RIGIDITY OF COMMENSURATORS AND IRREDUCIBLE LATTICES

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§0. Introduction and discussion of the main results

I. Introduction. The celebrated theorems of Mostow and Margulis, proven in the early 70's, were followed by two decades of remarkable developments. A rigidity theory of semisimple groups and their discrete subgroups emerged, incorporating powerful ideas and tools from diverse areas. More recently, however, some new directions have been explored, when it became clear that the wealth of techniques can be applied in a wider scope. The general setting of CAT(0), and especially CAT(-1) spaces, whose geometric and boundary structure had become well understood, gave rise to a natural framework in which the rigidity theory could ripen. Roughly speaking, these activities revolve around two closely related cores: the construction of new non positively curved spaces, often "singular", admitting discrete co-compact isometric actions, and the rigid behaviour of those co-compact discrete groups acting on them.

The first and foremost example of such a space is a tree, whose resembleness to the hyperbolic plain, especially in terms of properties of the automorphism group, attracted much attention. Tits [Ti] proved the simplicity of the automorphism group of a tree under very general conditions, and both uniform [BK] and nonuniform [Ca] lattices were constructed under natural assumptions. (The forthcoming book [BaLu] is especially devoted to this subject.) Density of commensurators of uniform tree lattices, a property modeled after the linear arithmetic ones and of particular interest to us in the present paper, was established in [Liu], verifying a conjecture of Bass and Kulkarni. Bass-Serre theory and the connection with amalgam splittings, motivated rigidity results for group actions on trees, which ultimately led to general theorems for actions on CAT(-1) spaces (cf., [Mar2]-[Gao], and [LMZ]-[BM1]; see also [Ad]). Interesting examples of lattices in other CAT(-1) spaces were considered by Bourdon [Bou], who proved for them a Mostow-type rigidity theorem. Some of these constructions were recently put in a fascinating family of automorphism groups of certain CAT(0) spaces, which are simple, locally compact, non-discrete and non-linear, admitting uniform lattices [HP]. Density of the commensurator of some of these lattices was established in [Ha]. We remark that unlike "nice" automorphism groups of trees, the simple groups constructed by Haglund and Paulin in [HP] do not have a structure so similar to that of simple Lie groups; for instance, they fail to have the Howe-Moore property. Many constructions of simple locally compact groups arise also from the work of Burger and Mozes [BM4]. Other interesting examples come from groups of Kac-Moody type over a finite field. As shown by Carbone and Garland [CG], these locally compact groups contain lattices, both uniform and nonuniform.

Recently, the developments of the theory culminated in the work of Burger and Mozes ([BM3], [BM4] -see [BM2] and [Moz]), who constructed many lattices in products of automorphism groups of trees, which are the first simple groups known to posses (simultaneously) various finiteness properties, such as finite presentability, torsion freeness, finite cohomological dimension and more. Their simplicity is based heavily on an adaptation of Margulis' deep normal subgroup theorem, together with the additional flexibility enabled by changing the ambient group to a non-linear one. The latter achievement, many examples and results like those mentioned above, and other constructions which will undoubtedly appear as the theory evolves, provide good motivation for the general setting we consider in this paper. We shall address here many ingredients of the well developed linear rigidity theory, such as superrigidity, strong-rigidity, arithmeticity, normal subgroup structure and others, in two general situations: lattices in products of (general) topological groups, and commensurators of lattices in topological groups. The approach we take for this purpose is, naturally, different from previous ones. Of course, our results apply also in the conventional linear framework. Some of them are new even in that case, and others provide an alternative, sometimes more natural approach, to several well-known results (including a new proof of Margulis' arithmeticity and commensurator-arithmeticity theorems for a certain class of lattices).

II. The setup. As mentioned, we shall be concerned with rigidity in two general situations, and we now set notations to be used throughout the paper. The first framework is that of a lattice in a product of groups, and the precise assumptions and notations will be as follows:

(0.1) $G = G_1 \times ... \times G_n$ is a compactly generated topological group. $\Gamma < G$ is a discrete co-compact subgroup, whose projections to all the G_i's are dense. If every G_i is the k-rational points of a semisimple, almost k-simple, k-isotropic linear algebraic group, for some local field k, and Γ is an irreducible lattice in the usual sense, then Γ is allowed to be **nonuniform** as well.

Notice that the assumption that G contains a discrete co-compact subgroup already implies that it is locally compact and unimodular (in particular, Γ is a lattice). The compact generation assumption is redundant in many of the results, but we will not elaborate on this technicality. An important consequence of compact generation is the finite generation of Γ. In particular, by the assumption on dense projections, G is second countable. Note, however, that the assumption on dense projections is not restrictive, as we can always replace the G_i's by the closures of the projections of Γ, and one cannot expect in general rigid behaviour of Γ beyond that "envelop". Strictly speaking, we allow also one factor ($n = 1$) in **(0.1)**, but this means by our density assumption, that $G = \Gamma$, in which case nothing interesting will (or can) be said. Despite of the title of the paper, notice that when Γ is uniform we do not impose in **(0.1)** any irreducibility condition. Of course, in this generality some of the G_i's may be compact or discrete, and again Γ may identify with G. Consequently, in some (although not all) of our results, we shall indeed impose an irreducibility property, namely, that the canonical projections of Γ to subproducts of the G_i's are not discrete. When making such an assumption, we in particular mean that $n \geq 2$. Besides rather pathological situations, this is equivalent to the natural condition that no subgroup of finite index in Γ breaks into a direct product of lattices in two disjoint subproducts of the G_i's (this is the case, e.g., for any totally disconnected G, see also [Mar1, p.102 Lemma 6.4]).

The second framework in which we shall be interested, is that of commensurators of lattices. Recall first that if G is a group and $\Gamma < G$ a subgroup, then the **commensurator** of Γ in G, denoted $Comm_G(\Gamma)$, is the subgroup $\{g \in G |\ g\Gamma g^{-1} \cap \Gamma$ has finite index in both Γ and $g\Gamma g^{-1}\}$. Our general notation here will be as follows.

(0.2) The group G is locally compact and second countable. $\Gamma < G$ is a discrete co-compact subgroup, and $\Lambda < G$ is a dense subgroup with $\Gamma < \Lambda < Comm_G(\Gamma) < G$. The lattice Γ is allowed to be also **nonuniform**, either in the situation of **(0.1)**, or when G is a simple Lie group of rank one, not locally isomorphic to $SL_2(\mathbb{R})$ or $SL_2(\mathbb{C})$.

Actually, nonuniform lattices in $SL_2(\mathbb{C})$ will be covered as well, but they require a modification of our methods to the setting of certain uniformly convex Banach spaces, rather than Hilbert spaces (this accounts for the general framework we consider in the preliminaries of Section 1.I). See Section 10.4 below for the discussion of rigidity for the nonuniform lattices in **(0.2)**. As in **(0.1)**, the density assumption on Λ is not restrictive. We should mention here also that general rigidity results in the setup of **(0.2)**, which overlap some of the ones we shall present, appear in an unpublished manuscript of Margulis [Mar4] (cf. [Bur]), relying on a different technique.

III. Statement of the main results. The work on the present paper began in trying to understand better Margulis' normal subgroup theorem for a higher rank irreducible lattice Γ. The remarkable strategy of proof, showing the finiteness of Γ/N by proving that it is both amenable and Kazhdan, seems even more striking when implemented for the lattices which themselves fail to have property (T). These lattices appear only in the setting of **(0.1)** (see [Mar1, p.155] for details). The case of nonuniform lattices introduces significant technical difficulties, and is mentioned only in the original paper [Mar3]. Our approach will show that a general new rigidity theorem underlies Margulis' result on property (T) of quotients, and that when formulated appropriately, no assumption whatsoever is required on the G_i's.

0.1 Theorem (Property (T) for quotients of irreducible lattices). *Retain the notation in* **(0.1)**, *and let $N \lhd \Gamma$ be a normal subgroup. Then Γ/N has property (T) of Kazhdan if and only if the following conditions are satisfied:*

(i) *Every complex additive character $\varphi : G \to \mathbb{C}$ which vanishes on N, is identically zero.*
(ii) *For every $1 \leq i \leq n$, the quotient $G_i/\overline{pr_i(N)}$ has property (T), where pr_i denotes the canonical projection to G_i.*

The necessity of the conditions (i) and (ii) is obvious. For instance, if the G_i's are simple as topological groups, which is the case in Margulis' theorem, the conditions are trivially verified, as $\overline{pr_i(N)}$ must equal G_i. The example of $\Gamma = \mathbb{Z}[\sqrt{2}] < G = \mathbb{R} \times \mathbb{R}$ and $N = \mathbb{Z}$, shows that condition (i) is indeed required. The additional difficulty in treating nonuniform lattices does not vanish, and our proof in that case, as the proof of all the other results stated below, uses the theorem of Lubotzky Mozes and Raghunathan [LMR], regarding the comparison between the word and Riemannian metrics on these lattices (in particular, we rely on their arithmeticity). Using different methods, a special case of Theorem 0.1 is established in [Pa] and [BM3] (where it plays an important role in proving the simplicity of the groups constructed, see also [BM2], [Moz]).

The proof of Theorem 0.1 is based on a new superrigidity theorem for unitary representations, satisfying non vanishing of certain cohomology group (see Theorem 4.1 below). Due to the length of the paper, one technical ingredient of the proof -the injectivity of a map between certain cohomology groups- will be given in [Sh4] (with a different application). To deduce Theorem 0.1 from our superrigidity theorem, an additional independent result is needed (see Theorem 6.1), which as a corollary, yields also a proof of a conjecture of Vershik and Karpushev [VK]:

0.2 Theorem. *Let G be a locally compact second countable group, which is compactly generated. If G does not have property (T) of Kazhdan, then there exists an* **irreducible** *unitary G-representation π, such that $H^1(G, \pi)$.*

The notion of first cohomology above is the usual one (see Def. 1.4). The issue here is the irreducibility of π, as without this property, the theorem, due to Guichardet, is well known (the other direction holds as well, and is due to Delorme -see [HV, Ch. 4 Theorem 7]). In fact without requiring irreducibility, the compact generation assumption is redundant, but it is necessary for Theorem 0.2. As we indicate in §6, there are many examples of groups without property (T) (among them some semisimple Lie groups and their lattices), which have only finitely many, or even exactly one irreducible representation with non-vanishing first cohomology.

A strategy to establish Kazhdan's property, which have proven successful in various situations, is showing vanishing of the first cohomology for all unitary representations (cf., [HV Ch.9], [Zuk], [Mok], [Pa], [BM3]). In doing so, the fact that one may restrict to irreducible representations only, can be quite useful. We illustrate this aspect in §6.II, where a geometric proof of property (T), for those simple Lie groups sharing it, is presented, using the rigidity theory of harmonic maps.

We now turn to describe various rigidity properties of discrete subgroups in the framework of **(0.1)** and **(0.2)**. All of our results are deduced from the same superrigidity theorem for unitary representations (Theorem 4.1), on which Theorem 0.1 above relies. In the setting of commensurators, they are based on an analogous result– Theorem 10.3.

0.3 Theorem (Superrigidity for irreducible lattices). *Retain the notation in* **(0.1)**. *Let H be a locally compact non-amenable group, and assume it contains a uniform lattice which is isomorphic to a discrete subgroup of $SO(n,1)$ or $SU(n,1)$, for some n. Let $\varphi : \Gamma \to H$ be a homomorphism whose image has nonamenable closure. Then there exists a compact normal subgroup $M \lhd \overline{\varphi(\Gamma)}$, such that the composition homomorphism: $\Gamma \to \varphi(\Gamma) \to \overline{\varphi(\Gamma)}/M$ extends continuously to a homomorphism of G, which factors through a homomorphism of one of the G_i's.*

In the setting of commensurators we have:

0.3' Theorem (Superrigidity for commensurators). *Retain the notation in* **(0.2)**, *and let H be as in Theorem 0.3. Let $\varphi : \Lambda \to H$ be a homomorphism such that $\varphi(\Gamma)$ has nonamenable closure. Then there exists a compact normal subgroup $M \lhd \overline{\varphi(\Lambda)}$, such that the homomorphism: $\Lambda \to \varphi(\Lambda) \to \overline{\varphi(\Lambda)}/M$ extends continuously to a homomorphism of G.*

Although one may add the other rank one Lie groups to $SO(n,1)$ and $SU(n,1)$ in the assumption of the theorem, their treatment introduces difficulties which have to be dealt with in a more systematic study of uniformly bounded, rather than unitary, representations. We shall consider this issue elsewhere. The assumption on H can in fact be made more general, and requires only the existence of one unitary representation with certain properties. The condition stated, however, is easy to describe with no further technical notions, and is especially suitable for a strong rigidity theorem (stated below). Of course, Theorems 0.3 and 0.3' hold when H itself is (locally) isomorphic to $SO(n,1)$ or $SU(n,1)$. Notice also that the condition on H is satisfied if it admits a uniform lattice which is a free group; particularly, if H is a rank one algebraic group over a non archimedean local field. With a little additional effort (see Lemma 7.2), one can then deduce:

Corollary 0.4. *Let k be a local field, and H denote the k-points of a semisimple almost k-simple linear algebraic group, of adjoint type and k-rank 1. If $k = \mathbb{R}$ assume that H is not locally isomorphic to $Sp(n,1)$ or $\mathbb{F}_{4(-20)}$. Keep for this H the other notations of Theorems 0.3 and 0.3'. Assume further that $\varphi(\Gamma)$ (in the former case) or $\varphi(\Lambda)$ (in the latter), is Zariski dense. Then φ extends to a continuous homomorphism $\tilde{\varphi} : G \to H$ (factoring through one G_i in the first case).*

Superrigidity may be used to get arithmeticity. Here we obtain from Corollary 0.4 the following:

0.5 Theorem (Arithmeticity). *Retain the notation in* **(0.1)**. *Suppose further that each G_i is simple as a topological group (i.e., every closed proper normal subgroup is trivial), and that every projection of Γ to a proper subproduct of the G_i's is not discrete. Let F be some field with characteristic $p \neq 2$, and suppose Γ admits a homomorphism into $PGL_2(F)$, with infinite image.*

Then every G_i is isomorphic to $PSL_2(k_i)$, for some local field k_i with char $k_i = p$, in a way that makes Γ an S-arithmetic lattice in G. More precisely, there is a global field K of characteristic p, an absolutely simple linear algebraic group $\mathbb{G}$, abstractly isomporphic to PGL_2 and defined over K, and a set S of n (inequivalent) valuations on K, satisfying:

(i) *for every $\nu \in S$, $\mathbb{G}(K_\nu)$ is K_ν-isotropic (hence $\mathbb{G}(K_\nu)^+ \cong PSL_2(K_\nu)$)*
(ii) *the set S contains all the archimedean valuations on K for which $\mathbb{G}$ is K_ν-isotropic*

such that after replacing Γ with a finite index subgroup, the following holds:

(iii) *For every $1 \leq i \leq n$ there exists a topological group isomorphism between G_i and some $\mathbb{G}(K_{\nu_i})^+$, inducing a bijection between the set of simple factors G_i and the set S, and:*
(iv) *The image of Γ under the induced isomorphism $\tilde{\varphi} : G \to \prod_{\nu \in S} \mathbb{G}(K_\nu)^+$ has finite index in (the diagonal embedding of) the S-arithmetic subgroup $\mathbb{G}(K(S))$, where $K(S)$ is the ring of S-integers in K.*

For the relevant notions from the theory of algebraic and arithmetic groups, see Section 8. As mentioned earlier, there are by now various examples of simple, locally compact (non discrete) groups which admit a lattice, and constructions of irreducible lattices in non algebraic simple groups are known as well. By projecting the lattice to one of the factors, a homomorphism as assumed in Theorem 0.5 exists whenever **at least one** of the G_i's is itself isomorphic to $PSL_2(k)$, for some local field k. Theorem 0.5 then says that all the other G_i's must be isomorphic to some PSL_2 as well, and Γ must be $S-$arithmetic. Thus, Theorem 0.5 yields in particular a (new) proof of the arithmeticity of higher rank uniform irreducible lattices in products of SL_2's. Constructions in [BMZ] show that the situation for SL_3-target can be quite different from that of SL_2.

For commensurators we prove the following analogue, giving also a new proof in the case of SL_2, of Margulis' well-known theorem on the arithmeticity of lattices with dense commensurators:

0.5' Theorem (Arithmeticity for commensurators). *Retain the notation in* **(0.2)**, *and assume further that G is simple as a topological group. Let F be any field with characteristic $p \neq 2$, and suppose that Λ admits a homomorphism into $PGL_2(F)$ for which the image of Γ is infinite. Then there exists a local field k with char $k = p$, such that G is isomorphic to $PSL_2(k)$, in a way that makes Γ an arithmetic lattice.*

We remark that although Theorem 0.5' is confined to the case of SL_2 target, Margulis' theorem on the arithmeticity of lattices with dense commensurators is especially interesting for $SL_2(\mathbb{R})$, which admits a space of positive dimension of uniform lattices (all isomorphic as abstract groups). On the other hand, as mentioned earlier, there exist also non algebraic simple locally compact groups, which contain uniform lattices with a dense commensurator.

We now return to the general framework of locally compact groups, and discuss strong rigidity.

0.6 Theorem (Strong rigidity of irreducible lattices). *Retain the notation in* **(0.1)** *and assume that $n \geq 2$. Let $G^* = G_1^* \times ... \times G_m^*$, and assume that each G_i^* satisfies the condition on H in Theorem 0.3. Let $\Gamma^* < G^*$ be a discrete subgroup whose projection to every proper subproduct of* *the G_j^* is dense. If $\varphi : \Gamma \to \Gamma^*$ is an isomorphism, and G, G^* have no non-trivial normal compact subgroups, then $n \leq m$, Γ^* is co-compact as well, and φ extends continuously to an isomorphism $\tilde{\varphi} : G \to G^*$, which is induced by "local" isomorphisms of each G_i with a subproduct of some G_j^*'s.*

The point of this result is that within the general class of lattices in products, certain groups, such as irreducible lattices in products of $PSL_2(\mathbb{R})$'s, or in automorphism groups of trees, determine completely (modulo a compact normal subgroup) the ambient group (see Corollary 7.3). For example, when the G_i^*'s are groups of automorphisms of trees, the existence of a lattice Γ^* implies that each G_i^* separately has a uniform lattice, which must then be a (virtually) free group. Thus the condition on the G_i^*'s in 0.6 is met in this case. In particular, when the G_i's are groups of this type as well, Theorem 0.6 reduces to a strong rigidity theorem in [BMZ] (see [Moz, Theorem 8]).

0.6' Theorem (Strong rigidity for commensurators). *Retain the notation in* **(0.2)**. *Let G^* be a group which satisfies the condition on H in Theorem 0.3, and $\Gamma^* < \Lambda^* < Comm_{G^*}(\Gamma^*) < G^*$ be as in* **(0.2)**. *If both G, G^* have no non trivial normal compact subgroups, then any isomorphism $\varphi : \Lambda \to \Lambda^*$ such that $\varphi(\Gamma) \subset \Gamma^*$ is infinite, extends continuously to an isomorphism of G and G^*.*

Consider for instance a free group Γ realized as a (uniform) lattice in $G = PGL_2(\mathbb{Q}_p)$, a subgroup $\{e\} \neq \Gamma_0 < \Gamma$, and a commensurator $\Gamma < \Lambda < Comm_G(\Gamma)$. Although free groups are themselves highly non rigid, Theorem 0.6' says that the pair (Λ, Γ_0) determines $PSL_2(\mathbb{Q}_p)$ as the **unique possible** topological group (up to a normal compact subgroup), with the property that the pair is embedded as (dense) commensurator and lattice. In particular, if $[\Gamma : \Gamma_0] = \infty$ then no such topological group exists. Of course, we may find commensurators of free lattices inside some other $PGL_2(\mathbb{Q}_q)$, or in the general automorphism group of a tree. Any choice of Λ determines its closure as a **unique** "envelop" for the pair (Λ, Γ). (Notice that the property that Λ commensurates Γ is intrinsic for the pair, and does not depend on an ambient group.)

Let us discuss superrigidity for actions on spaces which are not necessarily locally compact. An advantage of our approach, improving results also for lattices in semisimple groups, is that it does not require minimality of the action, an assumption which can be quite restrictive in general (compare with [Mar2], [BM1], [Gao]). Although our methods apply in a more general framework of actions on so called "spaces with walls" (see [HP]), such treatment involves some technical notions and difficulties, which due to the length of the paper, we prefer to discuss elsewhere. The particularly interesting case of isometric actions on trees illustrates our technique with a relatively little additional effort. Recall first that an action on a tree is called non-elementary if it admits no invariant vertex, edge, or ray (where two rays are identified if they coincide from some point on).

0.7 Theorem (Superrigidity for actions on trees). *Retain the notation in* **(0.1)**. *Assume that Γ acts isometrically, non-elementarily, on a tree (not necessarily locally finite). Then there exists a non empty Γ-invariant subtree, on which the Γ-action extends to a G-action, which factors through a continuous action of one of the G_i's. In the situation of* **(0.2)**, *a similar result holds for an action of Λ, under the assumption that the restriction of the Λ-action to Γ is not elementary.*

In particular, in the situation of **(0.1)** one deduces from Theorem 0.7 (and Theorem 0.8 below) that if G is connected and its abelization $G/\overline{[G,G]}$ is compact, then every Γ-action on a tree admits a fixed vertex (or edge). More generally, Theorem 0.7 implies that every amalgam splitting of Γ is lifted from a decomposition of one of the G_i's (for higher rank irreducible lattices in semisimple groups, the latter consequence is due to Margulis [Mar2]). Notably, this applies to the lattices constructed by Burger and Mozes (mentioned earlier), and shows that they admit exactly two different amalgam decompositions. See also Corollary 9.3 for another application of Theorem 0.7 to lattices in semisimple groups.

