Triangular Bimodules, Quasi-hereditary Algebras, and Actions of Parabolic Subgroups in GL_n

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Abstract

Let R be a parabolic subgroup in GL_n. It acts on its unipotent radical R_u and on any unipotent normal subgroup U via conjugation. These actions can be described using matrices over certain bimodules over the path algebra $k\mathbb{A}_t$ of a directed Dynkin quiver of type $\mathbb{A}$. All those bimodules are upper triangular with respect to the natural order of $k\mathbb{A}_t$.

In our main theorem, we show that the category of matrices over an arbitrary upper triangular bimodule B is equivalent to the category of $\Delta-$filtered modules over a quasi-hereditary algebra $\mathcal{A}$. Thus, the orbits of the action of R on U are in natural bijection to the isomorphism classes of $\Delta-$filtered modules over $\mathcal{A}$ with dimension vector d, where d is the dimension vector of block sizes of the Levi factor R/R_u. Moreover the stabilizer of an orbit is isomorphic to the group of invertible elements in the endomorphism algebra of the corresponding $\Delta-$filtered $\mathcal{A}-$module.

In a second part we compute the quiver and relations of the quasi-hereditary algebra $\mathcal{A}$ corresponding to the action of the parabolic group R on U.

1 Introduction

Let k be a fixed ground field. Recently all instances of parabolic subgroups R in $\mathrm{GL}_n(k)$ acting with a finite number of orbits on its unipotent radical R_u were classified (see [HR1],[HR2]). More generally, let $R_u^{(l)}$ be the l-th member of the descending central series; we can also consider the action of R on the Lie-algebra $\mathfrak{r}_u^{(l)}$ of $R_u^{(l)}$. Also all instances of pairs (R, l), where R acts with a finite number of orbits on $\mathfrak{r}_u^{(l)}$ were classified in [BH]. One main step in the proof of the results above relates the orbits of the action of R on $\mathfrak{r}_u^{(l)}$ to a classification problem of modules over a certain quasi-hereditary algebra $\mathcal{A}$, where $\mathcal{A}$ depends only on the number of blocks of R/R_u and on l.

The aim of this paper is to obtain a general result that relates the orbits of certain algebraic groups with isomorphism classes of $\Delta-$filtered modules over some quasi-hereditary algebra. We consider therefore a finite-dimensional bimodule B over a finite-dimensional $k-$algebra Λ and investigate the category $\operatorname{mat} B$ of matrices over B (see section 2.1). Let $P_\Lambda(i)$ for $i = 1, \ldots, t$ denote a complete set of indecomposable, pairwise non-isomorphic, projective left $\Lambda-$modules. Then

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we consider for each dimension vector $d = (d_1, \ldots, d_t)$ the opposite group $R(d)$ of the group of automorphisms of $\oplus_{i=1}^t P_\Lambda(i)^{d_i}$. If Λ is a basic algebra, then the group $R(1, \ldots, 1)$ is isomorphic to the group of units in Λ.
The group $R(d)$ acts on the vector space of all matrices over B of dimension vector d, and the orbits correspond to the isomorphism classes of matrices. In this way, we obtain a large class of linear algebraic groups acting on a $k-$vector space of certain matrices. If Λ is the path algebra of the directed Dynkin quiver of type $\mathbb{A}_t$ and B the bimodule of radical morhisms, for instance, we obtain the action of the parabolic group R on its unipotent radical (see section 3).
In our main result, we require B to be upper triangular, i.e. the algebra Λ is directed and the bimodule structure of B is determined by an upper triangular matrix (see section 2.4). Examples of upper triangular bimodules are provided by posets and directed vector space categories as well as bimodules stemming from the action of a parabolic group R on any unipotent normal subgroup U of R (see section 3.2).

Theorem 1.1 *Let B be an upper triangular bimodule over a directed algebra Λ. Then the category of B-matrices* $\operatorname{mat} B$ *admits a projective generator P whose endomorphism algebra $\mathcal{A}$ is quasi-hereditary, and the functor $Hom(P, -)$ induces an equivalence between* $\operatorname{mat} B$ *and the category $\mathcal{F}(\Delta)$ of $\Delta-$filtered $\mathcal{A}^{op}-$modules. Moreover, any quasi-hereditary algebra whose category of $\Delta-$filtered modules is equivalent to* $\operatorname{mat} B$ *is Morita equivalent to $\mathcal{A}^{op}$.*

In case of directed vector space categories, this Theorem is proven (by different means) in [Ba]. Note that it allows results concerning $\Delta-$filtered modules over quasi-hereditary algebras (such as the existence of almost split sequences, behaviour of the Euler form, degeneration properties) to be applied to the category $\operatorname{mat} B$. Thus we obtain a uniform approach to various categories such as subspace categories of directed vector space categories, orbits of parabolic groups R on some unipotent normal subgroup U and matrices over upper triangular bimodules.

