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MOSTOW-MARGULIS RIGIDITY WITH LOCALLY COMPACT TARGETS

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Abstract. Let Γ be a lattice in a simple higher rank Lie group G. We describe all locally compact (not necessarily Lie) groups H in which Γ can be embedded as a lattice. For lattices Γ in rank one groups G (with the only exception of non-uniform lattices in $G \simeq \mathrm{SL}_2(\mathbb{R})$, which are virtually free groups) we give a similar description of all possible locally compact groups H, in which Γ can be embedded as a uniform lattice.

1. Introduction and Statement of the Main Results

Throughout this paper we use the following terminology: *(semi)simple Lie group* stands for (semi)simple, connected real Lie group with finite center and no non-trivial compact factors. If (semi)simple Lie groups G, G' are locally isomorphic we write $G \simeq G'$. *Locally compact groups* are assumed to be second countable, but otherwise may be very general. A countable subgroup Γ in a locally compact group G is said to form a *lattice* if it is discrete and G/Γ carries a finite G-invariant measure (note that any locally compact group which has a lattice is necessarily unimodular). A lattice is said to be *uniform* if G/Γ is compact. An embedding $\pi : \Gamma \longrightarrow G$ of a countable group in a locally compact group G is said to be a *lattice embedding* (resp. *uniform lattice embedding*) if $\pi(\Gamma)$ forms a lattice (resp. uniform lattice) in G.

The starting point of our discussion is Mostow's rigidity:

Strong Rigidity Theorem (Mostow, Prasad, Margulis)**.** *Let G and H be semisimple Lie groups, where $G \not\simeq \mathrm{SL}_2(\mathbb{R})$. Let $\Gamma \subset G$ be an irreducible lattice and $\pi : \Gamma \longrightarrow H$ be a lattice embedding. Then $G \simeq H$ and there exists an isomorphism $\bar\pi : \mathrm{Ad}\, G \longrightarrow \mathrm{Ad}\, H$ such that the following diagram commutes:*

$$\begin{array}{ccc} \Gamma & \xrightarrow{\pi} & \pi(\Gamma) \\ \Big\downarrow{\scriptstyle \mathrm{Ad}} & & \Big\downarrow{\scriptstyle \mathrm{Ad}} \\ \mathrm{Ad}\, G & \xrightarrow{\bar\pi} & \mathrm{Ad}\, H \end{array}$$

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Originally, Mostow proved this remarkable theorem for uniform lattices [Mo2]. Mostow's approach was extended by Prasad [Pr] to encompass non-uniform lattices in rank one groups. Finally, the remaining case of non-uniform (irreducible) lattices in higher rank (semi)simple Lie groups was obtained as one of the corollaries of Margulis's superrigidity [Ma1].

Motivated by the strong rigidity theorem above, we consider the following general

Problem. Given a lattice Γ in a simple Lie group G, classify all locally compact groups H which admit lattice embedding or uniform lattice embeddings of Γ.

Consider the following examples:

Example 1.1. Let Γ be a torsion free subgroup of finite index in $\mathrm{SL}_n(\mathbb{Z})$. The collection of all locally compact groups which admit (non-uniform) lattice embeddings of Γ obviously contains $G = \mathrm{SL}_n(\mathbb{R})$, $\mathrm{Ad}\, G = \mathrm{PSL}_n(\mathbb{R})$, $\mathrm{Aut}\, G$, groups of the form $G' \times K$ where K is a compact group and G' is as above, skew-products $\mathrm{Aut}\, G \ltimes_{\mathrm{Out}\, G} K$, where $\mathrm{Aut}\, G$ acts through the finite group $\mathrm{Out}\, G$ on K, and "almost direct" products: $H = (G' \times K)/C$, where C is a finite abelian group embedded diagonally in the centers of G' and the center of a compact K. An example of the latter type is $H = (\mathrm{SL}_n(\mathbb{R}) \times \mathrm{SO}_n)/\pm I \times I$, for even n.

Example 1.2. Let Γ be as above. Obviously it has a uniform lattice embedding (the identity map) in itself, in any discrete $\Gamma' \supset \Gamma$ with $[\Gamma' : \Gamma] < \infty$ (for example $\Gamma' = \mathrm{SL}_n(\mathbb{Z})$, or $\mathrm{PSL}_n(\mathbb{Z})$); as well as in direct products $\Gamma' \times K$ where K is any compact group; or in skew products with compact groups, such as $\mathrm{SL}_n(\mathbb{Z}) \ltimes \mathbb{T}^n$, where $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is the torus. Γ also has uniform lattice embeddings in "almost skew-products", described by an exact sequence

$$1 \longrightarrow F \longrightarrow \Gamma' \ltimes K \longrightarrow H \longrightarrow 1$$

where $\Gamma' \supset \Gamma$ (with $[\Gamma' : \Gamma] < \infty$) acts by automorphisms on a compact group K, and F is a finite abelian group diagonally embedded in the center of Γ' and as a normal subgroup in K which is pointwise fixed by the Γ'-action.

Roughly speaking, the locally compact groups H in Example 1.1 are built from the ambient Lie group G, while the groups H in Example 1.2 are built from the lattice itself. Non uniform lattices in $\mathrm{SL}_2(\mathbb{R})$ admit (uniform) lattice embeddings of a completely different nature:

Example 1.3. Finitely generated non-abelian free groups $\Gamma = F_k$ form (non-uniform) lattices in $\mathrm{SL}_2(\mathbb{R})$. These groups have uniform lattice embeddings in $\mathrm{SL}_2(\mathbb{Q}_p)$ and in $\mathrm{Aut}\,(T)$ - the group of automorphisms of the regular $2k$-tree T. Taking direct and skew-products with compact groups one obtains additional examples.

The last example suggests the following general

Construction 1.4. *Let Γ be a finitely generated group and let $X_{\Gamma,\Sigma}$ denote the (unlabeled) Cayley graph of Γ with respect to some finite set Σ of generators. Then Γ has a uniform lattice embedding in the totally disconnected locally compact group* $\mathrm{Aut}\,(X_{\Gamma,\Sigma})$ *of all automorphisms of this graph (indeed Γ acts simply transitively on the vertices of $X_{\Gamma,\Sigma}$, while the stabilizer of a vertex is a compact subgroup in* $\mathrm{Aut}\,(X_{\Gamma,\Sigma})$*).*

We assert that for lattices in higher rank simple Lie groups the only lattice embeddings are such as the ones described in Examples 1.1 and 1.2:

Theorem A. *Let Γ be a lattice in a simple higher rank Lie group G, let H be some locally compact group, which admits a lattice embedding $\pi : \Gamma \longrightarrow H$ of Γ. Then $\Lambda = \pi(\Gamma)$ is contained in a closed subgroup H_0 of finite index in H, where H_0 has one of the following forms:*

(1) *H_0 is a central extension of locally compact groups*

$$1 \longrightarrow C \longrightarrow H_0 \xrightarrow{p} \mathrm{Ad}\, G \times K \longrightarrow 1$$

where C is a compact abelian group and K is a compact group. More precisely, $H_0 \cong (G' \times K')/C$, where the compact abelian group C is diagonally embedded in the centers of a connected locally compact G' with $G'/C \cong \mathrm{Ad}\, G$, and a compact group K' with $K'/C \cong K$. If the universal covering $\tilde{G}$ of G has finite center, one can take C to be a finite abelian group, in which case G' is a simple Lie group locally isomorphic to G.

(2) *H_0 admits an exact sequence*

$$1 \longrightarrow F \longrightarrow \Gamma \ltimes K \longrightarrow H_0 \longrightarrow 1$$

where Γ acts by automorphisms on a compact group K, and F is a finite abelian group diagonally embedded in the center of Γ and as a normal subgroup of K, which is elementwise fixed by the Γ-action. If Γ has trivial center, then $H_0 \cong \Gamma \ltimes K$.

Moreover, denoting by $\Phi : H_0 \longrightarrow \mathrm{Ad}\, G$ the (continuous) homomorphism

(1) $\quad \Phi : H_0 \xrightarrow{p} \mathrm{Ad}\, G \times K \xrightarrow{pr_1} \mathrm{Ad}\, G$ *, or*

(2) $\quad \Phi : H_0 \longrightarrow \mathrm{Ad}\, \Gamma \times (K/F) \xrightarrow{pr_1} \mathrm{Ad}\, \Gamma \subset \mathrm{Ad}\, G$

corresponding to the above cases (here pr_1 is the projection on the first factor) the following diagram commutes

$$\begin{array}{ccc} \Gamma & \xrightarrow{\pi} & \Lambda \\ \big\downarrow \mathrm{Ad} & & \big\downarrow \subset \\ \mathrm{Ad}\, G & \xleftarrow{\Phi} & H_0 \end{array} \tag{1.1}$$

For uniform lattice embeddings, we have a similar result for lattices in other simple Lie groups:

Theorem B. *Let Γ be a lattice in a simple Lie group $G \not\simeq \mathrm{SL}_2(\mathbb{R})$ and H be some locally compact group which admits a uniform lattice embedding $\pi : \Gamma \longrightarrow H$ of Γ. Then the conclusions of Theorem A hold.*

For uniform lattices in $G \simeq \mathrm{SL}_2(\mathbb{R})$ similar results hold, except for the commutativity of the diagram (1.1) in case (1). More precisely:

Theorem C. *Let Γ be a uniform lattice in $G \simeq \mathrm{SL}_2(\mathbb{R})$ and let H be a locally compact group, which admits a uniform lattice embedding $\pi : \Gamma \longrightarrow H$ of Γ. Then $\Lambda = \pi(\Gamma)$ is contained in a closed subgroup H_0 of finite index in H, where H_0 has one of the forms (1) or (2) as in Theorem A. Furthermore, if $\Phi : H_0 \longrightarrow \mathrm{Ad}\, G$ denotes the (continuous) homomorphism as in Theorem A, then $\Phi(\Lambda) \cong \mathrm{Ad}\,\Gamma$.*

In view of the Construction 1.4, one can deduce from Theorems B and C the following

Corollary 1.5. *Let Γ be a lattice in a simple Lie group G. In case of $G \simeq \mathrm{SL}_2(\mathbb{R})$ assume that Γ is uniform in G. Let Σ be some finite generating set for Γ, and let $X_{\Gamma,\Sigma}$ denote the corresponding undirected unlabeled Cayley graph. Then $X_{\Gamma,\Sigma}$ admits at most finite number of outer automorphisms, i.e.*

$$[\mathrm{Aut}\,(X_{\Gamma,\Sigma})\ :\ \Gamma] < \infty$$

Moreover, the above index is bounded by some constant $i(\Gamma)$, which does not depend on the chosen generating set Σ for a given Γ.

We refer to the phenomenon described in Theorems A and B as Mostow-Margulis rigidity with locally compact targets. The remaining problem (corresponding to Prasad's result), of describing locally compact groups H which admit a non-uniform lattice embedding of a rank one lattice Γ, remains open.

Remarks 1.6. (a) Another related problem, posed by Zimmer, is "superrigidity with locally compact targets", namely classification of locally compact groups H, for which there exists a *homomorphism* $\pi : \Gamma \longrightarrow H$ with the image $\Lambda = \pi(\Gamma)$ being sufficiently "dense" in H, where the notion of "density" should replace Zariski density in semisimple Lie groups. Theorem A applies to the situation where $\pi(\Gamma)$ forms a lattice in H. Indeed, Margulis's Normal Subgroup Theorem [Ma2] states that either $\pi(\Gamma)$ is finite (so that H is compact), or π factors through an *embedding* $\pi' : \Gamma' \longrightarrow H$ where Γ' is a lattice with $\mathrm{Ad}\,\Gamma' \cong \mathrm{Ad}\,\Gamma$, to which Theorem A applies.

(b) In principle, using structure Theorems A (respectively B), one can also classify not only the targets H of (uniform) lattice embeddings of Γ, but rather (uniform) lattice embeddings $\pi : \Gamma \longrightarrow H$ themselves, up to $\mathrm{Aut}\, H$. We have not addressed this question here.