0.8 Theorem (Superrigidity for characters of irreducible lattices). *Retain the notation in* **(0.1)**. *Every additive complex character $\varphi : \Gamma \to \mathbb{C}$ extends continuously to G.*

When every G_i has a compact subgroup K_i such that (G_i, K_i) is a Gelfand pair, Theorem 0.8 was established by Berenstein and Kazhadn [BeK]. Theorem 0.8 is discussed in §5, and is in fact an ingredient in the proof of Theorem 0.1; it enables one to deduce from condition (i) there, that the commutator of Γ/N has finite index. The analogue of Theorem 0.8 for commensurators reveals an interesting phenomenon. Although it has some counterpart in Margulis' superrigidity, it seems that it has not been noticed even in the case of lattices in semisimple groups.

0.8' Theorem (Superrigidity for characters of commensurators). *Retain the notation in* **(0.2)**. *Let $\varphi : \Lambda \to \mathbb{C}$ be an additive complex character. Then $\varphi|_\Gamma$ extends continuously to G.*

In other words, not every Γ-character extends to G or to Λ (e.g., when $G = SL_2(\mathbb{R})$), and if it does extend to Λ, it need not extend further to G (e.g., the trivial character of $\mathbb{Z}$ extends to $\mathbb{Z}[\sqrt{2}] < \mathbb{R}$ by $m+n\sqrt{2} \to n$, but cannot be extended further continuously to $\mathbb{R}$). However, Theorem 0.8' says that if a Γ-character extends to Λ, then there exists a possibly different continuous extension of it, to G. In particular, we deduce that if G has no nonzero characters (e.g., G is semisimple), then the commutator of Λ contains a subgroup of finite index in Γ. Theorems 0.8 and 0.8' are used in the proofs of some of the previous results, and may be used to sharpen others. For instance, if the abelization of (every subgroup of index 2 in) G is compact, then in Theorem 0.7 it suffices to assume only that the Γ-action on the tree does not fix an edge or a vertex.

IV. On the approach and the structure of the paper. It is clear from the outset that our results cannot rely on any special features of the groups involved. Unlike most rigidity theorems, which are based in an essential way on an intimate knowledge and a rich structure, e.g., of semisimple groups, our approach uses heavily one basic property: the commutativity between the factors in G. In studying the proofs of Margulis' superrigidity theorem and its various extensions, it is always instructive to see where the commutativity inside the group comes into play. This feature often looks innocent, although of course, it is quite powerful. The role of commutativity is even less transparent in the geometric approach to superrigidity. Underlying our analysis is a basic observation, showing that commuting isometric actions of groups on a Hilbert space must exhibit a certain kind of "degeneracy". When formulated appropriately, this phenomenon applies to actions on general CAT(0) spaces, although the linear structure is currently indispensable for our applications. This so-called "degeneracy" of commuting actions is easily, and even better seen, in actions on CAT(-1) spaces, and our result may be viewed as an extension of that phenomenon to non-positively curved spaces. The reader is encouraged at this point to turn to Proposition 3.3, which describes this phenomenon in a precise manner. That proposition is the heart of the superrigidity theorem for cohomology of unitary representations (Theorem 4.1), from which all the rigidity results described above follow. The geometric interpretation of first cohomology in terms of affine actions will turn out to fruitful, giving rise to an interplay between the algebraic and geometric setting. We believe that the phenomenon alluded to above, or variants of it, may appear and be useful in other situations of rigidity as well.

The treatment of rigidity of commensurators requires an additional idea. Here we shall use the action of the commensurator to construct a "generalized Hecke operator" on the space of L^2-sections of the induced G-bundle, an operator which commutes with the G-action (see (10.3) in Section 10). Then, a result similar to Proposition 3.3, this time for a group action on a Hilbert space which commutes with one affine operator (our "Hecke operator"), will enable us to apply the same strategy as before (see Theorem 10.1). Thus, we shall easily see how the group "gains" commutativity, and much before getting into the precise details of proofs, be able to "explain" the well known phenomenon that in the presence of commensurators, the lattice "behaves like" it is higher rank. This approach to the rigidity of commensurators is different from previous ones, in which it is always the uniqueness of a certain Γ-equivariant map which is utilized to deduce that it is actually a Λ-equivariant map, eventually leading to superrigidity. We remark that although the framework and arguments we consider are different, the idea of using Hecke operators was much inspired by their ingenious use in [BuSa].

The coarse structure of the paper and proofs is as follows. The analysis of commuting actions on a Hilbert space, mentioned above, is carried out in Section 3, and in Section 4 it is applied to establish superrigidity for the so-called first reduced cohomology group of unitary representations–Theorem 4.1 (the analogue for commensurators is proven in Theorem 10.3). The proof of the rigidity results described in the previous subsection are presented in the proceeding sections, where in each case one typically chooses an appropriate cohomological representation (according to the situation at hand), apply Theorem 4.1, and use the information it yields to derive the required result. To implement this strategy some complementary results are established, as in the case of (a strengthening of) Theorem 0.2, which is used in proving Theorem 0.1. The reader is encouraged, however, to read first Section 10 of the paper, which deals with rigidity of commensurators. Several technicalities become easier in that case, and the approach is better illustrated. The main part of this section is the proof of a superrigidity theorem for the reduced cohomology, after introducing the Hecke operator mentioned earlier. For the derivation of the various rigidity theorems from it, the reader is referred to the sections in which they are similarly discussed in the framework of **(0.1)**. The treatment of the nonuniform lattices in **(0.1)** and **(0.2)** is identical to the uniform ones, once some computation is verified (this is explained in §1.II). The verification in the case of **(0.1)** is the contents of §2, whereas for **(0.2)** we rely on computations made in [Sh2].

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Some of the results of the present paper were announced in [Sh1], but many of them are strengthened here. Other results are new, including the whole discussion of commensurators.

§1. Preliminaries

I. Cohomology and reduced cohomology. Although we will be mainly interested in Hilbert spaces, we shall benefit also from considering other uniformly convex Banach spaces. As we currently loose no generality by working in that setting, we assume throughout this subsection that $\mathcal{H}$ is a (separable) uniformly convex Banach space. We begin by recalling a well known geometric property of such a space:

1.1 Proposition. *Let C_i be a decreasing sequence of nonempty, bounded, closed convex subsets of $\mathcal{H}$. Then $\cap C_i$ is not empty.*

As a consequence, one has the well known notion of "center of mass" for bounded subsets:

1.2 Proposition. *Let $A \subset \mathcal{H}$ be a non empty bounded subset. Then there exists a unique point $x_0 \in \mathcal{H}$ which minimizes the function $f(x) = \sup_{a \in A} \|x - a\|$.*

The existence in 1.2 follows readily from 1.1 by looking at the intersection of all the sets C_α, $\alpha \geq 0$, for which $C_\alpha = \{x \in \mathcal{H} | f(x) \leq \alpha\}$ is not empty. The uniqueness follows of course by uniform convexity. An immediate (well known) consequence of the uniqueness is the following:

1.3 Corollary. *Any isometric group action on $\mathcal{H}$ with a bounded orbit, admits a global fixed point.*

Let now G be a locally compact group and $(\pi, \mathcal{H})$ a strongly continuous linear G-representation on $\mathcal{H}$, by isometries. Denote by $Z^1(G, \pi)$ the vector space of continuous G-**1-cocycles** with values in π, namely, the continuous functions $b : G \to \mathcal{H}$ satisfying

$$b(gh) = b(g) + \pi(g)b(h) \qquad \forall g, h \in G\ . \tag{1.0}$$

The subspace of **co-boundaries**: $b(g) = \pi(g)v - v$ for some $v \in \mathcal{H}$, is denoted by $B^1(G, \pi)$.

1.4 Definition. *The* **first cohomology group** *of G with coefficients in π, is the quotient space $H^1(G, \pi) = Z^1(G, \pi)/B^1(G, \pi)$. Two elements in $Z^1(G, \pi)$ are called* **cohomologous** *if their difference lies in $B^1(G, \pi)$ or, equivalently, they represent the same cohomology class in $H^1(G, \pi)$.*

Given a cocycle $b \in Z^1(G, \pi)$, one can deform the linear representation π to an affine isometric action of G on the space $\mathcal{H}$. More precisely, define an action ρ of G on $\mathcal{H}$, by $\rho(g)v = \pi(g)v + b(g)$.

The cocycle identity (1.0) readily implies that this is an action, which is obviously affine and isometric: $\|\rho(g)v - \rho(g)u\| = \|v-u\|$ for all $g \in G$ and $v, u \in \mathcal{H}$. Conversely, given any affine isometric action of G on a Banach space $\mathcal{H}$, it is easy to see that it is necessarily obtained from an isometric linear representation π on $\mathcal{H}$, and a cocycle $b \in Z^1(G,\pi)$ as above. The following result is now a straightforward consequence of the definitions:

1.5 Lemma. *If ρ, π and b are as above, then $b \in B^1(G,\pi)$ if and only if the associated G-action ρ has a fixed point.*

On $Z^1(G,\pi)$ we introduce the topology of uniform convergence on compact subsets of G. This makes $Z^1(G,\pi)$ a Fréchet space, but $B^1(G,\pi)$ is not always a closed subspace. The following well known result tells us when it is.

1.6 Proposition. *Assume that π does not have non-zero invariant vectors. Then $B^1(G,\pi)$ is closed in $Z^1(G,\pi)$ if and only if π does not admit almost invariant vectors, namely, there is a compact subset $Q \subset G$ and $\varepsilon > 0$, such that for every $v \in \mathcal{H}$ there is some $g \in Q$ with $\|\pi(g)v - v\| \geqslant \varepsilon\|v\|$.*

Proof. The more interesting implication, the "only if" part, was proven by Delorme, using the closed graph theorem for Fréchet spaces (see [HV, p.48] for details). Since the "if" implication will be essential for us, we recall the simple argument. Let $Q \subset G$ and $\varepsilon > 0$ be as in the assumption. If $\pi(g)v_n - v_n \to b(g)$ uniformly, set $\alpha = \max_{g\in Q}\|b(g)\|$, and note that $\limsup\|v_n\| \leqslant \alpha/\varepsilon$. Therefore $\|b(g)\| \leqslant 2\alpha/\varepsilon$ for **all** $g \in G$, that is, $b(g)$ is bounded. Since the isometric action ρ corresponding to a bounded cocycle has bounded orbits, by 1.3 it admits a fixed point, hence the cocycle b is a coboundary (Lemma 1.5). $\square$

Denoting by $-$ closure of sets in the above topology on $Z^1(G,\pi)$, we define:

1.7 Definition. *The* **first reduced cohomology group** *of G with coefficients in π is defined as the quotient $\overline{H^1}(G,\pi) = Z^1(G,\pi)/\overline{B^1(G,\pi)}$. Two elements in $Z^1(G,\pi)$ are called* **almost cohomologous**, *if their difference lies in $\overline{B^1(G,\pi)}$ or, equivalently, they represent the same element in $\overline{H^1}(G,\pi)$. A cocycle $b \in Z^1(G,\pi)$ is said to be* **almost a coboundary**, *if $b \in \overline{B^1(G,\pi)}$.*

We note the following immediate consequence of Proposition 1.6:

1.8 Corollary. *If π does not admit almost invariant vectors, then $\overline{H^1}(G,\pi) = H^1(G,\pi)$.*

To observe the geometric interpretation of the notion of reduced cohomology, we first need:

1.9 Definition. *We call an action of a locally compact group G on a metric space (X,d)* **uniform**, *if it does not admit almost fixed points, namely, there is some compact subset $Q \subset G$ and $\varepsilon > 0$, such that for every $x \in X$ there is some $g \in Q$ with $d(gx,x) > \varepsilon$.*

The following analogue of Lemma 1.5 now follows directly from the definitions:

1.10 Lemma. *With notations as in 1.5, the G-action ρ is* **not** *uniform if and only if $b \in \overline{B^1(G,\pi)}$.*

The geometric interpretation of the ordinary and reduced first cohomology, enable one to verify easily a useful property of them, which is not shared in general by higher degree cohomology.

1.11 Lemma. *Let Γ be a discrete group, φ a homomorphism of Γ, $(\pi,\mathcal{H})$ a representation of $\varphi(\Gamma)$, and $b \in Z^1(\varphi(\Gamma),\pi)$. Consider π^φ – the Γ-representation on $\mathcal{H}$ factoring through φ, and $b^\varphi \in Z^1(\Gamma,\pi^\varphi)$ – the Γ-cocycle with coefficients in π^φ, factoring via φ. Then:*

(1) *$b^\varphi \in B^1(\Gamma,\pi^\varphi)$ if and only if $b \in B^1(\varphi(\Gamma),\pi)$.*
(2) *$b^\varphi \in \overline{B^1(\Gamma,\pi^\varphi)}$ if and only if $b \in \overline{B^1(\varphi(\Gamma),\pi)}$.*

We leave the verification of this simple fact to the reader.

II. Lattices and induction of cohomology. In this subsection we examine induction from the lattice to the ambient group, at the level of the first cohomology. This construction will be central in what follows. For simplicity, we assume throughout this subsection that all the representations are **unitary** (so spaces are Hilbert).

Let G be a locally compact group, and $\Gamma < G$ a discrete co-compact subgroup. Fix a Haar measure μ on G, normalized so that the induced measure on G/Γ has total mass one. There are two well known equivalent ways to induce (unitarily) representations from Γ to G, and we shall need both of them in the sequel. For the first, we take a bounded fundamental domain X: $G = X\Gamma$, and consider the associated cocycle: $\alpha: G \times X \to \Gamma$, defined by

$$\alpha(g,x) = \gamma \quad \text{iff} \quad gx\gamma \in X. \tag{1.1}$$

If $(\pi,\mathcal{H})$ is a unitary Γ-representation, then the induced representation $Ind_\Gamma^G\pi$ is realized on the space $L^2(X,\mathcal{H})$, of $\mathcal{H}$-valued L^2-functions on X, where G acts by:

$$gf(x) = \pi \circ \alpha(g^{-1},x) f(g^{-1}\cdot x) \tag{1.2}$$

We have put the dot notation in $g^{-1}\cdot x$ to emphasize that this multiplication is the G-action on $X \cong G/\Gamma$, but we suppress this notation henceforth.

Next, let $b \in Z^1(\Gamma,\pi)$ and consider, for every $g \in G$, the function $\tilde{b} : G \to L^2(X,\mathcal{H})$ defined by:

$$\tilde{b}(g)(x) = b \circ \alpha(g^{-1},x). \tag{1.3}$$

It is easy to verify that (1.3) satisfies the formal cocycle identity (1.0). However to assert that indeed $\tilde{b} \in Z^1(G, Ind_\Gamma^G\pi)$, one first has to check the integrability condition:

$$\int_X \|\tilde{b}(g)(x)\|^2 d\mu(x) < \infty \qquad \forall g \in G. \tag{1.4}$$

Since we assume that Γ is co-compact, (1.4) is easily verified. If Γ is a **nonuniform** lattice, (1.4) may fail for any choice of X (e.g., for certain lattices in $SL_2(\mathbb{F})$, $\mathbb{F} = \mathbb{R},\mathbb{C}$, where π is the trivial representation). However, one can show that an intrinsic property of a (fundamental domain X for) Γ, ensures that (1.4) holds for **all** the unitary Γ-representations. To state this property we first need to assume that Γ is finitely generated, say by a finite set S. Let $l(\gamma) = l_S(\gamma)$ denote the word length corresponding to S. The condition alluded to is then the following:

$$\int_X \ell(\alpha(g,x))^2 d\mu(x) < \infty \qquad \forall g \in G \tag{1.5}$$

We refer to [Sh2, §3.III] for details, references, and an elaborate discussion on these as well as other claims stated here without proofs. We remark that it is shown in [Sh2] that (1.5) holds for all nonuniform lattices in rank one simple Lie groups, not locally isomorphic to $SL_2(\mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}$, and in the next section we show that it holds also for all the other nonuniform lattices we shall consider in this paper, namely, the ones in **(0.2)**. Once (1.5) is satisfied, which we assume hereafter, all the results concerning the first cohomology discussed in the sequel, continue to hold. In fact, all the results of the present paper are valid for nonuniform lattices in the framework of **(0.1)** and **(0.2)**, as long as they satisfy (1.5) (see also the modifications for lattices in $SL_2(\mathbb{C})$, in §10.4).

The map $b \to \tilde{b}$ described in (1.3), is easily seen to induce a map between the first cohomology, and first reduced cohomology groups, of the corresponding representations. The treatment of the former is easier, and the resulting map is an isomorphism. In the latter case this map is also an isomorphism, but the result is technically more involved, and we consider it in [Sh4]. The issue here is the injectivity of the map $b \to \tilde{b}$. We emphasize however, that using Corollary 1.8 one can transfer the whole discussion to the framework of ordinary cohomology, whenever the representation π does not admit almost invariant vectors. This will be the case in all the proofs of the rigidity theorems stated in the introduction, besides Theorem 0.1 (see also the discussion at the ending paragraph of Section 10). Although for most of the applications the ordinary notion of cohomology is enough, it will become evident that the right framework is that of reduced cohomology. Summarizing the above discussion, we have:

Proposition 1.12. *For a unitary Γ-representation π, when Γ is any of the lattices in* **(0.1)** *or* **(0.2)***, the map $b \to \tilde{b}$ of (1.3) defines isomorphisms:*

$$H^1(\Gamma, \pi) \cong H^1(G, Ind_\Gamma^G \pi) \qquad \overline{H^1}(\Gamma, \pi) \cong \overline{H^1}(G, Ind_\Gamma^G \pi)$$

We now recall the second model for induction of representations. Given a unitary Γ-representation $(\pi, \mathcal{H})$, the representation space for $Ind_\Gamma^G \pi$ is the space $L^2(G, \mathcal{H})$ of functions $f: G \to \mathcal{H}$ satisfying the two properties:

$$f(g\gamma) = \pi(\gamma)^{-1} f(g) \quad \forall \gamma \in \Gamma \qquad \text{and} \quad \int_{G/\Gamma} \|f(g)\|^2 d\mu(g) < \infty \tag{1.6}$$

where G operates from the left: $gf(x) = f(g^{-1}x)$. This description is somewhat more natural in the setting of representations, as it does not involve a choice of a fundamental domain.

Let us develop an explicit formula for the induced cocycle, analogous to (1.3). The point of view of isometric actions becomes here very useful. By Lemma 1.5 and the discussion preceding it, 1-cocycles correspond to isometric actions, and induction of isometric actions is easily describe in this model. Indeed, suppose $b \in Z^1(\Gamma, \pi)$ and let $\rho(\gamma)v = \pi(\gamma)v + b(\gamma)$ be the affine isometric Γ-action corresponding to b. Consider the space of measurable, locally bounded functions $f : G \to \mathcal{H}$, satisfying

$$f(g\gamma) = \rho(\gamma)^{-1} f(g) \qquad \forall \gamma \in \Gamma \tag{1.7}$$

and let $\tilde{L^2}(G, \mathcal{H})$ be its completion under the metric D defined by:

$$D^2(f, \varphi) = \int_{G/\Gamma} \|f(g) - \varphi(g)\|^2 d\mu(g) \tag{1.8}$$

As before, G operates on $\tilde{L^2}(G, \mathcal{H})$ isometrically, by translations from the left. Notice that by (1.7) the integrand in the right hand side of (1.8) is indeed Γ-invariant. If Γ is co-compact, then the finiteness of (1.8) for any two locally bounded functions is obvious. In the case of a non co-compact subgroup Γ, there is no natural choice of a "pre-Hilbert" subspace, like the locally bounded functions. However, one can take a fundamental domain X, fix an "origin" f_0 defined by the unique extension of the zero function on X to a function satisfying the Γ-equivariance condition, and define $\tilde{L^2}(G, \mathcal{H})$ to be the space of functions with finite L^2-distance from f_0. In general, this space will not be preserved under the G-action, a property which is equivalent to having **one** function whose G-translations stay in the space (since G-preserves distances). Spelling out the condition for the function f_0 and using the same reasoning as in [Sh2, §3], shows that condition (1.5) is sufficient for this property to be satisfied as well. Thus, when (1.5) is satisfied, we may continue the discussion safely.

The space $\tilde{L^2}(G, \mathcal{H})$ is an **affine Hilbert space** (see [HV, p.44]). By fixing an arbitrary "origin": $f_0 \in \tilde{L^2}(G, \mathcal{H})$, the map $f \to f - f_0$ is an isometry between $\tilde{L^2}(G, \mathcal{H})$ and the Hilbert space $L^2(G, \mathcal{H})$ defined in (1.6). The G-translation action on the former corresponds via this map to an affine action on the latter. More precisely, for any $f \in L^2(G, \mathcal{H})$ and $g \in G$, we define the action $\tilde{\rho}$ by:

$$\tilde{\rho}(g)(f) = g(f + f_0) - f_0 = f \circ g^{-1} + f_0 \circ g^{-1} - f_0 = \sigma(g) f + b'(g) \qquad b'(g) = f_0 \circ g^{-1} - f_0$$

where σ denotes the linear G-representation in the induced representation. Once we fix a fundamental domain X for Γ, identify $L^2(G, \mathcal{H})$ with $L^2(X, \mathcal{H})$, and take f_0 as in the preceding paragraph, the cocycle b' identifies with the cocycle $\tilde{b}$ of (1.3). In particular, we deduce from Proposition 1.12 and Lemma 1.10:

1.13 Lemma. *If the affine isometric action of Γ on $\mathcal{H}$ is uniform, then so is the induced isometric G-actions on $\tilde{L^2}(G, \mathcal{H})$ and $L^2(G, \mathcal{H})$.*

Although this relation between the two models of induction is simple, their interplay will turn out to be very useful in the sequel.