Given an upper triangular bimodule B, we describe in section 2.3 explicitely how to construct a projective generator P of the category $\operatorname{mat} B$. However, one can in general not expect to get quiver and relations of the endomorphism algebra $\mathcal{A}$ of P directly from B. But, we succeed in obtaining such a description of $\mathcal{A}$ for the problem described in the beginning: a parabolic subgroup R acts on a unipotent normal subgroup U. This is our main example, which is considered in detail in section 3. If the normal subgroup U is in addition abelian, then we can also consider another action, which admits the same orbits (see 3.1, part 2), but with different stabilizers; hence, we also obtain a different quasi-hereditary algebra. This shows that the abstract equivalence in Theorem 1.1 preserves not only the orbits, but much more structure. For results related to the geometry of the orbits, we also refer to [BHRZ].

Basic Notation. We denote by k a fixed field. Our algebras are always finite dimensional $k-$algebras with unit, but in general non-commutative. Modules are supposed to be finitely generated left modules, in particular, $\operatorname{mod} k$ is the category of finite dimensional $k-$vector spaces and $\operatorname{mod} \Lambda$ is the category of finitely generated left modules over an algebra Λ. All groups are linear algebraic groups defined over k, where we consider only the $k-$rational points.

Also all categories are $k-$categories and an equivalence preserves the underlying $k-$structure.
For more details on quasi-hereditary algebras, we refer to [DR], and our basic reference for representations of quivers is [R]. We also mention the text book [GR] for an introduction to representation theory, in particular to matrix problems. For more information on representation theory of posets and vector space categories, we also refer to [S].

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2 Matrices over Bimodules and Standardization

2.1 Bimodules

Let Γ be a finite dimensional $k-$algebra and $X_1, \dots X_t$ be a finite number of pairwise non-isomorphic indecomposable $\Gamma-$modules. We denote by Λ the opposite algebra of $\operatorname{End}_\Gamma(\oplus_{i=1}^t X_i)$. If the algebra Γ is representation finite and X_i is a set of representatives of the isomorphism classes of indecomposable $\Gamma-$modules, then Λ is the Auslander algebra of Γ. More generally, we denote for any vector $d = (d_1, \dots, d_t)$ of non-negative integers d_i the opposite algebra of $\operatorname{End}_\Gamma(\oplus_{i=1}^t X_i^{d_i})$ by $\Lambda(d)$.

DEFINITION. A *bimodule B over* $\operatorname{add} X_i$ is a $k-$bilinear functor $(\operatorname{add} X_i)^{op} \times \operatorname{add} X_i \to \operatorname{mod} k$. This is equivalent to having a bimodule B over Λ.

The bifunctors $\operatorname{Hom}_\Gamma(-,-), \operatorname{Ext}^i_\Gamma(-,-)$ and $\operatorname{rad}_\Gamma(-,-)$ provide natural examples of bimodules over $\operatorname{add} X_i$. Here $\operatorname{rad}_\Gamma(X,Y)$ denotes the $k-$vector space of radical morphisms from X to Y (cf. [GR,§3.2] or [Ri,§2.2]).
With these examples in mind, we use for any bimodule B over $\operatorname{add} X_i$ the following notation: If $\phi : X' \to X$ and $\psi : Y \to Y'$ are morphisms in $\operatorname{add} X_i$ and m is an element of $B(X,Y)$, then we denote the element $B(\phi,\psi)(m)$ of $B(X',Y')$ by $\phi m \psi$.

DEFINITION. Let B be a bimodule over $\operatorname{add} X_i$. A *matrix over B* is a pair (X,m), where $X \in \operatorname{add} X_i$ and $m \in B(X,X)$. The matrices over B form an exact category $\operatorname{mat} B$ (see Lemma 2.1) whose morphism set from (X,m) to (X',m') consists of all morphisms $\phi : X \to X'$ in $\operatorname{add} X_i$ satisfying $\phi m = m' \phi$.

Note that the equality $\phi m = m'\phi$ is defined in the vector space $B(X,X')$. The notation $m\phi$ for $B(X,\phi)(m)$ suggests to view this equation as a commutative diagram, keeping in mind that ϕ is a morphism, whereas m and m' are elements of $B(X,X)$ and $B(X',X')$, respectively.

$$\begin{array}{ccc} X & \xrightarrow{m} & X \\ \phi\downarrow & & \downarrow\phi \\ X' & \xrightarrow{m'} & X' \end{array}$$

We call the objects (X,m) matrices since we identify $B(X,X)$ with $\bigoplus_{i,j} B(X_i,X_j)$ and m with $(m_{ij})_{i,j}$ when $X = \bigoplus_i X_i$. This notation follows [D] and [C-B].

2.2 Exact Structures

Fix a bimodule B over $\operatorname{add} X_i$. Let $\mathcal{E}$ denote the class of pairs (i,d) of composable morphisms in $\operatorname{mat} B$

$$0 \longrightarrow (X', m') \xrightarrow{i} (X, m) \xrightarrow{d} (X'', m'') \longrightarrow 0$$

such that the induced sequence of $\Gamma-$modules

$$0 \longrightarrow X' \xrightarrow{i} X \xrightarrow{d} X'' \longrightarrow 0$$

is split exact in $\operatorname{add} X_i$.