(c) The proofs of Theorems A and B give effective bounds on the index $[H : H_0]$ in terms of $|\mathrm{Out}\,(G)| < \infty$ in case (1), and the index of $\mathrm{Ad}\,\Gamma$ in a maximal lattice

Γ_* in $\mathrm{Aut}\,(\mathrm{Ad}\,G)$, containing $\mathrm{Ad}\,\Gamma$ in case (2). The latter gives the bound $i(\Gamma)$ on the index $[\mathrm{Aut}\,(X_{\Gamma,\Sigma}) : \Gamma]$ in Corollary 1.5.

(d) In Section 3.5 we show that the description of H_0 in case (1) of Theorem C, as an almost direct product $H_0 \cong (G' \times K)/C$ over a *compact* abelian center C, cannot be replaced by an almost direct product over a *finite* abelian center C (as it is claimed in Theorems A and B for the case where the universal covering $\tilde{G}$ has a finite center). However, it is unclear to the author, whether *compact* center C can be replaced by a *finite* center in case (1) of Theorems A, B.

About the proofs. Having very similar appearance, Theorems A and B, C have quite different proofs. The proof of Theorem A uses measure-theoretic aspects of (semi)simple Lie group actions, such as Zimmer's superrigidity for cocycles (subsection 2.1) which is a generalization of Margulis's superrigidity, and an argument (subsection 2.2), which has a lot in common with the smoothness of algebraic actions on spaces of measures. On the other hand, the proof of Theorem B (and that of C) is based on, by now well developed, theory of quasi-isometries, which has Mostow's proof of strong rigidity as one of its origins. The proof of Theorem C also uses some special features of the group of homeomorphisms of the circle (see Appendix).

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2. Proof of Theorem A

Outline of the proof. The essential idea of the proof is to establish a homomorphism $\Phi : H \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\,G)$, for which the following diagram commutes:

$$\begin{tikzcd} \Gamma \arrow[r, "\pi"] \arrow[d, "\mathrm{Ad}"] & \Lambda \arrow[d, "\subset"] \\ \mathrm{Aut}\,(\mathrm{Ad}\,G) & H \arrow[l, "\Phi"'] \end{tikzcd} \tag{2.1}$$

This is done in two steps:

- Using superrigidity for measurable cocycles, a measurable bi-Γ-equivariant map $\Phi : H \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\,G)$, satisfying (2.1), is constructed (subsection 2.1).
- It is proved that any such bi-Γ-equivariant measurable map coincides a.e. with a continuous homomorphism (subsection 2.2).

The image $\Phi(H) \subseteq \mathrm{Aut}\,(\mathrm{Ad}\,G)$ is shown either to contain $\mathrm{Ad}\,G$ or to form a lattice $\Gamma_* \supset \mathrm{Ad}\,\Gamma$ in $\mathrm{Aut}\,(\mathrm{Ad}\,G)$, in which case $[\Gamma_* : \Gamma] < \infty$ (subsection 2.3), while the kernel $K = \mathrm{Ker}(\Phi)$ is a compact normal subgroup in H. Passing to the finite index subgroup $H_0 := \Phi^{-1}(\mathrm{Ad}\,G)$ or $H_0 := \Phi^{-1}(\mathrm{Ad}\,\Gamma)$, descriptions (1) or (2) are obtained in subsection 2.4.

2.1. Construction of a measurable equivariant map $\Phi : H \to \mathrm{Aut}\,(\mathrm{Ad}\,G)$.

Theorem 2.1. *There exists a unique measurable map $\Phi : H \to \mathrm{Aut}\,(\mathrm{Ad}\,G)$ such that*

$$\Phi(\pi(\gamma_1) \cdot h \cdot \pi(\gamma_2)) = \mathrm{Ad}\,\gamma_1 \cdot \Phi(h) \cdot \mathrm{Ad}\,\gamma_2 \tag{2.2}$$

for every $\gamma_1, \gamma_2 \in \Gamma$ and a.e. $h \in H$.

The group structure of H is immaterial for this statement. It holds whenever H is an (infinite) measure space with two commuting, measure preserving, free, finite covolume actions of a higher rank lattice Γ (here the actions are from the left and from the right). This was proved in [Fu] Theorem 4.1. For the sake of completeness we give a

Sketch of the proof. Consider the finite measure Lebesgue space H/Λ with the measure preserving (transitive) left H-action on it. Choose a (Lebesgue) measurable cross section $s : H/\Lambda \longrightarrow H$ for the natural projection $H \longrightarrow H/\Lambda$, and define a measurable cocycle

$$\alpha_0 : H \times H/\Lambda \longrightarrow \Lambda, \qquad \text{by} \qquad \alpha_0(h_1, h\Lambda) := s(h_1 h\Lambda)^{-1} h_1\, s(h\Lambda)$$

Let $\alpha : \Gamma \times H/\Lambda \longrightarrow \mathrm{Ad}\,\Gamma \subset \mathrm{Ad}\,G$ be the measurable cocycle, defined by

$$\alpha(\gamma, h\Lambda) := \mathrm{Ad}\ \alpha_0(\pi(\gamma), h\Lambda)$$

Working separately on each of the Γ-ergodic components, one checks that α is Zariski dense in $\mathrm{Ad}\,G$ ([Fu] Lemma 4.2, see also [Zi] p. 99). At this point we use the assumption that G is a simple Lie group of higher rank to apply the superrigidity for cocycles theorem ([Zi] 5.2.5), which gives an existence of a measurable $\phi : H/\Lambda \longrightarrow \mathrm{Ad}\,G$ and $\rho \in Hom(\Gamma, \mathrm{Ad}\,G)$ with

$$\alpha(\gamma, h\Lambda) = \phi(\pi(\gamma) h\Lambda)^{-1} \rho(\gamma) \phi(h\Lambda)$$

In the above formula, $\rho(\gamma)$ and $\phi(h\Lambda)$ can be replaced by $g^{-1}\rho(\gamma)\,g$ and $g^{-1}\,\phi(h\Lambda)$ for any $g \in \mathrm{Ad}\,G$. Allowing ϕ to take values in $\mathrm{Aut}\,(\mathrm{Ad}\,G) \supset \mathrm{Ad}\,G$ (which enables to use $g \in \mathrm{Aut}\,g$), and using Margulis's superrigidity ([Ma1]), one can always assume that $\rho(\gamma) = \mathrm{Ad}\,\gamma$. Fixing ρ this way, the map ϕ turns out to be uniquely determined (Remark 2.4.(a)). Now one reassembles the definition of ϕ on the ergodic components to a single (still measurable) map $\phi : H/\Lambda \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\,G)$. Extending ϕ to $\Phi : H \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\,G)$ in a Γ-equivariant (from the right) way, namely by

$$\Phi(h) := \phi(h\Lambda)\,\mathrm{Ad}\,(s(h\Lambda)^{-1} h)$$

one verifies that Φ satisfies the relation (2.2). □

Remark 2.2. The above argument is the only place in the proof, where the assumption that G is a *higher rank simple* Lie group, is used. To apply this construction to an *irreducible* lattice Γ in a higher rank *semisimple* $G = \Pi G_i$, one needs to know that each of the simple factors G_i acts ergodically on the skew-product $G/\Gamma \times X$, where X is a Λ-ergodic component of H/Λ. This would follow, for example, if one knew that Λ is mixing on each of the Λ-ergodic components of H/Λ.

2.2. **Equivariant measurable map Φ is a continuous homomorphism.** The following Theorem is formulated for our particular setup, but it might have an independent interest:

Theorem 2.3. *Let Γ be a lattice in a semisimple Lie group G, and Λ be a lattice in a locally compact group H. Assume that there exists an epimorphism $\rho : \Lambda \longrightarrow \mathrm{Ad}\,\Gamma$, and a measurable (with respect to the Haar measures) map $\Phi : H \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\,G)$ which satisfies*

$$\Phi(\lambda_1\, h\, \lambda_2) = \rho(\lambda_1)\, \Phi(h)\, \rho(\lambda_2)$$

for all $\lambda_1, \lambda_2 \in \Lambda$ and m_H-a.e. $h \in H$. Then the map Φ coincides m-a.e. with a continuous homomorphism $\Phi_0 : H \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\,G)$ with $\Phi_0|_\Lambda = \rho$, the image $L = \Phi_0(H)$ is a closed subgroup of $\mathrm{Aut}\,(\mathrm{Ad}\,G)$ *for which the pushforward measure $\Phi_*(m_H)$ gives the Haar measure m_L on L.*

Proof. Consider the measurable function $F : H \times H \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\,G)$ defined by

$$F(h_1, h_2) := \Phi(h_1)\, \Phi(h_1^{-1} h_2)\, \Phi(h_2)^{-1}$$

Observe that for any $\lambda \in \Lambda$ and a.e. h_1, h_2 one has

$$\begin{array}{rcl} F(h_1\lambda,\, h_2) & = & F(h_1, h_2) \\ F(h_1,\, h_2\lambda) & = & F(h_1, h_2) \\ F(\lambda\, h_1, \lambda\, h_2) & = & \rho(\lambda)\, F(h_1, h_2)\, \rho(\lambda)^{-1} \end{array} \tag{2.3}$$

Hence F descends to a measurable function f on a probability space $X := H/\Lambda \times H/\Lambda$. This f satisfies

$$f(\lambda \cdot x) = \rho(\lambda)\, f(x) \rho(\lambda)^{-1}$$

where $\lambda : x \mapsto \lambda \cdot x$ denotes the measure preserving diagonal left Λ-action on $H/\Lambda \times H/\Lambda$.

Let $\mu := f_*(m_{H/\Lambda} \times m_{H/\Lambda})$ denote the push forward measure on $\mathrm{Aut}\,(\mathrm{Ad}\,G)$. Then μ is a probability measure which is invariant under conjugation by elements of $\mathrm{Ad}\,\Gamma = \rho(\Lambda)$. We claim that such μ has to be concentrated on the identity: $\mu = \delta_e$. Indeed, it is well known that if γ is a regular semisimple element of $\mathrm{Ad}\,G \subset \mathrm{Aut}\,(\mathrm{Ad}\,G)$, and $g \in \mathrm{Aut}\,(\mathrm{Ad}\,G)$ does not commute with γ, then $\gamma^n g \gamma^{-n} \to \infty$ for $n \to \infty$ or $n \to -\infty$. Poincare recurrence (for μ under conjugation by γ) implies that μ is supported by the centralizer of γ. Since, by Borel's density theorem, regular elements of $\mathrm{Ad}\,\Gamma$ are Zariski dense in $\mathrm{Ad}\,G$, we conclude that μ is supported by the the center of $\mathrm{Aut}\,(\mathrm{Ad}\,G)$, which is trivial.

Hence almost everywhere $F(h_1, h_2) = e$, so that

$$\Phi(h_1^{-1}h_2) = \Phi(h_1)^{-1}\Phi(h_2), \qquad \text{a.e. on } H \times H$$

This means that Φ is an a.e. homomorphism and $\Phi(H)$ is an a.e. group with respect to the pushforward measure $m_* = \Phi_*(m_H)$. This is known to imply (cf. [Zi] Appendix B) that m_* is the Haar measure m_L of a closed subgroup $L \subseteq \mathrm{Aut}\,(\mathrm{Ad}\,G)$ and $\Phi(h) = \Phi_0(h)$ a.e. where $\Phi_0 : H \longrightarrow L \subseteq \mathrm{Aut}\,(\mathrm{Ad}\,G)$ is a continuous homomorphism. In particular, for a fixed $\lambda \in \Lambda$ and a.e. $h \in H$:

$$\rho(\lambda)\,\Phi(h) = \Phi(\lambda h) = \Phi_0(\lambda h) = \Phi_0(\lambda)\,\Phi_0(h) = \Phi_0(\lambda)\,\Phi(h)$$

so that $\Phi_0(\lambda) = \rho(\lambda)$ for $\lambda \in \Lambda$. □

Remarks 2.4. (a) Under the conditions of Theorem 2.3 with a fixed $\rho : \Lambda \longrightarrow \Gamma$ there exists *at most* one bi-equivariant measurable map Φ, and in particular a unique extension Φ_0 of ρ. The proof is similar to the above one: if Φ_1, Φ_2 are two bi-equivariant measurable maps, then $F(h) := \Phi_1(h)\,\Phi_2(h)^{-1}$ have properties similar to (2.3), so that $F_*(m_{H/\Lambda})$ is a probability measure on $\mathrm{Aut}\,(\mathrm{Ad}\,G)$ which is invariant under the conjugation by $\gamma \in \mathrm{Ad}\,\Gamma$, and hence is δ_e.