§2. Nonuniform higher rank lattices

The purpose of this section is to show that the nonuniform lattices in **(0.1)** satisfy condition (1.5) in Section 1. Thus, the first cohomology of their unitary representations can always be "induced" to the ambient group, and our rigidity results apply to them as well (see the discussion in §1.II). The result is technical in nature, and relies primarily on a deep theorem of Lubotzky Mozes and Raghunathan [LMR]. Hence the reader may prefer to skip this section at first reading.

We begin by recalling facts concerning S-arithmetic lattices, following [Mar1, Ch. VIII p. 258]. By Margulis' arithmeticity theorem, and after replacing Γ with a commensurable subgroup, we can assume that there exists a global field K, a finite set S of valuations on K, and a connected linear almost K-simple K-group $\mathbb{G}$, such that $G = \mathbb{G}_S = \prod_{\nu \in S} \mathbb{G}(K_\nu)$, and $\Gamma = \mathbb{G}(K(S))$

We view $\mathbb{G}$ as a K-subgroup of SL_n. For every $\nu \in S$, $y = (y_1, \dots, y_n) \in K_\nu^n$, $B \in End(K_\nu^n)$, and $g \in \mathbb{G}_S$, we define the norms

$$\|y\|_\nu = \sum_{i=1}^{n} |y_i|_\nu \qquad \|B\|_\nu = \sup_{0 \neq y \in K_\nu^n} \|By\|_\nu / \|y\|_\nu \qquad \|g\| = \max_{\nu \in S} \|pr_\nu(g)\|_\nu$$

where $|\ |_\nu$ is an absolute value associated with ν, and $pr_\nu : \mathbb{G}_S \to \mathbb{G}(K_\nu)$ is the natural projection.

The reduction theorem of Borel-Harish-Chandara-Behr-Harder exhibits a fundamental domain Y for Γ, which is a finite union of "generalized Siegel domains". It has the property that

$$\int_Y \ln^p \|g\| d\mu(g) < \infty \tag{2.1}$$

for every $p < \infty$, and in particular for $p = 2$. See [Mar1, p.260 Proposition 1.2] for details. Although it is asserted there only that $\int \ln \|g\| d\mu(g) < \infty$, the references cited actually show that (2.1) hold. (The situation of $SL_2(\mathbb{Z}) < SL_2(\mathbb{R})$, where (2.1) reduces to the integrability condition $\int_{[1,\infty)} \ln^p y/y^2 dy < \infty$ is good to keep in mind. The general case is even "better".)

Let us replace the fundamental domain Y with a "Dirichlet" fundamental domain X:

$$X = \{g \in G |\ \|g\gamma\| \geq \|g\| \quad \forall \gamma \in \Gamma\}. \tag{2.2}$$

We can now establish the assertion claimed at the beginning of the section. Let $\alpha : G \times X \to \Gamma$ be the associated cocycle, namely, for $g \in G$ and $x \in X$, write the unique decomposition $gx = x_1\gamma$ with $x_1 \in X$, $\gamma \in \Gamma$, so $\alpha(g, x) = x_1$. Notice that from (2.2) it follows that $\|x_1\| \leq \|gx\| \leq \|g\| \cdot \|x\|$. Since $\mathbb{G} \subset SL_n$, the entries of x_1^{-1} can be expressed as polynomials in the entries of x_1, and hence there exist global constants c, d, such that $\|x_1\|^{-1} \leq c\|x_1\|^d$. Putting all this together yields:

$$\|\alpha(g, x)\| = \|\gamma\| = \|x_1^{-1} gy\| \leq \|x_1^{-1}\| \cdot \|g\| \cdot \|x\| \leq c\|x_1\|^d \|g\| \cdot \|x\| \leq c \cdot \|g\|^{d+1} \|x\|^{d+1}. \tag{2.3}$$

Our interest is in the word length $\ell(\gamma)$ with respect to some fixed generating set of Γ. The relation between $\ell(\gamma)$ and $\|\gamma\|$ for higher rank (S-)arithmetic groups, is described by the deep result of Lubotzky-Mozes-Raghunathan [LMR], which says that there exist global constants a, b such that for every $\gamma \in \Gamma$:

$$\ell(\gamma) \leq a \ln \|\gamma\| + b \tag{2.4}$$

Therefore we can conclude from (2.3) and (2.4) that for a global constant C, we have for every fixed $g \in G$:

$$\int_X \ell^2(\alpha(g,x))d\mu(x) \leq \int_X (a \ln \|\alpha(g,x)\| + b)^2 d\mu(x) \leq$$

$$C \int_X (\ln^2 \|x\| + \ln \|x\| + \ln^2 \|g\| + \ln \|g\| + 1)d\mu(x). \tag{2.5}$$

If the integration in (2.5) was taken over the original fundamental domain Y, then the finiteness of (2.5) obviously follows from (2.1). However, from the definition of X, it is clear that the expression in (2.5) for X is bounded from above by the same integral over any other fundamental domain, in particular, the domain Y. This completes the proof. $\square$

§3. The reduced cohomology for products of groups

In this section we establish the fundamental ingredient in the proof of the cohomology rigidity theorem (stated in the next section- see Theorem 4.1). The result here provides a complete description of the first **reduced** cohomology of unitary representations for products of groups, in terms of the cohomology of the individual factors. Throughout this section, all the groups are assumed to be locally compact and second countable (lcsc). Here is our main result:

3.1 Theorem. *Let $(\pi, \mathcal{H})$ be a unitary representation of $G = G_1 \times \ldots \times G_n$, and $b \in Z^1(G, \pi)$. Then b is almost cohomologous (Def. 1.7) to a cocycle of the form $b_0 + \ldots + b_n$ with the following properties: b_0 takes values in the subspace of G-invariant vectors, and each b_i, for $1 \leq i \leq n$, is a cocycle depending only on G_i, and takes values in a G-invariant subspace for which the G-action factors through an action of G_i.*

The main ingredient in the proof of Theorem 3.1 is the following result:

3.2 Proposition. *Let $(\pi, \mathcal{H})$ be a continuous unitary representation of $G = G_1 \times G_2$. Then either $\overline{H^1}(G, \pi) = 0$ (Def. 1.7), or there exists a nonzero invariant vector in $\mathcal{H}$ for one of the G_i's.*

Remark. $\overline{H^1}$ cannot be replaced by H^1 in general, neither in 3.1 nor in 3.2.

In view of Lemma 1.10, Proposition 3.2 may be reformulated geometrically as follows:

3.3 Proposition (Equivalent formulation of 3.2). *Suppose that $G = G_1 \times G_2$ acts continuously by affine isometries on a Hilbert space $\mathcal{H}$. If the G-action is uniform (Def. 1.9), then there is a non zero invariant vector for the linear action of one of the G_i's.*

We start by concentrating on the proof of Proposition 3.3, and preface its proof with a general claim, whose relevance to our discussion will be revealed shortly:

3.4 Lemma. *Let G be a lcsc group, $Q \subset G$ a compact subset, and C a set of indices. Suppose that to every $x \in C$ corresponds a continuous function $0 \leq f_x : Q \to \mathbb{R}$ such that the following two conditions are satisfied:*

(i) *The f_x are uniformly continuous, namely, for every $\varepsilon > 0$ there exists a neighborhood U of the identity $e \in G$, such that if $g, h \in Q$ and $g^{-1}h \in U$, then for every $x \in C$: $|f_x(g) - f_x(h)| < \varepsilon$.*

(ii) *There exists $\lambda < 1$ such that for every $h \in Q$ and every $x \in C$, there exists $y \in C$ satisfying both $f_y(h) \leq \lambda f_x(h)$, and $f_y(g) \leq f_x(g)$ for all $g \in Q$.*

Then, for every $\varepsilon > 0$ there is $x \in C$, such that $f_x(g) < \varepsilon$ for all $g \in Q$.

Proof. Fix some f_{x_0} for $x_0 \in C$, and let L be a number such that $f_x(g) < L$ for all $g \in Q$. The set of all f_x's satisfying this bound is not empty, uniformly bounded, and by (i), uniformly continuous. By Arzela-Ascolli its closure is compact in the uniform convergence topology, and hence contains a (non negative) function which minimizes the maximum value. It only remains to show that this function must be identically zero. If not, one easily verifies that using condition (ii), a function f_y which decreases the maximum value can be obtained. We leave the details to the reader. $\square$

Proof of Proposition 3.3. Assuming there is **no** invariant vector for the linear action of each G_i (separately), we show that the G-action on $\mathcal{H}$ is **not** uniform. The proof proceeds in two steps:

I. Under the above assumption, the action of each G_i is not uniform.

II. If G acts by isometries on a Hilbert space and both G_i-actions are not uniform, then so is the G-action.

Let us start with II, and notice first the following general useful fact. If a group L acts isometrically, non uniformly, on a Hilbert space $\mathcal{H}$, then in **any** closed, convex, L-invariant subset $C \subset \mathcal{H}$, one can find a sequence of almost fixed points (in the usual sense). In other words, the uniformity fails already inside any such C. Indeed, recall that in a Hilbert space every closed convex subset C admits a projection $P_C : \mathcal{H} \to C$, defined by the condition that $P_C(x)$ is the unique point in C which is closest to x. Furthermore, P_C is a semi-contraction: $||P_C(x) - P_C(y)|| \leq ||x - y||$. By our assumption on L-invariance of C, we clearly have $P_C(gx) = gP_C(x)$, $\forall g \in L$. From the fact that P_C is a semi-contraction, we get for every $g \in L$ and $x \in \mathcal{H}$: $d(P_C(gx), P_C(x)) \leq d(gx, x)$. Thus if x is (Q, ε)-invariant, then so is $P_C(x) \in C$, hence our claim.

Returning to the proof of step II, let $\varepsilon > 0$ be given, and $Q_1 \times Q_2 \subset G_1 \times G_2 = G$ be compact subsets. Denote by $C \subset \mathcal{H}$ the set of (Q_1, ε) invariant points in $\mathcal{H}$, namely, the x's for which $d(g_1x, x) \leq \varepsilon$, $\forall g_1 \in Q_1$. By our assumption C is not empty. It is clearly closed and convex. Moreover, it is G_2-invariant: if $x \in C$ then $d(g_1g_2x, g_2x) = d(g_2g_1x, g_2x) = d(g_1x, x) \leq \varepsilon$ for all $g_2 \in G_2$ and $g_1 \in Q_1$. Since we assume the G_2-action is also not uniform, from the discussion in the previous paragraph it follows that we can find in C a (Q_2, ε)-invariant point. Finally, if $g_1 \in Q_1$, $g_2 \in Q_2$ and $x \in C$, we have $d(g_1g_2x, x) \leq d(g_1g_2x, g_1x) + d(g_1x, x) = d(g_2x, x) + d(g_1x, x) \leq 2\varepsilon$, showing that there are almost fixed points for the action of $Q_1 \times Q_2$. This proves II.

Let us discuss the proof of I. The argument we present is based on a suggestion of G. A. Margulis, to whom we extend our gratitude. Although it is less transparent than our original one (see [Sh1 p.839]), it is completely geometric, and eliminates the use of a spectral theorem. Actually, it yields a more general result which we shall exploit in the discussion of non uniform lattices in $SL_2(\mathbb{C})$ in Section 10 below. In that section we shall also reproduce an argument similar to our original one (using a spectral theorem) for the proof of step I (see Theorem 10.1). We remark further that while the Hilbert space structure was essential in the proof of step II, giving rise to projections on convex subsets, this is not the case here. For later use, but also for clarity of the exposition, we mention already at this point that the only property of Hilbert spaces we shall use is that they are Banach spaces which satisfy the following uniform convexity condition, for some global $t < 3$:

$$\|v + w\| + \|v - w\| \leq t < 3 \qquad \text{for all } v, w \text{ with } \|v\| = \|w\| = 1 \tag{3.1}$$

Of course, a Hilbert space satisfies (3.1) for every $2\sqrt{2} \leq t < 3$, and we fix such t in what follows.

Proof of Step I. Without loss of generality, we may assume that there is no non-zero G_1-invariant vector for its linear action in $\mathcal{H}$, and for any given compact subset $Q \subset G_2$, and $\varepsilon > 0$, we need to find a (Q, ε) invariant point. To minimize notation, we continue to denote the action, as before, simply by $(g, x) \to gx$. However, to distinguish this action from the linear part of it, we shall denote the latter by π.

Fix some $x_0 \in \mathcal{H}$ and denote $C = \text{Conv } G_1 x_0$, the (not necessarily closed) convex hull of its orbit under G_1. For every $x \in C$ define the function $f_x : Q \to \mathbb{R}$ by $f_x(g) = \|gx - x\|$. We shall show that the f_x's always satisfy condition (i) in Lemma 3.4, and that with our assumption that the G_1-action has no invariant vector, they also satisfy (ii) for any λ with $\frac{1}{2}(t-1) < \lambda < 1$ (where t is taken from 3.1). Notice that the conclusion of Lemma 3.4 then translates exactly to our claim, namely, that G_2 admits almost fixed points, thereby proving Proposition 3.3.

In order to show (i), let $\varepsilon > 0$ be given and take $e \in U \subset G_2$ such that $\|gx_0 - x_0\| \leq \varepsilon$ for all $g \in U$ (continuity of the G_2-action at x_0). We claim that this U satisfies the required condition for all the functions f_x, at all the points g. First let us check it for f_{x_0}. Indeed, for every $g, h \in Q$:

$$|f_{x_0}(g) - f_{x_0}(h)| = |\|gx_0 - x_0\| - \|hx_0 - x_0\|| \leq \|gx_0 - hx_0\| = \|h^{-1}gx_0 - x_0\|$$

Next, we check this uniform of continuity for every $x = g_1 x_0$, where $g_1 \in G_1$ is any element. Indeed, since the G_1 and G_2-actions commute, we have

$$f_{g_1 x_0}(g) = \|gg_1 x_0 - g_1 x_0\| = \|g_1 g x_0 - g_1 x_0\| = \|gx_0 - x_0\| = f_{x_0}(g) \tag{3.2}$$

Finally, it is easy to verify that passing to convex combinations of points in $G_1 x_0$ will still leave us with the same uniform continuity.

We now come to the main argument of the proof, which is showing that condition (ii) in 3.4 is satisfied, for any $\frac{1}{2}(t-1) < \lambda < 1$. Let $x \in C$ and $h \in Q \subset G_2$ be given, and by rescaling the norm, we may assume without loss of generality, that $f_x(h) = \|hx - x\| = 1$ (note that the result we seek is indeed invariant under rescaling). We examine two possibilities:

1. There exists $g_1 \in G_1$ with $\|(hx - x) - (g_1 hx - g_1 x)\| \geq 1$

2. For every $g_1 \in G_1$ the opposite inequality holds.

We shall see that in case **2** holds, there is a non-zero G_1-invariant vector (which contradicts our assumption), and that **1** implies the validity of condition (ii) in 3.4. Suppose first that **2** holds. Then denoting $v = hx - x$ we have by our normalization $\|v\| = 1$. For every $g_1 \in G_1$ we get: $g_1hx - g_1x = \pi(g_1)(hx - x) = \pi(g_1)v$, hence $\|v - \pi(g_1)v\| \leq 1$. Since v is a unit vector which belongs to $\pi(G_1)v$, the latter set contains a vector which has distance 1 from 0. But v has distance at most 1 from every element of $\pi(G_1)v$, so by the uniqueness the center of mass, it cannot be 0 (see Proposition 1.2), and we found a non zero invariant vector for the linear G_1-action.

We therefore conclude that condition **1** must hold. Take $g_1 \in G_1$ as given by **1**, denote again $v = hx - x$, and set $w = g_1hx - g_1x$. We will see that $y = \frac{1}{2}(x + g_1x)$ is a point as required in condition (ii) of 3.4 (with $\lambda = \frac{1}{2}(t-1) < 1$), thereby completing the whole proof of Proposition 3.3. Indeed, notice that

$$hy - y = h(\frac{1}{2}x + \frac{1}{2}g_1x) - \frac{1}{2}(x + g_1x) = \frac{1}{2}(hx + hg_1x - x - g_1x) = \frac{1}{2}[(g_1hx - g_1x) + (hx - x)] = \frac{1}{2}(w + v)$$

By definition of t we have $\|\frac{1}{2}(v + w)\| + \|\frac{1}{2}(v - w)\| < \frac{1}{2}t$. As the left hand side in the inequality of condition **1** is exactly $\|v - w\|$, we deduce

$$\|hy - y\| = \frac{1}{2}\|v + w\| < \frac{1}{2}t - \frac{1}{2}\|v - w\| \leq \frac{1}{2}t - \frac{1}{2}$$

Notice that y is a convex combination of x and g_1x, so as noted above, $f_y(g) \leq f_x(g)$ for all $g \in Q$. Thus, $\lambda = \frac{1}{2}(t - 1)$ satisfies the required condition in (ii) of 3.4. This completes the proof of Proposition 3.3 (and 3.2). $\square$

The argument in the proof of step I shows that if $G = G_1 \times G_2$ acts affinely and isometrically on $\mathcal{H}$, and the G_2-action is uniform, then there is a G_1-invariant vector for the linear action. A careful examination of the proof shows that the same result holds if G_1 is a cyclic semi-group of semi-contractions, namely, $G_1 = \{T^n \mid n \in \mathbb{N}\}$, where $Tv = T_0v + v_0$ is an affine operator with $\|T_0\| \leq 1$. Indeed, an inequality in (3.2) continues to hold in this setting. The dichotomy in the proof between the two possibilities **1** and **2**, should now be made when the value 1 in the inequality is replaced by some $\alpha < 1$. This dichotomy leads to a similar conclusion: when **2** holds for every $g_1 = T^n$, one gets a unit vector v satisfying $\|T_0^n v - v\| \leq \alpha < 1$, and any weak limit point of the sequence $\frac{1}{n}(T_0v + ... + T_0^n v)$ is a non zero T_0-invariant vector. In the situation of **1**, we derive in a similar way non uniformity of the G_2-action. To summarize, we state for later reference:

3.5 Corollary. *If a locally compact group G acts isometrically, affinely, and uniformly (Def. 1.9) on a Banach space $\mathcal{H}$ satisfying (3.1) (in particular, on a Hilbert space), then any affine semi-contarcting operator T on $\mathcal{H}$ which commutes with the G-action, admits a non zero fixed vector for its linear action.* $\square$

We mention another independent, immediate consequence of step I in the proof of Proposition 3.3:

3.5 Corollary. *If G acts isometrically affinely and uniformly on a Banach space which satisfies condition (3.1), then its center $Z(G)$ admits a non-zero invariant vector for its linear action.*

Proof. View the G-action as one of $G \times Z(G)$ factoring through the homomorphism $(g,z) \to gz$. $\square$

We are now able to complete the proof of Theorem 3.1

Proof of Theorem 3.1. We consider first the case $n=2$, i.e., we remain in the situation $G = G_1 \times G_2$. Let $(\pi, \mathcal{H})$ be a unitary G-representation, denote by $\mathcal{H}_0$, $\mathcal{H}_1$, $\mathcal{H}_2$ the subspaces consisting of the G-invariant vectors, G_1-invariant but not G_2-invariant vectors, and G_2-invariant but not G_1-invariant vectors, respectively. Clearly all the three are G-invariant subspaces and we denote by $\mathcal{H}^-$ the orthogonal complement to their sum. Let $b \in Z^1(G,\pi)$ be a cocycle. By Proposition 3.2 it follows that the projection of b to $\mathcal{H}^-$ must be almost a coboudary (Def. 1.7). Let us consider now the projection of b to $\mathcal{H}_1$, which we denote by b_1. We wish to show that b_1 must factor through G_2, namely, it vanishes on G_1. One can give also here a geometric argument, but the algebraic one is particularly simple: By the cocycle identity we have for every $g_1 \in G_1$ and $g_2 \in G_2$:

$$b_1(g_2) + \pi(g_2)b_1(g_1) = b_1(g_2 g_1) = b_1(g_1 g_2) = b_1(g_1) + \pi(g_1)b_1(g_2)$$

Since $\pi(G_1) = I$, we can cancel $b_1(g_2)$ from both sides to get $\pi(g_2)b_1(g_1) = b_1(g_1)$. In other words, each $b_1(g_1)$ is fixed by all $\pi(g_2)$, so by our assumption on $\mathcal{H}_1$ it must be zero. Of course, the same argument applies to show that the projection b_2 of the cocycle b to $\mathcal{H}_2$ depends only on G_1. This completes the proof of Theorem 3.1 in the case $n=2$. The general case $G = G_1 \times \ldots \times G_n$ is established by successive iterations of the above, first dealing with $G = G_1 \times G^1$ (where $G^1 = G_2 \times \ldots \times G_n$), by applying the result in the case $n=2$, and then continuing with the subrepresentation and cocycle which factor through G^1, as before. $\square$

3.6 Corollary. *Let G be a locally compact group, $Z(G)$ its center, and $(\pi,\mathcal{H})$ a unitary G-representation without non-zero invariant vectors. Then every cocycle $b \in Z^1(G,\pi)$ is almost cohomologous to a cocycle which factors through $G/Z(G)$, and takes values in a subrepresentation where the G-action factors through $G/Z(G)$.*

Proof. Apply an argument similar to that in the proof of Corollary 3.5. $\square$

In Theorem 6.1 below we show that if a compactly generated group G fails to have property (T), then it has a unitary representation with $\overline{H^1}(G,\pi) \neq 0$. This and Corollary 3.6 easily yield the following result of Serre (which is useful in constructing Kazhdan groups with infinite center -see [HV, p.28, p.41]): If G is compactly generated, $G/\overline{[G,G]}$ is compact, and $G/Z(G)$ has property (T), then G has property (T) as well. In a different direction, Theorem 3.1 provides a general point of view at the following result of Bekka and Valette, which can be used to show that the first ℓ^2-Betti number of a product of any two infinite groups vanishes (see [BV]): If $G = G_1 \times G_2$ is non amenable, and both G_i's are non compact, then $H^1(G, L^2(G)) = 0$. Indeed, all that we use is the fact that $L^2(G)$ does not admit almost invariant vectors, and that it does not contain a non-zero vector fixed by one of the G_i's.

