$ or $\dim(W^{\lambda^{-1}}) = 1$. Clearly, the case n is odd follows from Proposition 3.2, since $GL_n(3) = SL_n(3) \times \mathbb{Z}_2$ when n is odd.

Now we assume that n is even. Let α, resp. β, be the character afforded by V, resp. W. For definiteness we assume that $\dim(V^\lambda) = 1$, whence α is one of the two Weil characters of S by Lemma 3.1. By the virtue of Proposition 3.2, we may also assume that β is not extendible to G. In the rest of the proof we assume that $r = 0$.

First we consider the case $\alpha(1) = (3^n - 1)/3$. For any $g \in S$ let $d(g)$ denote the dimension of the fixed point subspace of g on the natural module of S. Assume g is unipotent. Then $d(g) \geq 1$. Following the proof of [TZ, Lemma 4.1], one can show that $\alpha(g) = (3^{d(g)} - 1)/2 \geq 1$. Also, it is easy to show that α extends to G. By our assumption, β^G is an irreducible complex character of G, and $\beta^G|_S = \beta + \beta'$ for some G-conjugate β' of β. Let T be a Sylow 3-subgroup of S. By a result of Zelevinsky [Z], there is a character λ of T such that the multiplicity of λ in $\beta^G|_T$ is one. From this it follows that λ enters exactly one of the two characters $\beta|_T$ and $\beta'|_T$, whence $\beta|_T \neq \beta'|_T$. Thus there is a unipotent element $g \in S$ such that $\beta(g) \neq \beta'(g)$. Since $\alpha(g) \geq 1$, $\alpha \otimes \beta \neq \alpha \otimes \beta'$. We have seen that the pair (α, β) satisfies the asumption of Lemma 2.4 (with $H = S$). By Lemma 2.4 and Proposition 3.2, $\alpha \otimes \beta$ cannot be irreducible, a contradiction.

Next we consider the case $\alpha(1) = (3^n - 3)/2$. Let $R = A : L$ denote the first maximal parabolic subgroup of S with $A := \mathbb{Z}_3^{n-1}$ and $L := GL_{n-1}(3)$. Then $\alpha = (1_R)^S - 1_S$, whence $(\alpha|_R, 1_R)_R = 1$. We claim that $\beta|_R$ is irreducible. (For, suppose that $\beta|_R$ is reducible: $\beta|_R = \gamma + \gamma'$ for some R-characters γ and γ' with $\gamma(1) \leq \beta(1)/2$. Then $(\alpha \otimes \beta, \gamma^S)_S = (\alpha|_R \otimes \beta|_R, \gamma)_R \geq 1$, but $\alpha \otimes \beta$ is irreducible by assumption, hence

$$\frac{3^n - 3}{2}\beta(1) = \alpha(1)\beta(1) \leq \gamma^S(1) = (S : R)\gamma(1) \leq (S : R)\beta(1)/2 = \frac{3^n - 1}{4}\beta(1),$$

a contradiction.) Next, the transitivity of L on $\mathrm{Irr}(A) \setminus \{1_A\}$ and the irreducibility of $\beta|_R$ imply that $(3^{n-1} - 1)|\beta(1)$. Let ℓ be a primitive prime divisor (cf. [Zs]) of $3^{n-1} - 1$, and let ℓ^a be the largest power of ℓ dividing $3^{n-1} - 1$. Then ℓ^{2a} divides $(\alpha \otimes \beta)(1)$ and does not divide $|S|$. Therefore $\alpha \otimes \beta$ cannot be irreducible, again a contradiction. $\square$

Thus we obtain a complete result for the complex case:

Theorem 3.5. *Suppose $n \geq 2$ and $(n, q) \neq (3, 4)$. Then the groups $SL_n(q)$ and $GL_n(q)$ have property $\mathcal{P}(0)$.* $\square$

4. Unitary Groups

Let $G = U_n(q)$, $S = SU_n(q)$ with $q = p^f$, $n \geq 3$, $(n, q) \neq (3, 2)$, $(4, 2)$, $(4, 3)$, $(4, 9)$, $(6, 2)$. Let $\mathbb{F}_{q^2}^n$ be the natural module for G, endowed with a Hermitian form with Gram matrix $\mathrm{diag}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, I_{n-2}\right)$ in a basis $(e_1, \ldots, e_n)$. We consider the parabolic subgroup $P = St_S(\langle e_1\rangle_{\mathbb{F}_{q^2}})$ of S. Then $Q = O_p(P)$ is a group of order q^{2n-3} of special type. The centre $Z(Q)$ has order q and contains $q-1$ transvections t_c, where $c \in \mathbb{F}_{q^2}^\bullet$, $c + c^q = 0$. It is easy to see that Q has a complement

$$\left\{\mathrm{diag}(a, a^{-q}, X) \mid a, b \in \mathbb{F}_{q^2}^\bullet, X \in U_{n-2}(q), \det(X) = a^{q-1}\right\},$$

and all the transvections t_c are conjugate in P. Let P' be the (unique) normal subgroup of type $Q : SU_{n-2}(q)$ in P. Then $Z(P') = Z(Q)$. Our assumption on (n, q) guarantees that S is the universal cover of $PSU_n(q)$, and that the group $SU_{n-2}(q)$ is perfect and acts faithfully on Q.

Let V, W be any irreducible $\mathbb{F}S$-module of dimension > 1. Then by Lemma 2.2,

$$V|_{P'} = V^0 \oplus \sum_{\lambda \in \Omega(V)} E^\lambda \otimes V^\lambda, \quad W|_{P'} = W^0 \oplus \sum_{\lambda \in \Omega(W)} E^\lambda \otimes W^\lambda$$

in the notation of (1), where $\dim(E^\lambda) = q^{n-2}$ and $\Omega(V) = \Omega(W) = \mathrm{IBr}_r(Z(Q)) \setminus \{1_Q\}$. The subgroup $\left\{\mathrm{diag}(a, a^{-q}, a^{q-1}, I_{n-3}) \mid a \in \mathbb{F}_{q^2}^\bullet\right\}$ permutes the summands $E^\lambda \otimes V^\lambda$, $\lambda \in \Omega(V)$, transitively, hence all V^λ are of the same dimension.