(b) Note that Theorem 2.3, with essentially the same proof, holds under a weaker assumption that $\rho(\Lambda)$ is Zariski dense in $\mathrm{Ad}\,G$, rather than forms a lattice there.

(c) Consider the following example: let $G = H = \mathrm{PSL}_2(\mathbb{R})$, Λ_1 be an infinite cyclic subgroup of the diagonal group, $\Lambda_2 \subset G$ be a (uniform) lattice and $\rho = \mathrm{Id}$. Then the Λ_1-action on G/Λ_2 is measurably isomorphic to the Bernoulli shift, which is known to have a vast family of measurable automorphisms. These automorphisms give rise to many measurable maps $\Phi : G \longrightarrow G$ which satisfy a.e. on G:

$$\Phi(\lambda_1 g \lambda_2) = \lambda_1\,\Phi(g)\,\lambda_2, \qquad (\lambda_1 \in \Lambda_1,\ \lambda_2 \in \Lambda_2)$$

This example shows that the conclusion of Theorem 2.3 does not follow just from the *ergodicity* of $\rho(\Lambda_1) \times \rho(\Lambda_2)$ on G.

Applying Theorem 2.3 with $\rho = \pi^{-1}$ to the measurable map $\Phi : H \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\,G)$, constructed in Theorem 2.1, we conclude that Φ (possibly, after an adjustment on a null set) is a continuous homomorphism satisfying

$$\Phi \circ \pi = \mathrm{Ad} \quad : \Gamma \longrightarrow \mathrm{Ad}\,\Gamma \subset \mathrm{Ad}\,G \subset \mathrm{Aut}\,(\mathrm{Ad}\,G) \tag{2.4}$$

Hereafter Φ denotes this continuous homomorphism.

2.3. **The Image and the Kernel of $\Phi : H \to \mathrm{Aut}\,(\mathrm{Ad}\, G)$.** Since $\mathrm{Ad}\,\Gamma$ is Zariski dense in $\mathrm{Ad}\, G$ the connected component of the closed subgroup $\Phi(H) \subseteq \mathrm{Aut}\,(\mathrm{Ad}\, G)$, which contains $\mathrm{Ad}\,\Gamma$ is either $\mathrm{Ad}\, G$, or trivial. In the former case $\mathrm{Ad}\, G$ has finite index in $\Phi(H)$ (recall that $[\mathrm{Aut}\,(\mathrm{Ad}\, G) : \mathrm{Ad}\, G] < \infty$), and in the latter case $\Phi(H)$ is a discrete group containing $\mathrm{Ad}\,\Gamma$, necessarily as a finite index subgroup. We shall now restrict our attention to the closed subgroup $H_0 \subseteq H$ of finite index, defined by $H_0 := \Phi^{-1}(\mathrm{Ad}\, G)$ or $H_0 := \Phi^{-1}(\mathrm{Ad}\,\Gamma)$, corresponding to the above cases. Note also that (2.4) shows that $\Lambda \subseteq H_0$.

Next we claim that the kernel $K = \mathrm{Ker}(\Phi) \lhd H_0$ is compact. First consider the case of $\Phi(H_0) = \mathrm{Ad}\, G$. Let $F \subset \mathrm{Ad}\, G$ be a measurable subset which forms an $\mathrm{Ad}\,\Gamma$-fundamental domain. Observe that the equivariance of Φ implies that $E := \Phi^{-1}(F)$ forms a Λ-fundamental domain in H_0, and we can normalize the Haar measure m_{H_0} (and hence m_*) so that $m_{H_0}(E) = m_*(F) = 1$. Disintegration of $m_{H_0}|_E$ (and then all of m_{H_0}) with respect to $m_*|_F$ (respectively m_*) gives a family of *probability* measures $\{\mu_g\}_{g\in \mathrm{Ad}\, G}$ supported on the fibers $K_g := \Phi^{-1}(\{g\})$ which are K-cosets. The uniqueness of the Haar measures (normalized as above) as (left) invariant measures, together with the uniqueness of the disintegration procedure, give for every $h \in H$ and m_*-a.e. $g \in \mathrm{Ad}\, G$:

$$h\,\mu_g = \mu_{\Phi(h)g}$$

In particular for each $k \in K$ and m_*-a.e. g, one has $k\,\mu_g = \mu_g$, and, by a standard Fubini argument, one concludes that μ_e is a left K-invariant *probability* measure on K, i.e. K is compact.

The case of $\Phi(H_0) = \mathrm{Ad}\, G$ is even simpler: the measure on the fiver $K = \Phi^{-1}(\{e_{\mathrm{Ad}\,\Gamma}\})$ is finite and K-invariant, so K is compact.

2.4. **The structure of H_0.** By now we have shown, that given $\pi : \Gamma \longrightarrow \Lambda \subseteq H$, there exists a (unique) continuous homomorphism $\Phi : H \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\, G)$ satisfying (2.1). Moreover, H_0 is described either by the exact sequence

$$1 \longrightarrow K \longrightarrow H_0 \xrightarrow{\Phi} \mathrm{Ad}\, G \longrightarrow 1 \tag{2.5}$$

or by the exact sequence

$$1 \longrightarrow K \longrightarrow H_0 \xrightarrow{\Phi} \mathrm{Ad}\,\Gamma \longrightarrow 1 \tag{2.6}$$

In both cases K denotes a compact group.

Case (1): $\Phi(H_0) = \mathrm{Ad}\, G$**.**

Theorem 2.5. *Suppose that a locally compact H admits an exact sequence of continuous homomorphisms*

$$1 \longrightarrow K \longrightarrow H \xrightarrow{\Phi} L \longrightarrow 1$$

where K is a compact normal subgroup of H and L is a connected group. Then H contains a closed normal connected subgroup L', which centralizers K, and forms a central extension of L by a compact abelian group $A := L' \cap K$:

$$1 \longrightarrow A \longrightarrow L' \xrightarrow{\Phi} L \longrightarrow 1$$

The group H is then described by the exact sequence

$$1 \longrightarrow A \xrightarrow{j} L' \times K \longrightarrow H \longrightarrow 1$$

where $j(a) := (a, -a)$ is a diagonal embedding of the compact abelian A in the centers of L' and K.

Proof. Let us denote by $C_H(K)$ the centralizer of K in H, i.e.

$$C_H(K) := \{h \in H \mid [h, k] = 1 \ \forall k \in K\}$$

$C_H(K)$ is a closed normal subgroup of H. We claim that $H = K \cdot C_H(K)$.

Let us assume first that K has a faithful finite dimensional representation, i.e. it is a subgroup of some $U(n)$. In this case $\mathrm{Out}\, K$ is known to be finite. The conjugations $h : k \mapsto h^{-1} k\, h$ gives a continuous homomorphism

$$i : H \longrightarrow \mathrm{Aut}\, K \quad \text{which descents to} \quad j : L \longrightarrow \mathrm{Out}\, K$$

Since L is connected, this j must be trivial, so $i(H) = i(K) = \mathrm{Inn}\, K \subseteq \mathrm{Aut}\, K$. Note that $i(h) = 1$ iff h lies in the centralizer $C_H(K)$ of K in H. Hence $H = K \cdot C_H(K)$ as asserted.

Now consider the general case of a compact K. Let $\pi \in \hat{K}$ be an irreducible (unitary) K-representation. The H-action on K moves π to $\pi^h \in \hat{K}$ defined by

$$\pi^h(k) := \pi(h^{-1} k\, h) \qquad (k \in K,\ h \in H,\ \pi \in \hat{K})$$

The corresponding continuous H-action on the characters $\chi_\pi(k) = \mathrm{Tr}\, \pi(k)$ is trivial on K, and therefore descends to the action of L. Since $\{\chi_\pi\}_{\pi \in \hat{K}}$ are orthogonal, the latter action of the connected group L is trivial. Thus $\chi_{\pi^h} = \chi_\pi$, implying $\pi^h \sim \pi$. Hence $\mathrm{Ker}(\pi)$ is normal not only in K but also in H.

Let us enumerate by π_i, $i = 0, 1, \ldots$ all the irreducible K-representations and form the increasing sequence of finite-dimensional unitary K-representations

$$\sigma_n = \oplus_{i=0}^n \pi_i$$

Since $\mathrm{Ker}(\sigma_n) = \bigcap_0^n \mathrm{Ker}(\pi_i)$ is a closed normal subgroup of H, the exact sequence of (continuous homomorphisms)

$$1 \longrightarrow K \longrightarrow H \longrightarrow L \longrightarrow 1$$

descends to the exact sequence

$$1 \longrightarrow K_n \longrightarrow H_n \longrightarrow L \longrightarrow 1$$

where $K_n = K/\mathrm{Ker}(\sigma_n)$ and $H_n = H/\mathrm{Ker}(\sigma_n)$. The discussion of the finite dimensional case above yields that $H_n = K_n \cdot C_{H_n}(K_n)$. This means that given a fixed element $h \in H$ we can find $k_n \in K$ and $z_n \in H$ so that

$$h = k_n\, z_n \qquad \text{and} \qquad \forall k \in K \;:\; [z_n, k] \in \mathrm{Ker}(\sigma_n)$$

Passing to a subsequence n_i, we can assume that $k_{n_i} \to k_* \in K$ and hence $z_{n_i} \to z_* = h\, k_*^{-1} \in H$. Since $\mathrm{Ker}(\sigma_{n_i})$ form a decreasing sequence of compact subsets, we have

$$h = k_*\, z_* \qquad \text{and} \qquad \forall k \in K \;:\; [z_*, k] \in \mathrm{Ker}(\sigma_{n_i}),\ \forall i$$

Recall that $\{\pi_n\}_n$ separate points of K, and so do $\{\sigma_{n_i}\}_i$, hence $\bigcap_i \mathrm{Ker}(\sigma_{n_i}) = \{1\}$ and therefore $z_* \in C_H(K)$. This shows that $H = K \cdot C_H(K)$.

Now let C denote the center of K, i.e. $C = K \cap C_H(K)$. Observe that $C_H(K)$ is a central extension of L by C:

$$L = H/K = K \cdot C_H(K)/K \cong C_H(K)/K \cap C_H(K) = C_H(K)/C$$

Taking L' to be the connected component of $C_H(K)$ and $A := L' \cap K$ one obtains the assertion of the Theorem. □

Applying Theorem 2.5 to the sequence (2.5) one obtains the first assertion of Theorem A, case (1). In the case of G with a *finite* universal covering $\tilde{G}$, the description of H_0 as $(G' \times K')/C$ where a compact C is diagonally embedded in the centers of G' and K', can be reduced to a *finite* C, using the following:

Lemma 2.6. *Let G' be a locally compact group which is a (topological) central extension*

$$1 \longrightarrow C \longrightarrow L \xrightarrow{p} G \longrightarrow 1 \tag{2.7}$$

of a semisimple Lie group G by a compact abelian group C. Assume that the center of G is trivial, and the center Z of its universal covering $\tilde{G}$ is finite. Then there exists a closed connected subgroup $G' \subseteq L$ with a finite center $Z' = G' \cap C$, so that the finite covering

$$1 \longrightarrow Z' \longrightarrow G' \xrightarrow{p} G \longrightarrow 1$$

is a local isomorphism of the semisimple Lie groups $G' \simeq G$.

Proof. First consider the situation, where C is totally disconnected. Then (2.7) is a fiber bundle with a totally disconnected fiber C, over the base G which is a connected manifold. Hence any point of G can be connected by a path in G, starting at e_G; such path has a unique lifting to a path in L, starting at e_L; while homotopic paths in G lift to homotopic paths in L. Define G' to be the path-connected component of the identity e_L in L. Then $G' \cap C$ is a homomorphic image of $\pi_1(G, e_G) = Z$ which is finite. Hence G' is a finite covering of G. It is closed in L, since the covering map $p|_{G'} : G' \longrightarrow G$ is finite to one.