Lemma 2.1 ([T,§2]) *The pair* $(\operatorname{mat} B, \mathcal{E})$ *is an exact category in the sense of* [Qu].

We note that $\operatorname{Ext}^2(-,-)$ vanishes for all objects in $\operatorname{mat} B$ (see e.g. [T]). Thus, we write Ext for the first extension group in the category $\operatorname{mat} B$.

Proposition 2.2 ([Qu],[GR,§9.1]) *If* $(\mathcal{C}, \mathcal{E})$ *is a skeletally small exact category, then there is an equivalence* $G : \mathcal{C} \to \mathcal{D}$ *onto a full subcategory* $\mathcal{D}$ *of an abelian category such that* $\mathcal{D}$ *is closed under extensions and that* $\mathcal{E}$ *is formed by the composable pairs* (i,d) *inducing exact sequences*

$$0 \longrightarrow GX \xrightarrow{Gi} GY \xrightarrow{Gd} GZ \longrightarrow 0.$$

We also refer to [K, App. A] for a proof.

2.3 Triangular Bimodules

Definition. The bimodule B over $\operatorname{add} X_i$ is *triangular* if $B(X_i, X_j) = 0$ whenever $i \geq j$.

Note that this definition depends on the chosen numeration of the X_i's. When B is a triangular bimodule, we identify the $\Gamma-$module X_i with the $B-$matrix $(X_i, 0)$. Let $\mathcal{F}(X)$ denote the full subcategory of all $B-$matrices which have a filtration by elements in $\{X_1, \ldots, X_t\}$.

Lemma 2.3 *Let B be a triangular bimodule over* $\operatorname{add} X_i$. *Then*

(i) $\operatorname{mat} B = \mathcal{F}(X)$;

(ii) $\operatorname{Ext}(X_i, X_j) = 0$ *for* $i \geq j$;

(iii) $\operatorname{mat} B$ *has enough projectives.*

Proof. (i). Let (X,m) be a $B-$matrix with $X \neq 0$. Let $i \in \{1, \ldots, t\}$ be the smallest index such that X_i is a direct summand of X. Then (X,m) is isomorphic to a $B-$matrix

$$\left(X' \oplus X_i, \begin{bmatrix} m' & * \\ 0 & 0 \end{bmatrix} \right),$$

where $m' \in B(X', X')$ and $* \in B(X_i, X')$. Therefore we obtain a short exact sequence

$$0 \longrightarrow (X', m') \xrightarrow{i} (X, m) \xrightarrow{d} (X_i, 0) \longrightarrow 0.$$

The desired filtration of (X, m) is constructed by induction on the number of direct summands of X. Consequently $\operatorname{mat} B \subset \mathcal{F}(X)$. The reversed inclusion is obvious.
(ii). Recall that $\operatorname{Ext}(X_i, X_j)$ is formed by short exact sequences

$$0 \longrightarrow (X_j, 0) \xrightarrow{i} (X, m) \xrightarrow{d} (X_i, 0) \longrightarrow 0.$$

As discussed in part (i), the $B-$matrix (X, m) is isomorphic to

$$\left(X_j \oplus X_i, \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \right),$$

where $* \in B(X_i, X_j)$.
(iii). Recall that an object P is projective if $\operatorname{Ext}(P, X) = 0$ for all objects X. Since we have an additive exact category with simple objects $X_i = (X_i, 0)$, this is equivalent to showing $\operatorname{Ext}(P, X_i) = 0$ for all simple objects X_i. We construct an object P_i inductively in the following way: $X_i^0 := X_i$ and X_i^l is the universal extension of X_i^{l-1} by simple objects:

$$0 \longrightarrow \bigoplus_r \operatorname{Ext}(X_i^{l-1}, X_r) \otimes X_r \longrightarrow X_i^l \longrightarrow X_i^{l-1} \longrightarrow 0.$$

If we apply $\operatorname{Hom}(-, X_j)$ to this sequence, then we obtain a long exact sequence

$$0 \to \operatorname{Hom}(X_i^{l-1}, X_j) \to \operatorname{Hom}(X_i^l, X_j) \to \bigoplus_r \operatorname{Hom}(X_r, X_j) \otimes \operatorname{Ext}(X_i^{l-1}, X_j)$$

$$\to \operatorname{Ext}(X_i^{l-1}, X_j) \to \operatorname{Ext}(X_i^l, X_j) \to \operatorname{Ext}(X_r, X_l) \to 0.$$