§4. The superrigidity theorem for reduced cohomology

In this section we state and prove the superrigidity theorem for reduced cohomology, which will be essential to our rigidity results for irreducible lattices. The proof relies on the study of the reduced cohomology in the previous section, subsumed by Theorem 3.1. The result is as follows:

4.1 Theorem (Superrigidity for reduced cohomology). *Let Γ, G_i and G be as in* **(0.1)**. *Let $(\pi, \mathcal{H})$ be a unitary Γ-representation and $b \in Z^1(\Gamma, \pi)$. Then b is* **almost cohomologous** *(Def. 1.7) to a cocycle of the form $b_0 + b_1 + \ldots + b_n$, where b_0 takes values in the subspace of Γ-invariant vectors, and each b_i, for $1 \leq i \leq n$, takes values in a Γ-invariant subspace $\mathcal{H}_i \subseteq \mathcal{H}$, on which the Γ-representation extends continuously to a G-representation which factors through a representation of G_i. Furthermore, each b_i extends continuously to a cocycle depending only on G_i for $1 \leq i \leq n$, and b_0 extends continuously to a cocycle of G.*

The following result is, in light of Proposition 1.6, an immediate consequence of Theorem 4.1:

4.2 Corollary. *Retain the notation in* **(0.1)**. *Let π be a unitary Γ-representation for which the orthogonal complement to the invariant vectors does not contain weakly the trivial representation. Then every cocycle $b \in Z^1(\Gamma, \pi)$ is* **cohomologous** *to a cocycle of the form $b_0 + b_1 + \ldots + b_n$, as in Theorem 4.1.*

A weaker result in the spirit of Corollary 4.2 was recently established by Louvet [Lo] (independently and simultaneously to [Sh1]). We begin with a basic result on superrigidity of homomorphisms of irreducible lattices. The weak assumption on the target will enable us to apply the result in various general settings.

4.3 Proposition. *Retain the notation of* **(0.1)**. *Let H be a group and $\mathfrak{T}$ a Hausdorff topology on H such that the following completeness condition is satisfied: For any sequence $h_k \in H$ such that $\lim_{i,j\to\infty} h_j^{-1}h_i = 1_H$, there exists an element $h \in H$ with $\lim h_k = h$. Let $\varphi : \Gamma \to H$ be a homomorphism satisfying the following condition: There exists one $1 \leqslant i \leqslant n$, such that whenever $\gamma_k \in \Gamma$ is a sequence with $pr_i(\gamma_k) \to 1_{G_i}$ (where pr_i is the projection to G_i), we have $\varphi(\gamma_k) \to 1_H$. Then φ extends to a continuous homomorphism of G, factoring through G_i. More precisely, there exists a continuous homomorphism $\widetilde{\varphi} : G_i \to H$ such that for every $\gamma \in \Gamma : \varphi(\gamma) = \widetilde{\varphi} \circ pr_i(\gamma)$.*

Proof. We shall define the homomorphism $\widetilde{\varphi}$ and show it satisfies the required condition. For simplicity denote $pr = pr_i$, and $G_1 = G_i$. Let $g \in G_1$ and choose, by density of the projection of Γ to G_1, a sequence $\gamma_k \in \Gamma$ with $pr(\gamma_k) \to g$. Notice that the sequence $h_k = \varphi(\gamma_k)$ satisfies the condition in 4.3, namely, $\lim h_j^{-1}h_i = 1_H$, since $h_j^{-1}h_i = \varphi(\gamma_j^{-1}\gamma_i)$ and clearly $pr(\gamma_j^{-1}\gamma_i) \to 1_{G_1}$. By the assumption of 4.3 it follows that there exists $h \in H$ with $\lim \varphi(\gamma_k) = h$, and we define $\widetilde{\varphi}(g) = h$. That $\widetilde{\varphi}$ is well defined, i.e., does not depend on the choice of γ_k, follows from the above argument by concatenating any two sequences, and it is also easy to check that it is a homomorphism, which is continuous. Next, notice that Ker $\varphi \subset$ Ker pr. Indeed, if $pr(\gamma) = 1_{G_1}$ then by considering the constant sequence $\gamma_k = \gamma$ we get $\varphi(\gamma) = \varphi(\gamma_k) \to 1_H$, and since $\mathfrak{T}$ is Hausdorff, $\varphi(\gamma) = 1_H$. It follows that as an abstract homomorphism, φ factors through the projection pr, and in light of the construction of $\widetilde{\varphi}$ it is now clear that for every $\gamma \in \Gamma$, we have $\varphi(\gamma) = \widetilde{\varphi} \circ pr(\gamma)$. $\square$

The following result will enable us to verify the condition of 4.3 in the case we shall need:

4.4 Proposition. *Retain the notation of* **(0.1)**. *Let ρ denote an (affine) isometric Γ–action on a Hilbert space $\mathcal{H}$, and assume that the induced isometric G–action $Ind_\Gamma^G\rho$ (see (1.7) in section 1.II) admits a point f fixed by some $\check{G}_i = G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n$. Then there exists a point $v \in \mathcal{H}$ such that for any sequence $\gamma_k \in \Gamma$ with $pr_i(\gamma_k) \to 1_{G_i}$, one has $\rho(\gamma_k)v \to v$.*

Proof. The argument is standard (cf. [Mar1, p.138 Theorem 6.3] or [BL Lemma 1], and we sketch it for completeness. We adapt the model for $Ind_\Gamma^G\rho$ described in (1.7) of Section 1.II. Notice that the set of functions f with the property assumed in 4.4 is invariant under the whole G-action. Also observe that for any continuous nonnegative function $\varphi : G \to \mathbb{R}$, having compact support and satisfying $\int \varphi = 1$, the convolution by φ defines an affine operator on $Ind_\Gamma^G\rho$. Thus, we may assume that the function $f : G \to \mathcal{H}$ satisfying $f(g\gamma) = \rho(\gamma)^{-1}f(g)$ for all $\gamma \in \Gamma$, is continuous. Set $v = f(e) \in \mathcal{H}$. For any $\gamma \in \Gamma$ we then have $f(\gamma^{-1}) = f(e\gamma^{-1}) = \rho(\gamma)f(e) = \rho(\gamma)v$. Since both the function f as well as the G-action on it factor through the projection to G_i, it follows from the continuity of f that whenever $pr_i(\gamma_k) \to 1_{G_i}$, we have $\rho(\gamma_k)v = f(\gamma_k^{-1}) \to f(e) = v$. $\square$

To extend continuously (cohomologous) cocycles in Theorem 4.1, we will use the following:

4.5 Lemma. *Retain the notation in* **(0.1)**. *Let $(\pi, \mathcal{H})$ be a unitary Γ-representation, $b \in Z^1(\Gamma, \pi)$, and $\gamma v = \pi(\gamma)v + b(\gamma)$ the corresponding affine Γ-action on $\mathcal{H}$. Suppose that for some $1 \le i \le n$, there exists an affine Γ-invariant subspace, say, $w + \mathcal{H}_i$, on which the Γ-action extends continuously to G, factoring through an action of G_i. Then:*

1. *The linear subspace $\mathcal{H}_i$ is invariant under the linear Γ-action π, and this Γ-action extends continuously to a unitary representation of G, factoring through G_i.*
2. *The cohomologous cocycle $b(\gamma) + \pi(\gamma)w - w$ takes values in $\mathcal{H}_i$, and extends continuously to a cocycle of G, factoring through G_i.*

Proof. For $v \in \mathcal{H}_i$ we have: $\gamma(v + w) - \gamma w = \pi(\gamma)v + \pi(\gamma)w + b(\gamma) - \pi(\gamma)w - b(\gamma) = \pi(\gamma)v$. Thus the linear Γ-action on $\mathcal{H}_i$ may be retrieved from its affine action, and by the assumption it is clear that this action extends to G_i (unitarily, by density of the Γ projection to G_i). This proves (1).

Next, the difference between any two points in the affine subspace lies in $\mathcal{H}_i$. Thus, using the assumption we see that the function $\gamma \to \gamma w - w \in \mathcal{H}_i$ extends continuously to G (factoring through G_i). But $\gamma w - w = b(\gamma) + \pi(\gamma)w - w$, hence the result. $\square$

With these preliminaries, we may now turn to the proof of Theorem 4.1.

Proof of Theorem 4.1. Since a 1-cocycle with coefficients in a trivial representation is just a homomorphism to the Hilbert space (viewed as an additive abelian group), the part of the theorem concerning the subspace of invariant vectors and the cocycle b_0 is equivalent (by writing such homomorphism in, say, orthonormal coordinates), to the claim that any Γ- additive character to $\mathbb{C}$

extends continuously to G. We shall prove this result in Section 5 below (see Corollary 5.2). By decomposing a general representation $\pi = \pi_0 \oplus \pi_1$, where π_0 is the representation on the Γ-invariant vectors in $\mathcal{H}$, and π_1 on its orthogonal complement, and writing $b = p_0 \circ b + p_1 \circ b$, where p_0, p_1 are the orthogonal projections to π_0 and π_1 resp., we may (and shall) assume here that π does not contain non zero Γ-invariant vectors.

Denote the induced unitary representation $\sigma = Ind_\Gamma^G \pi$. Let $\tilde{b} \in Z^1(G, \pi)$ be the cocycle corresponding to b under the map in (1.3) of Section 1.II. Let $b'_0 + ... + b'_n$ be the cocycle almost cohomologous to b, as guaranteed by Theorem 3.1. Since π does not have invariant vectors, so does σ, hence $b'_0 = 0$. By the isomorphism between the ordinary cohomology groups (Proposition 1.12), we can find cocycles $b_i \in Z^1(\Gamma, \pi)$, such that each $\tilde{b_i}$ is cohomologous to b'_i. Since the map in (1.3) is injective also on the **reduced** first cohomology groups (see again Proposition 1.12), $b_1 + ... + b_n$ is almost cohomologous to b (indeed, both of them are mapped to the same element in $\overline{H^1}(G, \sigma)$, represented by $\tilde{b}$).

Finally, we show that each b_i is cohomologous to a cocycle as required in Theorem 4.1, thereby completing the proof of the Theorem. Fix some i, denote $\gamma v = \pi(\gamma)v + b_i(\gamma)$ -the isometric Γ-action corresponding to b_i, and $gf = \sigma(g)f + \tilde{b}_i(g)$ -the isometric induced G-action (see Section 1.II). The relation between b'_i and $\tilde{b}_i$ tells us that for some vector f_0 in the space of induced representation, we have $\tilde{b}_i(g) + \sigma(g)f_0 - f_0 = b'_i(g)$, where by definition, the latter cocycle vanishes on $\check{G}_i$ (with notations as in Proposition 4.4). Since $gf_0 = \sigma(g)f_0 + \tilde{b}_i(g) = b'_i(g) + f_0$, we see that f_0 is fixed by $\check{G}_i$. We are now exactly in position to apply Proposition 4.4, noticing that the set of points $v \in \mathcal{H}$ satisfying its conclusion is a closed, affine, Γ-invariant subspace. By Proposition 4.3 it now follows that the Γ-action on that subspace extends continuously to a G-action, factoring through an action of G_i. Lemma 4.5 then completes the proof. $\square$

§5. Cohomology of finite dimensional representations

In this section we discuss various rigidity aspects of irreducible lattices, related to cohomology of finite dimensional representations. Our first purpose is to complete the proof of the cohomology rigidity theorem (Theorem 4.1). Recall that we are left with the case of the trivial representation, namely, we need to show that every cocycle $b \in \overline{H^1}(\Gamma, 1) = H^1(\Gamma, 1)$ extends continuously to a cocycle $\tilde{b} \in H^1(G, 1)$. Since cocycles with coefficients in the trivial representation are merely additive characters to $\mathbb{C}$, this particular case is of special interest. In [Sh1 Theorem 4] a proof of this fact was presented based on the fundamental observation of Zimmer regarding the connection between cocycles and extensions of homomorphisms. We shall return to this idea in the framework of commensurators in Section 10 (see the proof of Theorem 0.8'). Here we take a more elementary approach, and observe that our method can actually give a result in a greater generality.

5.1 Theorem. *Let $\Gamma < G$ be a uniform lattice in a lcsc, compactly generated group G. Denote by $L^2_0(G/\Gamma)$ the regular G-representation on the space of zero mean functions on G/Γ. If $\overline{H^1}(G, L^2_0(G/\Gamma)) = 0$, then any homomorphism $\varphi : \Gamma \to \mathbb{C}$ may be (uniquely) extended continuously to a homomorphism $\tilde{\varphi} : G \to \mathbb{C}$.*

Of course, $\mathbb{C}$ may be replaced by $\mathbb{R}$. The compact generation assumption is redundant, but will be used in the proof we shall present here (compare with the proof in [Sh1 Theorem 4]). We emphasize that the vanishing of $\overline{H^1}$ is a considerably weaker condition than that of H^1, and is often easy to verify, as we shortly see. For instance, if G has only finitely (or countably) many **irreducible** unitary representations with non vanishing first cohomology (a situation which is not rare -see the discussion preceding Lemma 6.3 below), then the failure of the condition in 5.1 implies that one of these representations must appear **discretely** in $L_0^2(G/\Gamma)$. Before proving Theorem 5.1, let us observe two applications, the first of which completes the proof of Theorem 4.1.

5.2 Corollary (Theorem 0.8). *With the notations of* **(0.1)**, *the assumption of Theorem 5.1 is always satisfied. Therefore, every character* $\varphi : \Gamma \to \mathbb{C}$ *extends continuously to* G.

Proof of 5.2. Follows directly from Proposition 3.2 and the fact that Γ projects densely to every G_i (implying that the product of the other factors does not admit a nonzero invariant vector in $L_0^2(G/\Gamma)$ -cf. [Zi1 2.2.3]). $\square$

The following consequence of Theorem 5.1 is of a more elementary nature, however it may highlight the usefulness of the approach suggested by this result:

5.3 Corollary. *Let* $\Gamma < G$ *be as in Theorem 5.1, and assume that* G *is nilpotent. Then the assumption of 5.1 is satisfied. Therefore, every character* $\varphi : \Gamma \to \mathbb{C}$ *extends continuously to* G.

Proof of 5.3. Applying Corollary 3.6 and an inductive argument, it easily follows that for a nilpotent group G, one has $\overline{H^1}(G,\pi) = 0$ for **any** unitary G-representation π, not containing the trivial representation. $\square$

We now turn to the proof of Theorem 5.1.

Proof of Theorem 5.1. As mentioned earlier, we identify additive homomorphisms to $\mathbb{C}$ with elements of the first cohomology of the trivial representation. Since the natural restriction map from $\mathrm{Hom}(G,\mathbb{C})$ to $\mathrm{Hom}(\Gamma,\mathbb{C})$ is obviously injective, and by the compact generation assumption these spaces have finite dimension, it is enough to produce an injection $i : H^1(\Gamma,1) \hookrightarrow H^1(G,1)$. Recall that if Γ and G are as in the theorem, then for every unitary representation π one has an isomorphism $H^1(\Gamma,\pi) \cong H^1(G, Ind_\Gamma^G \pi)$ (see Proposition 1.12). Applying this when π is the trivial representation yields $H^1(\Gamma,1) \cong H^1(G, L^2(G/\Gamma)) \cong H^1(G,1) \oplus H^1(G, L_0^2(G/\Gamma))$. It is therefore enough to show the vanishing of $H^1(G, L_0^2(G/\Gamma))$. However, since $L_0^2(G/\Gamma)$ does not contain weakly 1_G (see [Mar1, p. 112] for all uniform lattices, and [Be] for the nonuniform lattices in **(0.1)**), this follows from Corollary 1.8 and the assumption. $\square$

Before discussing general finite dimensional representations, let us make a slight digression. In light of the results of the present paper, but also independently, it is natural to ask which (finitely generated) groups Γ may appear as an irreducible co–compact subgroup in a product of two locally compact (necessarily compactly generated) groups. To avoid trivial constructions, where one of the groups is commensurable with Γ and the other is finite, the following notion seems natural.

5.4 Definition. A uniform lattice $\Gamma < G_1 \times G_2$ is called **strongly irreducible** if both G_i's are nontrivial, and every finite index subgroup of Γ projects densely to both factors.

This notion is of interest mainly for residually finite groups. For example, it can be shown that if Γ is word hyperbolic, then Γ cannot be a strongly irreducible lattice. In fact, being a strongly irreducible lattice is in some sense an opposite to the hyperbolicity property. Corollary 5.2 yields another intrinsic property of a group which prevents it from being a strongly irreducible lattice.

5.5 Corollary. *If there exists a subgroup of finite index $\Gamma_0 < \Gamma$ with $\beta_1(\Gamma) < \beta_1(\Gamma_0)$, namely, $\dim(\mathrm{Hom}(\Gamma, \mathbb{C})) < \dim(\mathrm{Hom}(\Gamma_0, \mathbb{C}))$, then Γ cannot be a strongly irreducible lattice.*

For example, if Γ is a non-abelian a free group, or a product of such groups, then the condition in 5.5 is satisfied. On the other extreme, there are solvable groups satisfying 5.5 as well. The strong rigidity Theorem for irreducible lattices (Theorem 0.6) shows that if Γ is a discrete nonamenable subgroup in a product of two rank one groups, then Γ can be a strongly irreducible lattice **only** in the product of the closures of its canonical projections (see Corollary 7.3). In particular, $SL_2(\mathbb{Z}[\frac{1}{p}])$ cannot be a strongly irreducible (uniform) lattice. Results of Zimmer [Zi2] on rigidity of orbit equivalence, show that lattices in simple Lie groups also cannot be strongly irreducible lattices. We remark here that in an analogous way one may ask which pairs (Λ, Γ) can appear as a commensurator and (uniform) lattice in a locally compact group, and our results in that setting similarly yield strong obstructions.

Let us proceed to discuss briefly general finite dimensional unitary representations. Since for them weak and proper containment of the trivial representation coincide, Corollary 4.2 yields:

5.6 Theorem. *Retain the notation of* **(0.1)**. *Let π denote a finite dimensional unitary irreducible Γ-representation with $H^1(\Gamma, \pi) \neq 0$. Then π extends to a G–representation which factors through one of the G's. In particular, if no G_i has non–trivial finite dimensional representations, then $H^1(\Gamma, \pi) = 0$ for every finite dimensional unitary Γ-representation.* □

5.7 Corollary. *If every G_i is totally disconnected, and any open normal subgroup of it is cocompact, then $H^1(\Gamma, \pi) = 0$ for every finite dimensional unitary Γ-representation.*

Proof. Use 5.6 and the fact that a compact open subgroup $K_i < G_i$ must have finite image in any finite dimensional unitary representation. □

Our interest in the vanishing of the first cohomology for finite dimensional representations stems from Weil's well known observation regarding its connection with local rigidity. As noted in [Sh1, Proof of Theorem 5] (see also the references therein), one can now deduce:

5.8 Theorem. *Retain the notation of* **(0.1)**. *Assume that no G_i has a nontrivial finite dimensional unitary representation (e.g., the condition in 5.7 is satisfied). Then Γ has only finitely many inequivalent unitary representations in any finite dimension. The image of Γ in any such representation can be conjugated so that all matrices have entries in some number field.*

§6. Normal subgroups and property (T)

I. Proof of Theorems 0.1 and 0.2. Our proof of Theorem 0.1 relies on Theorem 4.1, and the following result, whose proof will occupy most of our attention in this (sub-)section:

6.1 Theorem. *Assume that the locally compact second countable group G is compactly generated. If G does not have property (T) of Kazhdan, then there exists a continuous unitary G-representation π satisfying $\overline{H^1}(G,\pi) \neq 0$ (Def. 1.7).*

Before we discuss the proof of the theorem, let us see why it implies Theorems 0.1 and 0.2 (stated in the introduction), and observe another application (Theorem 6.2 below). As noted in [Sh1], the derivation of Theorem 0.1 from Theorems 4.1 and 6.1 is easy, and we recall it for completeness. With the two assumptions and notations of the theorem, assume that Γ/N fails to have property (T). Since Γ is finitely generated, Γ/N has this property as well. We now consider a unitary representation and a cocycle b which is not almost a coboundary (Def. 1.7) of the group Γ/N, as guaranteed by Theorem 6.1, view them as Γ-representation and cocycle which factor through N, and invoke Theorem 4.1 (keeping in mind Lemma 1.11). The fact that each $G_i/\overline{pr_i(N)}$ has property (T), and that N acts trivially, readily imply that in the decomposition $b = b_0 + b_1 + ... + b_n$ provided by Theorem 4.1, all the b_i's with $i \geq 1$ are bounded, and hence b_0 cannot be zero. However by Theorem 4.1, b_0 extends to a cocycle of G with trivial coefficients, but since the cocycle is trivial on N this contradicts condition (i) in Theorem 0.1. Next, we consider the proof of Theorem 0.2.