Proposition 4.1. *Suppose $n \geq 3$, and $(n, q) \neq (4, 3)$, $(4, 9)$. Then the groups $SU_n(q)$ with $q \geq 4$, and $U_n(q)$ with $q \geq 3$, have property $\mathcal{P}(r)$ for any $r \not| q$.*

Proof. First we assume that $G = U_n(q)$, $q \geq 3$, has irreducible $\mathbb{F}$-representations V, W of dimension > 1 such that $V \otimes W$ is irreducible. Consider the parabolic subgroup $\hat{P} = St_G(\langle e_1\rangle_{\mathbb{F}_{q^2}})$ of G. Then $Q = O_p(\hat{P})$, and Q has a complement

$$D := \left\{\mathrm{diag}(a, a^{-q}, X) \mid a \in \mathbb{F}_{q^2}^\bullet, X \in U_{n-2}(q)\right\}$$

in $\hat{P}$. Consider the following subgroups

$$A := \{\mathrm{diag}(1, 1, X) \mid X \in U_{n-2}(q)\}, \quad B := \left\{\mathrm{diag}(a, a^{-q}, I_{n-2}) \mid a \in \mathbb{F}_{q^2}^\bullet\right\}$$

of D, and let $C = Q : A$. Clearly, C normalizes Q and centralizes $Z(Q)$. By Lemma 2.3, C has a simple submodule T inside $V \otimes W$ of dimension at most $\dim(V^\lambda) \cdot \dim(W^{\lambda^{-1}})$, which is acted on trivially by Q. Observe that $\hat{P} = CB$, and B centralizes C/Q. Hence T extends to a $\hat{P}$-submodule inside $V \otimes W$, which we denote by the same symbol T. Thus $0 \neq \mathrm{Hom}(T, (V \otimes W)|_{\hat{P}}) \simeq \mathrm{Hom}(T^G, V \otimes W)$. But $V \otimes W$ is irreducible, hence $\dim(V) \cdot \dim(W) \leq (G : \hat{P}) \cdot \dim(V^\lambda) \cdot \dim(W^{\lambda^{-1}})$. Thus

$$q^{2n-4}(q-1)^2 \leq \frac{\dim(V) \cdot \dim(W)}{\dim(V^\lambda) \cdot \dim(W^{\lambda^{-1}})} \leq \frac{(q^n - (-1)^n)(q^{n-1} - (-1)^{n-1})}{q^2 - 1},$$

a contradiction since $q \geq 3$ and $n \geq 3$.

Next we assume that $S = SU_n(q)$, $q \geq 4$, has irreducible $\mathbb{F}$-representations V, W of dimension > 1 such that $V \otimes W$ is irreducible. Consider the complement

$$D := \left\{\mathrm{diag}(a, a^{-q}, X) \mid a \in \mathbb{F}_{q^2}^\bullet, X \in U_{n-2}(q), \det(X) = a^{q-1}\right\}$$

of Q in P. Consider the following subgroups

$$A := \left\{\mathrm{diag}(a,a,X) \mid a^{q+1}=1, \det(X)=a^{-2}\right\}, \quad B := \left\{\mathrm{diag}(a,a^{-1},I_{n-2}) \mid a^{q-1}=1\right\}$$

of D, and let $C = Q : A$. Clearly, C normalizes Q and centralizes $Z(Q)$. By Lemma 2.3, C has a simple submodule T inside $V \otimes W$ of dimension at most $\dim(V^\lambda)\cdot\dim(W^{\lambda^{-1}})$, which is acted on trivially by Q. Since B centralizes C/Q, T extends to a BC-submodule inside $V\otimes W$, which we denote by the same symbol T. Thus $0 \neq \mathrm{Hom}(T,(V\otimes W)|_{BC}) \simeq \mathrm{Hom}(T^S, V\otimes W)$. But $V\otimes W$ is irreducible, hence $\dim(V)\cdot\dim(W) \leq (S:BC)\cdot\dim(V^\lambda)\cdot\dim(W^{\lambda^{-1}})$. Thus

$$q^{2n-4}(q-1)^2 \leq \frac{\dim(V)\cdot\dim(W)}{\dim(V^\lambda)\cdot\dim(W^{\lambda^{-1}})} \leq \frac{(q^n-(-1)^n)(q^{n-1}-(-1)^{n-1})}{q^2-1}\cdot\gcd(2,q-1),$$

a contradiction since $q \geq 4$ and $n \geq 3$. □

Lemma 4.2. *Keep the above notation. If $G = SU_4(9)$, then G has property $\mathcal{P}(r)$.*

Proof. First we observe that $\dim(V)\cdot\dim(W)$ exceeds the degree of the largest irreducible unless $\dim(V^\lambda)$, $\dim(W^{\lambda^{-1}}) \leq 2$. That means that the group A acts trivially on V^λ and on $W^{\lambda^{-1}}$ and thus C has a trivial submodule inside of $V\otimes W$. Thus BC stabilizes some one-dimensional subspace of $V\otimes W$. Again we get the same contradiction as in the proof of Proposition 4.1. □

5. Symplectic Groups in Odd Characteristic

Let $G = CSp_{2n}(q)$, $S = Sp_{2n}(q)$ with $q = p^f$, $n \geq 2$, $p > 2$, and $(n,q) \neq (2,3)$, $(2,9)$. Let $\mathbb{F}_q^{2n}$ be the natural module for G, endowed with a symplectic form with Gram matrix $\mathrm{diag}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}, \begin{pmatrix} 0 & I_{n-2} \\ -I_{n-2} & 0\end{pmatrix}\right)$ in a basis $(e_1,\ldots,e_{2n})$. We consider the parabolic subgroup $P = St_S(\langle e_1\rangle_{\mathbb{F}_q})$ of S. Then $Q = O_p(P)$ is a group of order q^{2n-1} of special type. The centre $Z(Q)$ has order q and contains $q-1$ transvections t_c, where $c \in \mathbb{F}_q^\bullet$ and t_c has matrix $\left(\begin{pmatrix} 1 & c \\ 0 & 1\end{pmatrix}, I_{2n-2}\right)$ in the given basis. It is easy to see that Q has a complement

$$\left\{\mathrm{diag}(a,a^{-1},X) \mid a \in \mathbb{F}_q^\bullet, X \in Sp_{2n-2}(q)\right\}.$$

Let P' be the (unique) normal subgroup of type $Q : Sp_{2n-2}(q)$ in P. Then $Z(P') = Z(Q)$. Our assumption on (n,q) guarantees that S is the universal cover of $PSp_{2n}(q)$, and that the group $Sp_{2n-2}(q)$ is perfect and has trivial Schur multiplier.

The transvections t_c are conjugate in G, but they lie in two conjugacy classes in S. Let V, W be any irreducible $\mathbb{F}S$-module of dimension > 1. Then

$$V|_{P'} = V^0 \oplus \sum_{\lambda \in \Omega(V)} E^\lambda \otimes V^\lambda, \quad W|_{P'} = W^0 \oplus \sum_{\lambda \in \Omega(W)} E^\lambda \otimes W^\lambda$$

in the notation of (1), where $\dim(E^\lambda) = q^{n-1}$. We can identify $\mathrm{IBr}_r(Z(Q))$ with $\mathbb{F}_q$ by assigning to each $a \in \mathbb{F}_q$ the character

$$\lambda_a \quad : \quad t_c \mapsto \epsilon^{\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(ac)},$$

where $\epsilon = \exp(2\pi i/p)$. Let Ω_+, resp. Ω_-, be the set of characters corresponding to the squares, resp. non-squares, in $\mathbb{F}_q^\bullet$. Then $\Omega(V)$, $\Omega(W)$ can be either Ω_+, Ω_-, or $\Omega := \mathrm{IBr}_r(Z(Q)) \setminus \{1_Q\} = \Omega_+ \cup \Omega_-$.

If V is extendible to G, then clearly $\Omega(V) = \Omega$. Actually we can say more about the case where $\Omega(V) \neq \Omega$.