By construction of the universal extension, the connecting homomorphism is surjective, thus $\operatorname{Ext}(X_i^l, X_j) \simeq \operatorname{Ext}(X_r, X_l)$. Applying induction we obtain that the process of extending X_i^l must stop, since B is triangular, and we have (ii): thus $X_i^l = X_i^{l_0}$ for some index l_0 and all $l \geq l_0$. In fact $l_0 \leq t$. We denote the object $X_i^{l_0}$ by P_i'. It is a projective object by construction, however *not* indecomposable in general. By P_i we denote the unique indecomposable direct summand which contains X_i in its filtration. Thus P_i is indecomposable and projective.
It remains to show, that we have enough projectives. Let (X, f) be a matrix. Then $(X, 0) \simeq (X_i^{a_i}, 0)$ for certain natural numbers a_i. There exists a map $\oplus P_i^{a_i} \longrightarrow (X, f)$ sending the top of P_i to $(X_i, 0)$, thus this map yields a surjection. $\square$

We note that the last statement was already proven in [T]. However, since we use a different language in this note, we prove this statement again for the convenience of the reader.
Since we have enough projective objects in $\operatorname{mat} B$ when the bimodule B is triangular, we can construct a finite dimensional algebra $\mathcal{A}$, so that $\operatorname{mat} B$ is a full subcategory of the category $\operatorname{mod} \mathcal{A}$ of left finitely generated $\mathcal{A}$ modules. The category $\operatorname{mat} B$ consists of those modules which have a filtration by the images of $(X_i, 0)$ for the various i.

Lemma 2.4 *Let $\mathcal{A}$ be the opposite algebra of the endomorphism algebra of a projective generator of* $\operatorname{mat} B$*. Then* $\operatorname{mat} B$ *is a full subcategory of* $\operatorname{mod} \mathcal{A}$*.*

PROOF. This is a standard result. We fix a projective generator P. Then the functor $X \mapsto \operatorname{Hom}(P, X)$ defines a full exact embedding of $\operatorname{mat} B$ into $\operatorname{mod} \operatorname{End}(P)^{op}$. □

In this way, we obtain a particular module category in which we can embed our exact category, not just an abstract existence result as in Proposition 2.2. We note that in all our examples in section 3 the algebra Λ is a subalgebra of $\mathcal{A}$, which simplifies many computations. However, in general the algebra Λ is only a quotient of some subalgebra $\overline{\Lambda}$ of $\mathcal{A}$:

Proposition 2.5 *There exists a subalgebra $\overline{\Lambda}$ of $\mathcal{A}$ such that Λ is a quotient of $\overline{\Lambda}$.*

PROOF. We first construct the algebra $\overline{\Lambda}$. Let $P \longrightarrow \oplus_{i=1}^t X_i$ be a surjection of the projective generator P onto the direct sum of simple objects in $\operatorname{mat} B$, and let U be the kernel. Thus we have an exact sequence

$$0 \longrightarrow U \longrightarrow P \longrightarrow \oplus_{i=1}^t X_i \longrightarrow 0.$$

We define $\overline{\Lambda}$ to be the opposite algebra of the algebra of all endomorphisms f of P satisfying $f(U) \subset U$. Recall that Λ is the opposite algebra of $\operatorname{End}(\oplus_{i=1}^t X_i)$. Since $f \in \overline{\Lambda}$ maps U into itself, it induces an endomorphism of $\oplus_{i=1}^t X_i$. In this way, we obtain an algebra homomorphism $\overline{\Lambda} \longrightarrow \Lambda$. We claim that this algebra homomorphism is surjective: let g be an element of $\operatorname{End}(\oplus_{i=1}^t X_i)$. Then g induces a map $g' \in \operatorname{Hom}(P, \oplus_{i=1}^t X_i)$ which lifts to a map in $\operatorname{End}(P)$ since P is projective and $P \longrightarrow \oplus_{i=1}^t X_i$ is surjective. This shows the claim. □

2.4 Upper Triangular Bimodules and Quasi-Hereditary Algebras

DEFINITION. The bimodule B over add X_i is *upper triangular* if the algebra Λ is directed (i.e. $\operatorname{rad}_\Lambda(X_i, X_j) = 0$ whenever $i \geq j$) and if $B(X_i, X_j) = 0$ whenever $i \geq j$.

REMARK. Lemma 2.3 can be reformulated as follows: if the bimodule B is upper triangular, then $X_1, \ldots, X_t$ is standardizable in the sense of [DR,§3] and $\operatorname{mat} B = \mathcal{F}(X)$. Thus each upper triangular bimodule B defines a quasi-hereditary algebra $\mathcal{A} = \mathcal{A}(B)$, unique up to Morita equivalence. The aim of this part is to make this correspondence more precise (Theorem 2.7). We need the following result.

Theorem 2.6 ([DR,§3]) *Let $X_1, \ldots, X_t$ be a standardizable set of objects of an abelian category $\mathcal{D}$. Then there exists a quasi-hereditary algebra $\mathcal{A}$, unique up to Morita equivalence, such that the subcategory $\mathcal{F}(X)$ of $\mathcal{C}$ and the category of all $\Delta-$filtered $\mathcal{A}-$modules are equivalent.*

If we combine Proposition 2.2 and Theorem 2.6, then we immediately obtain our main result, which is a restatement of Theorem 1.1. We denote the images of X_i under the embedding of Proposition 2.2 by $\Delta(i)$. Moreover, $\mathcal{F}(\Delta)$ denotes the category of $\Delta-$filtered $\mathcal{A}-$modules, that is the category of those modules which have a filtration by the various $\Delta(i)$.