Next consider the general case of a compact C. The connected component C_0 of the identity in C, is a closed (compact) subgroup of the center of L. Dividing (2.7) by C_0, one arrives in the situation discussed above, in which case L/C_0 contains a closed connected finite covering G_0' of G. Let $L' \subseteq L$ be the preimage of $G_0' \subseteq L/C_0$. It is a closed connected subgroup in L, which is a topological central extension

$$1 \longrightarrow C_0 \longrightarrow L' \xrightarrow{p'} G_0' \longrightarrow 1$$

of the semisimple Lie group G_0' by a connected compact group C_0. This central extension is an inverse limit of central extensions of Lie groups

$$1 \longrightarrow C_{0,n} \longrightarrow L_n' \xrightarrow{p_n'} G_0' \longrightarrow 1$$

where $C_{0,n}$ are finite dimensional tori. Locally, these extensions are trivial, since the second cohomology $H^2(Lie(G_0'), \mathbb{R}^d)$ vanishes. Hence L_n' contains a closed subgroup G_n' for which $p_n' : G_n' \longrightarrow G_0'$ is a finite covering. Since the degree of this covering is bounded by $|Z| < \infty$, the subgroups G_n' stabilize (in the sense that $G_{n+1}' \longrightarrow G_n'$ is one-to-one for large n), so that the limit group L' contains a closed subgroup G' for which $p' : G' \longrightarrow G_0'$ is a finite covering. $\square$

This completes the proof of Theorem A for Case (1).

Case (2): $\Phi(H_0) = \operatorname{Ad}\Gamma$. The compact group K is normal in $H_0 \supset \Lambda \cong \Gamma$, so Γ acts on K by automorphisms:

$$\gamma : k \mapsto k^\gamma := \pi(\gamma)^{-1}\, k\, \pi(\gamma) \qquad (k \in K,\ \gamma \in \Gamma)$$

The corresponding skew product $\Gamma \ltimes K$ maps homomorphically into H_0 by

$$p : \Gamma \ltimes K \longrightarrow H_0 \quad \text{defined by} \quad p(\gamma, k) := \pi(\gamma) \cdot k$$

Since the homomorphism

$$\Gamma \ltimes K \xrightarrow{p} H_0 \xrightarrow{\Phi} \operatorname{Ad}\Gamma$$

coincides with the natural homomorphism

$$\Gamma \ltimes K \longrightarrow \Gamma \xrightarrow{\operatorname{Ad}} \operatorname{Ad}\Gamma$$

it follows that p is onto, and that $Z = \operatorname{Ker}(p)$ consists of elements $z = (\gamma_z, k_z)$ with $\gamma_z \in \operatorname{Ker}(\operatorname{Ad} : \Gamma \longrightarrow \operatorname{Ad}\Gamma)$ - the finite abelian center of Γ, while $\pi(\gamma_z) = k_z^{-1} \in \Lambda \cap K$. Observe that

- $K_Z := \{k_z \in K \mid z \in Z\}$ is a finite abelian group, isomorphic to Z;
- K_Z is normal in K, because $\{1\} \times K$ and Z are normal in $\Gamma \ltimes K$, so that

$$\{1\} \times K_Z = (\{1\} \times K) \cap Z \ \lhd\ \Gamma \ltimes K$$

- for each $\gamma \in \Gamma$, one has $\gamma^{-1}\gamma_z\gamma = \gamma_z$, so that $k_z^\gamma = k_z$, for each $z \in Z$.

Hence H_0 is described by the exact sequence

$$1 \longrightarrow Z \xrightarrow{j} \Gamma \ltimes K \xrightarrow{p} H_0 \longrightarrow 1$$

with $j(z) = (\gamma_z, k_z)$, where $z \mapsto \gamma_z$ is an embedding of Z in the finite abelian center of Γ; and $K_Z = \{k_z\}_{z\in Z}$ forms a normal subgroup in K whose elements are fixed by the action of Γ.

Clearly, if Γ has trivial center, then H_0 is just isomorphic to the skew-product $H_0 \cong \Gamma \ltimes K$.

The proof of Theorem A is completed. □

3. Proof of Theorems B and C

Outline of the proof. As in the proof of Theorem A, the essential step is to construct a continuous homomorphism $\Phi : H \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\, G)$. If $G \not\simeq \mathrm{SL}_2(\mathbb{R})$ the constructed Φ will satisfy (2.1); while in the case $G \simeq \mathrm{SL}_2(\mathbb{R})$ we shall only obtain that

$$\Phi \circ \pi : \Gamma \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\, G)$$

takes Γ to a lattice in $\mathrm{PSL}_2(\mathbb{R}) \subset \mathrm{Aut}\,(\mathrm{Ad}\, G)$, isomorphic to $\mathrm{Ad}\,\Gamma$.

The homomorphism Φ is constructed using the quasi-isometry groups. For any finitely generated group Γ, there is an associated quasi-isometry group $\mathrm{QI}\,(\Gamma)$, which we shall consider as an abstract group with no topology. The quasi-isometry group has the property that any uniform lattice embedding π of Γ in a locally compact group H, gives rise to a homomorphism (of abstract groups) $\Phi_\pi : H \longrightarrow \mathrm{QI}\,(\Gamma)$ (see subsection 3.1).

Consider the case where Γ is a uniform lattice in a simple Lie group G of rank one. Then Γ is quasi-isometric to the symmetric space $X = G/K$, which has strictly negative Riemannian curvature (X is real, complex, quaternionic hyperbolic space or the Cayley plane), in which case there exists an embedding

$$\beta : \mathrm{QI}\,(\Gamma) \longrightarrow \mathrm{Homeo}\,(\partial X)$$

in the group of homeomorphisms of the boundary ∂X of X. This gives an abstract homomorphism of groups $\Psi_\pi : H \longrightarrow \mathrm{Homeo}\,(\partial X)$

$$\Psi_\pi : H \xrightarrow{\Phi_\pi} \mathrm{QI}\,(\Gamma) \xrightarrow{\beta} \mathrm{Homeo}\,(\partial X)$$

which turns out to be *continuous* (Proposition 3.6). The given uniform lattice embedding $\sigma : \Gamma \longrightarrow G$ gives rise to the homomorphism of abstract groups Φ_σ : $\mathrm{Aut}\,(\mathrm{Ad}\, G) \longrightarrow \mathrm{QI}\,(\mathrm{Ad}\,\Gamma) \cong \mathrm{QI}\,(\Gamma)$, and to another continuous homomorphism

$$\Psi_\sigma : \mathrm{Aut}\,(\mathrm{Ad}\, G) \xrightarrow{\Phi_\sigma} \mathrm{QI}\,(\mathrm{Ad}\,\Gamma) \xrightarrow{\beta} \mathrm{Homeo}\,(\partial X)$$

which is known to form a *continuous isomorphic embedding* with a *continuous* inverse Ψ_σ^{-1}. The following is crucial

Claim 3.1. *The images of* $\Psi_\pi(H)$ *and* $\Psi_\sigma(\mathrm{Aut}\,(\mathrm{Ad}\,G))$ *in* $\mathrm{Homeo}\,(\partial X)$, *are locally compact groups, which contain* $\Psi_\pi(\Lambda)$ *and* $\Psi_\sigma(\mathrm{Ad}\,\Gamma)$ *as uniform lattices. From the construction one has*

$$\Psi_\pi(\pi(\gamma)) = \Psi_\sigma(\mathrm{Ad}\,\gamma), \qquad (\gamma \in \Gamma)$$

Moreover,

- *If* $G \not\simeq \mathrm{SL}_2(\mathbb{R})$ *then* $\Psi_\pi(H) \subseteq \Psi_\sigma(\mathrm{Aut}\,(\mathrm{Ad}\,G))$ *and*
$$\Phi := \Psi_\sigma^{-1} \circ \Psi_\pi \;:\; H \longrightarrow \mathrm{Homeo}\,(\partial X) \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\,G)$$
is a continuous homomorphism, satisfying (1.1).
- *If* $G \simeq \mathrm{SL}_2(\mathbb{R})$, *then for some* $f \in \mathrm{Homeo}\,(\partial X)$, *one has* $f^{-1}\,\Psi_\pi(H)\,f \subseteq \Psi_\sigma(\mathrm{Aut}\,(\mathrm{Ad}\,G))$, *in which case the continuous homomorphism*
$$\Phi := \Psi_\sigma \circ f^{-1}\Psi_\pi\, f : H \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\,G)$$
maps $\Lambda = \pi(\Gamma)$ *onto a subgroup* $\Phi(\Lambda) \subset \mathrm{Aut}\,(\mathrm{Ad}\,G)$, *which is isomorphic (but not necessarily conjugate) to* $\mathrm{Ad}\,\Gamma$.

¿From this claim the proof of Theorems B and C can be completed as in subsections 2.3 and 2.4.

In the case of a *uniform* lattice embedding of a *non-uniform* lattice Γ in G, we rely on recent deep results of R. Schwartz, which identify the quasi-isometry group $\mathrm{QI}\,(\Gamma)$ with the commensurator of Γ in $\mathrm{Aut}\,(\mathrm{Ad}\,G)$. This commensurator is countable, and it readily follows that H has a finite index subgroup, which is an almost skew-product of Γ with a compact group.

3.1. **Quasi-isometries.** Recall the notion of quasi-isometries (see [Gr], [GP] for detailed discussions): a map $q : X_1 \longrightarrow X_2$ between proper metric spaces (X_1, d_1) and (X_2, d_2) is said to be a *quasi-isometric embedding* if there exist constants M, A such that for all $x, y \in X_1$:

$$\frac{1}{M} \cdot d_1(x,y) - A \le d_2(q(x),\, q(y)) \le M \cdot d_1(x,y) + A \tag{3.1}$$

Sometimes such q is called an (M, A)-quasi-isometric embedding to emphasize that (3.1) is satisfied for specific M and A.

A quasi-isometric embedding $q : X_1 \longrightarrow X_2$ with an image $q(X_1)$ which is within bounded distance from all of X_2, is called a *quasi-isometry*, and the spaces (X_1, d_1), (X_2, d_2) are said to be *quasi-isometric*. Two quasi-isometries (or quasi-isometric embeddings) $q, q' : X_1 \longrightarrow X_2$ are said to be *equivalent* (notation: $q \sim q'$) if $d_2(q(x), q'(x))$ is uniformly bounded on X_1.

Now consider the collection of all self quasi-isometries of a fixed proper metric space (X, d). It forms a semi-group with respect to composition (which respects the equivalence relation $\sim$). Modulo this relation, the semi-group of actual quasi-isometries becomes a group, called the *quasi-isometry group* of (X, d) and denoted by $\mathrm{QI}\,(X, d)$. Quasi-isometric spaces have isomorphic quasi-isometry groups. Therefore, for a finitely generated group Γ it makes sense to talk about its quasi-isometry group $\mathrm{QI}\,(\Gamma)$ without specifying any particular left invariant word metric. Note also, that the isometric left Γ-action on itself (with respect to some/any left-invariant word metric) defines a natural homomorphism

$$\rho : \Gamma \longrightarrow \mathrm{QI}\,(\Gamma)$$

3.2. **Uniform lattice embeddings and quasi-isometries.** We claim that the notions of quasi-isometries appear naturally in our situation, where a given finitely generated group Γ is embedded as a *uniform lattice* in an (unknown) locally compact group H.

Construction 3.2. *Choose some open subset $E \subset H$ with compact closure, such that $H = \cup_{\gamma\in\Gamma} \pi(\gamma)E$, and fix some (typically discontinuous) function $p : H \longrightarrow \Gamma$, satisfying*

$$h \in \pi(p(h))\, E \qquad (h \in H)$$

With this choice of E and p define a family of maps $\{q_h : \Gamma \longrightarrow \Gamma\}_{h\in H}$ by the rule

$$q_h(\gamma) := p(h\pi(\gamma))$$

If E is a neighborhood of the unit e, we can choose p with $p(\pi(\gamma)) = \gamma$, $\gamma \in \Gamma$, so that

$$q_\gamma(\gamma') = \gamma\gamma' \tag{3.2}$$

Lemma 3.3. *Let E, p and $\{q_h\}_{h\in H}$ be as above. Then*

(a) *Each $q_h : \Gamma \longrightarrow \Gamma$ is a quasi-isometry of Γ and its equivalence class $[q_h] \in \mathrm{QI}\,(\Gamma)$ depends only on $h \in H$ (and not on the choice of E and p).*

(b) *The map $\Phi_\pi : H \longrightarrow \mathrm{QI}\,(\Gamma)$, given by $\Phi_\pi(h) = [q_h]$, is a homomorphism of (abstract) groups, such that the composition*

$$\Gamma \xrightarrow{\pi} H \xrightarrow{\Phi_\pi} \mathrm{QI}\,(\Gamma)$$

coincides with the standard homomorphism $\rho : \Gamma \longrightarrow \mathrm{QI}\,(\Gamma)$.