Lemma 5.1. *Let V be an $\mathbb{F}S$-module. Let $S_{n-1} \simeq Sp_{2n-2}(q)$ be the pointwise stabilizer of the 2-subspace $\langle e_{2n-1}, e_{2n}\rangle_{\mathbb{F}_q}$. Suppose that $\Omega(V) \neq \Omega$. Then $\Omega(U) \neq \Omega$ for any irreducible constituent U of $V|_{S_{n-1}}$.*

Proof. Clearly $Z(Q) < S_{n-1}$. The assumption on $\Omega(V)$ means that the set of nontrivial characters of $Z(Q)$ occuring in $V|_{Z(Q)}$ is either $\emptyset$, Ω_+, or Ω_-. The same is true for $U|_{Z(Q)}$, where U is any irreducible constituent of $V|_{S_1}$. This forces $\Omega(U)$ to be different from Ω. □

Corollary 5.2. *Let V be an irreducible $\mathbb{F}S$-module of characteristic zero such that $\Omega(V) \neq \Omega$. Then V is either trivial, or one of Weil modules of S.*

Proof. We proceed by induction on $n \geq 1$. If $n = 1$ then our statement can be checked directly. Suppose $n > 1$. By Lemma 5.1 and by induction hypothesis, any irreducible constituent U of $V|_{S_{n-1}}$ is either trivial, or one of Weil modules of S_{n-1}. This implies that V itself is either trivial or one of Weil modules of S_n, according to the main result of [TZ]. □

In view of Corollary 5.2 we call a nontrivial $\mathbb{F}S$-module such that $\Omega(V) \neq \Omega$ a *module of Weil-type*. We remark that it seems very like that a module of Weil-type is in fact a Weil module.

Proposition 5.3. *Suppose $n \geq 2$, $p > 2$, $p \neq \mathrm{char}(\mathbb{F})$, and $(n, q) \neq (2, 9)$. Let V, W be irreducible $\mathbb{F}S$-modules of dimension > 1 such that $V \otimes W$ is irreducible.*

(i) *If $q \geq 7$, then both of V, W are Weil-type modules.*

(ii) *If $q = 5$, then at least one of V, W is a Weil-type module, and both of them are not invariant under $CSp_{2n}(q)$.*

Proof. (i) Assume the contrary, say W is not a Weil-type module. Then $\Omega(W) = \Omega$ and so we can choose a $\lambda \in \Omega(V)$ such that $\lambda^{-1} \in \Omega(W)$. Consider the complement

$$D := \left\{\mathrm{diag}(a, a^{-1}, X) \mid a \in \mathbb{F}_q^\bullet, X \in Sp_{2n-2}(q)\right\}$$

of Q in P. Consider the following subgroups

$$A := \{\mathrm{diag}(1,1,X) \mid X \in Sp_{2n-2}(q)\}, \quad B := \left\{\mathrm{diag}(a, a^{-1}, I_{n-2}) \mid a \in \mathbb{F}_q^\bullet\right\}$$

of D. Then $P' = Q : A$ and $P = P'B$. By Lemma 2.3, P' has a simple submodule T inside $V \otimes W$ of dimension at most $\dim(V^\lambda) \cdot \dim(W^{\lambda^{-1}})$, which is acted on trivially by Q. Since B centralizes P'/Q, T extends to a P-submodule inside $V \otimes W$, which we denote by the same symbol T. Thus $0 \neq \mathrm{Hom}(T, (V \otimes W)|_P) \simeq \mathrm{Hom}(T^S, V \otimes W)$. But $V \otimes W$ is irreducible, hence $\dim(V) \cdot \dim(W) \leq (S : P) \cdot \dim(V^\lambda) \cdot \dim(W^{\lambda^{-1}})$. Thus

$$q^{2n-2}(q-1)^2/4 \leq \frac{\dim(V) \cdot \dim(W)}{\dim(V^\lambda) \cdot \dim(W^{\lambda^{-1}})} \leq \frac{q^{2n}-1}{q-1},$$

a contradiction since $q \geq 7$.

(ii) First we consider the case where $\Omega(V) = \Omega(W) = \Omega$. Then we can choose a $\lambda \in \Omega(V)$ such that $\dim(W^{\lambda^{-1}})$ is smallest among all $\dim(W^\mu)$, $\mu \in \Omega(W)$. Clearly, $\dim(V)/\dim(V^\lambda) \geq q^{n-1}(q-1)/2$ and $\dim(W)/\dim(W^{\lambda^{-1}}) \geq q^{n-1}(q-1)$. Using the same arguments as in (i) we come to the inequality

$$q^{2n-2}(q-1)^2/2 \leq \frac{\dim(V) \cdot \dim(W)}{\dim(V^\lambda) \cdot \dim(W^{\lambda^{-1}})} \leq \frac{q^{2n}-1}{q-1},$$

a contradiction since $q = 5$.

Next we consider the case where W is invariant under $CSp_{2n}(q)$. Then $\Omega(W) = \Omega$ and $\dim(W^\mu)$ is the same for all $\mu \in \Omega(W)$, hence $\dim(W)/\dim(W^{\lambda^{-1}}) \geq q^{n-1}(q-1)$. Now we can use the same arguments as in (i) and again come to the contradiction that

$$q^{2n-2}(q-1)^2/2 \leq \frac{\dim(V) \cdot \dim(W)}{\dim(V^\lambda) \cdot \dim(W^{\lambda^{-1}})} \leq \frac{q^{2n}-1}{q-1}.$$

$\square$

The same proof as above shows

Proposition 5.4. *Suppose $n \geq 1$, $p > 2$, $q = p^f \geq 5$, and $(n,q) \neq (2,9)$. Then $CSp_{2n}(q)$ has property $\mathcal{P}(r)$ for any r.* $\square$

In view of Proposition 5.3, one has to study the tensor product of Weil modules of $S = Sp_{2n}(q)$. Recall that S has four complex irreducible Weil modules, whose characters are denoted by ξ, ξ^* (of degree $(q^n+1)/2$), and η, η^* (of degree $(q^n-1)/2$). If $r \neq 2, p$, then the reduction modulo r of these four characters are absolutely irreducible, and distinct (look at the character values at transvections). If $r = 2$, then $\xi(\mathrm{mod}\, 2) = 1 + \eta(\mathrm{mod}\, 2)$, $\xi^*(\mathrm{mod}\, 2) = 1 + \eta^*(\mathrm{mod}\, 2)$, and $\eta(\mathrm{mod}\, 2)$ and $\eta^*(\mathrm{mod}\, 2)$ are absolutely irreducible and distinct. If V is any irreducible module, then let $S^2(V)$, resp. $\wedge^2(V)$, be the quotient (of codimension 0 or 1) of $\mathrm{Sym}^2(V)$, resp. of $V \wedge V$, by the fixed point subspace. If V is not self-dual, then let $A(V)$ be the natural submodule of codimension 1 in $V \otimes V^*$. Let

$$\mathcal{M} = \left\{V \otimes W, S^2(V), \wedge^2(V), A^2(V) \mid V, W \text{ Weil modules of } Sp_{2n}(q)\right\}.$$