Theorem 2.7 *Let B be an upper triangular bimodule over* $\operatorname{add} X_i$ *and $\mathcal{A}$ be the opposite algebra of the endomorphism algebra of a projective generator P of* $\operatorname{mat} B$. *Then the algebra $\mathcal{A}$ is quasi-hereditary and the functor $Hom(P,-)$ induces an equivalence between* $\operatorname{mat} B$ *and $\mathcal{F}(\Delta)$. Moreover, any quasi-hereditary algebra whose category of $\Delta-$filtered modules is equivalent (as an exact category) to* $\operatorname{mat} B$ *is Morita equivalent to $\mathcal{A}$.*

PROOF. The proof of the existence and uniqueness up to Morita equivalence of the algebra $\mathcal{A}$ follows from Proposition 2.2 and the previous Theorem: We embed the category $\operatorname{mat} B$ in an abelian category $\mathcal{D}$, as stated in Proposition 2.2. Then we apply the Theorem above. Assume we have two different embeddings into categories $\mathcal{D}$ and $\mathcal{D}'$ as in Proposition 2.2. Then the corresponding images are equivalent exact categories. By construction (see the proof of Theorem 2.4 in [DR]), we obtain Morita equivalent quasi-hereditary algebras $\mathcal{A}$ and $\mathcal{A}'$. The particular form of $\mathcal{A}$ follows by construction. $\square$

REMARK. Thus we have a functor $\operatorname{Hom}(P,-)$ which identifies $\operatorname{mat} B$ with $\mathcal{F}(\Delta) \subset \operatorname{mod} \mathcal{A}$. This functor maps the simple object X_i to the simple object $\Delta(i)$ of $\mathcal{F}(\Delta)$ and maps the indecomposable projective object $P(i)$ to the corresponding projective $\mathcal{A}-$module.
We note that it is natural to work with proj Λ instead of add X_i. Then Λ is a quotient of a subalgebra of $\mathcal{A}$ by Proposition 2.5. In several examples (in fact this holds for all examples in the last section), we obtain Λ as a subalgebra of $\mathcal{A}$, and, moreover, the category of $\Delta-$filtered modules coincides with those modules which are projective over Λ. However, we do not have a characterization of those bimodules, or algebras $\mathcal{A}$, with this property.

3 Actions of Parabolic Subgroups in GL_n

3.1 Actions of Parabolic Subgroups and Matrix Problems

Recall that a standard parabolic subgroup in GL_n has an upper triangular block structure. We denote the size of the blocks by $d = (d_1, \ldots, d_t)$ for $d_i \geq 1$ and all $i = 1, \ldots, t$. Thus for $d = (1, \ldots, 1)$, the corresponding standard parabolic subgroup $P(d)$ is isomorphic to the automorphism group of the algebra $k\mathbb{A}_t$ (here $\mathbb{A}_t$ is the directed Dynkin quiver) and for general d, it is isomorphic to the automorphism group of the Morita equivalent upper triangular block matrix algebra

$$\Lambda(d) = \begin{array}{c|c|c|c|c|l} \multicolumn{5}{c}{} & \\ \hline * & * & * & \cdots & * & d_1 \\ \hline 0 & * & * & \cdots & * & d_2 \\ \hline 0 & 0 & * & \cdots & * & d_3 \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \\ \hline 0 & 0 & 0 & \cdots & * & d_t \\ \hline d_1 & d_2 & d_3 & & d_t & \end{array}$$

.

Now we consider the action of $P(d)$ on a unipotent normal subgroup U via conjugation. Thus U has also a block structure: the entries in the diagonal are always unit matrices, and the set of blocks containing non-zero entries is contained in the upper part and is closed under left and upper shift. Hence, U is determined by a subset $I \subset \{(i,j) \mid i < j; i,j = 1,\ldots,t\}$, where for all $(i,j) \in I$ we have $(i-1,j), (i,j+1) \in I$. The group U will be denoted by $U(I)$, or if necessary by $U(I)(d)$, since it depends on both, d and I. In the next subsection, we construct a bimodule $B = B(I)$ over Λ, such that the orbits of the action of $P(d)$ on $U(I)(d)$ coincide with isomorphism classes of matrices over $B(I)$ with dimension vector d. Moreover, we determine the quasi-hereditary algebra $\mathcal{A} = \mathcal{A}(I)$ such that the category of matrices over B and the category of $\Delta-$filtered modules over $\mathcal{A}$ are equivalent according to Theorem 2.7.