Proof of Theorem 0.2. Let G be as in the theorem and take a unitary G-representation π with $\overline{H^1}(G,\pi) \neq 0$, as guaranteed by Theorem 6.1. Consider a direct integral decomposition of π into irreducibles $\pi = \int \pi_x \, dx$. It was shown by Blanc (see [Gui Ch.3 §2]) that if for a.e. x one has $\overline{H^1}(G,\pi_x) = 0$, then also $\overline{H^1}(G,\pi) = 0$. (It may be good to keep in mind that the latter property does not hold for the ordinary cohomology in general, even for "nice" groups such as simple Lie groups.) This proves the theorem, and moreover shows that G admits an irreducible representation with non vanishing first **reduced** cohomology. $\square$

Remark. Blanc's theorem applies to cohomology in any degree. One can avoid his result, or applying direct integral decomposition, using the geometric interpretation of first cohomology, by considering an appropriate cone with compact base of negative definite functions, and taking an extreme ray (provided by Krein-Milman Theorem), which corresponds to an irreducible representation. See below for the relation between negative definite functions and first cohomology.

We note that there are various families of groups without property (T) which admit only **finitely many irreducible** representations with non vanishing first cohomology. Perhaps the best known examples are the simple Lie groups $SO(n,1)$ $(n \geq 3)$ and $SU(n,1)$ $(n \geq 1)$ which have exactly one or two such representations, respectively. An amenable example is the group of rigid motions of $\mathbb{C}^n$, for which only the natural (finite dimensional) unitary representation has this property. Another class is that of locally compact nilpotent groups (in particular the discrete ones), for which only the trivial representation has nonvanishing cohomology, and as an example of a discrete non amenable group with this property, one can take an irreducible lattice Γ in $SO(4,1) \times SO(3,2)$. Γ has a

unique irreducible unitary representation with $H^1 \neq 0$, which is the one factoring through the unique cohomological representation of the first factor. On the other hand, we should mention the well known fact that for free non abelian groups, **any** unitary representation has non vanishing first cohomology. These examples illustrate both the subtle nature of Theorem 0.2, and suggest a more systematic study of this subject.

Finally, before turning to the proof of Theorem 6.1, we mention another consequence of Theorem 0.2. Using the main result of [VK], which asserts that irreducible representations with non trivial first cohomology are not Hausdorff separated from the trivial representation $\mathbf{1}$ in the Fell topology (i.e., they lie in the closure of any neighbourhood of $\mathbf{1}$), Theorem 0.2 yields a new (cohomological free) characterization of property (T), which we prefer to formulate in a negative form:

6.2 Theorem. *A locally compact second countable group G fails to have property (T) of Kazhdan, if and only if at least one of the following conditions is satisfied:*

(1) *G is not compactly generated.*
(2) *G admits a non zero additive complex character.*
(3) *G admits a nontrivial irreducible unitary representation which is not Hausdorff separated from the trivial representation (in the Fell topology).*

Neither one of the three conditions in Theorem 6.2 can be omitted.

We now turn to discuss the proof of Theorem 6.1, starting with the following observation:

6.3 Lemma. *Let $(\pi, \mathcal{H})$ be a unitary representation of a lcsp group G which is compactly generated. Assume that π contains weakly, but not properly, the trivial representation 1_G. Let $Q \subset G$ be a compact generating set with nonempty interior, and set $\delta(v) = \max\{\|\pi(g)v - v\| \; g \in Q\}$ for every $v \in \mathcal{H}$. Then for every $M < \infty$ one can find a vector $v_M \in \mathcal{H}$ with the properties $\delta(v_M) = 1$, and $\delta(u) > 1/6$ for every vector u with $\|u - v_M\| < M$. Furthermore, the vectors v_M can be chosen such that the family of functions $g \to \|\pi(g)v_M - v_M\|$ is uniformly continuous on compact subsets.*

Proof. Given $M < \infty$, let us show first that there exists a vector ω_M with the following property: there is some $r > 0$ and an open ball of radius r around ω_M, $B_r(\omega_M)$, such that $\delta(\omega_M) = \frac{r}{M}$, and $\delta(u) > \frac{r}{2M}$ for every $u \in B_r(\omega_M)$. Indeed, since $1_G \prec \pi$, there exists a vector ω_1 with $\|\omega_1\| > 1$ and $\delta(\omega_1) = \frac{1}{2M}$. Now, if in an open ball of radius $1/2$ around ω_1 there is no vector u with $\delta(u) \leqslant \frac{1}{4M}$, we are done, by taking ω_1 and $r = 1/2$. Otherwise, there is such vector, and by continuity of the function δ we may always choose this vector ω_2 with $\delta(\omega_2) = \frac{1}{4M}$. Next, if in an open ball of radius $1/4$ around ω_2 there is no vector u with $\delta(u) \leqslant \frac{1}{8M}$, again we are done, by taking ω_2 and $r = 1/4$. Continuing this process we see that if the claim above fails, we get a sequence of vectors $\{\omega_i\}_{i=1}^{\infty}$ with the properties $\delta(\omega_i) = (2^i M)^{-1}$ and $\|\omega_{i+1} - \omega_i\| \leqslant 2^{-i}$, where $\|\omega_1\| > 1$. The sequence $\{\omega_i\}$ is obviously converging, say to v, and notice that $v \neq 0$. It is easy to see that v is a (non-zero) vector which is invariant under Q, hence under G, contradicting the assumption of the lemma.

Next, if for $M < \infty$ the vector ω_M is as in the preceding paragraph, set $v'_M = \frac{M}{r}\omega_M$. Then $\delta(v'_M) = 1$ and $\delta(u) \geqslant \frac{1}{2}$ for all $u \in B_M(v'_M)$. Had G been discrete, we would now be done, and to establish the result in the general case we shall perform a standard smoothing operation. Take a continuous nonnegative function $f : G \to \mathbb{R}$ whose support is contained in the interior of Q, and satisfies $\int f(g)d\mu(g) = 1$, where μ is a left G-invariant Haar measure. Consider the vectors $v_M = \int f(g)\pi(g)v'_M d\mu(g)$. We have

$$\|v_M - v'_M\| = \left\| \int f(g)\pi(g)v'_M d\mu - \int f(g)v'_M d\mu \right\| = \left\| \int f(g)(\pi(g)v'_M - v'_M)d\mu \right\| \leqslant 1$$

Since for all $g \in G$ one has $\|\pi(g)v_M - v_M\| \leqslant \|\pi(g)v_M - \pi(g)v'_M\| + \|\pi(g)v'_M - v'_M\| + \|v'_M - v_M\|$, we deduce that $\delta(v_M) \leqslant 1 + 1 + 1 = 3$. Furthermore, for every u with $\|u - v_M\| \leqslant M - 1$ we have $\|u - v'_M\| \leqslant M$, hence $\delta(u) \geqslant \frac{1}{2}$. Thus, once we observe that the functions $g \to \|\pi(g)v_M - v_M\|$ are uniformly continuous on compact subsets of G, the result would follow, by replacing the sequence v_M with an appropriate normalization $c_M v_M$, where $1/3 \leq c_M \leq 2$ is chosen such that $\delta(c_M v_M) = 1$.

To see the uniform continuity let $\varepsilon > 0$ be given, and choose some neighborhood U of $e \in G$ so that $\|f(h_0^{-1}g) - f(g)\|_1 < \varepsilon$, and $f(h_0^{-1}g)$ is supported on Q, for all $h_0 \in U$. Then for every M, every $g \in G$, and every $h_0 \in U$, we get

$$\Big| \|\pi(g)\pi(h_0)v_M - v_M\| - \|\pi(g)v_M - v_M\| \Big| \leqslant \|\pi(g)\pi(h_0)v_M - \pi(g)v_M\| =$$

$$\|\pi(h_0)v_M - v_M\| = \left\| \int f(x)\pi(h_0)\pi(x)v'_M d\mu(x) - \int f(x)\pi(x)v'_M d\mu(x) \right\| =$$

$$\left\| \int f(h_0^{-1}g)\pi(g)v'_M d\mu(g) - \int f(g)\pi(g)v'_M d\mu(g) \right\| = \left\| \int (f(h_0^{-1}g) - f(g))\pi(g)v'_M d\mu(g) \right\|$$

$$= \left\| \int (f(h_0^{-1}g) - f(g))(\pi(g)v'_M - v'_M) d\mu(g) \right\| \leqslant \|f(h_0^{-1}g) - f(g)\|_1 \cdot \delta(v'_M) \leqslant 3\varepsilon \qquad \square$$

Once Lemma 6.3 is established, the idea of proof of Theorem 6.1 can easily be described. First, by Lemma 1.10 we need to construct a uniform action of G on a Hilbert space (Def. 1.9). Starting with a continuous unitary G-representation π satisfying $1_G \prec \pi$ but not $1_G \subseteq \pi$, we obtain a sequence of vectors v_M as in Lemma 6.3. We shall then view the linear action of G in π as an **affine** action, noticing that in a ball of radius M around v_M, the action looks like the uniform action that we seek, namely, does not posses almost invariant vectors (in a precise quantitative way). Then, taking a "limit" of some subsequence of the affine actions over the larger and larger balls, yields the required affine action on a Hilbert space, having at least the same uniformity constant. That the limiting action does not "blow up" , or "degenerate", follows from the fact that all of our actions are constrained by the condition $\delta(v_M) = 1$. The device that we shall use in order to carry this limiting process rigorously, is the notion of negative definite function.

6.4 Definition. Let G be a lcsc group. A continuous function $\psi : G \to \mathbb{R}$ is called **negative definite** if $\psi(e) = 0$, and for every n real numbers $\lambda_1, \dots, \lambda_n$ with $\Sigma\lambda_i = 0$, and every $g_1, \dots, g_n \in G$, one has

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i\lambda_j\psi(g_i^{-1}g_j) \leqslant 0 \ . \tag{6.1}$$

If a group G acts continuously and affinely on a (say, real) Hilbert space $\mathcal{H}$, then it is easily verified that for every $v \in \mathcal{H}$ the function $\psi(g) = \|gv - v\|^2$ is negative definite. Conversely, it can be shown that every such function arises in this way. (see [HV, p.63 Prop. 14] for this and more on this notion). In fact, if GV spans affinely the space, then all the information on the affine action is in principle encoded in the corresponding negative definite function (in much of the same way that one matrix coefficient determines an irreducible representation). Our next purpose is to make this principle precise in the quantitatively way we shall need.

6.5 Lemma. *Suppose that the group G acts (continuously and) affinely on the real Hilbert space $\mathcal{H}$, and $v \in \mathcal{H}$. Let $\psi(g) = \|gv - v\|^2$ be the corresponding negative definite function. Let $g_1, \dots, g_n \in G$, and let $a_1, \dots, a_n \in \mathbb{R}$ be some nonnegative numbers satisfying $\Sigma a_i = 1$. Set $u = \sum\limits_{i=1}^{n} a_i g_i v$. Then for every $g \in G$ one has:*

$$\|gu - u\|^2 = \sum_{i,j} a_i a_j(\psi(g_j^{-1}gg_i) - \psi(g_j^{-1}g_i)) \tag{6.2}$$

Proof. We have

$$\begin{aligned}
||gu - u||^2 &= \Big\langle g\sum_{i=1}^{n} a_i g_i v - \sum_{i=1}^{n} a_i g_i v,\ g\sum_{j=1}^{n} a_j g_j v - \sum_{j=1}^{n} a_j g_j v\Big\rangle \\
&= \Big\langle \sum_{i=1}^{n} a_i g g_i v - \sum_{i=1}^{n} a_i g_i v,\ \sum_{j=1}^{n} a_j g g_j v - \sum_{j=1}^{n} a_j g_j v\Big\rangle \\
&= \Big\langle \sum_{i=1}^{n} a_i(gg_i v - v) - \sum_{i=1}^{n} a_i(g_i v - v),\ \sum_{j=1}^{n} a_j(gg_j v - v) - \sum_{j=1}^{n} a_j(g_j v - v)\Big\rangle \\
&= \sum_{i,j} a_i a_j\big\langle gg_i v - v, gg_j v - v\big\rangle + \sum_{i,j} a_i a_j\big\langle g_i v - v, g_j v - v\big\rangle \\
&\quad - 2\sum_{i,j} a_i a_j\big\langle gg_i v - v, g_j v - v\big\rangle \ .
\end{aligned}$$

All the terms above involve expressions of the form $\langle \alpha v - v, \beta v - v\rangle$ where $\alpha, \beta \in G$. Observe that, using the equality $\langle u, \omega\rangle = \frac{1}{2}\ (\|u\|^2 + \|\omega\|^2 - \|u - \omega\|^2)$, we have

$$\begin{aligned}
&\big\langle \alpha v - v, \beta v - v\big\rangle = \frac{1}{2}(\|\alpha v - v\|^2 + \|\beta v - v\|^2 - \|\alpha v - \beta v\|^2) = \\
&\frac{1}{2}(\|\alpha v - v\|^2 + \|\beta v - v\|^2 - \|\beta^{-1}\alpha v - v\|^2) = \frac{1}{2}(\psi(\alpha) + \psi(\beta) - \psi(\beta^{-1}\alpha)) \ .
\end{aligned}$$

substituting this in the final step of the computation above yields (6.2). □

We have not elaborated on the final part of the calculation above, as the precise expression in (6.2) will be of no importance to us. The fact that $\|gu-u\|^2$ may be expressed in a formula which depends on the values of ψ only, is the information that we shall extract from Lemma 6.5. We have chosen to work throughout the whole proof of Theorem 6.1 over $\mathbb{R}$ instead of $\mathbb{C}$, because of the simplicity of the computation above over that field.

6.6 Corollary. *Suppose that the group G is generated by the compact subset $Q \subseteq G$, and that G acts affinely on a real Hilbert space $\mathcal{H}$. Then the following conditions are equivalent:*

(1) *The G-action is uniform (Def. 1.9).*
(2) *There exists* **at least one** *negative definite function $\psi(g)$ associated with the action (i.e. $\psi(g) = \|gv-v\|^2$ for some $v \in \mathcal{H}$), and some $\varepsilon > 0$, such that the following holds:*
For every choice of n nonnegative numbers with $\sum_{i=1}^{n} a_i = 1$, and n elements $g_1, \dots, g_n \in G$, there exists some $g \in Q$ with

$$\sum_{i,j} a_i a_j (\psi(g_j^{-1} g g_i) - \psi(g_j^{-1} g_i)) \geqslant \varepsilon \,. \tag{6.3}$$

Proof. In light of equality (6.2) in Lemma 6.5, it is obvious that if the G-action is uniform, then the condition in (2) must hold for **all** the negative definite functions associated with the action. In fact, $\varepsilon > 0$ is then independent of the choice of ψ, being the infimum of the function $\delta^2(v) = \delta_Q^2(v)$ defined in Lemma 6.3. In the other direction, fix any $v \in \mathcal{H}$ and consider the associated negative definite function $\psi(g) = \|gv-v\|^2$. Assuming that the G-action is not uniform, we will show that (6.3) fails for every $\varepsilon > 0$, for an appropriate choice of a_i's and g_i's. Indeed, consider the set of points $C = \sum_{i=1}^{n} a_i g_i v$ for all choices of $n \in \mathbb{N}$, a_i's and g_i's, as in condition (2). The set C, and hence also its closure $\overline{C}$, is convex and G-invariant. By the discussion at the second paragraph of the proof of Step I in Proposition 3.3, we deduce that if there exist almost fixed points for the G-action on $\mathcal{H}$, then there are such points in $\overline{C}$. Since C is dense in $\overline{C}$, we get also almost fixed points in C, which by the equality in (6.2) means that (6.3) is violated for any $\varepsilon > 0$, for some choice of a_i's and g_i's. □

With these preliminaries, we can now turn to the proof of Theorem 6.1.

Proof of Theorem 6.1. Notice first that we may work with orthogonal representations on real Hilbert spaces. That is, if G fails to have property (T) then it admits an orthogonal representation which contains weakly, but not strongly, 1_G (see [HV, p.68 Thm. 20]). Once we construct a uniform affine action on a real Hilbert space, then the same cocycle defines a uniform affine action on the complexification, and Lemma 1.10 completes the proof. Let then $(\pi, \mathcal{H})$ be an orthogonal representation of G on the Hilbert space $\mathcal{H}$, and let $v_n \in \mathcal{H}, n \geqslant 1$, be a sequence of vectors satisfying the conclusion of Lemma 6.3 when $n = M$, with respect to some preselected compact generating set $Q \subseteq G$ (having nonempty interior). Let $\psi_n(g) = \|\pi(g)v_n - v_n\|^2$ be the associated negative definite functions. By the last part of Lemma 6.3, the functions $\psi_n(g)$ are uniformly continuous on compact subsets of G. We also claim that $\psi_n(g)$ are uniformly bounded on every compact subset $K \subseteq G$. Indeed, if the function $\delta(v_n) = \max\{\|gv_n - v_n\| \mid g \in Q\}$ is as in Lemma 6.3, and m is some number such that $K \subset Q^m$, then successive iterations of the triangular inequality easily show, using the fact that $\delta(v_n) = 1$, that $\sqrt{\psi_n(g)} = \delta(v_n) \leqslant m$ for all $g \in K$. It now follows from the Arzela–Ascolli theorem that after passing to a subsequence, we may assume that the sequence $\psi_n(g)$ converges, uniformly on compacta in G, to a continuous function $\psi(g)$. It is obvious that inequality (6.1) in Definition 6.4 is preserved under convergence, hence $\psi(g)$ is a continuous negative definite function on G. As mentioned earlier, there exists then a continuous affine G-action on a real Hilbert space, such that $\psi(g) = \|g\omega - \omega\|^2$ for some vector ω [HV, p.63]. By Corollary 6.6 it is enough to show that $\psi(g)$ satisfies condition (6.2) there. Indeed, we shall show that inequality in (6.3) holds for every $\varepsilon < (1/6)^2$, and we choose, say, $\varepsilon = 1/50$. Suppose that (6.3) is not satisfied for this ε, and take $g_1, \dots, g_k \in G$, and a_i's, with $\sum_{i=1}^{k} a_i = 1$, such that (6.3) fails when $\varepsilon = 1/50$, for **every** $g \in Q$. Since all the elements of the form $g_j^{-1} g g_i, g_j^{-1} g_i$ are confined in some fixed compact subset of G when $g \in Q$ and $g_i g_j$ are taken from the sequence $g_1, \dots, g_k$ above, and $\psi_n(g) \to \psi_n(g)$ uniformly on compacta, we deduce that (6.3) fails for $\psi = \psi_n$ for all n large enough, say for $n \geqslant L$, when $\varepsilon = 1/49$. Let $m \in \mathbb{N}$ be a number such that every g_i $(1 \leqslant i \leqslant k)$ may be expressed as a product of at most m elements in Q, and let $M > \max\{m, L\}$. By the above discussion (6.3) fails for the function ψ_M when $\varepsilon = 1/49$, which implies by (6.2), that the vector $u = \sum_{i=1}^{k} a_i g_i v_M$ satisfies $\|gu - u\|^2 \leqslant 1/49$ for all $g \in Q$. In other words, we have $\delta(u) \leqslant 1/7$. However, since by the construction of v_M in Lemma 6.3 we have $\delta(v_M) = 1$, by the triangular inequality and the definition of m we have $\|g_i v_M - v_M\| \leqslant m$ for all $1 \leqslant i \leqslant k$. Thus all the vectors $g_i v_M$ lie in the ball of radius m around v_M, denoted $B_m(v_M)$, and so is their convex combination $u = \sum a_i g_i v_M$. It follows that $u \in B_m(v_M) \subset B_M(v_M)$ satisfies $\delta(u) < 1/6$, which contradicts the construction of v_M in Lemma 6.3. □

II. Property (T) and harmonic maps. An important class of groups for which Theorem 0.2 was known (originally due to Delorme [De]), is the simple Lie groups. All previous proofs, however, use heavily the representation theory of semisimple groups (see also [HV, Ch. 9] and [BW, p.163]). In this section we apply our general result, Theorem 0.2, in this particular case, together with the rigidity theory of harmonic maps, to present a unified geometric proof of property (T) for **all** the simple Lie groups sharing it. Bochner-type formulae, originally developed for geometric superrigidity, were applied also in [Mok] and [Pa] for a proof of property (T), by showing vanishing of appropriate 1-forms. However, our approach is quite different, and shows that not only the theory of harmonic maps is applicable in the study of property (T), but that in fact it arises remarkably naturally, and connects the latter directly with geometric superrigidity.

We begin with the following observation.