Proposition 5.5. (i) *Suppose $q \geq 11$ and $r \neq p$. Then no $M \in \mathcal{M}$ can be irreducible.*

(ii) *Suppose $q = 7, 9$. Then the only modules $M \in \mathcal{M}$ which can be irreducible are $S^2(\eta)$, $S^2(\eta^*)$, $\wedge^2(\xi)$, and $\wedge^2(\eta^*)$. Moreover, these modules are irreducible if $r = 0$.*

(iii) *Suppose $q = 5$. Then the only modules $M \in \mathcal{M}$ which can be irreducible are $S^2(\xi) = S^2(\xi^*)$, $\wedge^2(\xi)$, $\wedge^2(\xi^*)$, $S^2(\eta)$, $S^2(\eta^*)$, $\wedge^2(\eta) = \wedge^2(\eta^*)$, $\xi\eta = \xi^*\eta^*$. Moreover, these modules are irreducible if $r = 0$.*

(iii) *Suppose $q = 3$. Then the only modules $M \in \mathcal{M}$ which can be irreducible are $A(\xi)$, $A(\eta)$, $\xi\eta^*$, $\xi^*\eta$, $S^2(\xi) = S^2(\xi^*)$, $\wedge^2(\xi)$, $\wedge^2(\xi^*)$, $S^2(\eta)$, $S^2(\eta^*)$, $\wedge^2(\eta) = \wedge^2(\eta^*)$, $\xi\eta = \xi^*\eta^*$. Moreover, these modules are irreducible if $r = 0$.*

Proof. The discussion before this proposition shows that it is enough to prove this proposition for complex Weil modules.

We can embed S naturally in $A := GL_{2n}(q)$. The group A has a so-called (reducible) rational Weil character τ of degree q^{2n}, and $\tau|_S$ is just the permutation character of S acting on the natural module $\mathbb{F}_q^{2n}$, in particular, $(\tau|_S, \tau|_S)_S = 2q+2$. Furthermore, τ_S is the sum of twice the trivial character, an irreducible character of degree $(q^n-1)(q^n+q)/2(q-1)$ (of multiplicity 1), an irreducible character of degree $(q^n+1)(q^n-q)/2(q-1)$ (of multiplicity 1), two irreducible characters of degree $(q^{2n}-1)/2(q-1)$ (each of multiplicity 1), and $(q-3)/2$ irreducible characters of degree $(q^{2n}-1)/(q-1)$ (each of multiplicity 2) (cf. [T]).

We can also embed S naturally in $B := U_{2n}(q)$. The group B has a so-called (reducible) $\mathbb{Q}$-valued Weil character σ of degree q^{2n}, and $(\sigma|_S, \sigma|_S)_S = 2q+2$. Furthermore, σ_S is the sum of an irreducible character of degree $(q^n+1)(q^n+q)/2(q+1)$ (of multiplicity 1), an irreducible character of degree $(q^n-1)(q^n-q)/2(q+1)$ (of multiplicity 1), two irreducible characters of degree $(q^{2n}-1)/2(q+1)$ (each of multiplicity 1), and $(q-1)/2$ irreducible characters of degree $(q^{2n}-1)/(q+1)$ (each of multiplicity 2) (cf. [T]).

We distinguish two cases.

1) $q \equiv 1 (\mathrm{mod} 4)$. In this case the characters ξ, ξ^* are real, and the characters η, η^* are real-valued but have Schur index 2 over $\mathbb{R}$.

One can show that $\tau|_S = (\xi+\eta)^2 = (\xi^*+\eta^*)^2$ and $\sigma|_S = (\xi+\eta)(\xi^*+\eta^*)$ (cf. [T]). The biggest dimension of irreducible constituents of σ_S is $(q^{2n}-1)/(q+1)$, which is less than $(q^n-1)^2/4$, hence all the characters $\xi\xi^*$, $\eta\eta^*$, $\xi\eta^*$, and $\xi^*\eta$ are not irreducible.

Next we suppose $q \geq 9$. Then the biggest dimension of irreducible constituents of τ_S is $(q^{2n}-1)/(q-1)$, which is less than the dimension of $S^2(\xi)$ and $\xi\eta$. Also, the dimension of $\wedge^2(\eta)$ cannot be equal to the dimension of any irreducible constituents of $\tau|_S$. Finally, the only case when the dimension of $S^2(\eta)$ and $\wedge^2(\xi)$ is equal to the dimension of an irreducible constituent of $\tau|_S$ is $q = 9$. In fact, if $q = 9$, then one can work out the following decomposition of the characters in $\mathcal{M}$ in terms of the irreducible constituents of $\tau|_S$:

$$\xi\eta = \overline{\frac{q^{2n}-1}{q-1}} + \overline{\frac{q^{2n}-1}{q-1}}, \quad S^2(\xi) = \overline{\frac{q^{2n}-1}{2(q-1)}} + \overline{\frac{(q^n-1)(q^n+q)}{2(q-1)}},$$

$$\wedge^2(\xi), S^2(\eta) = \overline{\frac{q^{2n}-1}{q-1}}, \quad \wedge^2(\eta) = \overline{\frac{q^{2n}-1}{2(q-1)}} + \overline{\frac{(q^n+1)(q^n-q)}{2(q-1)}},$$

where $\overline{d}$ denotes an irreducible character of degree d.

Finally, suppose $q = 5$. Then we have the obvious decomposition

$$\tau_S = (\xi+\eta)^2 = S^2(\xi) + \wedge^2(\xi) + S^2(\eta) + \wedge^2(\eta) + 2\xi\eta + 2\cdot 1_S.$$

Since $(\tau_S, \tau_S)_S = 12$, each summand in this decomposition is irreducible. Here are the degrees of these summands:

$$(\xi\eta)(1) = \frac{q^{2n}-1}{q-1}, \quad S^2(\xi)(1) = \frac{(q^n-1)(q^n+q)}{2(q-1)},$$

$$\wedge^2(\xi)(1) = S^2(\eta)(1) = \frac{q^{2n}-1}{2(q-1)}, \quad \wedge^2(\eta)(1) = \frac{(q^n+1)(q^n-q)}{2(q-1)}.$$

2) $q \equiv 3(\mathrm{mod} 4)$. Then $\xi^* = \overline{\xi}$, $\eta^* = \overline{\eta}$, and both ξ, η are non-real.