(c) *$\{q_h\}_{h\in H}$ are (M, A)-quasi-isometries for some fixed M and A, depending just on E, p, and independent of $h \in H$.*

Remarks 3.4. (a) The collection of (M, A)-quasi-isometries $\{q_h\}_{h\in H}$ is not closed under composition: $q_{h_1} \circ q_{h_2}$ is equivalent, but typically is not equal, to $q_{h_1h_2}$.

(b) If H carries a left invariant metric quasi-isometric to some word metric on Λ, then Lemma 3.3 follows just from the standard homomorphism

$$H \longrightarrow \mathrm{QI}\,(H) \cong \mathrm{QI}\,(\Lambda) \cong \mathrm{QI}\,(\Gamma)$$

Proof. Fix some finite set S of generators for Γ, and let d denote the corresponding left invariant word metric on Γ. Observe that, for any two compact sets $Q, Q' \subset H$ there is at most finite number of $\gamma \in \Gamma$, so that $Q\,\pi(\gamma)\,\cap\, Q' \neq \emptyset$. This implies that for any sets $E, E' \subset H$ with compact closures, there exists an integer $A = A(E, E')$, so that

$$\pi(\gamma)\,E\,\cap\,\pi(\gamma')\,E' \neq \emptyset \quad \Longrightarrow \quad d(\gamma, \gamma') \leq A \tag{3.3}$$

In particular, if q_h and q'_h are constructed from E, p and E', p' respectively, then for some $A = A(E, E')$ any fixed $h \in H$ and all $\gamma \in \Gamma$:

$$h\,\pi(\gamma) \in \pi(q_h(\gamma))\,E\,\cap\,\pi(q'_h(\gamma))E' \quad \Longrightarrow \quad d(q_h(\gamma), q'_h(\gamma)) \leq A \tag{3.4}$$

Denote by B_n the n-ball in Γ centered at the origin. The set $E\,B_1 = \cup_{\gamma \in B_1} E\,\pi(\gamma)$ has a compact closure in H, and therefore $E\,B_1 \subset B_M\,E$ for some sufficiently large integer $M = M(E)$. Then for all $n > 1$, we have:

$$E\,\pi(B_n) \subset \pi(B_M)\,E\,\pi(B_{n-1}) \subset \cdots \subset \pi(B_{M\cdot n})E$$

Take some fixed $h \in H$ and any $\gamma_1, \gamma_2 \in \Gamma$. Denoting $n = d(\gamma_1, \gamma_2)$, we have

$$\begin{array}{rcl} h\,\pi(\gamma_1) & \in & q_h(\gamma_1)\,E \qquad \text{and} \qquad h\,\pi(\gamma_2) \in q_h(\gamma_2)\,E \\ h\,\pi(\gamma_2) & = & h\,\pi(\gamma_1\,(\gamma_1^{-1}\gamma_2)) \in \pi(q_h(\gamma_1))\,E\,\pi(\gamma_1^{-1}\gamma_2) \\ & & \subset \pi(q_h(\gamma_1))\,E\,\pi(B_n) \subset \pi(q_h(\gamma_1)B_{M\cdot n})E \end{array}$$

Thus $\pi(q_h(\gamma_2))\,E$ intersects $\pi(q_h(\gamma_1)B_{M\cdot n})E$, and using (3.3), we deduce that

$$d(q_h(\gamma_1), q_h(\gamma_2)) \leq M \cdot d(\gamma_1, \gamma_2) + A$$

Similar arguments show the other properties of quasi-isometries. Hence q_h is a quasi-isometry of Γ. Its class $[q_h] \in \mathrm{QI}\,(\Gamma)$, which we shall denote by $\Phi_\pi(h)$, does not depend on the choice of E, p as (3.4) shows. This proves (a).

Next take $h_1, h_2 \in H$ and observe that for each $\gamma \in \Gamma$, $h_i\,\pi(\gamma) \in \pi(q_{h_i}(\gamma))\,E$ for $i = 1, 2$, and therefore

$$(h_1\,h_2)\,\pi(\gamma) \in h_1\,\pi(q_{h_2}(\gamma))\,E \subset \pi(q_{h_1}(q_{h_2}(\gamma)))\,E \cdot E$$

Since $E' := E \cdot E$ has a compact closure in H, (3.3) implies that

$$q_{h_1} \circ q_{h_2} \sim q_{h_1 h_2} \qquad \text{i.e.} \qquad \Phi_\pi(h_1) \cdot \Phi_\pi(h_2) = \Phi(h_1 h_2)$$

Similarly $q_h \circ q_{h^{-1}} \sim \mathrm{Id}_\Gamma$, so that Φ_π is a homomorphism of abstract groups. Note also, that since Φ_π does not depend on the choice of E, p, we could choose them so that (3.2) holds. This shows that $\Phi_\pi \circ \pi = \rho$. Hence (b) is proved.

Statement (c) follows from the proof of (a) above, where M and A were constructed independently of $h \in H$. $\square$

In our particular situation Γ is a lattice in a simple Lie group G. Quasi-isometries of lattices have been thoroughly analyzed by several people over a series of many papers, and by now a comprehensive understanding of these objects has been achieved. The reader is referred to the survey papers by Gromov and Pansu [GP] and Farb [Fa] for details and further references for this beautiful theory. In what follows we shall only briefly mention some of the facts needed for the proof of Theorems B and C. We need to consider several cases.

3.3. **Γ is a uniform lattice in a simple G of rank one.** A uniform lattice Γ in G is quasi-isometric to G itself and to its symmetric space $X = G/K$, so that $\mathrm{QI}(\Gamma) \cong \mathrm{QI}(X)$, where the latter isomorphism is induced by some quasi-isometries

$$X \xrightarrow{j'} \Gamma \xrightarrow{j} X \quad \text{with} \quad j' \circ j \sim \mathrm{Id}_X,\ j \circ j' \sim \mathrm{Id}_\Gamma \tag{3.5}$$

It will be convenient to define $j : \Gamma \longrightarrow X$ to be $j(\gamma) := \gamma\, x_*$ where $x_* \in X$ is some fixed base point. Denote by $J : \mathrm{QI}(\Gamma) \cong \mathrm{QI}(X)$ the isomorphism corresponding to these j, j', which will be fixed for the rest of the discussion.

The group $\mathrm{Isom}(X)$ of isometries of X is (continuously) isomorphic to $\mathrm{Aut}(\mathrm{Ad}\, G)$, and it contains $\mathrm{Ad}\,\Gamma$ as a uniform lattice. It is well known (e.g. see [GP] 3.3.A') that the natural $\mathrm{Isom}(X)$-action on X gives an *injective* homomorphism $\mathrm{Isom}(X) \longrightarrow \mathrm{QI}(X)$, and therefore $\mathrm{Isom}(X)$ can be identified (as an abstract group) with a subgroup of $\mathrm{QI}(X)$. Identifying $\mathrm{QI}(\Gamma)$ with $\mathrm{QI}(X)$ via J, we have

$$\Phi_\sigma(\gamma) = \mathrm{Ad}\,\gamma, \qquad (\gamma \in \Gamma) \tag{3.6}$$

due to our choice of the quasi-isometry $j : \Gamma \longrightarrow X$ above. (Recall, that we denote by $\sigma : \Gamma \longrightarrow G$ the inclusion).

The symmetric space X of a rank one simple Lie group is a simply connected, complete Riemannian manifold with uniformly bounded from zero negative curvature. The crucial, for our purposes, property of such spaces is

Lemma 3.5 (Morse). *Fix a base point $x_* \in X$. Then any quasi-geodesic ray (i.e. a quasi-isometric embedding $\rho : [0, \infty) \to X$) is within bounded distance from a unique x_*-based geodesic ray (i.e. an isometric embedding $R : [0, \infty) \to X$ with $R(0) = x_*$). Moreover the distance*

$$\mathrm{dist}(\rho, R) := \sup\{\mathrm{dist}(\rho(t), R(t)) \mid t \geq 0\}$$

is bounded in terms of the quasi-isometric constants of ρ and the distance $\mathrm{dist}(\rho(0), x_*)$.

Morse Lemma gives rise to a natural identification between the *visual boundary* ∂X of X (which can be defined as the space of all x_*-based *geodesic rays*), with the *ideal boundary* (which consists of the equivalence classes of all *quasi-geodesic rays* in X, modulo the relation $\rho \sim \rho'$ if $\mathrm{dist}(\rho(t), \rho'(t))$ is bounded). The natural action of $\mathrm{QI}(X)$ on the (ideal) boundary, defines an *injective* homomorphism (e.g. see [GP] 3.5):

$$\beta : \mathrm{QI}(X) \longrightarrow \mathrm{Homeo}(\partial X)$$

Proposition 3.6. *The homomorphism*

$$\Psi_\pi : H \xrightarrow{\Phi_\pi} \mathrm{QI}\,(\Gamma) \cong \mathrm{QI}\,(X) \xrightarrow{\beta} \mathrm{Homeo}\,(\partial X) \tag{3.7}$$

is continuous with respect to the uniform convergence in $\mathrm{Homeo}\,(\partial X)$. *The image* $\Psi_\pi(H)$ *is a locally compact subgroup of* $\mathrm{Homeo}\,(\partial X)$, *containing* $\Psi_\pi(\Lambda)$ *as a uniform lattice. Moreover,* $\Psi_\pi(\pi(\gamma)) = \Psi_\sigma(\mathrm{Ad}\,\gamma)$ *for* $\gamma \in \Gamma$, *i.e.* $\Psi_\pi(\pi(\gamma))$ *acts as* γ *on* ∂X.

Proof. Continuity of Ψ_π. Fix a base point $x_* \in X$, and identify boundary points $\xi \in \partial X$ with the x_*-based geodesic rays $R_\xi : [0,\infty) \to X$, which approach ξ at infinity. The boundary ∂X is homeomorphic to the unit sphere $\{R_\xi(1)\}_{\xi \in \partial X}$ in X centered at x_*. We have to show that if $h_n \to e$ in H, then

$$\mathrm{dist}(R_{\Psi_\pi(h_n)\xi}(1), R_\xi(1)) \to 0$$

as $n \to \infty$, uniformly in $\xi \in \partial X$. Using the divergence of geodesic rays in X, it suffices to show that there is a $K < \infty$, such that for an arbitrarily large T, there is an integer $N(T)$, s.t.

$$\mathrm{dist}(R_{\Psi_\pi(h_n)\xi}(T), R_\xi(T)) \leq K \qquad (\xi \in \partial X,\ n \geq N(T)) \tag{3.8}$$

Let $\{q_h\}_{h\in H}$ be (M,A)-quasi-isometries of Γ, as in the Construction 3.2 and Lemma 3.3.(c). The quasi-isometries

$$q'_h := j' \circ q_h \circ j \ :\ X \longrightarrow X, \qquad (h \in H)$$

are (M',A')-quasi-isometries of X, where M' and A' are some constants depending on M, A, j and j'.

Applying such a quasi-isometry q'_h to an x_*-based ray R_ξ, we obtain an (M',A')-quasi-geodesic ray $\rho_{h,\xi} := q'_h \circ R_\xi$. By Morse Lemma, there exists a unique x_*-based ray R_ζ which is within bounded distance from $\rho_{h,\xi}$. In fact, by the definition (3.7), the ideal end point ζ is just $\Psi_\pi(h)\xi$. Moreover, Morse Lemma asserts that the distance

$$D := \sup\{\mathrm{dist}(R_{\Psi_\pi(h)\,\xi}(t), \rho_{h,\xi}(t)) \mid t \geq 0\} \tag{3.9}$$

is bounded in terms of M' and A' (and $\mathrm{dist}(\rho_{h,\xi}(0), x_*)$, which is bounded by A' as well).