6.7 Lemma. *Let G be a locally compact, compactly generated group, and $K < G$ a compact subgroup for which (G,K) forms a Gelfand pair, namely, the convolution algebra $G//K$ of L^1 bi-K-invariant functions on G is commutative. Let $(\pi,\mathcal{H})$ be a non trivial, irreducible, unitary G-representation, and $b \in Z^1(G,\pi)$ a cocycle which is not a co-boundary. Then the corresponding affine isometric G-action on $\mathcal{H}$: $\rho(g)v = \pi(g)v + b(g)$, admits a* **unique** *$K$-fixed point.*

Proof. As is well known, the assumption on (G,K) implies that for any irreducible unitary G-representation $(\pi,\mathcal{H})$, the dimension of $\mathcal{H}^K$ -the subspace of K-fixed vectors- is at most one. (This follows by showing that for such representation, the associated action of $G//K$ on $\mathcal{H}^K$ must be irreducible as well. If $G//K$ is abelian, $\mathcal{H}^K$ can be at most one dimensional.) It follows that if π is irreducible and non-trivial, then it cannot contain weakly the trivial representation, as a sequence of almost invariant unit vectors can always be taken to be invariant under the action of a given compact subgroup. Hence, if μ is any symmetric, continuous, generating probability measure on G, the convolution operator $\pi(\mu)$ acting in $\mathcal{H}$ satisfies: $\|\pi(\mu)\| < 1$ (cf. [Mar1 p.155]).

For any probability, compactly supported measure μ on G, denote: $\rho(\mu)v = \int \rho(g)v\, d\mu(g)$. Fix some symmetric, continuous, generating measure, μ_0. By the preceding discussion $\|\pi(\mu_0)\| = \lambda < 1$, so for all vectors v, w: $\|\rho(\mu_0)v - \rho(\mu_0)w\| = \|\pi(\mu_0)(v-w)\| \leq \lambda\|v-w\|$. Hence, being a strict contraction on a complete metric space, $\rho(\mu_0)$ admits a unique fixed point v_0. Assume further that μ_0 was taken to be bi$-K$-invariant. Since by our assumption, the convolution semi-group $P^K(G)$ of Haar-continuous, bi-K-invariant, compactly supported probability measures on G, is commutative, it follows that for every $\mu \in P^K(G)$, the point $\rho(\mu)v_0$ is fixed by $\rho(\mu_0)$ as well. By uniqueness, v_0 is fixed by $\rho(\mu)$ for all $\mu \in P^K(G)$ (in particular, v_0 is fixed by K). Let dk denote the normalized Haar measure on K. Then for any $g \in G$, approximating the Dirac measure δ_g by smooth measures ν_n, we deduce from the invariance of v_0 under each convolution $dk{*}\nu_n{*}dk$, that it is invariant under $dk{*}\delta_g{*}dk$ as well.

We shall now see that v_0 is the unique K-fixed point in $\mathcal{H}$, as required in the theorem. Of course, every K-fixed point v satisfies $\int \rho(k)v\, dk = v$, so it is enough to show that for every $v \in \mathcal{H}$, $\int \rho(k)v\, dk = v_0$. First, notice that the set of points v satisfying the latter property contains $\rho(g)v_0$ for all $g \in G$, since from the previous paragraph we have

$$\int \rho(k)\rho(g)v_0\, dk = \int\int \rho(k_1)\rho(g)\rho(k_2)v_0\, dk_1 dk_2 = \rho(dk{*}\delta_g{*}dk)v_0 = v_0$$

Furthermore, the above property is preserved under limits and affine closure. Since π is irreducible, the only G-invariant affine (closed) subspaces of $\mathcal{H}$ are either one point, or the whole space. But $\rho(G)v_0$ cannot consist of one point, as this would mean that v_0 is fixed by G, contradicting our assumption on the cocycle b. The possibility remaining is the required one. □

The conclusion of Lemma 6.7 is equivalent to saying that π admits no non-zero K-invariant vectors. From the point of view of Lie algebra cohomology (which is avoided here), this is clear for simple Lie groups, and holds for irreducible representations with non vanishing cohomology in any degree. Indeed, as is well known, the infinitesimal character of such representation must be trivial, and in the class one spectrum this happens only for the trivial representation. Our purpose, however, is to show that property (T) for simple Lie groups can be deduced only from the general statement of Theorem 0.2, and the rigidity theory of harmonic maps. As we now show, such a map arises naturally as the orbital map of the unique fixed point given by Lemma 6.7.

Henceforth we let G be a simple Lie group (with K maximal compact), and assume it fails to have property (T). From Theorem 0.2 we then know that there is a (non trivial) unitary G-representation π satisfying the assumption (and hence the conclusion) of Lemma 6.7. We retain the notations in that lemma, except that we suppress the notation ρ for the affine action, writing simply $\rho(g)w = gw$. The unique K-fixed point for the latter action will be denoted by v. We proceed to show how the rigidity theory of harmonic maps, for the groups it applies (fully), leads to a contradiction of our assumption.

Recall that dk denotes the normalized Haar measure on K. For any vector $w \in \mathcal{H}$ we have:

$$v = \int kw \, dk \tag{6.4}$$

(This follows from Lemma 6.7 and the fact that the right hand side is fixed by K.) Consider the function $F : G \to \mathcal{H}$ defined by $F(g) = gv$. Since $F(gK) = F(g)$, the function F descends to a function on the symmetric space G/K. We will show that as such, F is a smooth ($= C^\infty$) function which is **harmonic** in the sense that $\Delta F = 0$, where Δ is the Laplace-Beltrami operator on G/K. To this end, we first notice that for every left K-invariant probability measure μ on G, with compact support, one has

$$\int F(gh) \, d\mu(h) = F(g) \tag{6.5}$$

Indeed, using the fact that the G-action preserves convex combinations (which was used also in (6.4) above), the left K-invariance of μ, and (6.4), we have:

$$\begin{aligned}\int F(gh) \, d\mu(h) &= \int \int F(gkh) \, dk d\mu(h) = \int \int gkhv \, dk d\mu(h) = \\ &= g \int \int khv \, dk d\mu(h) = g \int kw \, dk = gv = F(g)\end{aligned}$$

Next, for $g_1 \in G$ and any function f on G (or G/K), we denote: $f^{g_1}(g) = f(g_1 g)$. Then, for any $g_1 \in G$, F satisfies:

$$\int F^{g_1}(kg) \, dk = constant \tag{6.6}$$

Indeed, by the same reasoning as above we have:

$$\int F^{g_1}(kg) \, dk = \int F(g_1 kg) \, dk = \int g_1 kgv \, dk = g_1 \int kgv \, dk = g_1 v = constant$$

Fix now any vector $u \in \mathcal{H}$, and consider the complex valued (right K-invariant) function $f(g) = f_u(g) = \langle F(g), u\rangle$. From (6.5) and (6.6) we get:

$$\int f(gh)\, d\mu(h) = f(g) \tag{6.7}$$

$$\int f^{g_1}(kg)\, dk = constant \tag{6.8}$$

By choosing a smooth (K-invariant) measure μ we readily deduce from (6.7) that if f is in C^i, then the left hand side of (6.7) is in C^{i+1}, so f is in fact C^∞. We may thus regard f as a smooth function on G/K, and we shall keep the same notation, writing $f(x)$ instead of $f(g)$ in that setting. Notice that (6.8) continues to hold, when we replace $g \in G$ by $x \in G/K$. Using an argument similar to that in [Fu, Theorem 4.4], we shall now see that $\Delta f = 0$. To that end we need only to recall that Δ, the Laplace-Beltrami operator, is a differential operator which annihilates scalars, and commutes with the G-action: $(\Delta\psi)^g = \Delta(\psi^g)$ for all smooth functions ψ, and $g \in G$. Let x_0 denote the (unique) K-fixed point in G/K. We first show that $\Delta f(x_0) = 0$:

$$\Delta f(x_0) = \Delta f(kx_0) = (\Delta f)^k(x_0) = \int (\Delta f)^k(x_0)\, dk$$

To show the vanishing of the latter at x_0, we find it easier to show its vanishing everywhere:

$$\int (\Delta f)^k(x)\, dk = \int \Delta(f^k)(x)\, dk = \Delta \int f^k(x)\, dk = \Delta \int f(kx)\, dk = \Delta(constant) = 0$$

Thus $\Delta f(x_0) = 0$. The only property of f we used is that $\int f(kp)dk = constant$. This follows from (6.8) when $g_1 = e$, which in fact yields this property for every f^{g_1}, $g_1 \in G$. Therefore the same argument shows that: $0 = \Delta(f^{g_1})(x_0) = (\Delta f)^{g_1}(x_0) = \Delta f(g_1 x_0)$. Since G acts transitively on G/K and g_1 is arbitrary, it follows that $\Delta f = 0$, as required.

Summing up, we have shown that for every $u \in \mathcal{H}$, the function $f_u(x) = \langle F(x), u\rangle$ on G/K is smooth and harmonic. Thus $F : G/K \to \mathcal{H}$ is smooth and harmonic as well. Notice also that F is equivariant with respect to the isometric G-action on $\mathcal{H}$, for which we henceforth retain the notation ρ, namely, $F(gx) = \rho(gx)v = \rho(g)\rho(x)v = \rho(g)F(x)$. In particular, F is equivariant with respect to the action of a uniform lattice $\Gamma < G$, which we fix in what follows.

Our arguments so far apply to any simple Lie group. However, we now observe that we encountered a similar situation to that in the geometric proof of superrigidity, where a Γ-equivariant harmonic map to the target symmetric space is constructed from a Γ-homomorphism (this is again general, and applies to any G). Then, for G not locally isomorphic to $SO(n,1)$ or $SU(n,1)$, a Bochner-type formula shows that every Γ-equivariant harmonic map from G/K to a non-positively curved manifold, on which Γ acts by isometries, is either constant (implying that the original homomorphism was bounded), or after rescaling, a totally geodesic isometric embedding (which implies superrigidity). This remarkable phenomenon was discovered by Corlette [Co] in the case of $Sp(n,1)$ and $F_{4(-20)}$. It was extended in [GS] to other "singular" targets (thereby providing arithmeticity), whereas the Bochner-type formula itself was extended in [MSY] and [JY] to all the other higher rank simple Lie groups as well. As noted in [Jo Theorem 4.1], the Bochner-type formula yields this rigidity property of harmonic maps when the target space is any complete, possibly infinite dimensional Riemannian manifold, of non positive curvature operator. Our target space is the simplest example of such a manifold, namely, a flat Hilbert space, and the equation $\Delta F = 0$ says that F is a harmonic map between the manifolds G/K and $\mathcal{H}$. Thus we conclude that in the cases of $Sp(n,1)$, $F_{4(-20)}$ and the higher rank simple Lie groups, the harmonic map $F: G/K \to \mathcal{H}$ must be either constant, which means that G fixes a point in $\mathcal{H}$ and the cocycle b is a coboundary, or the symmetric space G/K can be isometrically embedded in a Hilbert space, which is again impossible, as it is not flat. $\square$

§7. Superrigidity and strong rigidity of irreducible lattices

Superrigidity. Following the statement of Theorem 0.3, we mentioned that the condition imposed on the group H can be made more general, and requires only the existence of one unitary representation with certain properties. Such properties are provided by the following result:

7.1 Lemma. *Let H be a group as in Theorem 0.3. There exists a unitary H-representation $(\pi, \mathcal{H})$ with the following two properties:*

(i) *π is mixing, i.e., for all $u, v \in \mathcal{H}$, $\langle \pi(h)u, v\rangle \to 0$ when $h \to \infty$ in H.*
(ii) *For every countable subgroup $L < H$ with nonamenable closure, one has $\overline{H^1}(L, \pi|_L) \neq 0$.*

Proof. Let $\Gamma < H$ be a lattice as assumed in Theorem 0.3. We identify Γ with its image in $SO(n,1)$ or $SU(n,1)$. It was shown in [Sh2 Theorem 4.5] that Γ has a unitary representation σ which is strongly L^p for some $p < \infty$, and satisfies $H^1(\Gamma, \sigma) \neq 0$. (Recall that a unitary representation is called strongly L^p if it admits a dense subspace of vectors for which the associated matrix coefficients are in L^p of the group. The Γ-representation σ alluded to, is obtained by restricting an irreducible representation with nontrivial cohomology of the ambient Lie group.) In fact, the cocycle $b \in Z^1(\Gamma, \sigma)$ so obtained satisfies $\|b(\gamma)\| \to \infty$ whenever $\gamma \to \infty$, that is, b is a **proper** cocycle in the terminology of [Sh2 Lemma 5.6]. Now, consider the H-representation $\pi = Ind_{\Gamma}^{G} \sigma$, and the cocycle $\tilde{b}$ induced from b (see (1.3) in §1.II). By [Sh2, Lemma 5.7], $\tilde{b}$ is a proper G-cocycle as well, so its restriction to any noncompact subgroup of H is unbounded; in particular, it cannot be a coboundary. Thus, for any L as in (ii), we have $H^1(L, \pi|_L) \neq 0$. To show that one has in fact $\overline{H^1}(L, \pi|_L) \neq 0$, it is enough, by Corollary 1.8, to know that $\pi|_L$ does not have almost invariant vectors. However, since σ was a strongly L^p Γ-representation, it follows from [Sh2, Lemma 4.10] that π is strongly L^p (for the same $p < \infty$). The proof of [Sh3, Theorem C] shows that the restriction of such a representation to any subgroup L as in (ii), cannot contain weakly the trivial representation. This establishes property (ii), whereas (i) is obvious since every strongly L^p-representation ($p < \infty$) is mixing. $\square$

We are now able to present the proof of the superrigidity theorem.

Proof of Theorem 0.3. We retain the notation of the theorem. Let $(\pi, \mathcal{H})$ be a unitary H-representation as in Lemma 7.1, and set $L = \varphi(\Gamma)$. Restricting π to L defines through φ a unitary

Γ–representation, which we continue to denote by π, satisfying $\overline{H^1}(\Gamma,\pi)\neq 0$ (see also Lemma 1.11). From Theorem 4.1 it now follows that the Γ-representation on some nonzero Γ-invariant subspace $\mathcal{H}_\circ\subset\mathcal{H}$ extends to a G-representation, factoring through the projection to, say, G_1.

Denote by $p:G\to G_1$ the natural projection. By Proposition 4.3 it is enough to show that for any sequence $\gamma_i\in\Gamma$ with $p(\gamma_i)\to 1_{G_1}$, one has $\varphi(\gamma_i)\to 1_H$. Although this is not quite true in general, for such sequence, $\varphi(\gamma_i)$ must be bounded. Indeed, otherwise, taking any nonzero $v_0\in\mathcal{H}_0$, we deduce from the fact that π is mixing, that $\liminf\|\pi(\varphi(\gamma_i))v_0-v_0\|>0$, contradicting the facts that $p(\gamma_i)\to 1_{G_1}$, and the Γ-action on v_0 extends continuously to G_1.

Let now $M<H$ denote the set of all limit points of sequences of the form $\varphi(\gamma_i)$ where $p(\gamma_i)\to 1$. $M<H$ is a closed subgroup, which is compact by the previous discussion. It is easy to check that it is normalized by $\varphi(\Gamma)$ (hence by $\overline{\varphi(\Gamma)}$ as well), and the above strategy may now be applied to complete the proof of the Theorem. □

Proof of Corollary 0.4. The Corollary follows readily from Theorem 0.3 and the following result:

7.2 Lemma. *Let* $\mathbf{H}$ *be a connected semisimple, almost k-simple, linear algebraic group, over a local field k. Let $\Lambda<\mathbf{H}(k)$ be a Zariski dense, unbounded subgroup. If $M<\mathbf{H}(k)$ is a closed amenable subgroup which is normalized by Λ, then M is finite and central.*

Proof. By the assumption on Λ and [Mar1, p.211 Lemma 4.4.5], there exists a nontrivial k-rational representation of $\mathbf{H}$ on a vector space W which is ($k-$)irreducible, and an element $h\in\Lambda$ such that the transformation $\rho(h)$ is proximal on $W(k)\cong k^n$. In particular, its image in $PGL(W(k))$ is non-compact (3.3.5, 3.3.6 *loc. cit.*). Since $\rho(M)$ is amenable, its action on the corresponding projective space admits an invariant measure μ. As $\rho(h^n)$ normalizes $\rho(M)$ for all n, the translations of μ under all the projective transformations corresponding to $\rho(h^n)$ remain $\rho(M)$–invariant, and thus so is any limit measure ν of them. By Furstenberg's Lemma (cf. [Zi1, 3.2.1]), ν is supported on a union of two proper subspaces of $W(k)$ (in fact one of them is a line since $\rho(h)$ is proximal, but this is not important to us). Let $V_1\cup\ldots\cup V_n\subset W(k)$ be the unique minimal quasi-linear subvariety whose projective image supports the measure ν. By the above, all the V_i's are proper subspaces, and by uniqueness, every element of $\rho(M)$ must permute them. It follows that a subgroup of finite index in $\rho(M)$ preserves each of the V_i's (set-wise). Since ρ is irreducible, $\rho(M)$ cannot be Zariski dense in $\rho(\mathbf{H}(k))$. Therefore M is not Zariski dense in $\mathbf{H}$, and its Zariski closure $\overline{M}$ is a proper algebraic k-subgroup, normalized by $\overline{\Lambda}=\mathbf{H}$. By almost k-simplicity, we are done. □

Strong rigidity. As we now observe, the strong rigidity is an easy consequence of superrigidity.

Proof of Theorem 0.6. Denote by p_j $(1\le j\le m)$ the canonical projection from G^* to G^*_j. Since every G^*_j is not amenable, and the projections of Γ^* are dense, we may apply Theorem 0.3 to every homomorphism $p_j\circ\varphi$. By the assumption on normal compact subgroups in G^*_j, every such homomorphism extends continuously to G, factoring through some G_i, where $i=i(j)$. Combining together all the homomorphisms yields a continuous extension of φ, $\tilde\varphi:G\to G^*$. Since Γ is co-compact and $\tilde\varphi|_\Gamma=\varphi$ is injective with discrete image, it follows that $\tilde\varphi$ is proper, i.e., if $g\to\infty$

in G, then $\tilde{\varphi}(g) \to \infty$ in G^*. In particular, $\tilde{\varphi}$ has compact kernel (hence a trivial one, by our assumption on G). Observe that every G_i "participated" in this process of extension of φ. For if not, then $\tilde{\varphi}$ factors through some proper subproduct of the G_i's. Hence the projection to this subproduct must be proper as well, which means that it has compact kernel, contradicting the assumption on no compact normal subgroups. To complete the proof, it only remains to show that $\tilde{\varphi}$ is onto. For this, we pick some i, and consider all the j's for which $i = i(j)$ (in the preceding process of extension). The above extension argument yields an injective continuous homomorphism $\tilde{\varphi}_i$ of G_i, into the subproduct of the G_j^*'s for which $i = i(j)$. Since $\tilde{\varphi}$ is proper, it is clear that $\tilde{\varphi}_i$ must be proper as well. In particular, its image is closed. But by its construction, this image contains the projection of Γ^* to that subproduct, which is assumed to be dense. Therefore each $\tilde{\varphi}_i$ is onto, the same holds for $\tilde{\varphi}$, and the proof is complete. $\square$

Notice that although the discreteness of Γ^* in G^* was essential in the proof, we did not assume, nor use, co-compactness. From this general feature one readily sees the phenomenon of strong rigidity alluded to in the introduction, and the fact, mentioned after Definition 5.4, that some discrete groups cannot be uniform lattices in products of (non compact) topological groups. We highlight this point of view in the following concrete setting:

7.3 Corollary. *Let $G^* = G_1^* \times G_2^*$, be a product of groups, each of which isomorphic to $SO(n,1)$, $SU(n,1)$, rank one algebraic group over a non archimedean local field, or the automorphism group of some locally finite tree. Let $\Gamma^* < G^*$ be* **any** *finitely generated discrete subgroup, whose projections to both G_i^*'s have non amenable closure. If Γ^* is embedded as a co-compact subgroup in a product of two topological groups, and the closures of its canonical projections in this embedding are not compact and denoted by G_1, G_2, then possibly after rearranging the G_i^*'s, each G_i is isomorphic to the closure of the projection of Γ^* to G_i^* (modulo a compact normal subgroup), in a way that extends the embedding of Γ^* in $G_1 \times G_2$. In other words, the product of the closures of the projections of Γ^* in the* **original** *group G^*, is the* **only** *nontrivial product of topological groups (up to a normal compact subgroup), in which it can be co-comapct (with dense projections). If this is the case, that product is the "unique minimal envelop" for Γ^* in the class of products of groups: every other such embedding extends to it continuously.*

Our notation of G and G^* continues here to be consistent with that in Theorems 0.3 and 0.6, where G is the "unknown" product of groups, and G^* is "motivated by" some standard examples. Notice that we do not impose here any kind of irreducibility of Γ^*. The result holds (although it is of little interest) also when $\Gamma^* = \Gamma_1^* \times \Gamma_2^*$, which is its own "envelop".

§8. Arithmeticity

In this section we prove Theorem 0.5 stated in the introduction. As a reference for the material related to algebraic and arithmetic groups, we refer the reader to the useful account given in [Mar1, Ch.1 §1,2,3]. We shall first need some preliminary discussion and auxiliary results.