One can show that $\tau|_S = (\xi+\eta)(\xi^*+\eta^*)$ and $\sigma|_S = (\xi+\eta)^2 = (\xi^*+\eta^*)^2$ (cf. [T]). First we look at $\tau|_S$. Suppose $q \geq 7$. Then the biggest dimension of irreducible constituents of τ_S is $(q^{2n}-1)/(q-1)$, which is less than $(q^n-1)(q^n-3)/4$, hence all the characters $A(\xi)$, $A(\eta)$, $\xi\eta^*$, and $\xi^*\eta$ are not irreducible. Next suppose $q = 3$. Then we have the obvious decomposition

$$\tau_S = A(\xi) + A(\eta) + \xi\eta^* + \xi^*\eta + 2\cdot 1_S.$$

Since $(\tau_S, \tau_S)_S = 8$, each summand in this decomposition is irreducible. Here are the degrees of these summands:

$$A(\xi)(1) = \frac{(q^n-1)(q^n+q)}{2(q-1)}, \quad A(\eta)(1) = \frac{(q^n+1)(q^n-q)}{2(q-1)},$$

$$(\xi\eta^*)(1) = (\xi^*\eta)(1) = \frac{q^{2n}-1}{2(q-1)}.$$

Next we consider $\sigma|_S$ and suppose $q \geq 7$. Then the biggest dimension of irreducible constituents of σ_S is $(q^{2n}-1)/(q+1)$, which is less than the dimension of $S^2(\xi)$ and $\xi\eta$. Also, the dimension of $\wedge^2(\eta)$ cannot be equal to the dimension of any irreducible constituents of $\sigma|_S$. Finally, the only case when the dimension of $S^2(\eta)$ and $\wedge^2(\xi)$ is equal to the dimension of an irreducible constituent of $\sigma|_S$ is $q = 7$. In fact, if $q = 7$, then one can work out the following decomposition of the characters in $\mathcal{M}$ in terms of the irreducible constituents of $\sigma|_S$:

$$\xi\eta = \overline{\frac{q^{2n}-1}{q+1}} + \overline{\frac{q^{2n}-1}{q+1}}, \quad S^2(\xi) = \overline{\frac{q^{2n}-1}{2(q+1)}} + \overline{\frac{(q^n+1)(q^n+q)}{2(q+1)}},$$

$$\wedge^2(\xi), S^2(\eta) = \overline{\frac{q^{2n}-1}{q+1}}, \quad \wedge^2(\eta) = \overline{\frac{q^{2n}-1}{2(q+1)}} + \overline{\frac{(q^n-1)(q^n-q)}{2(q+1)}}.$$

Finally, suppose $q = 3$. Then we have the obvious decomposition

$$\sigma_S = (\xi + \eta)^2 = S^2(\xi) + \wedge^2(\xi) + S^2(\eta) + \wedge^2(\eta) + 2\xi\eta.$$

Since $(\sigma_S, \sigma_S)_S = 8$, each summand in this decomposition is irreducible. Here are the degrees of these summands:

$$(\xi\eta)(1) = \frac{q^{2n}-1}{q+1}, \quad S^2(\xi)(1) = \frac{(q^n+1)(q^n+q)}{2(q+1)},$$

$$\wedge^2(\xi)(1) = S^2(\eta)(1) = \frac{q^{2n}-1}{2(q+1)}, \quad \wedge^2\,(\eta)(1) = \frac{(q^n-1)(q^n-q)}{q+1}.$$

$\square$

Let $G = Sp_4(9)$. Then the largest degree of complex irreducibles of G is 8200. Meanwhile, $\chi(1) \geq 40$ if χ is a nontrivial complex irreducible character of G, and if in addition χ is not a Weil character then $\chi(1) \geq 288$, cf. [TZ]. If χ, χ' are two Weil characters, then $\chi\chi'$ is not irreducible by Proposition 5.5. Thus $Sp_4(9)$ has property $\mathcal{P}(0)$. This observation and Propositions 5.3 and 5.5 altogether imply

Corollary 5.6. *Suppose $n \geq 2$, $p > 2$, and $q \geq 7$. Then the group $Sp_{2n}(q)$ has property $\mathcal{P}(0)$.* $\square$

We expect that Corollary 5.6 holds for all r since we expect Corollary 5.2 to hold for all r.

6. Orthogonal Groups In Characteristic 2

Let $S = \Omega^\epsilon_{2n}(q)$ with $q = 2^f > 2$, $n \geq 4$, $\epsilon = \pm$. Let $\mathbb{F}_q^{2n}$ be the natural module for S, endowed with a non-degenerate quadratic form, whose bilinear form has Gram matrix $\mathrm{diag}\left(\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_{n-2} \\ I_{n-2} & 0 \end{pmatrix}\right)$ in a basis $(e_1, \dots, e_{2n})$. We may assume that $\langle e_1, e_2 \rangle_{\mathbb{F}_q}$ is totally singular. Consider the parabolic subgroup $P = St_S(\langle e_1, e_2\rangle_{\mathbb{F}_q})$ of S. Then $Q = O_p(P)$ is a group of order q^{4n-7} of special type. The center $Z(Q)$ has order q and contains $q-1$ elements t_c, where $c \in \mathbb{F}_q^\bullet$. It is easy to see that Q has a complement

$$D := \left\{\mathrm{diag}\left(A, {}^tA^{-1}, X\right) \mid A \in GL_2(q), X \in \Omega^\epsilon_{2n-4}(q)\right\}$$

in P, and all the elements t_c are conjugate in P. Let P' be the (unique) normal subgroup of type $Q : (SL_2(q) \times \Omega^\epsilon_{2n-4}(q))$ in P. Then $Z(P') = Z(Q)$. Our assumption on (n, q) guarantees that S is the universal cover of $\Omega^\epsilon_{2n}(q)$, and that the group $SL_2(q) \times \Omega^\epsilon_{2n-4}(q)$ is perfect and acts faithfully on Q.

Let V, W be any irreducible $\mathbb{F}S$-module of dimension > 1. Then by Lemma 2.2,

$$V|_{P'} = V^0 \oplus \sum_{\lambda \in \Omega(V)} E^\lambda \otimes V^\lambda, \quad W|_{P'} = W^0 \oplus \sum_{\lambda \in \Omega(W)} E^\lambda \otimes W^\lambda$$

in the notation of (1), where $\dim(E^\lambda) = q^{2n-4}$ and $\Omega(V) = \Omega(W) = \mathrm{IBr}_r(Z(Q)) \setminus \{1_Q\}$. The subgroup $\{\mathrm{diag}\,(a, 1, a^{-1}, 1, I_{2n-4}) \mid a \in \mathbb{F}_q^\bullet\}$ permutes the summands $E^\lambda \otimes V^\lambda$, $\lambda \in \Omega(V)$, transitively, hence all V^λ are of the same dimension.