Next we consider the simultaneous action of two parabolic groups $P(d^+)$ and $P(d^-)$ on an invariant vector space V contained in the tensor product of the two natural representations W^+ and W^- of $P(d^+)$ and $P(d^-)$ respectively. The orbits of this action correspond in a natural way to the orbits of the action of the parabolic group $P(d^+, d^-)$, that is just take the dimension vector $(d^+, d^-) = (d_1^+, \ldots, d_{t^+}^+, d_1^-, \ldots, d_{t^-}^-)$, on a normal subgroup $U(I)$ of the abelian unipotent subgroup $U(\{1,\ldots,t^+\} \times \{1,\ldots,t^-\})$. In other words we consider the action of two parabolic groups $P(d^+)$ and $P(d^-)$ on the vector space of block matrices $M(J)$ which have nonzero entries only in the $(i,j)-$blocks for (i,j) in J. As in the previous problem, J must be closed under shift to the top and to the right, that is whenever $(i,j) \in J$, then $(i-1,j)$ and $(i,j+1)$ are also in J. Again we want to construct a bimodule $B = B(J)$ over $\Lambda(d^+)$ from the left and $\Lambda(d^-)$ from the right such that the orbits of the action are in natural one-to-one correspondence with the isomorphism classes of matrices over B. If we require in addition, that the stabilizer of a point $x \in V$ of the action is exactly the automorphism group of the endomorphism algebra of the corresponding matrix, then we get different bimodules for the two different actions (see Theorem 3.1 and Theorem 3.2), however the isomorphism classes of matrices coincide. Thus we restrict for a given set J as above to the first problem, the action of $P(d^+) \times P(d^-)$, since the other problem is already considered before.

3.2 The Action of a Parabolic Group via Conjugation

We keep the notation from the previous part: $P(d)$ denotes a standard parabolic subgroup consisting of upper triangular block matrices, and $U(I)$ is a normal unipotent subgroup which has arbitrary entries in all $(i,j)-$blocks for (i,j) in I. We associate to I a set I^{gen} consisting of all blocks $(i,j) \in I$ which are minimal with respect to a shift to the left and the bottom: that is $(i,j) \in I^{gen}$ precisely when $(i+1,j)$ and $(i,j-1)$ are both *not* in J. Moreover, we define a set $I' \subset I$ consisting of all (i,j) satisfying $(i+1,j+1) \in I$ but $(i+1,j) \notin I$, and an index j' satisfying $(1,j') \notin J$ but $(1,j'+1) \in J$. We illustrate these sets in the following figure: the elements in I are denoted by " $*$ ".

	*	*	*	*	*	*
				*	*	*
				*	*	*
					*	*

$I = \{(1,2),(1,3),(1,4),(1,5),(1,6),(2,5),(2,6),$
$(2,7),(3,5),(3,6),(3,7),(4,6),(4,7)\}$

$I^{gen} = \{(1,2),(3,5),(4,6)\}$

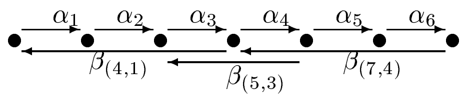
$I' = \{(1,4),(3,5),(4,7)\}, \quad j' = 1$

To obtain the unipotent subgroup $U = P_u^{(l)}$ considered in the introduction, for instance, one has to consider the set I which is generated by

$$I^{gen} = \{(i, i+l+1) \mid i = 1, \dots, t-l-1\}.$$

Let Λ be the basic algebra $\Lambda(1,\dots,1)$. We define a bimodule $B(I)$ over Λ by the following data: a $k-$basis of $B(I)$ consists of elements $b_{(i,j)}$ for $(i,j) \in I$. Choose a basis of Λ consisting of elements $\lambda_{(i,j)}$ for $1 \le i \le j \le t$. We define a bimodule structure on the vector space $B(I)$ via $\lambda_{(i,j)} b_{(i',j')} \lambda_{(i'',j'')} = \delta_{j,i'}\delta_{j',i''} b_{(i,j'')}$. Thus $B(I)$ is the natural bimodule defined by ordinary matrix multiplication of the matrix of upper triangular matrices on the set of block-matrices with entries in the $(i,j)-$block for (i,j) in I. Note that $B(I)$ is generated as a bimodule over Λ by the elements $b_{(i,j)}$ for $(i,j) \in I^{gen}$.

Next we define a quasi-hereditary algebra $\mathcal{A} = \mathcal{A}(I)$. This algebra does not depend on the entries in the dimension vector d, it only depends on I. First we define the quiver $Q(I)$ of $\mathcal{A}(I)$: it has vertices $Q_0 = \{1,\dots,t\}$, and arrows $\alpha_i : i \longrightarrow i+1$ and $\beta_{(j,i)} : j \longrightarrow i$ whenever $(i,j) \in I'$. Next we define relations $\alpha^r\beta = \beta\alpha^s$ whenever possible and a zero relation $\alpha^r\beta$ starting in j' and including the first possible arrow β. Then we define the algebra $\mathcal{A}$ to be the path algebra of the quiver $Q(I)$ modulo the ideal generated by these relations. Again we illustrate the construction in a figure containing the quiver and the relations for the problem in the previous figure.