In what follows we consider an absolutely simple linear algebraic group of adjoint type, $\mathbb{G}$, abstractly isomorphic to $PGL_2 = GL_2/Z(GL_2)$. Although we shall not depend on it, to make our discussion more concrete we recall that if $\mathbb{G}$ is defined over a field F and is F-isotropic, then, as is well known, $\mathbb{G}$ is isomorphic over F to PGL_2. In particular, $\mathbb{G}(F)^+ = PSL_2(F)$ (see [Mar1, p.45 1.5] for the notion of G^+. We caution the reader that in general $\mathbb{G}(F)$ strictly contains $PGL_2(F)$ -a notational difficulty). For an element $g \in \mathbb{G}$ the trace $tr(g)$ will always denote the trace of the linear transformation $Ad\,g$ acting on the Lie algebra. More concretely, in the case of PGL_2, the conjugation action of GL_2 on the vector space of traceless 2×2 matrices, factors through an action of PGL_2, and the trace of the transformation corresponding to $g \in PGL_2$ will be denoted by $tr(g)$.

The condition that the characteristic be different from 2 in Theorem 0.5 will always be assumed in the sequel, enabling us to avoid several technical difficulties, as in the following useful result of Vinberg (see [Mar1, p. 284 Prop. 3.22], and the proof of step 2 in [Mar1, p.305]):

8.1 Lemma. *Let $\Gamma < PGL_2$ be a Zariski dense subgroup. Denote by F_0 the field generated by $tr(\gamma)$ for all $\gamma \in \Gamma$. Then there exists an algebraic group $\mathbb{G}$ defined over F_0, and an isomorphism ρ of algebraic groups $\rho : PGL_2 \to \mathbb{G}$, such that $\rho(\Gamma) < \mathbb{G}(F_0)$.*

Next, we shall need information regarding the behavior of the traces under topological isomorphisms, using also the so called "standardness" of the group $\mathbb{G}$ (see [Mar1, p.282 3.15]):

8.2 Theorem (Borel-Tits, cf. [Mar1, p.48, 1.8.1-1.8.2]). *Assume that $\mathbb{G}$ is defined over a local field F and F-isotropic, and that $\mathbf{H}$ is an algebraic group defined over a local field K, which is also abstractly isomorphic to PGL_2. Let $\varphi : \mathbb{G}(F)^+ \to \mathbf{H}(K)^+$ be an isomorphism of topological groups. Then there exists a continuous field isomorphism $\sigma : F \to K$, and an algebraic group K-isomorphism $\psi : {}^\sigma\mathbb{G} \to \mathbf{H}$, such that if $\sigma : \mathbb{G}(F) \to {}^\sigma\mathbb{G}(K)$ denotes the obvious homomorphism, then $\varphi(g) = \psi(\sigma(g))$ for all $g \in \mathbb{G}(F)^+$.*

As a consequence of Theorem 8.2 we deduce (see [Mar1, p.43 1.4.8]):

8.3 Corollary. *In the situation and notation of Theorem 8.2, for every $g \in \mathbb{G}(F)^+$ one has $tr(\varphi(g)) = \sigma(tr(g))$.*

The proof of the following result relies heavily, in the positive characteristic case, on a recent deep result of Pink [Pi].

8.4 Proposition. *Suppose that L be a simple, locally compact group, and assume that $\mathbb{G}$ is defined and isotropic over a local field F. Let $\varphi : L \to \mathbb{G}(F)$ be a continuous homomorphism with Zariski dense image, such that $tr(\varphi(L))$ generates F as a closed field. Then φ is an isomorphism of L with $\mathbb{G}(F)^+$.*

Proof. Since L is simple, it is either connected or totally disconnected. In the first case F must be either $\mathbb{R}$ or $\mathbb{C}$, and $\mathbb{G}(F)^+$ is a simple Lie group (isomorphic to $PSL_2(F)$). The result is then easily verified and we leave the details to the reader (the proof in the totally disconnected case can be modified to hold in that case as well, and is distinguished for conceptual convenience).

Assume that L (and hence also F) is totally disconnected, and let $M < L$ be a compact open subgroup. Then $\varphi(M)$ is a compact subgroup of $\mathbb{G}(F)$. By a result of Pink [Pi Theorems 0.2, 0.7] there exists a closed subfield $E \subset F$ and an algebraic group $\mathbf{H}$ defined over E, such that $\mathbf{H}$ is isomorphic to $\mathbb{G}$ over F, and under this isomorphism $\varphi(M)$ is open in $\mathbf{H}(E)$. Notice that $\varphi(M)$ is Zariski dense in $\mathbb{G}$, since it is commensurated by $\varphi(L)$, which is itself Zariski dense (so the connected component of $\overline{\varphi(M)}$ must be normalized by $\overline{\varphi(L)} = \mathbb{G}$). We claim that the image of $\varphi(L)$ in $\mathbf{H}$ is also contained in $\mathbf{H}(E)$. Indeed, an element h in this image commensurates a Zariski dense subgroup of $\mathbf{H}(E)$, and in particular conjugates a Zariski dense subset of $\mathbf{H}(E)$ into another such subset. Therefore $Ad(h)$ is defined over E, but $\mathbf{H}$ is of adjoint type, $Ad : \mathbf{H} \to \mathbf{H}$ is an isomorphism over E, hence $h \in \mathbf{H}(E)$ (cf. [Mar1, p.43 1.4.11]). By Corollary 8.3 and the assumptions on the traces, it follows that $E = F$, and hence $\varphi(M)$ is open in $\mathbb{G}(F)$. Finally, we see that $\varphi(L)$ is an open subgroup of $\mathbb{G}(F)$ which is not compact (for otherwise $\varphi(M)$ is of finite index in $\varphi(L)$ so L itself is compact, contradicting the fact that it is totally disconnected and simple). However, as is well known, every open unbounded subgroup contains $\mathbb{G}(F)^+$ (as a normal subgroup), therefore $\varphi(L) = \mathbb{G}(F)^+$. $\square$

We are now able to present the proof of the arithmeticity theorem.

Proof of Theorem 0.5. Let $\varphi : \Gamma \to PGL_2(F)$ be the assumed homomorphism, where char $F \neq 2$. First, notice that the image of Γ is Zariski dense. Indeed, otherwise by passing to a subgroup of finite index, the image may assumed to be solvable, so the commutator has infinite index, contradicting Theorem 5.2 and simplicity. To keep notations simple, we shall henceforth identify Γ with its image in $PGL_2(F)$ (although we do not really need it, we remark that from Theorem 0.1, the simplicity of the G_i's, and the well known fact that every Kazhdan subgroup of $PGL_2(F)$ is finite, it follows that φ must have trivial kernel). Let $R \subset F$ denote the ring generated by the traces of the elements of Γ, and K its quotient field. Then by 8.1 we find an algebraic group $\mathbb{G}$ (abstractly isomorphic to PGL_2), defined over K, such that Γ is embedded in $\mathbb{G}(K)$, and the ring R generated by the traces of the elements of Γ, has quotient field K.

Claim. K is a global field.

Proof of claim. First, as the image of Γ in $PGL_2(F)$ was infinite, K is infinite as well. Next, since Γ is finitely generated and the entries of product of matrices are polynomials with integral coefficients in the entries of the factors, it follows that the ring R, and hence also the field K, are finitely generated. Denote by K_0 the prime subfield of K. Then, if K is not a global field, there exists $\gamma \in \Gamma$ such that $tr(\gamma)$ is transcendental over K_0, and as in [Mar1, p. 303 Lemma 2.9], there are infinitely many inequivalent valuations ν on K such that the image of γ in $\mathbb{G}(K_\nu)$ (hence also that of the whole group Γ), is unbounded. We will show that this is impossible. In fact, we will see that such a set has cardinality at most n (where n is the number of G_i's in **(0.1)**). Indeed, if ν is such a valuation, then the natural embedding $\varphi_\nu : \Gamma \to \mathbb{G}(K_\nu)$ extends by the superrigidity theorem (Corollary 0.4) to a continuous homomorphism $\widetilde{\varphi}_\nu : G_i \to \mathbb{G}(K_\nu)$, where G_i is one of the simple factors. By Proposition 8.4 it follows that $\widetilde{\varphi}_\nu$ is an isomorphism with

$\mathbb{G}(K_\nu)^+$. If there are more than n inequivalent valuations, then we may find two: ν, μ, such that the same simple factor G_{i_0} appeared in the extension of φ_ν and φ_μ. We then obtain through G_{i_0} a continuous isomorphism $\varphi : \mathbb{G}(K_\nu)^+ \to \mathbb{G}(K_\mu)^+$, which is equivariant with respect to the two natural embeddings of Γ. Let $\sigma : K_\nu \to K_\mu$ be the field isomorphism corresponding to φ, given by Theorem 8.2. Then by Corollary 8.3, we have for every $\gamma \in \Gamma$:

$$tr(\varphi_\mu(\gamma)) = tr(\varphi \circ \varphi_\nu(\gamma)) = \sigma(tr(\varphi_\nu(\gamma))) \ .$$

As the traces of the elements of Γ generate K as a field, we have $\sigma(K) = K$. Since σ is continuous (which follows from the continuity of φ), we deduce that μ and ν are equivalent. This completes the proof of the claim. $\square$

Let S be the set of inequivalent valuations ν on K such that the image of Γ in $\mathbb{G}(K_\nu)$ is unbounded. We have already seen that S is finite, in fact $|S| \leqslant n$. We claim that for every **archimedean** valuation $\nu \notin S$, the group $\mathbb{G}$ is K_ν-anisotropic, i.e. $\mathbb{G}(K_\nu)$ is compact. Indeed, since the image of Γ in $\mathbb{G}(K_\nu)$ is bounded, its closure is compact and hence all the traces of its elements are real (this is of course a general fact, which the reader is invited to verify directly in the special case of $SU(2) < SL_2(\mathbb{C})$, which is of interest to us). Since these traces generate K, necessarily $K_\nu = \mathbb{R}$. But any compact real matrix group is algebraic, and since Γ is Zariski dense, it follows that $\mathbb{G}(K_\nu)$ is compact.

Let $K(S)$ denote the ring of S-integers in K [Mar1, p.28 0.32]. Then $\mathbb{G}(K(S))$ is embedded diagonally in $\prod_{\nu \in S} \mathbb{G}(K_\nu)$ as a lattice (using the discussion in the preceding paragraph when char$K = 0$ -see [Mar1, p.63 3.2.5]). By definition of S, after passing to a subgroup of finite index in Γ we may assume that $\Gamma < \mathbb{G}(K(S))$ (in particular, S is not empty). For every $\nu \in S$ the embedding of Γ in $\mathbb{G}(K_\nu)$ has unbounded and Zariski dense image, hence by Corollary 0.4 this embedding extends to a continuous homomorphism $\varphi_\nu : G \to \mathbb{G}(K_\nu)$ factoring through some simple factor $G_{i(\nu)}$. By 8.4 it follows that φ_ν is an isomorphism with $\mathbb{G}(K_\nu)^+$. The proof of the theorem will be complete once we show that the map $\nu \to i(\nu)$ is a bijection, for then we have a topological isomorphism $\varphi : G \to \prod \mathbb{G}(K_\nu)^+$ and $\varphi(\Gamma)$ must be a lattice (in $\prod \mathbb{G}(K_\nu)$ as well), so its index in $\mathbb{G}(K(S))$ is finite. Since we already know that $|S| \leqslant n$, to show that $\nu \to i(\nu)$ is a bijection it is enough to show that it is onto. However if it weren't, then the extension of the homomorphism $\Gamma \to \mathbb{G}(K(S))$ to G factors through a proper subproduct of the G_i's, and since the projection of Γ to any proper subproduct is non discrete, this contradicts the discreteness of $\mathbb{G}(K(S))$ in $\prod \mathbb{G}(K_\nu)$. $\square$

§9. Isometric actions on trees

In this section we prove Theorem 0.7 in the case of irreducible lattices. We begin with the following result, which will enable us to identify elementary actions.

9.1 Proposition. *Let Γ be a locally compact group acting (continuously and) isometrically on a tree $T = T(V, E)$, where V, E denote the set of vertices and edges resp. Then the following* *conditions are equivalent:*

(1) *The* Γ*–action on* T *is elementary.*
(2) *There exists a* Γ*–invariant* **finitely additive** *probability measure on* V*.*
(3) *The natural* Γ*-representation on* $\ell^2(V)$ *admits almost invariant vectors.*

Proof. $1 \to 3$: If there exists a fixed edge or vertex, then $\ell^2(V)$ clearly contains a non zero Γ-invariant vector. If r is a Γ-invariant ray emanating, say, from $v_0 \in V$, then it is easy to verify that the sets $\{v \in r | d(v, v_0) \leqslant n\}$ form a sequence of almost invariant ("Fölner") sets, and their characteristic functions are almost invariant in $\ell^2(V)$.
$3 \to 2$: If $f_n \in \ell^2(V)$ is a sequence of Γ–almost invariant normalized vectors in $\ell^2(V)$, then it is easy to verify that so are $|f_n|$, and then $|f_n|^2 \in \ell^1(V)$ are Γ–almost invariant as well. Viewing each $|f_n|^2$ as a density of a probability measure on V, we observe that any weak limit of these measures in the compact set of finitely additive probability measures on V, is Γ-invariant.
$2 \to 1$: This is the nontrivial part (the equivalence $2 \leftrightarrow 3$ does not depend on the tree structure). Denote by μ the assumed Γ-invariant probability measure on V. For any vertices $u, w \in V$ set:

$$V(u, w) = \{v \in V | d(v, u) < d(v, w)\} \qquad \mu(u, w) = \mu\{V(u, w)\}$$

where d is the combinatorial tree metric. Notice that if u and w are adjacent, then V is a disjoint union of $V(u, w)$ and $V(w, u)$, hence $\mu(u, w) + \mu(w, u) = 1$. In particular, at least one of these summands is not less than $1/2$. The fact that μ has total mass one, together with the tree structure, easily imply the following.

Claim 1. Every $w \in V$ has at most one adjacent vertex u with $\mu(u, w) > \mu(w, u)$, and at most two with $\mu(u, w) = \mu(w, u)$. In the latter case, both vertices satisfy $\mu(u, w) = 1/2$. □

Call an edge e connecting the vertices u, w **even** if $\mu(u, w) = \mu(w, u)$.

Claim 2. If e_1 and e_2 are even, then so are all the edges in the geodesic connecting them.

Proof. We may assume that μ has no atoms, for then the set of vertices with a given positive weight is finite and Γ-invariant, so the action is elementary. Thus, if $v_1, v_2, \ldots, v_n$ from (in that order) the set of vertices along the geodesic in the claim, we have $V(v_{i-1}, v_i) \subset V(v_i, v_{i+1})$ for all i, hence $\mu(v_{i-1}, v_i) \leqslant \mu(v_i, v_{i+1})$. However by assumption $\mu(v_1, v_2) = \mu(v_{n-1}, v_n) = 1/2$, hence this equality must hold all the way. □

For the proof of $2 \to 1$ let us distinguish between two cases:

(i) There exist adjacent vertices u, w with $\mu(u, w) = 1/2$.
(ii) For all adjacent vertices u, w one has $\mu(u, w) \neq \mu(w, u)$.

Assume first that (i) holds. Since μ is Γ-invariant, the set of even edges in E forms a nonempty Γ-invariant subtree S (by Claim 2). Moreover, by the second part of Claim 1, S is either finite, or a ray, or a two regular subtree. In any of these cases, the Γ–action is elementary. Therefore we may assume that (ii) holds. Then on every edge $e \in E$ connecting the vertices u, w, one can define an orientation by saying that e goes out from u, or into w, iff $\mu(u,w) < \mu(w,u)$. Of course, since μ is Γ-invariant, this orientation is preserved under the Γ-action. By claim 1 for every vertex u there is at most one edge going out from u. A careful examination of the situation now shows that one of the following must occur: either there is a **unique** vertex u such that **no** edge goes out from u, in which case all the edges in the tree are "directed" to u and this vertex must be fixed by Γ, or every vertex u has exactly one edge going out. In the latter case we get a "current" (or a "non-vanishing vector field") on the tree, which can be seen to define a Γ-invariant ray. This completes the proof of Proposition 9.1. $\square$

We remark in passing the following straightforward consequence of Proposition 9.1:

9.2 Corollary. *Let Γ be a finitely generated group. Suppose that $\Gamma = A \star_C B$ is an amalgamated product of its proper subgroups A, B over C, and $[A : C] \geq 2$, $[B : C] \geq 3$. Then the Γ-representations $\ell^2(\Gamma/A), \ell^2(\Gamma/B)$ (and $\ell^2(\Gamma/C)$) do not admit almost invariant vectors.*

We can now turn to the proof of Theorem 0.7, which goes along similar lines as the sketch in [Sh1 p.841], with some modifications.

Proof of Theorem 0.7 for the case of irreducible lattices. By performing a barycentric subdivision we may assume that all elements of Γ act without inversions. Notice that it is enough to find one vertex $v_0 \in V$ and a factor G_i, having the following property: if $\gamma_n \in \Gamma$ is any sequence with $pr_i(\gamma_n) \to 1_{G_i}$ (where pr_i is the projection to G_i), then eventually $\gamma_n v_0 = v_0$. Indeed, the set S of all such vertices is easily seen to be connected and Γ–invariant. Putting on $Aut(S)$ the topology $\mathfrak{T}$ of point-wise convergence, we then deduce from Proposition 4.3 that the Γ-action on S extends continuously to G, factoring through G_i.

Let $\widetilde{E}$ denote the set of oriented edges in T. A well known, yet key observation for us, is the fact that the natural Γ-representation $\pi = \ell^2(\widetilde{E})$ has non-vanishing first cohomology if Γ has an infinite orbit in T (cf. [Sh2 Section 7]). Since we may clearly assume the latter property, we have $H^1(\Gamma, \ell^2(\widetilde{E})) \neq 0$. We claim that by the non-elementary assumption, $\ell^2(\widetilde{E})$ does not admit almost invariant vectors, and hence by Corollary 1.8 we actually have $\overline{H^1}(\Gamma, \ell^2(\widetilde{E})) \neq 0$. Indeed, otherwise a general standard argument shows that the Γ-action on $\widetilde{E}$ admits almost invariant ("Fölner") sets, and it is easily verified that the sets of vertices incident to the edges in these sets are almost Γ-invariant as well, which by 9.1 implies that the Γ-action is elementary.

Thus $\overline{H^1}(\Gamma, \ell^2(\widetilde{E})) \neq 0$, and we may invoke Theorem 4.1. It follows that there exists a nonzero Γ-invariant subspace $\mathcal{H}_0 \subset \ell^2(\widetilde{E})$ and a factor G_i, such that the Γ–representation on $\mathcal{H}_0$ extends continuously to G, factoring through G_i. Define on $\widetilde{E}$ an equivalence relation: $\tilde{e}_1 \sim \tilde{e}_2$ iff $f(\tilde{e}_1) = f(\tilde{e}_2)$ for all $f \in \mathcal{H}_0$. Let $\widetilde{E}'$ be the set of (oriented) edges on which **some** $f \in \mathcal{H}_0$ does not vanish. Then $\widetilde{E}'$ is not empty, Γ-invariant, and breaks under $\sim$ into equivalence classes which are all finite (since $\mathcal{H}_0$ consists of ℓ^2–functions), and permuted by the Γ-action. Let R be one such class and $M < \Gamma$ its stabilizer. Then M preserves the barycenter (of the barycenters of the edges) of R, and since we assume that every element acts without inversions, M fixes a vertex v_0.

Finally, we show that v_0 is a vertex as required at the beginning of the proof. Given any sequence

$\gamma_n \in \Gamma$ with $pr_i(\gamma_n) \to 1$ in G_i, we need to show that eventually $\gamma_n \in M$. Indeed, otherwise we may assume one of two possibilities: (i) $\gamma_n \neq \gamma_m \mod M$ for all $n \neq m$; (ii) $\bar{\gamma}_n = \bar{\gamma} \mod M$ for some $\gamma \notin M$, and all n. Assume first that (i) holds and choose some $f \in \mathcal{H}_0$, such that f does not vanish on some $\tilde{e} \in R$. Notice that $\gamma_n^{-1}\tilde{e}$ and $\gamma_m^{-1}\tilde{e}$ belong to different equivalence classes for $n \neq m$, so $f(\gamma_n^{-1}\tilde{e}) \to 0$ (as f is square sumable). It follows that $\liminf \|\gamma_n f - f\| > 0$. However, since the Γ-action on f_0 extends continuously to G_i, and $pr_i(\gamma_n) \to 1$ in G_i, we should have had $\|\gamma_n f - f\| \to 0$, a contradiction. If (ii) holds, we obtain a contradiction in a similar way, choosing $f \in \mathcal{H}_0$ which separates the classes R and γR. □

Notice that when one of the G_i's in the setup of **(0.1)**, or G itself in the setup of **(0.2)**, is connected, Theorem 0.7 yields a phenomenon of "relative property (FA) of Serre" (e.g., of $SL_2(\mathbb{Z}[\frac{1}{p}])$ with respect to $SL_2(\mathbb{Z})$). For example, using a recent result of Haglund Paulin and Valette [HPV], one can deduce:

9.3 Corollary. *Let Γ be a higher rank irreducible lattice of a semisimple group, in which at least one (non compact) simple factor is connected. Then any homomorphism of Γ into a Coxeter group has finite image.*

§10. Rigidity of commensurators

In this section we discuss the various results in the introduction concerning rigidity of commensurators. The following result will be fundamental in what follows, and accounts essentially for all the rigidity phenomena we shall exhibit in this framework. It highlights again the relation between commutativity and rigidity.