Proposition 6.1. *Suppose $q = 2^f \geq 4$ and $n \geq 4$. Then the group $\Omega^\epsilon_{2n}(q)$ has property $\mathcal{P}(r)$ for any $r \neq 2$.*

Proof. Assume that $S = \Omega^\epsilon_{2n}(q)$ has irreducible $\mathbb{F}$-representations V, W of dimension > 1 such that $V \otimes W$ is irreducible. Consider the subgroup

$$B := \{\mathrm{diag}\,(aI_2, a^{-1}I_2, I_{2n-4}) \mid a \in \mathbb{F}_q^\bullet\}$$

in P. By Lemma 2.3, P' has a simple submodule T inside $V \otimes W$ of dimension at most $\dim(V^\lambda) \cdot \dim(W^{\lambda^{-1}})$, which is acted on trivially by Q. Observe that $P = P'B$, and B centralizes P'/Q. Hence T extends to a P-submodule inside $V \otimes W$, which we denote by the same symbol T. Thus $0 \neq \mathrm{Hom}(T, (V \otimes W)|_P) \simeq \mathrm{Hom}(T^S, V \otimes W)$. But $V \otimes W$ is irreducible, hence $\dim(V) \cdot \dim(W) \leq (S : P) \cdot \dim(V^\lambda) \cdot \dim(W^{\lambda^{-1}})$. Thus

$$q^{4n-8}(q-1)^2 \leq \frac{\dim(V) \cdot \dim(W)}{\dim(V^\lambda) \cdot \dim(W^{\lambda^{-1}})} \leq \frac{(q^n - \epsilon)(q^{2n-2} - 1)(q^{n-2} + \epsilon)}{(q-1)(q^2-1)},$$

a contradiction since $q \geq 4$ and $n \geq 4$. $\square$

7. Orthogonal Groups In Odd Characteristic

Let $S = Spin^\epsilon_n(q)$ with $q = p^f \geq 5$, $p > 2$ a prime, $n \geq 7$, $\epsilon = \pm$ and $(n, q) \neq (7, 9), (8, 9)$. Let $\mathbb{F}_q^n$ be the natural module for $\bar{S} := \Omega^\epsilon_n(q)$, endowed with a non-degenerate quadratic form, whose bilinear form has Gram matrix $\mathrm{diag}\left(\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, J\right)$ in a basis $(e_1, \ldots, e_n)$ for a non-degenerate diagonal matrix J. We may assume that $\langle e_1, e_2 \rangle_{\mathbb{F}_q}$ is totally singular. Consider the parabolic subgroup $\bar{P} = St_{\bar{S}}(\langle e_1, e_2 \rangle_{\mathbb{F}_q})$ of $\bar{S}$. Then $Q = O_p(\bar{P})$ is a group of order q^{2n-7} of special type. Namely, Q is the subgroup

$$\left\{ [A, B] := \begin{pmatrix} I_2 & B & -{}^t\!A \cdot J \\ 0 & I_2 & 0 \\ 0 & A & I_{n-4} \end{pmatrix} \;\middle|\; \begin{matrix} A \in M_{n-4,2}(\mathbb{F}_q), B \in M_{2,2}(\mathbb{F}_q) \\ B + {}^t\!B + {}^t\!AJA = 0 \end{matrix} \right\}.$$

Each matrix $[A, B] \in Q$ can be identified with the triple $[X, Y, z]$, where $A = \mathrm{diag}(X, -2J^{-1}Y)$ and $B = z\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \frac{1}{2}\,{}^t\!AJA$, $X, Y \in \mathbb{F}_q^{n-4}$ and $z \in \mathbb{F}_q$. This identification shows that Q is isomorphic to the group described before Lemma 2.2 with $m = n-4$. The centre $Z(Q)$ has order q and contains $q-1$ elements t_c, where $c \in \mathbb{F}_q^\bullet$. Let $GL_2^{\bullet 2}(q)$ be the subgroup of all $A \in GL_2(q)$ with $\det(A)$ being a square. By [KL, 2.7.2], the element $\mathrm{diag}(A, {}^tA^{-1}, I_{n-4})$, $A \in GL_2(q)$, belongs to $\Omega_n^\epsilon(q)$ if and only if $\det(A)$ is a square. Hence Q has a complement D in $\bar{P}$, which is isomorphic to $\left(GL_2^{\bullet 2}(q) \times \Omega_{n-4}^\epsilon(q)\right) \cdot 2$, and all the elements t_c are conjugate in P. Let C be the (unique) normal subgroup of type $SL_2(q) \times \Omega_{n-4}^\epsilon(q)$ in D and let $\bar{P}' = Q : C$.

Let P and P' be complete inverse images of $\bar{P}$ and $\bar{P}'$ in S. Then P is the parabolic subgroup corresponding to the second node of the Dynkin diagram of S. S and P certainly contain a subgroup $N \simeq Spin_{n-4}^\epsilon(q)$ (which corresponds to the subdiagram containing all but the first and the second nodes). If a root α corresponds to the first node, then S and P also contain the subgroup M (generated by the root subgroups) corresponding to $\pm\alpha$. It is known that $M \simeq SL_2(q)$, since the rank of S is bigger than 1. Clearly $[M, N] = 1$ and $M \cap N = 1$ (as S is of simply connected type). Since M and N are perfect, they are contained in P'. Also, we can lift Q to $O_p(P)$, which we also denote by Q. Now it is clear that $P' = Q : (M \times N)$. Observe that $M \times N$ acts trivially on $Z(Q)$. Furthermore, M acts faithfully on Q and it is perfect, and N has trivial Schur multiplier. Therefore, Lemma 2.2 applies to P'.

Let V, W be any irreducible $\mathbb{F}S$-module of dimension > 1. Then

$$V|_{P'} = V^0 \oplus \sum_{\lambda \in \Omega(V)} E^\lambda \otimes V^\lambda, \quad W|_{P'} = W^0 \oplus \sum_{\lambda \in \Omega(W)} E^\lambda \otimes W^\lambda$$

in the notation of (1), where $\dim(E^\lambda) = q^{n-4}$ and $\Omega(V) = \Omega(W) = \mathrm{IBr}_r(Z(Q)) \setminus \{1_Q\}$. The complete inverse image of the subgroup $\left\{\mathrm{diag}\left(a, 1, a^{-1}, 1, I_{n-4}\right) \mid a \in \mathbb{F}_q^\bullet\right\}$ in S permutes the summands $E^\lambda \otimes V^\lambda$, $\lambda \in \Omega(V)$, transitively, hence all V^λ are of the same dimension.

Proposition 7.1. *Suppose $q = p^f \geq 5$, $n \geq 7$ and $(n, q) \neq (7, 9), (8, 9)$. Then the group $Spin_n^\epsilon(q)$ has property $\mathcal{P}(r)$ for any $r \neq p$.*

Proof. Assume that $S = Spin_n^\epsilon(q)$ has irreducible $\mathbb{F}$-representations V, W of dimension > 1 such that $V \otimes W$ is irreducible. Consider the subgroup L of S generated by the root subgroups corresponding to the first and second nodes of the Dynkin diagram of S. Then $L \simeq SL_3(q)$, and L contains M. Inside L we can choose a torus $B \simeq \mathbb{Z}_{q-1}$ which centralizes M and N. By Lemma 2.3, P' has a simple submodule T inside $V \otimes W$ of dimension at most $\dim(V^\lambda) \cdot \dim(W^{\lambda^{-1}})$, which is acted on trivially by Q. Since B centralizes P'/Q and B is abelian, by Lemmas 2.5 and 2.6 T extends to a $P'B$-submodule inside $V \otimes W$, which we denote by the same symbol T. Thus $0 \neq \mathrm{Hom}(T, (V \otimes W)|_{P'B}) \simeq \mathrm{Hom}(T^S, V \otimes W)$. But $V \otimes W$ is irreducible, hence $\dim(V) \cdot \dim(W) \leq (S : P'B) \cdot \dim(V^\lambda) \cdot \dim(W^{\lambda^{-1}})$. Also, $P/(P'B) = \mathbb{Z}_2$. Consequently,

$$q^{4k-8}(q-1)^2 \leq \frac{\dim(V) \cdot \dim(W)}{\dim(V^\lambda) \cdot \dim(W^{\lambda^{-1}})} \leq \frac{2(q^k - \epsilon)(q^{2k-2} - 1)(q^{k-2} + \epsilon)}{(q-1)(q^2-1)}$$

if $n = 2k$, and

$$q^{4k-6}(q-1)^2 \leq \frac{\dim(V) \cdot \dim(W)}{\dim(V^{\lambda}) \cdot \dim(W^{\lambda^{-1}})} \leq \frac{2(q^{2k}-1)(q^{2k-2}-1)}{(q-1)(q^2-1)}$$

if $n = 2k+1$. Both of these inequalities cannot hold since $q \geq 5$ and $n \geq 7$. $\square$

