BROWN REPRESENTABILITY AND FLAT COVERS

HENNING KRAUSE

In this note we exhibit a surprising connection between the following two concepts:

- Brown representability which arises in stable homotopy theory, and
- flat covers which arise in module theory.

Let $\mathcal{T}$ be a compactly generated triangulated category, for example the stable homotopy category of CW-spectra, and denote by $\mathcal{F}$ the full subcategory of compact objects (an object X in $\mathcal{T}$ is *compact* if the representable functor $\mathrm{Hom}(X,-)$ preserves arbitrary coproducts). We prove that Brown representability holds for $\mathcal{T}$ if and only if for every functor $\mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ a flat cover can be constructed in a canonical way. The proof also shows that Brown representability for objects and morphisms is a consequence of Brown representability for objects and isomorphisms.

Let us recall the relevant definitions. For every object X in $\mathcal{T}$ consider the functor

$$H_X = \mathrm{Hom}(-,X)|_{\mathcal{F}} \colon \mathcal{F}^{\mathrm{op}} \longrightarrow \mathrm{Ab}.$$

This gives rise to a functor $\mathcal{T} \to (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ into the category of additive functors $\mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$. Every functor of the form H_X is *exact*, that is, it sends triangles in $\mathcal{F}$ to exact sequences in Ab. In some cases also the converse is true. More precisely, one says that *Brown representability* holds for $\mathcal{T}$, if

- every exact functor $\mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ is isomorphic to H_X for some object X in $\mathcal{T}$, and
- every natural transformation $H_X \to H_Y$ is of the form H_f for some map $f \colon X \to Y$.

A classical theorem due to Brown and Adams states that Brown representability holds for the stable homotopy category [1]. This has been generalized in recent work by Neeman, which also contains an example of a compactly generated triangulated category where Brown representability fails [8]. For the relevance of Brown representability in the representation theory of finite groups we refer to recent work of Benson and Gnacadja [3].

Now consider the functor category $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. Recall that there exists a *tensor product*

$$(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab}) \times (\mathcal{F}, \mathrm{Ab}) \longrightarrow \mathrm{Ab}, \quad (F,G) \to F \otimes_{\mathcal{F}} G$$

where for any functor $F \colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$, the tensor functor $F \otimes_{\mathcal{F}} -$ is determined by the fact that it preserves colimits and $F \otimes_{\mathcal{F}} \mathrm{Hom}(X,-) \cong F(X)$ for all X in $\mathcal{F}$. A functor $F \colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ is *flat* if the tensor functor $F \otimes_{\mathcal{F}} -$ is exact. Our interest in flat functors is motivated by the elementary fact that a functor is flat if and only if it is exact [7]. A map $p \colon E \to F$ between functors is called a *flat cover* of F, if

- E is flat and every map $E' \to F$ such that E' is flat factors through p;
- every endomorphism $e \colon E \to E$ satisfying $pe = p$ is an isomorphism.

This is a useful concept because flat covers usually exist whereas projective covers rarely exist. In fact, Enochs conjectured in [5] that every module over an associative ring has a flat cover. Note that a module category is a special case of a functor category. In any case, Enochs' conjecture seems to be hard; it is still open [9]. Let us construct for every functor

Date: May 1999.

$F\colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ an object $X \in \mathcal{T}$ and a map $p_F\colon H_X \to F$ which is a good candidate for being a flat cover of F. Choose a minimal injective copresentation of the form

$$0 \longrightarrow F \longrightarrow H_I \xrightarrow{H_f} H_J$$

in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. This is possible by Lemma 1 below. The map f can be completed to a triangle

$$X \xrightarrow{e} I \xrightarrow{f} J \xrightarrow{g} \Sigma X,$$

and the map $H_e\colon H_X \to H_I$ induces a map $p_F\colon H_X \to F$. Note that p_F is uniquely determined by F, up to isomorphism. We are now in a position to state the main result of this note.