Now, take a sequence $h_n \to e$ in H. First consider the picture in Γ. For any bounded (i.e. finite) set $F \subset \Gamma$, we have $h_n\pi(\gamma) \to \pi(\gamma)$ as $n \to \infty$, uniformly on F. At the same time

$$h_n\pi(\gamma) \in \pi(q_{h_n}(\gamma))\, E$$

Thus $\pi(q_{h_n}(\gamma)^{-1}\gamma) \in E$, $\gamma \in F$, for all $n \geq N(T)$. This places a bound $B = B(E)$, which is independent of T, on the displacements

$$d(q_{h_n}(\gamma), \gamma) \leq B, \qquad (\gamma \in F,\ n \geq N(T))$$

Applying the quasi-isometries $X \xrightarrow{j} \Gamma \xrightarrow{j'} X$, we obtain that there exists a constant B', which depends only on B, j, j', such that

$$\mathrm{dist}(q'_{h_n}(x), x) \leq B' \qquad (x \in B_T,\ n \geq N(T))$$

This, together with (3.9), gives the condition (3.8) for the continuity of Ψ.

Note that 3.6 means that for $\gamma \in \Gamma$

$$\Psi_\pi(\pi(\gamma)) = \beta(\mathrm{Ad}\,\gamma) = \Psi_\sigma(\mathrm{Ad}\,\gamma)$$

So the $\Psi_\pi(\Lambda)$ action on ∂X coincides with the Γ-action (factored through $\mathrm{Ad}\,\Gamma$). Hence $\Psi_\pi(\Lambda)$ is discrete in $\mathrm{Homeo}\,(\partial X)$.

Now take a compact subset $Q \subset H$ with $Q\,\Lambda = H$. By continuity, $\Psi_\pi(Q)$ is compact and its translations by the discrete group $\Psi_\pi(\Lambda)$ give all of $\Psi_\pi(H)$. This easily implies that $\Psi_\pi(H)$ is locally compact (in the topology of the uniform convergence) and $\Psi_\pi(\Lambda)$ is a uniform lattice in $\Psi_\pi(H)$. □

Next consider the different families of rank one groups G.

Case $G \simeq \mathrm{SL}_2(\mathbb{R})$ (Theorem C). Let H be a locally compact group, admitting an uniform lattice embedding $\pi : \Gamma \longrightarrow H$, where Γ is a uniform lattice in $\Gamma \subset G \simeq \mathrm{SL}_2(\mathbb{R})$ (so $\sigma : \Gamma \longrightarrow G$ is a uniform lattice embedding as well). Applying Proposition 3.6 one obtains a locally compact group

$$L = \Psi_\pi(H) \subset \mathrm{Homeo}\,(S^1)$$

where the circle S^1 is the boundary of the hyperbolic plane. Moreover $\Psi_\pi(\Lambda)$, where $\Lambda = \pi(\Gamma)$, forms a uniform lattice in L, and it coincides with $\mathrm{Ad}\,\Gamma \cong \Psi_\sigma(\Gamma) \subset \mathrm{Homeo}\,(S^1)$. The latter group acts minimally and strongly proximally on S^1, hence so does all of L. Applying Corollary 4.2 from the Appendix we deduce that either $L \subset \mathrm{Homeo}\,(S^1)$ itself is discrete, or L is conjugate (in $\mathrm{Homeo}\,(S^1)$) to either $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{Ad}\,G$ or $\mathrm{Aut}\,\mathrm{PSL}_2(\mathbb{R})$ (in our present notation the corresponding subgroups of $\mathrm{Homeo}\,(S^1)$ are $\Psi_\sigma(\mathrm{PSL}_2(\mathbb{R}))$ and $\Psi_\sigma(\mathrm{Aut}\,\mathrm{PSL}_2(\mathbb{R}))$).The latter cases directly give Claim 3.1. In the case of a discrete $L \subseteq \mathrm{Homeo}\,(S^1)$, we should show that L is conjugate to a lattice Γ_* in $\mathrm{Aut}\,\mathrm{PSL}_2(\mathbb{R}) \cong \Psi_\sigma(\mathrm{Aut}\,(\mathrm{Ad}\,G))$. Note that since $\Psi_\pi(\Lambda) = \Psi_\sigma(\mathrm{Ad}\,\Gamma)$ forms a (uniform) lattice in the discrete L, it is just a finite index subgroup of L. A discrete group $L \subset \mathrm{Homeo}\,(S^1)$ which contains a Fuchsian group as a finite index subgroup, is conjugate to a Fuchsian group. This follows, for example from the Gabai's theorem [Ga] about convergence groups on the circle.

This proves Claim 3.1 and Theorem C follows.

Case $G \simeq \mathrm{SO}(n, 1)$, $n > 2$. For this and the following cases of Γ being uniform in a rank one group $G \not\simeq \mathrm{SL}_2(\mathbb{R})$, we shall use some known facts about the conformal structures on ∂X (ideas which are going back to Mostow).

For $G \simeq \mathrm{SO}(n,1)$, $n > 2$, the the boundary $\partial X \cong S^{n-1}$ of the real hyperbolic space $X = \mathbf{H}^n$, $n > 2$, carries a natural conformal structure (e.g. see [GP] 3.6). Any quasi-isometry class $\phi \in \mathrm{QI}\,(X)$ defines a quasi-conformal map $\beta(\phi)$ of S^{n-1} with the quasi-conformal constant bounded in terms of the quasi-isometric constant M of ϕ. In fact $\beta : \mathrm{QI}\,(X) \longrightarrow \mathcal{QC}(\partial X)$ is an isomorphism (of abstract groups). The subgroup of isometries $\mathrm{Isom}\,(X)$ is mapped isomorphically by β onto the group $\mathcal{C}(\partial X)$ of conformal maps. Hence, we have

$$\Psi_\pi(H) \subseteq \mathcal{QC}(\partial X) \subset \mathrm{Homeo}\,(\partial X), \qquad \Psi_\sigma(\mathrm{Aut}\,(\mathrm{Ad}\,G)) = \mathcal{C}(\partial X)$$

The fact that all $\{\Phi_\pi(h)\}_{h\in H}$ can be represented by quasi-isometries $\{q_h\}_{h\in H}$ with a uniform bound on the quasi-isometric constant M (Lemma 3.3.(c)) means that $\Psi_\pi(H)$ is *uniformly quasi-conformal* subgroup of $\mathcal{QC}(S^{n-1})$. This group has an additional important property (introduced by Tukia): its action on the *space of distinct triples*

$$\partial^3 X := \{(\xi_1, \xi_2, \xi_3) \in \partial X \times \partial X \times \partial X \mid \xi_i \neq \xi_j,\ i \neq j\}$$

is *cocompact*. This follows from the fact that already the action of $\Psi_\sigma(\mathrm{Ad}\,\Gamma) = \Psi_\pi(\Lambda) \subset \Psi_\pi(H)$ on the triple space is cocompact. Tukia [Tu] proved, that any uniformly quasi-conformal subgroup of $\mathcal{QC}(S^{n-1})$, $n > 2$, whose action on the triple space is cocompact, is conjugate into $\mathcal{C}(S^{n-1})$. Hence, there exists an $f \in \mathcal{QC}(\partial X)$, so that

$$f^{-1}\,\Psi_\pi(H)\,f \subset \mathcal{C}(\partial X) = \Psi_\sigma(\mathrm{Aut}\,(\mathrm{Ad}\,G))$$

However, no conjugation is really needed. Since both $\Psi_\pi(\Lambda) = \Psi_\sigma(\mathrm{Ad}\,\Gamma)$ and $f^{-1}\Psi_\pi(\Lambda)f$ are in $\mathcal{C}(\partial X)$, Mostow's rigidity implies that for some $g \in \mathcal{C}(\partial X)$, one has

$$(g \circ f)^{-1}\,\Psi_\pi(\lambda)\,(g \circ f) = \Psi(\lambda), \qquad (\lambda \in \Lambda)$$

In other words, $g \circ f$ is a Λ-equivariant continuous (actually quasi-conformal) map of ∂X to itself. However, the Λ-action (i.e. the $\mathrm{Ad}\,\Gamma$-action) on ∂X is known to be minimal and proximal. It is easy to see that such actions have no non-trivial equivariant continuous maps. Hence $f = g^{-1} \in \mathcal{C}(\partial X)$, and one has $\Psi_\pi(H) \subseteq \mathcal{C}(\partial X) = \Psi_\sigma(\mathrm{Aut}\,(\mathrm{Ad}\,G))$. This proves Claim 3.1 for uniform lattices Γ in $G \simeq \mathrm{SO}(n,1)$, $n > 2$.

Case $G \simeq \mathrm{SU}(n,1)$, $n > 1$. Here the symmetric space X is the complex hyperbolic spaces $\mathbf{CH}^n$, and the appropriate conformal structure on ∂X corresponds to the Carnot-Caratheodory metric. As in the previous case, the group $\Psi_\pi(H)$ is uniformly quasi-conformal and its action on the triple space is cocompact. Following Tukia, Chow [Ch] proved that any such subgroup is conjugate into $\mathcal{C}(\partial X)$. As in the previous case, the fact that $\Psi_\pi(\Lambda)$ is already in $\mathcal{C}(\partial X)$, together with Mostow Rigidity, imply that no conjugation is needed. Thus

$$\Psi_\pi(H) \subseteq \mathcal{C}(\partial X) = \Psi_\sigma(\mathrm{Aut}\,(\mathrm{Ad}\,G))$$

and Claim 3.1 is proved.

Case $G \simeq \mathrm{Sp}(n,1)$ or F_4^{-20}. The symmetric spaces in these cases are the quaternionic hyperbolic space $X = \mathbf{QH}^n$ (corresponding to $G \simeq \mathrm{Sp}(n,1)$) and the Cayley plane $X = \mathbf{CaH}^2$ (corresponding to $G \simeq \mathrm{F}_4^{-20}$). These spaces are very rigid: Pansu [Pa] proved that for these spaces $\mathcal{QC}(\partial X) = \mathcal{C}(\partial X)$, while the latter again coincides with $\Psi_\sigma(\mathrm{Aut}\,(\mathrm{Ad}\,G))$, and Claim 3.1 follows.

Hence Claim 3.1 is proven and the proof of Theorem B can be completed exactly as in the Subsections 2.3 and 2.4 (the compactness of the kernel $\mathrm{Ker}(\Phi : H \longrightarrow \mathrm{Aut}\,(\mathrm{Ad}\,G)$ in these cases can be established by geometric arguments).

3.4. **Γ is non-uniform in $G \not\simeq \mathrm{SL}_2(\mathbb{R})$.** Non-uniform irreducible lattices Γ in semisimple Lie groups $G \not\simeq \mathrm{SL}_2(\mathbb{R})$ have only "algebraic" quasi-isometries, i.e. the quasi-isometry group $\mathrm{QI}\,(\Gamma)$ coincides with the commensurator group $\mathrm{Comm}\,(\Gamma)$ of Γ in $\mathrm{Isom}\,(X) \cong \mathrm{Aut}\,(\mathrm{Ad}\,G)$, defined by:

$$\mathrm{Comm}\,(\Gamma) := \{g \in \mathrm{Aut}\,(\mathrm{Ad}\,G) \mid [\Gamma : g^{-1}\Gamma g \cap \Gamma] < \infty\}$$

This remarkable fact was first discovered by Schwartz [S1]

The commensurator $\mathrm{Comm}\,(\Gamma)$ is a countable (dense) subgroup of $\mathrm{Aut}\,(\mathrm{Ad}\,G)$. Clearly, $\Phi_\pi^{-1}(\{e\})$ is a measurable subset of H with a finite but positive Haar measure:

$$0 < c := m_H(\Phi_\pi^{-1}(\{e\})) \le m_H(H/\Lambda) < \infty$$

Hence, the push forward measure $(\Phi_\pi) * m_H$ on $\mathrm{Aut}\,(\mathrm{Ad}\,G)$ is an atomic measure, equally distributed on some subgroup $\Gamma_* \supset \mathrm{Ad}\,\Gamma$. Moreover, $\mathrm{Ad}\,\Gamma$ has finite index in Γ_*, for

$$c \cdot [\Gamma_* : \mathrm{Ad}\,\Gamma] = m_H(H/\Gamma) < \infty$$

Hence Γ_* is a lattice in $\mathrm{Aut}\,(\mathrm{Ad}\,G)$, and taking $H_0 := \Phi_\pi^{-1}(\mathrm{Ad}\,\Gamma)$ we conclude the proof as in Subsection 2.4, Case (2).