8. Exceptional Groups of Lie Type

In this section we consider exceptional groups G of Lie type, i.e. of type ${}^2B_2$, ${}^3D_4$, G_2, ${}^2G_2$, F_4, ${}^2F_4$, E_6, ${}^2E_6$, E_7, or E_8, defined over a finite field $\mathbb{F}_q$. Let $\ell(G)$ denote the smallest integer $t > 1$ such that G has a projective irreducible representation of degree t over a field of characteristic not dividing q. Lower bounds for $\ell(G)$ were found by Landázuri and Seitz in [LS] and improved by Seitz and Zalesskii in [SZ]. Recently, the Landázuri-Seitz-Zalesskii bounds have been improved in a number of cases, cf. [GT1], [Hof1], [Hof2]. Here we would like to exhibit the exact value of $\ell(G)$ (and thereby improve the bound to the best possible one) in the case $G = G_2(q)$ and q is divisible by 3.

Proposition 8.1. *Suppose that $G = G_2(q)$, where $3|q$. Then $\ell(G) = q^2(q^2+1)$ if $q > 3$, and 14 if $q = 3$.*

Proof. The statement is well-known if $q = 3$, cf. [Atlas] and [JLPW]. Suppose $q = 3^f > 3$ and r is any prime other than 2 and 3. The Brauer trees for cyclic r-blocks of G are described in [Sh]. All the non-cyclic r-blocks have maximal defect, and the decompostion numbers for these blocks are computed in [H]. These two results readily imply that $\varphi(1) \geq q^4+q^2+1$ for all nontrivial $\varphi \in \mathrm{IBr}_r(G)$. Finally, the 2-modular characters of G are determined in [HS], from which it follows that the smallest degree of a nontrivial 2-modular character is $q^2(q^2+1)$. $\square$

Observe that $\ell(G_2(q)) = q(q^2-1)$ if $3 \not| q$ (and $q \neq 2, 4$).

Proposition 8.2. *Let G be the universal cover of a finite simple exceptional group of Lie type, defined over a field $\mathbb{F}_q$. Then G has property $\mathcal{P}(r)$ for any $r \not| q$, except possibly for the following cases:*
(i) $q \leq 3$;
(ii) *G is of type F_4 or ${}^2F_4$, and $2|q$.*

Proof. 1) First we consider the case where G is of type G_2 (and $3|q$), ${}^2G_2$, or ${}^2B_2$. The subcase of ${}^2B_2(8)$ can be checked by hand using [Atlas] and [JLPW]. In all remaining subcases G is simple, and $\ell(G) = q^2(q^2+1)$ (by Proposition 8.1), $q(q-1)$, and $(q-1)\sqrt{q/2}$, respectively. On the other hand, the largest degree of any irreducible complex character (and therefore of any irreducible Brauer character in cross characteristic) of G is $(q+1)(q^2+q+1)(q^3+1)$, $(q^2-1)(q+\sqrt{3q}+1)$, and $(q-1)(q+\sqrt{2q}+1)$, respectively. Since this degree is clearly smaller than $\ell(G)^2$, G has property $\mathcal{P}(r)$.

2) Now we suppose that G is the universal cover of an exceptional group of Lie type $E_8(q)$, $E_7(q)$, $E_6(q)$, ${}^2E_6(q)$, $F_4(q)$ (and q is odd), ${}^3D_4(q)$, or $G_2(q)$ (and $3 \not| q$), respectively. Then G has a subgroup Q of special type, of order q^{1+56}, q^{1+32}, q^{1+20}, q^{1+20}, q^{1+14}, q^{1+8}, and q^{1+4}, respectively (cf. for instance Lemma 4.3 of [LS]). In all cases $Z(Q)$ is an elementary abelian group of order q. If $P = N_G(Q)$ and $P' = C_P(Z(Q))$, then P/Q contains an index $\gcd(2, q-1)$ subgroup $(P'/Q) * A$, where $A = \mathbb{Z}_{q-1}$ (or $\mathbb{Z}_{q+1}$ in ${}^2E_6(q)$-case). Correspondingly, P'/Q is the (simply connected) group of type $E_7(q)$, $Spin^+_{12}(q)$, $SL_6(q)$, $SU_6(q)$, $Sp_6(q)$, $SL_2(q^3)$, and $SL_2(q)$. Also, P acts transitively on $Z(Q) \setminus \{1\}$ (to see this observe that every long root group is contained in an $SL_3(q)$-subgroup). Let V and W be any two $\mathbb{F}G$-modules of dimension > 1 such that $V \otimes W$ is irreducible. Then the same arguments as the ones we used for $SL_n(q)$ in the proof of Proposition 3.2 show that

$$\dim(V) \cdot \dim(W) \leq (G : P) \cdot \dim(V^\lambda) \cdot \dim(W^{\lambda^{-1}}) \cdot \gcd(2, q-1)$$

(in the notation of (1). But $\dim(V) \geq (q-1) \cdot \dim(V^\lambda) \cdot \dim(E^\lambda)$ and $\dim(E^\lambda) = \sqrt{|Q|/q}$. Similarly for $\dim(W)$. Hence we get the following inequality

$$(q-1)^2 \cdot |Q| \leq q \cdot (G : P) \cdot \gcd(2, q-1),$$

which cannot happen in any of the above cases when $q \geq 4$. $\square$

Proposition 8.3. *Let $q = 2^f > 2$ and $G = F_4(q)$ or ${}^2F_4(q)$. Then G has property $\mathcal{P}(0)$.*

Proof. Suppose $\alpha, \beta \in \mathrm{Irr}(G)$ are of degree > 1 and such that $\gamma := \alpha \otimes \beta$ is irreducible. Let ϕ_m be the mth cyclotomic polynomial in q.

1) Suppose that G is of type F_4. By [S2, (2.1)], $\gamma(1) \leq (q^{12}-1)(q^8-1)(q^6-1)(q^2-1)/(q-1)^4$. On the other hand, $\alpha(1) \geq \ell(G) \geq q^7(q^3-1)(q-1)/2$. Hence

$$\beta(1) \leq \frac{2(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)}{q^7(q^3-1)(q-1)^5} < \phi_1\phi_3\phi_4\phi_6\phi_8\phi_{12},$$

since $q \geq 4$. The five smallest nontrivial degrees d_i, $1 \leq i \leq 5$, of complex irreducible characters of G are computed in [Lu], and the fifth degree d_5 is exactly $\phi_1\phi_3\phi_4\phi_6\phi_8\phi_{12}$. Hence $\beta(1)$ is one of the four smallest degrees. The same thing is true for $\alpha(1)$.