$\alpha_1\alpha_2\alpha_3\beta_{(4,1)} = 0$

$\alpha_4\beta_{(5,3)} = \beta_{(4,1)}\alpha_1\alpha_2\alpha_3$

$\alpha_5\alpha_6\beta_{(7,4)} = \beta_{(5,3)}\alpha_3$

Thus, for a given subset I, we have defined a bimodule $B = B(I)$ over Λ and a quasi-hereditary algebra $\mathcal{A} = \mathcal{A}(I)$. The next result states, that this quasi-hereditary algebra $\mathcal{A}$ is exactly the quasi-hereditary algebra of Theorem 2.7 associated to the bimodule B. We note that $\mathcal{A}$ has a unique structure of a quasi-hereditary algebra, so that the standard modules $\Delta(i)$ are exactly the projective $\Lambda-$modules $P(i)$ via the natural embedding $\Lambda \subset \mathcal{A}$.

Theorem 3.1 *The category of modules with $\Delta-$filtration over $\mathcal{A}(I)$ is equivalent to the category of matrices over $B(I)$. This equivalence is induced by sending a matrix $(\bigoplus_i P(i) \otimes k^{d(i)}, f)$ to the following representation: α_i is the inclusion to the first factor $k^{\Sigma_{l\le i} d(l)} \longrightarrow k^{\Sigma_{l \le i+1} d(l)}$ and $\beta_{(j,i)}$ is the restriction of f to $k^{\Sigma_{l\le j} d(l)}$.*

The proof uses the category of flags, which we introduce in section 3.4.

3.3 Actions of Parabolic Subgroups via Multiplication

We maintain the notation introduced in Section 3.1: Let $d^+ = (d_1, \ldots, d_{t^+})$ and $d^- = (d_1, \ldots, d_{t^-})$ be two dimension vectors of length t^+ and t^-, respectively, and J a subset in $\{1, \ldots, t^+\} \times \{1, \ldots, t^-\}$ which is closed under right and upper shift, as explained before. We also denote J by a diagram.

EXAMPLE. Let $J = \{(1,1), (1,2), (2,1), (2,2), (3,2)\}$. Then $J' = \{(3,2), (2,1)\}$, and $j' = 0$.

We define a quiver $Q(J)$ as follows:

$$Q_0 = \{(+,1), \ldots, (+,t^+), (-,1), \ldots (-,t^-)\} \text{ and}$$

$$Q_1 = \{(a,i) \xrightarrow{\alpha_{(a,i)}} (a,i+1) \mid a = +,-\} \cup \{(+,i) \xrightarrow{\beta_{(i,j)}} (-,j) \mid (i,j) \in J'\}.$$

Moreover, we consider the relations $\alpha^r\beta = \beta\alpha^s$, whenever possible, and a zero relation $\alpha^r\beta$ starting in j' and including the first possible arrow β. We define the algebra $\mathcal{A}(J)$ to be the path algebra of this quiver subject to the relations defined above.
Note that $\mathcal{A}(J)$ is quasi-hereditary with respect to several orders, however we only consider the order defined by $(a,i) \leq (b,j)$ precisely when $a = -, b = +$ or $a = b$ and $i \leq j$. This is equivalent to the property that the category of $\Delta-$filtered modules coincides with the category of all modules with $\alpha_{(a,i)}$ injective.

Theorem 3.2 *The category of matrices over $B(J)$ is equivalent to the category of modules with $\Delta-$filtration over the quasi-hereditary algebra $\mathcal{A}(J)$. This equivalence is induced by sending a matrix $(\bigoplus_i P(+,i) \otimes k^{d(+,i)} \oplus \bigoplus_j P(-,j) \otimes k^{d(-,j)}, f)$ to the following representation: $\alpha_{(a,i)}$ is the inclusion to the first factor: $k^{\Sigma_{l \leq i} d(a,l)} \longrightarrow k^{\Sigma_{l \leq i+1} d(a,l)}$ and $\beta_{(i,j)}$ is the restriction of f to $k^{\Sigma_{l \leq i} d(+,l)}$.*

The proof follows together with the proof of Theorem 3.1 in the following section and uses the category of flags associated to J.

3.4 Categories of Flags

In this section we introduce the category of flags $\mathcal{FL}(I)$ associated to a set I, and the category of separated flags $\mathcal{FL}^s(J)$ associated with a set J, where I and J are as introduced in the first part of this section. Using this technique we first prove Theorem 3.1; the proof of Theorem 3.2 is quite analogous.

Let V be a finite dimensional vector space together with a flag of length t $F : \{0\} = V_0 \subset V_1 \subset \ldots \subset V_t = V$ and an endomorphism $f : V \longrightarrow V$ satisfying $f(V_j) \subset V_i$, whenever $(i,j) \in I'$ and $f(V_{j'}) = 0$. The objects in the category $\mathcal{FL}(I)$ are those pairs (F, f). A morphism $\phi : (V, f) \longrightarrow (V', f')$ is a linear map $\phi : V \longrightarrow V'$ which preserves the flag, that is $\phi(V_i) \subset V_i'$, and commutes with the given endomorphisms, that is $g\phi = \phi f$. We note that for $I^{gen} = \{(i, i+l+1) \mid i = 1, \ldots, t-l-1\}$, this category was already considered in [BH] and for $l = 0$ in [HR2].