The proof of Theorem B is now completed. □

3.5. **An Exotic Example.** Let Γ be a surface group of a compact Riemann surface of genus g. As an abstract group Γ has the presentation

$$\Gamma = \langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid [a_1,b_1][a_2,b_2]\cdots[a_g,b_g] = 1\rangle$$

Such Γ can be embedded as a uniform lattice in $G = \mathrm{SL}_2(\mathbb{R})$. Let $\tilde{G}$ be the universal covering of G

$$1 \longrightarrow Z \longrightarrow \tilde{G} \xrightarrow{p} G \longrightarrow 1$$

with $Z \cong \mathbb{Z}$. The preimage $\Gamma' = p^{-1}(\Gamma)$ is a (non-trivial) central extension of Γ by Z, which can be presented as

$$\Gamma' = \langle a_1, \ldots, a_g, b_1, \ldots, b_g, z \mid [a_1,b_1][a_2,b_2]\cdots[a_g,b_g] = z\rangle$$

where z generates the center Z. The crucial observation is that Γ' is linear. Indeed, if ρ is a the Γ'-representation in the Heisenberg group

$$\rho(a_i)=\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix},\qquad \rho(b_i)=\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix},\qquad \rho(z)=\begin{pmatrix}1&0&g\\0&1&0\\0&0&1\end{pmatrix}$$

then $\mathrm{Ad}\,\times\rho$ is a faithful linear Γ'-representation. (I would like to thank Gregory Soifer for pointing this out to me).

Hence $\tilde{\Gamma}$ is residually finite. Let K' be a profinite completion of Γ'. The center $\overline{Z}$ of K' is a profinite completion of $Z\cong\mathbb{Z}$.

Now let G' be defined as the quotient of the direct product $\overline{Z}\times\tilde{G}$ by the diagonal embedding of Z. This embedding is discrete (due to the $\tilde{G}$-coordinate), so that G' becomes a locally compact group, which contains $\tilde{G}$ as a dense subgroup. Hence G' is connected, although it is neither locally connected, nor path-connected. The center of G' is a compact (profinite) abelian group, isomorphic to $\overline{Z}$. Observe, that Γ' embeds in G', via $\Gamma'\subset\tilde{G}\subset G'$, however its image is no longer discrete.

Next, consider the *locally compact* group $H:=(G'\times K')/\overline{Z}$, where $\overline{Z}$ is embedded diagonally in the centers of G' and K'. Denoting by π' the diagonal embedding of Γ' into $G'\times K'$, one observes that the homomorphism

$$\Gamma'\xrightarrow{\pi'}G'\times K'\longrightarrow H$$

has Z as its kernel, and therefore, can be written as

$$\Gamma'\xrightarrow{p}\Gamma\xrightarrow{\pi}H$$

where π is *injective*. We claim that $\pi(\Gamma)$ forms a *lattice* in H. Indeed, H contains two closed commuting subgroups, isomorphic to G' and K' (these are the projections of $G'\times\{e_{K'}\}$ and $\{e_{G'}\}\times K'$, respectively), which we shall again denote by G' and K', and $G',K'\lhd H$ intersect along their common center, which is isomorphic to $\overline{Z}$. Note that if $\Phi:H\longrightarrow H/K'\cong G$ denotes the projection, then

$$\Gamma\xrightarrow{\pi}H\xrightarrow{\Phi}G$$

coincides with the embedding $\Gamma\subset G$. Since $K'=\mathrm{Ker}(\Phi:H\longrightarrow G)$ is compact, we conclude that π is an embedding of Γ as a lattice in H.

Any *connected* subgroup of H, which projects onto G, has to contain $\tilde{G}\subset G'$, and any *closed connected* one contains G', which has a compact, but not finite, abelian center $\overline{Z}$.

Remark 3.7. The above construction does not work for other simple Lie groups G, which have universal covering with an infinite center. It is known that for such G, there does not exist a residually finite $\Gamma'\subset\tilde{G}$ which projects onto $\Gamma\subset G$. However, it is not clear to the author whether a risidually finite Γ' projecting on Γ exists in $G'=\tilde{G}\times\overline{Z(G)}/Z(G)$.

Remark 3.8. A construction similar to (but somewhat simpler than) the above one gives an embedding of a surface group Γ in a locally compact group $H \cong G' \times K'/Z$, where G' is a simple Lie group with a finite center Z, with G' being locally isomorphic but not isomorphic to G.

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4. Appendix: Locally Compact subgroups of Homeo(S^1)

In this appendix we consider subgroups of Homeo(S^1), which are locally compact with respect to the induced topology of uniform convergence, focusing on non discrete groups. We shall confine the discussion to orientation preserving case, namely locally compact subgroups of $\mathrm{Homeo}_+(S^1)$. The main example to have in mind is $\mathrm{PSL}_2(\mathbb{R})$, which acts (faithfully) on $S^1 = \mathbb{R} \cup \{\infty\}$. Indeed the standard $\mathrm{SL}_2(\mathbb{R})$-action by fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \; : \; x \;\mapsto\; \frac{a\,x+b}{c\,x+d} \tag{4.1}$$

factors through $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\pm I$. Hereafter, we consider $\mathrm{PSL}_2(\mathbb{R})$ as a fixed subgroup of $\mathrm{Homeo}_+(S^1)$ and refer to it as *the projective* subgroup of $\mathrm{Homeo}_+(S^1)$. The *affine subgroup* will refer to group of transformations $x \mapsto e^t\,x+b$, corresponding to $c=0$ in (4.1); and the *translation group* will refer to the subgroup $x \mapsto x+b$, corresponding to $c=0$ and $|a|=|b|=1$ in (4.1). Clearly, the affine group fixes the point at infinity $x_\infty \in S^1$. The image of $\mathrm{SO}_2 \subset \mathrm{SL}_2(\mathbb{R})$ in $\mathrm{PSL}_2(\mathbb{R})$ is still isomorphic to SO_2. This group acts simply transitively on S^1 by "rotation". Fixing some SO_2-invariant metric on S^1, this *rotation* action becomes isometric.

Theorem 4.1. *Let $G \subset \mathrm{Homeo}_+(S^1)$ be a locally compact group. Then one of the following mutually excluding possibilities occurs:*

(D) *G is a discrete subgroup of* $\mathrm{Homeo}_+(S^1)$.

(R) *G is conjugate in* $\mathrm{Homeo}_+(S^1)$ *to the rotation group* SO_2.

(P) *G is a simple Lie group locally isomorphic to* $\mathrm{PSL}_2(\mathbb{R})$*; the projection* $\mathrm{Ad} : G \longrightarrow \mathrm{PSL}_2(\mathbb{R})$ *is realized by a semi-conjugacy in* $\mathrm{Homeo}_+(S^1)$, *i.e. there exists an n-to-1 covering map $f : S^1 \longrightarrow S^1$ so that*

$$f \circ g = \mathrm{Ad}\,g \circ f, \qquad (g \in G)$$

(S) *The connected component G^0 of G is a two step solvable, or an abelian group; it embeds into a finite product $\mathbb{R}^k \times (\mathbb{R} \ltimes \mathbb{R})^l$; G^0-orbits consist of open intervals I_i, $1 \le i \le k$ (where $0 < k \le \infty$); while G^0 fixes pointwise the set $S^1 \setminus (\cup I_i)$; for each $1 \le i \le k$, there is a homeomorphism $f_i : \mathbb{R} \longrightarrow I_i$ s.t. $f_i^{-1} G^0 f_i$-acts $\mathbb{R}$ through one of the factors by translations ($\mathbb{R}$-factor) or by affine transformations ($\mathbb{R} \ltimes \mathbb{R}$-factor). The factor group G/G^0 is totally disconnected, it acts faithfully by permutations of the intervals I_i preserving their cyclic ordering.*

Recall that an action of a group G by homeomorphisms on a compact space X is said to be *minimal* if all orbits Gx are dense in X; and *strongly proximal* if for any probability measure μ on X the orbit $G\mu$ contains a point measure in its closure. In the paper we have used the following

Corollary 4.2. *Let $G \subset \mathrm{Homeo}\,(S^1)$ be a locally compact group, acting minimally and strongly proximally on the circle S^1. Then either G is discrete or G is conjugate in $\mathrm{Homeo}\,(S^1)$ to $\mathrm{PSL}_2(\mathbb{R})$, or to the index two covering of $\mathrm{PSL}_2(\mathbb{R})$ which is isomorphic to $\mathrm{Aut}\,\mathrm{PSL}_2(\mathbb{R})$.*

Proof. Observe the following general fact: if a group $G \subseteq \mathrm{Homeo}\,(X)$ acts minimally and strongly proximally on a compact space X, then G has no amenable normal subgroups. Indeed, if $A \lhd G$ is amenable, then the set V_A of all A-invariant probability measures on X is a non-empty convex compact (with respect to $C(X)^*$-topology) set, which is G-invariant. By strong proximality it contains point measures, and by minimality all point measures δ_x are in V_A. Hence A acts trivially, and therfore A itself is trivial.

Let G be as in the corollary, and $G_+ = G \cap \mathrm{Homeo}_+(S^1)$ - the orientation preserving subgroup (of index at most two in G). The above argument, applied to G_+, rules out possibilities (S) and (R), while in case (P) the group G_+ should have trivial center, therefore being conjugate to $\mathrm{PSL}_2(\mathbb{R})$. □

The following Lemma is the key observation for the analysis of locally compact subgroups of $\mathrm{Homeo}\,(S^1)$. I would like to thank Etienne Ghys for this crucial suggestion:

Lemma 4.3. *Let $K \subseteq \mathrm{Homeo}_+(S^1)$ be a compact subgroup. Then for some $f \in \mathrm{Homeo}_+(S^1)$, one has $f\,K\,f^{-1} \subseteq \mathrm{SO}_2$ and therefore K is isomorphic to the rotation group SO_2, or to one of its finite cyclic subgroups.*

Proof. Let m be the SO_2-invariant probability measure m on S^1. Consider the averaged probability measure $\tilde{m}$ on S^1, defined by

$$\int \phi(x)\, d\tilde{m}(x) := \int_K \int_{S^1} \phi \circ k\,(x)\, dm(x)\, dk, \qquad (\phi \in C(S^1))$$

Then $\tilde{m}$ is K-invariant and absolutely continuous with respect to m. Define a function $f : S^1 \longrightarrow S^1$ by the relation

$$m([x_0, f(x)]) = \tilde{m}([x_0, x])$$

where $[x_0, y]$ denotes the arc from x_0 to y on S^1 in the positive direction. Then f is a well defined orientation preserving homeomorphism (to see this, check that both m and $\tilde{m}$ assign positive measures to non-empty open intervals). By the definition $f_* m = \tilde{m}$, which means that the compact group $f^{-1}\,K\,f$ preserves m, and therefore acts on S^1 by isometries (with respect to the SO_2-invariant metric). Hence $f^{-1}\,K\,f \subseteq \mathrm{SO}_2$, but the only non-trivial closed subgroups of SO_2 are finite cyclic groups. □

Corollary 4.4. *A locally compact subgroup $G \subset \mathrm{Homeo}_+(S^1)$ contains no small subgroups. In particular, the connected component G^0 of the identity of G is a real Lie group.*

Proof. Clearly, small enough neighborhood of the identity in $\mathrm{Homeo}_+(S^1)$ does not contains neither (conjugates of) SO_2 nor finite cyclic subgroups. Hence by Lemma 4.3, there are no small subgroups in $G \subset \mathrm{Homeo}_+(S^1)$. By the results of Montgomery and Zippin [MZ], this implies that G^0 is a Lie group. □

Lemma 4.5. *The groups SO_2, $\mathrm{PSL}_2(\mathbb{R})$ coincide with their normalizers in $\mathrm{Homeo}_+(S^1)$.*

Proof. Let G denote the normalizer of SO_2 in $\mathrm{Homeo}_+(S^1)$, then $g : k \mapsto g^{-1}\, k\, g$ defines a G-action by orientation preserving automorphisms on SO_2. But the only such automorphism is the trivial one, which means that G centralizers SO_2. Since SO_2 acts (simply) transitively on S^1 for any $g \in G$ and arbitrary $x \in S^1$ there is a $k_{g,x} \in \mathrm{SO}_2$ such that $g(x) = k_{g,x}(x) \in S^1$. The fact that G centralizers SO_2, while the latter acts transitively, shows that the above $k_{g,x}$ does not depend on x, so that $g \in \mathrm{SO}_2$, and $G = \mathrm{SO}_2$.