Theorem. *Let $\mathcal{T}$ be a compactly generated triangulated category and denote by $\mathcal{F}$ the full subcategory of compact objects in $\mathcal{T}$. Then the following conditions are equivalent:*

1. *Brown representability holds for $\mathcal{T}$.*
2. *Every exact functor $\mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ is isomorphic to H_X for some object X in $\mathcal{T}$, and every isomorphism $H_X \to H_Y$ is of the form H_f for some map $f\colon X \to Y$ in $\mathcal{T}$.*
3. *A functor $F\colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ is flat if and only if $\mathrm{Ext}^2(-, F) = 0$.*
4. *For every functor $F\colon \mathcal{F}^{\mathrm{op}} \to \mathrm{Ab}$ the map $p_F\colon H_X \to F$ is a flat cover of F.*

For a local version of this result we refer to the proposition at the end of this note. The theorem is based on the interplay between the triangulated structure of $\mathcal{T}$ and the abelian structure of the functor category $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. A similar approach can be found in recent work of Christensen, Strickland, and Neeman [4, 8]. In fact, the connection between Brown representability and the vanishing of $\mathrm{Ext}^2(-, F)$ for exact functors F has been noticed by these authors. The new ingredient which leads to the connection with flat covers, is the existence of injective envelopes in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$, see [6]. Recall that a monomorphism $i\colon F \to Q$ in any abelian category is an *injective envelope* of F, if Q is injective and every endomorphism $e\colon Q \to Q$ satisfying $ei = i$ is an isomorphism.

The rest of this paper is devoted to proving the main result. To this end we need to recall our assumptions on the triangulated category $\mathcal{T}$:

- $\mathcal{T}$ has arbitrary coproducts;
- the isomorphism classes of compact objects in $\mathcal{T}$ form a set;
- $H_X = 0$ implies $X = 0$ for every object X in $\mathcal{T}$.

The proof of the theorem is divided into several lemmas. The first one is taken from [7]. We include the short proof for the convenience of the reader.

Lemma 1. *Let Q be an injective object in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. Then there exists, up to isomorphism, a unique object I in $\mathcal{T}$ such that $Q \cong H_I$. Moreover, the map*

$$\mathrm{Hom}(X, I) \longrightarrow \mathrm{Hom}(H_X, H_I), \quad f \mapsto H_f,$$

is an isomorphism for all X in $\mathcal{T}$.

Proof. Consider the exact functor $F\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$, $X \mapsto \mathrm{Hom}(H_X, Q)$. It sends coproducts in $\mathcal{T}$ to products in Ab and is therefore representable by Brown's classical theorem [2]. Thus $F \cong \mathrm{Hom}(-, I)$ for some object I in $\mathcal{T}$. The induced map $\mathrm{Hom}(I, I) \cong F(I) = \mathrm{Hom}(H_I, Q)$ sends id_I to a map $p\colon H_I \to Q$ which is an isomorphism since

$$H_I(C) = \mathrm{Hom}(C, I) \cong \mathrm{Hom}(H_C, Q) \cong Q(C)$$

for every compact object C by Yoneda's lemma. The inverse $p^{-1}\colon Q \to H_I$ induces an isomorphism

$$\mathrm{Hom}(X, I) \cong \mathrm{Hom}(H_X, Q) \cong \mathrm{Hom}(H_X, H_I)$$

which is precisely the map $f \mapsto H_f$. This finishes the proof. $\square$

Lemma 2. *Let $F \in (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$. Then $\mathrm{Ext}^2(-,F) = 0$ if and only if the canonical map $p_F\colon H_X \to F$ is an isomorphism.*

Proof. We fix the minimal injective copresentation

$$0 \longrightarrow F \longrightarrow H_I \xrightarrow{H_f} H_J$$

and denote by $e\colon X \to I$ the fiber of $f\colon I \to J$. We have $\mathrm{Ext}^2(-,F) = 0$ if and only if H_f is an epimorphism, and this is equivalent to H_e being a monomorphism since the fiber e gives rise to an exact sequence

$$H_{\Sigma^{-1}J} \xrightarrow{H_{\Sigma^{-1}g}} H_X \xrightarrow{H_e} H_I \xrightarrow{H_f} H_J \xrightarrow{H_g} H_{\Sigma X}.$$

Furthermore, H_e is a monomorphism if and only if H_X is a kernel of H_f. We conclude from the construction of the map p_F that $\mathrm{Ext}^2(-,F) = 0$ if and only if p_F is an isomorphism. ☐