10.1 Theorem. *Let G be a locally compact group acting isometrically, affinely and uniformly (Def. 1.9) on a Hilbert space $\mathcal{H}$. Let $T : \mathcal{H} \to \mathcal{H}$, $Tv = T^0v + v_0$, be an affine operator (where $v_0 \in \mathcal{H}$ is fixed), such that T^0 is self adjoint and satisfies $\|T^0\| \leq 1$. If T commutes with the action of G, then the linear operator T^0 admits a non-zero invariant vector in $\mathcal{H}$.*

Theorem 10.1 was actually established in the proof of Theorem 3.1 (see Corollary 3.5 and the discussion preceding it). In fact, that proof shows that the self adjointness of T^0 is redundant, and that the result actually holds for certain uniformly convex Banach spaces. However, the formulation in 10.1 will suffice for our purposes (besides the case of non uniform lattices in $SL_2(\mathbb{C})$, which we discuss in 10.4 below). Let us use the opportunity to give a short proof of 10.1, in the spirit of our original argument in [Sh1 p.839], based on the spectral theorem for self adjoint operators.

Proof of Theorem 10.1. By the correspondence between affine isometric G-actions and cocycles for the associated linear representation (see §1.I), we need to show the following: Let σ be a unitary G-representation and $b \in Z^1(G, \sigma)$ a 1-cocycle. Suppose that the corresponding affine G-action $gv = \sigma(g)v + b(g)$ commutes with an affine operator T as in the theorem. If the linear part T^0 has no non zero invariant vectors, then $b \in \overline{B^1(G, \sigma)}$ (see Def. 1.7 and Lemma 1.10).

Indeed, since T and G commute, we have for all $g \in G$: $gTo = Tgo$ (o=zero is taken only for convenience). Recalling that $Tv = T^0 v + v_0$, the latter equality reads:

$$\sigma(g)v_0 + b(g) = gv_0 = g(To) = T(go) = Tb(g) = T^0 b(g) + v_0 \tag{10.1}$$

Looking at the first and last terms in (10.1), we deduce:

$$(I - T^0)b(g) = v_0 - \sigma(g)v_0 \tag{10.2}$$

Since we assume that T^0 does not admit nonzero invariant vectors, $I - T^0$ is a positive self adjoint operator. By its commutation with $\sigma(G)$ and the spectral theorem, we can find a sequence of ("almost inverting") operators T_n, which commute with $\sigma(G)$ and satisfy $T_n(I - T^0)v \to v$ for every v (in fact, uniformly on v in compact subsets of $\mathcal{H}$). Therefore, applying T_n on both sides of (10.2) yields:

$$T_n v_0 - \sigma(g)T_n v_0 = T_n v_0 - T_n \sigma(g) v_0 = T_n(v_0 - \sigma(g)v_0) = T_n(I - T^0)b(g) \to b(g)$$

Hence $b(g)$ is a uniform limit the co-boundaries associated with $T_n v_0$, as required. $\square$

The following extension of Theorem 10.1 follows from it immediately:

10.2 Corollary. *With the notation of 10.1, every $b \in Z^1(G, \sigma)$ is almost cohomologous (Def. 1.7) to a cocycle taking values in the (G-invariant) subspace of T^0-invariant vectors.*

Proof. Let $\mathcal{H}_0$ denote the subspace of T^0-invariant vectors, and $\mathcal{H}_1$ its orthogonal complement. Both spaces are of course invariant under the actions of $\sigma(G)$ and T^0. Let p_0, p_1 denote the corresponding orthogonal projections. Then applying p_1 in both sides of (10.2), and continuing the argument as in the proof of Theorem 10.1 for the cocycle $p_1 \circ b$, yields that this cocycle is almost a coboundary. Since $b = p_0 \circ b + p_1 \circ b$, the result follows. $\square$

The uniform affine G-actions for which we shall apply Theorem 10.1 will be induced from actions of Γ. Then, an affine operator commuting with the G-action will arise from the fact that the original Γ-action is not arbitrary; it admits a further symmetry coming from the action of the commensurator on the same space. This will give rise to a "Hecke operator", acting in the space of induced representation, and commuting with the G-action. The conclusion of Theorem 10.1, that this operator fixes a non-zero vector, together with the density of the commensurator, then immediately yield a subspace of the **original** representation on which the action extends to G. The way to the rigidity results of the introduction then goes in the same path we paved in the case of irreducible lattices. This is the general scheme, and we now turn to elaborate on the details.

Recall that $\Gamma < \Lambda < Comm_G(\Gamma) < G$ and that we assume Λ is dense in G. The commensurability property implies that every Γ-orbit in its action on Λ/Γ is **finite**. Let $M \subset \Lambda/\Gamma$ denote a finite Γ-invariant subset, which we can always assume to be symmetric by replacing M with $M \cup M^{-1}$ (i.e., if $m \in \Lambda$ with $\overline{m} \in M$, then $\overline{m^{-1}} \in M$).

Let π be a unitary Λ-representation, and consider the G-representation σ which is induced from the restriction of π to Γ. In what follows, we adapt the second standard model of unitary induction, as described in (1.7) of Section 1.II. Corresponding to the finite set M we define the "Hecke operator" T_M^0 acting in the representation space $L^2(G, \mathcal{H})$ of Γ-equivariant functions:

$$T_M^0 f(x) = \frac{1}{|M|} \sum_{m \in M} \pi(m) f(xm) \tag{10.3}$$

Several properties should be verified in order for the operator T_M^0 to be well defined. First, for the expression in (10.3) to make sense, we require that each term $\pi(m)f(xm)$ does not depend on the choice of representative in the coset. Indeed, if $m_1 = m\gamma$, then

$$\pi(m_1)f(xm_1) = \pi(m\gamma)f(xm\gamma) = \pi(m)\pi(\gamma)\pi(\gamma)^{-1}f(xm) = \pi(m)f(xm)$$

Next, we check that $T_M^0 f(x)$ verifies the Γ-equivariance condition, $[T_M^0 f](x\gamma) = \pi(\gamma)^{-1}[T_M^0 f(x)]$:

$$T_M^0 f(x\gamma) = \frac{1}{|M|} \sum_{m \in M} \pi(m) f(x\gamma m) = \frac{1}{|M|} \sum_{m \in M} \pi(\gamma)^{-1}\pi(\gamma m) f(x\gamma m) = \tag{10.4}$$

$$\pi(\gamma)^{-1}\left[\frac{1}{|M|} \sum_{m \in M} \pi(\gamma m) f(x\gamma m)\right] = \pi(\gamma)^{-1}\left[\frac{1}{|M|} \sum_{m' \in \gamma M} \pi(m') f(xm')\right] = \pi(\gamma)^{-1}[T_M^0 f(x)]$$

It is also easy to verify that T_M^0 is self adjoint (as M is symmetric), and that it satisfies $\|T_M^0\| \leq 1$. (Applying norms and the triangular inequality, the latter reduces to the same assertion for the action of T_M^0 in $L^2(G/\Gamma)$, in which it is established by evaluating the integrals over a cover G/Γ_0, where $\Gamma_0 = \cap\, m\Gamma m^{-1} < \Gamma$ has finite index.)

Let now b be a Λ-1-cocycle for the representation π and consider the affine isometric Λ-action on $\mathcal{H}$ defined by: $\rho(\lambda)v = \pi(\lambda)v + b(\lambda)$. As described in (1.7) of Section 1.II, we may induce to G the affine action of Γ, to get an isometric G-action (by left translation) on the affine Hilbert space $\tilde{L}^2(G, \mathcal{H})$. Analogous to (10.3) we define the operator T_M on the latter space by:

$$T_M f(x) = \frac{1}{|M|} \sum_{m \in M} \rho(m) f(xm) = \frac{1}{|M|}\left(\sum_{m \in M} \pi(m) f(xm) + b(m)\right) \tag{10.5}$$

Notice that T_M is defined as an affine combination, and is itself an affine operator. The proof that it is well defined is the same as before, replacing π by ρ. In order to pull out the operator $\rho(\gamma)^{-1}$ in the second line of (10.4), use the fact that its action on $\mathcal{H}$ preserves convex combinations. Again, T_M commutes with the action of every $g \in G$. By fixing any $f_0 \in \tilde{L}^2(G, \mathcal{H})$ as an "origin", the isometry $\varphi \to \varphi - f_0$ from $\tilde{L}^2(G, \mathcal{H})$ onto $L^2(G, \mathcal{H})$ realizes the latter Hilbert space as the translation space for $\tilde{L}^2(G, \mathcal{H})$, and the actions of T_M and G transform to commuting affine actions on $L^2(G, \mathcal{H})$ (see §1.II). As one can check easily, the **linear** part of T_M on the latter Hilbert space is given by the linear operator T_M^0, acting as in (10.3). Using Theorem 10.1, we now deduce the following:

10.3 Theorem. *Let $(\pi, \mathcal{H})$ be a unitary Λ-representation and $b \in Z^1(\Lambda, \pi)$, a cocycle. If $b|_\Gamma$ is not almost a coboundary (Def. 1.7) or equivalently, the affine Γ-action it defines is uniform (see 1.9-1.10), then there exists a nonzero Λ-invariant subspace on which the linear Λ-action extends continuously to a unitary representation of G.*

Proof. First, since G is second countable we may assume that Λ is countable. Furthermore, we will assume that Λ is finitely generated, an assumption made only for convenience of the exposition. By this assumption we may find a Hecke operator T_M as above, such that M generates Λ. In general, the proof proceeds by working with an infinite convex combination of Hecke operators which exhaust the group Λ.

We induce to G the restriction to Γ of the affine action determined by the representation π and the cocycle b, recalling that by our assumption on b, the resulting affine G-action is uniform as well (see Lemma 1.13). The commutation of the G-action with the action of the Hecke operator T_M defined in (10.5), together with the foregoing discussion and Theorem 10.1, then readily imply that there exists a non zero function $f \in L^2(G, \mathcal{H})$ which is fixed by T_M^0. More precisely, $\frac{1}{M}\Sigma\pi(m)f(xm) = f(x)$ for almost every x. The set of functions fixed by T_M^0 is therefore a non-zero closed G-invariant subspace. In particular, it is preserved under applying any convolution operator $f \to \int \varphi(g)f(g^{-1}x)$, where φ is a smooth function on G with compact support. As G acts transitively on itself and $f \neq 0$, we may therefore assume that the function f is continuous and satisfies $f(e) \neq 0$. Hence, $T_M^0 f(x) = f(x)$ for every x. Observe that this implies that $\|f(x)\|$ must be constant. Indeed, $\|f(x)\|$ is a continuous function on the compact space G/Γ, and admits a maximal value, say in x_0. By unitarity of π and convexity, we must have $\|f(x_0m)\| = \|f(x_0)\|$ for all $m \in M$. As M generates the dense subgroup Λ, and f is continuous, we see that $\|f(x)\| = \|f(x_0)\|$ for all x. The same convexity argument now shows that we must have $\pi(m)f(xm) = f(x)$ for all $x \in G$ and $m \in M$; hence for all $m \in \Lambda$ as well. Finally, notice that the vector $v_0 = f(e) \neq 0$ satisfies $\pi(m)v_0 \to v_0$ whenever $m \to e$. Indeed, in that case we have $f(m) \to f(e) = v_0$ by continuity of f, but $f(m) = f(em) = \pi(m)f(e) = \pi(m)v_0$. The set of all vectors v_0 with this property is easily seen to form a closed, Λ-invariant subspace, and the same argument as in the proof of Proposition 4.3 shows that the Λ-representation on it extends continuously to G. □

Proof of Theorems 0.3', 0.6', and 0.4, 0.7 for commensurators. The proof of these results can now be deduced from Theorem 10.3 in exactly the same manner as Theorems 0.3, 0.6 and 0.4, 0.7 (for irreducible lattices) are obtained from Theorem 4.1. □

An argument different from the one given for irreducible lattices, is needed in the case of characters (compare with the proof of Theorem 0.8 in 5.2). We shall use Zimmer's fundamental observation regarding the connection between cocycles and extensions of homomorphisms.

Proof of Theorem 0.8'. Let $\varphi : \Lambda \to \mathbb{C}$ be as in the theorem. We view φ (and $\varphi|_\Gamma$) as an element of H^1 with coefficients in the trivial representation. Consider the G-cocycle $\tilde{b}$ induced to G from $\varphi|_\Gamma$, which takes values in the induced representation $L^2(G/\Gamma)$. We shall now use the first model of induction described in §1.II -see (1.3). Recall that $\tilde{b}$ is given by the formula:

$\tilde{b}(g)(x) = \varphi|_\Gamma \circ \alpha(g^{-1}, x)$, where $\alpha : G \times G/\Gamma \to \Gamma$ is a cocycle corresponding to a fundamental domain for Γ. By Corollary 10.2, where $T_0 = T_M^0$ is the Hecke operator (10.3), we deduce that $\tilde{b}$ is almost cohomologous to a cocycle taking values in the space of T_M^0-invariant vectors in $L^2(G/\Gamma)$, which by density of Λ, is easily seen to be the one dimensional subspace of (essentially) constant functions (any left convolution of such function is continuous and hence constant by an argument as in the proof of Theorem 10.3 above). However, since $L^2(G/\Gamma) = \mathbb{C}1 \oplus L_0^2(G/\Gamma)$, and the second summand does not admit almost invariant vectors [Mar1, p.112 1.10], we deduce from Corollary 1.8 that "almost cohomologous" is equivalent to "cohomologous" in this case. Our discussion thus amounts to the following: there exists some $f \in L^2(G/\Gamma)$ such that for every fixed $g \in G$, the cocycle $\varphi|_\Gamma \circ \alpha(g^{-1}, x) + f(g^{-1}x) - f(x)$ as a complex function of x, has (essentially) constant value, which can be denoted by $\tilde{\varphi}(g)$. It is easy to check using the cocycle identity, that $\tilde{\varphi}(g)$ is an additive (complex) character on G. Moreover, as shown by Zimmer [Zi1, 4.2.16], $\tilde{\varphi}(g)$ extends $\varphi|_\Gamma$ continuously, and the proof is complete. $\square$

Proof of the arithmeticity theorem for commensurators (Theorem 0.5'). The proof goes along similar lines as that of Theorem 0.5, relying mainly on Corollary 0.4. We only remark on some modifications needed due to the change of framework. The first observation required is that once the image $\varphi(\Gamma)$ is infinite (which is our assumption), then it is actually Zariski dense. Otherwise, since every proper algebraic subgroup of PGL_2 is virtually solvable, by passing to subgroups of finite index and conjugating (over the algebraic closure), we may assume that the image lies in the (say, upper) triangular matrices. Since by the commensurability property, the Zariski closure of $\varphi(\Lambda)$ must normalize the connected component of the Zariski closure of $\varphi(\Gamma)$, the former must also be (virtually) upper triangular. It is not difficult to derive from here a contradiction to Theorem 0.8' (and the simplicity of G), and we leave the details to the reader. The next step is to consider the field K generated by the traces of the elements of Γ and deduce, as in Section 8, that we may assume that (the image of) Γ is contained in $\mathbb{G}(K)$, where $\mathbb{G} \cong PGL_2$. Since $\varphi(\Lambda)$ commensurates the Zariski dense subgroup $\varphi(\Gamma)$, and $\mathbb{G}$ is of adjoint type, a standard argument as in the proof of Theorem 8.4 shows that also $\varphi(\Lambda) \subset \mathbb{G}(K)$. The rest of the proof is very similar to that described in Section 8. We only remark that once we take a valuation ν on K for which the image of $\varphi(\Gamma)$ in $\mathbb{G}(K_\nu)$ is unbounded (which always exists by our assumption that $\varphi(\Gamma)$ is infinite), the extension $\tilde{\varphi}$ to the simple group G (by Corollary 0.4) shows that the original homomorphism φ was injective. The proof of the claim in Section 8 shows that K is a global field, and that in fact, the valuation ν with the above property is unique. $\square$

10.4. On the modifications required when Γ is a nonuniform lattice. In light of our previous results concerning the rigidity of irreducible lattices, the only interesting case to understand in relation to rigidity of commensurators is that when G is of rank one. The main difficulty in extending our results to nonuniform lattices lies in the verification of condition (1.5) in Section 1.II, which enables one to induce the first cohomology of unitary representations from Γ to G. If G is a rank one simple Lie group with finite center, then it was shown in [Sh2] that this condition is indeed satisfied (for an appropriate choice of a fundamental domain), provided that G is not locally isomorphic to $SL_2(\mathbb{R})$ or $SL_2(\mathbb{C})$ (in the latter cases, there are indeed lattices for which there is no such fundamental domain!). We shall shortly return to discuss the interesting case of $SL_2(\mathbb{C})$, and show that nevertheless all of our results continue to hold. However before doing so, we should remark that the co-compactness of Γ was used elsewhere, in the course of the proof of Theorem 10.3. We used the compactness of G/Γ to deduce that the continuous function f satisfying $T^0_M(f)(x) = \frac{1}{M}\Sigma\pi(m)f(xm) = f(x)$ for all x, must have constant norm. Let us indicate an alternative argument. Denoting $F(x) = \|f(x)\|$, we see that $F \in L^2(G/\Gamma)$ is a continuous function satisfying for all x:

$$\frac{1}{|M|}\sum_{m\in M} F(xm) \geq \|\frac{1}{|M|}\sum_{m\in M} f(xm)\| = \|f(x)\| = F(x)$$

However the operator T^0_M has norm at most one on $L^2(G/\Gamma)$, so by positivity and continuity, equality must hold everywhere. Now, the fact that a continuous function $F \in L^2(G/\Gamma)$ satisfying $\frac{1}{M}\Sigma F(xm) = F(x)$ must be constant, follows from the proof of [BuSa, Lemma 2.2 (a)].

Another general fact about uniform lattices that we used, is that the G-representation $L^2_0(G/\Gamma)$ does not admit almost invariant vectors. This is not the case for general nonuniform lattices, but as showed by Bekka [Be], it does hold for all lattices in simple Lie groups.

The case of nonuniform lattices in $SL_2(\mathbb{C})$. As mentioned earlier, condition (1.5) in §1.II is not generally satisfied for these lattices. The computation in [Sh2, Section 3] shows that it is just barely missed, more precisely, keeping the notations as in there, the following is satisfied:

$$\int_X \ell(\alpha(g,x))^p d\mu(x) < \infty \quad \forall g \in G \tag{10.6}$$

for every $p < 2$, but not for $p = 2$ (in the case of $SL_2(\mathbb{R})$ it holds for every $p < 1$). We are interested in the value $p = 2$ because the representation space of the (unitarily) induced representation consists of L^2-functions. More generally, however, for every $p \geq 1$ one may induce a unitary Γ-representation $(\pi, \mathcal{H})$ to G by considering the usual G-action on the representation space $L^p(G, \mathcal{H})$, a space which is not Hilbert if $p \neq 2$. However, the preliminaries on first cohomology in Section 1.I cover all L^p spaces ($1 < p < \infty$). The same argument showing that condition (10.6) when $p = 2$ enables one to induce cocycles to $L^2(G, \mathcal{H})$, shows that this condition guarantees this property when we induce π to $L^p(G, \mathcal{H})$ for a general p (we remark that the injectivity of the map on the cohomology groups, which is all we use, is easy and completely general -cf. [Sh2, 3.5]. As we shortly observe, it is enough to consider here only the ordinary cohomology, not the reduced one). Note that Theorem 10.1 is the fundamental tool from which the results of the present section were deduced. Although the simple proof given above used the Hilbertian structure, in Corollary 3.5 we noticed that the same result holds for all Banach spaces $\mathcal{H}$ satisfying inequality (3.1) there. For the Banach space $\mathcal{H} = L^p(G, \mathcal{H})$, classical inequalities (see [Cl]) show that (3.1) holds with $t = 2^{1+\frac{1}{p}}$ if $p \leq 2$ (if $p \geq 2$, replace p by its conjugate q). Thus, for ϵ sufficiently small, (3.1) holds when $p = 2 - \epsilon$, and hence also Theorem 10.1 for affine G-actions on $L^p(G, \mathcal{H})$ is valid.

Finally, we should remark also on our use of reduced cohomology for representations on Banach spaces. By Proposition 1.6, for a linear (isometric) representation on any uniformly convex Banach space without almost invariant vectors, the notions of reduced and ordinary first cohomology coincide. As mentioned already, besides the results of Section 6 (which we do not discuss in the setting of commensurators), all the representations used in deducing the rigidity results from the cohomology rigidity theorem, had this property. Since that property is also preserved under the above L^p-induction operation from lattices (the proof in [Mar1, p.112 1.11] applies), we can implement the same strategy of proofs in this setting as well. For the proof of the superrigidity theorem for additive characters (Theorem 0.8'), we should also add that the condition that $L_0^2(G/\Gamma)$ does not admit almost invariant vectors, is equivalent to this condition in any other $L_0^p(G/\Gamma)$, for $1 \leq p < \infty$ (use [Be] to restrict the G-representation to some lattice, and apply [Sch, Prop. 2.3]).

We conclude by observing the interesting situation which arises in the case of nonuniform lattices $\Gamma < SL_2(\mathbb{C})$. For (at least some) such lattices, one can show that $H^1(G, L^2(G/\Gamma)) = 0$, but $H^1(G, L^p(G/\Gamma)) \neq 0$ for all $1 \leq p < 2$. This phenomenon never occurs for uniform lattices, and may be interesting to understand more throughly, for other rank one Lie groups as well.

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