Now let G denote the normalizer of $\mathrm{PSL}_2(\mathbb{R})$ in $\mathrm{Homeo}_+(S^1)$. The group $\mathrm{Aut}\,\mathrm{PSL}_2(\mathbb{R})$ contains $\mathrm{PSL}_2(\mathbb{R})$ as an index two subgroup. One can check that since $G \subset \mathrm{Homeo}_+(S^1)$ is orientation preserving, the homomorphism $\sigma : G \longrightarrow \mathrm{Aut}\,\mathrm{PSL}_2(\mathbb{R})$ defined by

$$\sigma(g) : h \mapsto g^{-1} \circ h \circ g, \qquad (g \in G,\ h \in \mathrm{PSL}_2(\mathbb{R}))$$

is never an outer automorphism. Hence for each $g \in G$ one can find an $h \in \mathrm{PSL}_2(\mathbb{R})$ so that $\sigma(g) = \sigma(h)$, which means that $g^{-1}h \in G$ centralizers $\mathrm{PSL}_2(\mathbb{R})$. But the centralizer of $\mathrm{PSL}_2(\mathbb{R})$ in $\mathrm{Homeo}_+(S^1)$ is trivial (use SO_2 above). Hence $g = h$, and $G = \mathrm{PSL}_2(\mathbb{R})$. □

Lemma 4.6. *Let $S \subset \mathrm{Homeo}_+(S^1)$ be a non-trivial connected solvable subgroup, which is not isomorphic to SO_2. Then the collection of S-orbits on S^1 consists of open intervals and fixed points, and there is at least one open interval and at least one fixed point.*

Proof. S is a connected real Lie group. Let $\{1_S\} = S_0 \lhd S_1 \lhd \cdots \lhd S_r = S$ be closed subgroups with $A_i := S_i/S_{i-1}$ being abelian. All A_i, S_i, $1 \le i \le r$ are connected abelian groups. Let U be some one parameter subgroup of S_1. Then U is isomorphic to either SO_2 or to $\mathbb{R}$.

If $U \cong \mathrm{SO}_2$ we can assume, after conjugation in $\mathrm{Homeo}_+(S^1)$, that $U = \mathrm{SO}_2$. Applying Lemma 4.5 to S_i for $i = 1, 2, \ldots$ one notes that all of S should be SO_2 contrary to the assumption.

Hence $U \cong \mathbb{R}$. Since the only closed subgroups of $\mathbb{R}$ are $\{0\}$, $a\mathbb{Z}$ and $\mathbb{R}$ itself, possible $\mathbb{R}$-orbits on S_1 are open intervals, the whole circle S^1, and U-fixed points. We have excluded already the second case (i.e. $S = U \cong \mathrm{SO}_2$). Hence all U-orbits are open intervals and fixed points. In particular, there is at least one fixed point and at least one open interval (otherwise U is trivial).

U is central in the connected group S_1 which maps U-invariant sets to U-invariant sets, and the continuity of S_1 yields that all U-orbits are S_1-invariant. Hence U-orbits coincide with S_1-orbits. Iterating this argument (using normality of S_i in S_{i+1} and continuity of the latter) one concludes that U-orbits coincide with the S-orbits. □

Lemma 4.7. *Let $S \subset \mathrm{Homeo}_+(S^1)$ be as in Lemma 4.6. If I is one of these intervals, then there exists a homeomorphism $f : \mathbb{R} \longrightarrow I$, such that the $f^{-1}\,S\,f$-action on $\mathbb{R}$ coincides either with the action of the translation group $\mathbb{R}$, or with the action of the affine group $\mathbb{R} \ltimes \mathbb{R}$.*

Proof. Let $S \subset \mathrm{Homeo}_+(S^1)$ be as in Lemma 4.6. Then S has a continuous embedding into a finite product $\Pi_1^r G_i$ where each of the G_i is isomorphic either to $\mathbb{R}$ or to $\mathbb{R} \ltimes \mathbb{R}$. For each open S-orbit (i.e. an interval) I there is a homeomorphism $f : \mathbb{R} \longrightarrow I$ and an index $1 \leq i \leq r$, so that the $f^{-1}Sf$-action on $\mathbb{R}$ factors through the G_i, acting by translations (if $G_i \cong \mathbb{R}$), or by affine transformations (if $G_i \cong \mathbb{R} \ltimes \mathbb{R}$). Note that there might be infinitely many open S-orbits on S^1, but among them only finitely many classes of actions, given by G_i-s. □

Lemma 4.8. *Let $G \subset \mathrm{Homeo}_+(S^1)$ be a subgroup, continuously isomorphic to $\mathrm{PSL}_2(\mathbb{R})$, then G is conjugate in $\mathrm{Homeo}_+(S^1)$ to the projective group $\mathrm{PSL}_2(\mathbb{R}) \subset \mathrm{Homeo}_+(S^1)$. The conjugation map is unique, up to the inner conjugations of $\mathrm{PSL}_2(\mathbb{R})$.*

Proof. Given a continuous isomorphism $\alpha : \mathrm{PSL}_2(\mathbb{R}) \longrightarrow G \subset \mathrm{Homeo}_+(S^1)$, consider the image $K := \alpha(\mathrm{SO}_2)$ - a compact subgroup in $G \subset \mathrm{Homeo}_+(S^1)$. By Lemma 4.3, for some $f_1 \in \mathrm{Homeo}_+(S^1)$ one has $f_1^{-1}\,K\,f_1 = \mathrm{SO}_2$, and since SO_2 has no non-trivial orientation preserving automorphisms, one has:

$$f_1^{-1}\,\alpha(k)\,f_1 = k, \qquad (k \in \mathrm{SO}_2)$$

Next consider the affine subgroup P_∞ of $\mathrm{PSL}_2(\mathbb{R})$, and its image $P := \alpha^{-1}(P_\infty)$ in G. Both P_∞ and P are connected solvable groups. Hence, by Lemma 4.6, P fixes at least one point $y \in S^1$, while P_∞ fixes $x_\infty \in S^1$. Using transitivity of SO_2, one can choose $k_1 \in \mathrm{SO}_2$ so that $y = k_1 \circ f_1\,(x_\infty)$. Finally define the injective homomorphism $\beta : \mathrm{PSL}_2(\mathbb{R}) \longrightarrow \mathrm{Homeo}_+(S^1)$ by

$$\beta(g) := (k_1 \circ f_1)^{-1}\,\alpha(g)\,(k_1 \circ f_1), \qquad (g \in \mathrm{PSL}_2(\mathbb{R}))$$

The crucial properties of β are that $\beta(k) = k$ for $k \in \mathrm{SO}_2$, and the subgroup $\beta(P_\infty) \subset \mathrm{Homeo}_+(S^1)$ still fixes the point x_∞.

To prove the Lemma it is enough to show that $\beta(g) = g$ on $\mathrm{PSL}_2(\mathbb{R})$, and for this it suffices to check that

$$\beta(g)(x) = g(x), \qquad (g \in \mathrm{PSL}_2(\mathbb{R}),\ x \in S^1)$$

Fix some $x \in S^1$ and $g \in \mathrm{PSL}_2(\mathbb{R})$. Choose $k_x \in \mathrm{SO}_2$, so that $k_x(x_\infty) = x$ and write the element $g\, k_x \in \mathrm{PSL}_2(\mathbb{R})$ in the form $k_{g,x}\, p_{g,x}$ where $k_{g,x} \in \mathrm{SO}_2$ and $p_{g,x} \in P_\infty$. Now, using the above mentioned properties of β, one calculates

$$\begin{aligned} \beta(g)(x) &= \beta(g) \circ k_x\,(x_\infty) = \beta(g\,k_x)\,(x_\infty) = \beta(k_{g,x}\,p_{g,x})\,(x_\infty) \\ &= \beta(k_{g,x}) \circ \beta(p_{g,x})\,(x_\infty) = \beta(k_{g,x})\,(x_\infty) = k_{g,x}(x_\infty) \\ &= k_{g,x} \circ p_{g,x}\,(x_\infty) = g \circ k_x\,(x_\infty) = g(x) \end{aligned}$$

The uniqueness of $f_\alpha = k_1 \circ f_1 \in \mathrm{Homeo}_+(S^1)$ defined above, given an α, follows from the fact that the centralizer of $\mathrm{PSL}_2(\mathbb{R})$ is trivial. □

Proof of Theorem 4.1. Consider the case of a general locally compact group $G \subset \mathrm{Homeo}_+(S^1)$. By Corollary 4.4, the connected component G^0 of the identity of G is a connected Lie group. Let $G^0 = R \ltimes S$ be a Levi decomposition of G^0 as a semi-direct product of a reductive group R and a solvable group S. Further R is an almost direct product of a semisimple L (with a finite center) by an abelian group Z, which forms the center of R. Since the maximal compact subgroup of L is isomorphic to SO_2 or is trivial (Lemma 4.3), one concludes that L is locally isomorphic either to $\mathrm{PSL}_2(\mathbb{R})$, or to SO_2 or is trivial.

Case $L \simeq \mathrm{PSL}_2(\mathbb{R})$. The center Z of L is a finite group, which is conjugate to a cyclic rotation (Lemma 4.3). Z-acting on S^1 can be considered as "deck transformation" group of a $|Z|$-fold covering

$$\pi : S^1 \longrightarrow S^1/Z \cong S^1$$

where $L \subseteq G$ acts on the covering, and this L-action descends via π to a $\mathrm{PSL}_2(\mathbb{R})$-action on the base, which can be conjugated (Lemma 4.8) to the standard projective one.

Next, let us check that $L \simeq \mathrm{PSL}_2(\mathbb{R})$ implies $L = G \subset \mathrm{Homeo}_+(S^1)$. Consider an element g in the center $Z(R)$ of the reductive group $R \supset L$. Then g commutes with Z and thus descends to some $\bar{g}$ a homeomorphism of S^1, commuting with the standard (up to conjugation) $\mathrm{PSL}_2(\mathbb{R})$-action on S^1. By Lemma 4.5, $\bar{g}$ is trivial, so $g \in Z$, and therefore $R = L$.

Let S be the radical of G^0. Since S is normalized by L, any S-fixed point is mapped by L to an S-fixed point. S has at least one fixed point (Lemma 4.7), and since L-acts transitively on S^1, the radical S is trivial and $L = G^0$.

The whole group G normalizers its connected component $G^0 = L$, and also normalizers the center Z of L (for the latter is invariant under any automorphisms of L). Since Z is cyclic (it is a quotient of the center $\mathbb{Z}$ of the universal covering $\widetilde{\mathrm{SL}}_2(\mathbb{R})$), and G acts by orientation preserving homeomorphisms of S^1, G permutes Z cyclically. This means that the G-action on S^1, descends to a G/Z-action on $S^1/Z \cong S^1$, and this latter action normalizers the L/Z-action, i.e. $\mathrm{PSL}_2(\mathbb{R})$. In view of Lemma 4.5, we conclude that G coincides with L. This proves case (P) of the Theorem.

Case $L \simeq \mathrm{SO}_2$. Then L is compact and just isomorphic to SO_2 (Lemma 4.3). The radical in such case is trivial, so that $G^0 \cong \mathrm{SO}_2$. G normalizers its connected component G^0, and Lemma 4.5 gives that $G = G^0$ conjugate of SO_2 - case (R) of the Theorem.

Case L is trivial. In this case the connected component G^0 of G is either trivial or solvable. If G^0 is trivial, then G is a totally disconnected group which has no small subgroups. This means that G is a discrete group - case (D) of the Theorem. If G^0 is a non-trivial connected solvable group, then Lemma 4.7 provides the description (S) of the Theorem.

This completes the proof of Theorem 4.1. □