Suppose that $\alpha(1) = d_4 = q^2\phi_3^2\phi_6^2\phi_{12}$. Since $|G|$ is not divisible by ϕ_{12}^2 and d_2 is divisible by ϕ_{12}, $\beta(1)$ is neither d_2 nor d_4. Since $|G|$ is not divisible by ϕ_6^3 and d_3 is divisible by ϕ_6, $\beta(1)$ cannot be d_3. Finally, since $|G|$ is not divisible by ϕ_3^3 and d_1 is divisible by ϕ_3, $\beta(1)$ cannot be d_1, a contradiction.

Thus $\alpha(1)$ is one of the three smallest degrees. In particular, $\alpha(1)$ is divisible by ϕ_8. By symmetry, the same thing is true for $\beta(1)$. Hence $\gamma(1)$ and $|G|$ are divisible by ϕ_8^2, again a contradiction.

2) Next suppose that $G = {}^2F_4(q)$. The three smallest nontrivial degrees d_i, $1 \leq i \leq 3$, of complex irreducible characters of G are computed in [Lu], where

$d_3 = \phi_1\phi_4^2\phi_{12}$ if $q \geq 32$, and $d_3 = 13778800$ if $q = 8$. By [S2, (2.1)], $\gamma(1) \leq (q^6+1)(q^4-1)(q^3+1)(q-1)/(q-\sqrt{2q}+1)^2$. On the other hand, $\alpha(1) \geq \ell(G) \geq q^4(q-1)\sqrt{q/2}$. Hence

$$\beta(1) \leq \frac{(q^6+1)(q^4-1)(q^3+1)(q-1)}{(q-\sqrt{2q}+1)^2 q^4(q-1)\sqrt{q/2}} < d_3,$$

since $q \geq 8$. This implies that $\beta(1)$ is d_1 or d_2. The same thing is true for $\alpha(1)$. But d_1 and d_2 are divisible by ϕ_6, hence $\gamma(1)$ and $|G|$ are divisible by ϕ_6^2, a contradiction. $\square$

One can show that if G is as in Proposition 8.3 and $\alpha, \beta \in \mathrm{IBr}_r(G)$ are nontrivial characters such that $\alpha \otimes \beta$ is irreducible, then both α and β are unipotent.

References

[A] M. Aschbacher, On the maximal subgroups of the finite classical groups, Invent. Math. **76** (1984), 469 – 514.

[Ar] Z. Arad, On products of conjugacy classes and irreducible characters in finite groups, *Proc. Symp. Pure Math.* **47** (1987), Pt. II, 3 – 10.

[Atlas] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, "An ATLAS of Finite Groups", Clarendon Press, Oxford, 1985.

[BK1] C. Bessenrodt and A. S. Kleshchev, On Kronecker products of complex representations of the symmetric and alternating groups, (preprint).

[BK2] C. Bessenrodt and A. S. Kleshchev, (in preparation).

[DD] R. Dipper and J. Du, Harish-Chandra vertices, *J. reine angew. Math.* **437** (1993), 101 – 130.

[F] W. Feit, "The Representation Theory of Finite Groups", North-Holland Publishing Company, Amsterdam, 1982.

[FT] P. Ferguson and A. Turull, Prime characters and factorizations of quasi-primitive characters, *Math. Z.* **190** (1985), 583 – 604.

[GM] D. Gluck and K. Magaard, Character and fixed space ratios of Brauer characters of finite classical groups, *J. Algebra* **204** (1998), 188 – 201.

[GK] R. Gow and A. S. Kleshchev, Connections between the representations of the symmetric group and the symplectic group in characteristic 2, (in preparation).

[GT1] R. M. Guralnick and Pham Huu Tiep, Low-dimensional representations of special linear groups in cross characteristics, *Proc. London Math. Soc.* (to appear).

[GT2] R. M. Guralnick and Pham Huu Tiep, (in preparation).

[H] G. Hiss, On the decomposition numbers of $G_2(q)$, *J. Algebra* **120** (1989), 339 – 360.

[HS] G. Hiss and J. Shamash, 2-blocks and 2-modular characters of the Chevalley groups $G_2(q)$, *Math. Computations* **59** (1992), 645 – 672.

[Hof1] C. Hoffman, Projective representations in cross characteristics for some classical groups, Preprint, 1997.

[Hof2] C. Hoffman, Low dimensional projective representations for some exceptional finite groups of Lie type, Preprint, 1997.

[HL] R. Howlett and G. Lehrer, On Harish-Chandra induction and restriction for modules of Levi subgroups, *J. Algebra* **165** (1994), 172 – 183.

[Hup] B. Huppert, "Endliche Gruppen I", Springer, Berlin, 1967.

[Is] I. M. Isaacs, (in preparation).

[JLPW] C. Jansen, K. Lux, R. A. Parker, and R. A. Wilson, "An ATLAS of Brauer Characters", Oxford University Press, Oxford, 1995.

[KL] P. Kleidman and M. Liebeck, "The Subgroup Structure of the Finite Classical Groups", London Math. Soc. Lecture Note Ser. no. 129, Cambridge University Press, 1990.

[LS] V. Landázuri and G. Seitz, On the minimal degrees of the projective representations of the finite Chevalley groups, *J. Algebra* **32** (1974), 418–443.

[Lu] F. Lübeck, Smallest degrees of complex characters of exceptional groups of Lie type, Preprint, 1997.

[MM] K. Magaard and G. Malle, Irreducibility of alternating and symmetric squares, *Manuscripta Math.* **95** (1998), 169 – 180.

[S1] G. M. Seitz, The maximal subgroups of classical algebraic groups, Memoirs Amer. Math. Soc. **67** (1987), no. 365.

[S2] G. M. Seitz, Cross-characteristic embeddings of finite groups of Lie type, *Proc. London Math. Soc.* (3) **60** (1990), 166 – 200.

[SZ] G. M. Seitz and A. E. Zalesskii, On the minimal degrees of projective representations of the finite Chevalley groups. II, *J. Algebra* **158** (1993), 233 – 243.

[Sh] J. Shamash, Blocks and Brauer trees for the groups $G_2(2^k)$, $G_2(3^k)$, *Comm. Algebra* **20** (1992), 1375 – 1387.

[T] Pham Huu Tiep, Dual pairs and low-dimensional representations of finite classical groups, (in preparation).

[TZ] Pham Huu Tiep and A. E. Zalesskii, Some characterizations of the Weil representations of the symplectic and unitary groups, *J. Algebra* **192** (1997), 130 – 165.

[Z] A. V. Zelevinsky, "Representations of finite classical groups: A Hopf algebra approach", Lecture Notes in Math. **869** (1981), Springer-Verlag, Berlin.

[Zs] K. Zsigmondy, Zur Theorie der Potenzreste, *Monath. Math. Phys.* **3** (1892), 265 – 284.