Lemma 3.3 *The map assigning to a flag (F, f) the representation $(V_i, \alpha_i, \beta_{(i,j)})$, where α is the injection $V_i \subset V_{i+1}$ and $\beta_{(i,j)}$ is the restriction of f, induces an equivalence between the category $\mathcal{FL}(I)$ and the category of $\Delta-$filtered $\mathcal{A}(I)-$modules.*

Proof. Note that an $\mathcal{A}(I)-$module has a $\Delta-$filtration precisely when all maps $\alpha_{(i,j)}$ are injective. Then the proof is elementary and left to the reader (it is proven for special cases in [BH] and [HR2] and the proof of this Lemma does not need any additional argument). $\square$

In the next step, we prove that the category of matrices $\operatorname{mat} B(I)$ is also equivalent to $\mathcal{FL}(I)$. Recall that an object in $\operatorname{mat} B(I)$ consists of a projective $\Lambda-$module X together with an element m. We can identify the module X with a representation of $\mathbb{A}_t$ with injective maps, also denoted by α_i. That is we can identify X with a flag $F : \{0\} = V_0 \subset V_1 \subset \ldots \subset V_t = V$. We choose a decomposition of the various $V_{i-1} \oplus V_i^c = V_i$. Thus we obtain a decomposition $V = \oplus V_i^c$, where $i = 1, \ldots, t$. With this decomposition the element m consists of a homomorphism $m_{i,j} : V_i^c \longrightarrow V_j^c$ satisfying the condition $m_{i,j} = 0$ whenever $B(X_i, X_j) = 0$. In this way we obtain for each flag (V, f) a matrix $(\oplus X_i \otimes V_i, f)$.

Lemma 3.4 *The category* $\operatorname{mat} B(I)$ *is equivalent to the category of flags* $\mathcal{FL}(I)$.

Proof. In fact the assignment above defines a flag for each matrix and, vice versa, for each matrix a flag. It is straightforward to check that the conditions $f(V_j) \subset V_i$ whenever $(i,j) \in I$ and $f(V_{j'}) = 0$ on the endomorphism f in the flag are satisfied precisely when the conditions $m_{(i,j)} = 0$ whenever $(i,j) \notin I$ on the matrix m are satisfied. $\square$

Proof of Theorem 3.1 From the two Lemmata above, we obtain the equivalence of the categories $\operatorname{mat} B$ and $\mathcal{F}(\Delta)$ in $\operatorname{mod} \mathcal{A}(I)$. Since $\mathcal{A}(I)$ is quasi-hereditary and the matrix $(X_i, 0)$ corresponds to $\Delta(i)$ under the equivalences in the two Lemmata above, we obtain our result by Theorem 2.7. $\square$

For the proof of Theorem 3.2 we need a modification of the category of flags: we consider the category of separated flags $\mathcal{FL}^s(J)$ corresponding to a set J, introduced in the beginning of this section. The objects are triples (F, G, f) consisting of two flags $F : \{0\} = V_0 \subset V_1 \subset \ldots \subset V_{t+} = V$ and $G : \{0\} = W_0 \subset W_1 \subset \ldots \subset W_{t-} = W$, and a linear map $f : V \longrightarrow W$ satisfying the conditions $f(V_j) \subset V_i$ for $(i,j) \in J$ and $f(V_{j'}) = 0$ for $j' \neq 0$. A morphism ϕ between two objects $(F, G, f) \longrightarrow (F', G', f')$ consists of two linear maps $\phi_1 : V \longrightarrow V'$ and $\phi_2 : W \longrightarrow W'$, which preserve the flags and commute with the given endomorphisms, that is $\phi_2 f = g\phi_1$.
Given such a separated flag (F, G, f), it is straightforward to define a representation of the algebra $\mathcal{A}(J)$ defined in the beginning of this section: the vector space in $(+, i)$ is V_i and the vector space in $(-, i)$ is W_i. The maps α are the inclusions of the flag, and the maps β are the restrictions of the map f. As before, it is easy to check the following Lemma.

Lemma 3.5 *The natural map defined above induces an equivalence between the category* $\mathcal{FL}^s(J)$ *and the category of* $\Delta-$*filtered* $\mathcal{A}(J)-$*modules.*

Also the second Lemma proven for the category of flags generalizes with appropriate modifications to the category of separated flags.

Lemma 3.6 *The category* $\operatorname{mat} B(J)$ *is equivalent to the category* $\mathcal{FL}^s(J)$.

Proof of Theorem 3.2. The proof is analogous to the proof of Theorem 3.1: the categories mat $B(J)$ and $\mathcal{F}(\Delta)$ are equivalent, $\mathcal{A}(J)$ is quasi-hereditary, and, by the uniqueness result of Theorem 2.7, we proved the Theorem. □

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