Lemma 3. *Let $X \xrightarrow{e} I \xrightarrow{f} J \xrightarrow{g} \Sigma X$ be a triangle in $\mathcal{T}$ and suppose that the induced map $\operatorname{Im} H_f \to H_J$ is an injective envelope. Then every endomorphism $u\colon X \to X$ satisfying $eu = e$ is an isomorphism.*

Proof. Choose a map $v\colon J \to J$ which completes the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "u"] & I \arrow[r, "f"] \arrow[d, equal] & J \arrow[r, "g"] & \Sigma X \arrow[d, "\Sigma u"] \\ X \arrow[r, "e"] & I \arrow[r, "f"] & J \arrow[r, "g"] & \Sigma X \end{tikzcd}$$

The assumption on u and f imply that H_v is an isomorphism. Thus v is an isomorphism, and we conclude that u is an isomorphism. ☐

Lemma 4. *Let $Y \in \mathcal{T}$ and $p = p_F\colon H_X \to H_Y$ be the canonical map for $F = H_Y$. Then the following are equivalent:*

(1) $\mathrm{Ext}^2(-,H_Y) = 0$.
(2) *Every isomorphism $H_X \to H_{Y \coprod Z}$ which corestricts to p is of the form H_u for some map $u\colon X \to Y \coprod Z$.*
(3) *There exists a map $v\colon X \to Y$ such that $p = H_v$.*

Proof. We denote by $c\colon Y \to I$ the map which induces the first map of the minimal injective copresentation

$$0 \longrightarrow H_Y \longrightarrow H_I \xrightarrow{H_f} H_J;$$

it exists by Lemma 1. The map c factors through the fiber $e\colon X \to I$ of f via some map $w\colon Y \to X$ since $fc = 0$. Note that $pH_w = \mathrm{id}_{H_Y}$.

(1) ⇒ (2) If $\mathrm{Ext}^2(-,H_Y) = 0$, then p is an isomorphism by Lemma 2. Moreover, H_w is an inverse for p, and therefore w is an isomorphism. The inverse $w^{-1}\colon X \to Y$ gives a map such that $H_{w^{-1}} = p$. This proves (2) since any isomorphism $H_X \to H_{Y \coprod Z}$ which corestricts to p has $Z = 0$.

(2) ⇒ (3) Complete the map $w\colon Y \to X$ to a triangle $Y \to X \to Z \to \Sigma Y$. This gives a split short exact sequence

$$0 \longrightarrow H_Y \longrightarrow H_X \longrightarrow H_Z \longrightarrow 0$$

since $pH_w = \mathrm{id}_{H_Y}$. We obtain an isomorphism $H_X \to H_{Y \coprod Z}$ which corestricts to p. Condition (3) is now an immediate consequence of (2).

(3) ⇒ (1) Let $p = H_v$. The composition wv is an endomorphism of X satisfying $ewv = e$. Therefore wv is an isomorphism by Lemma 3. In particular $p = H_v$ is a monomorphism.

On the other hand, p is an epimorphism since $pH_w = \mathrm{id}_{H_Y}$. It follows from Lemma 2 that $\operatorname{Ext}^2(-, H_Y) = 0$. ☐

Proposition. *The following conditions are equivalent for an object X in $\mathcal{T}$:*

(1) $\operatorname{Ext}^2(-, H_X) = 0$.
(2) *Every map $H_{X'} \to H_X$ is of the form H_u for some map $u\colon X' \to X$.*
(3) *Every split epimorphism $H_{X'} \to H_X$ is of the form H_u for some map $u\colon X' \to X$.*

Proof. (1) $\Rightarrow$ (2) Fix a map $H_{X'} \to H_X$. Using Lemma 1, we choose injective envelopes $H_e\colon H_X \to H_I$ and $H_{e'}\colon H_{X'} \to H_{I'}$. The maps e and e' can be completed to triangles

$$X \xrightarrow{e} I \xrightarrow{f} J \xrightarrow{g} \Sigma X \quad \text{and} \quad X' \xrightarrow{e'} I' \xrightarrow{f'} J' \xrightarrow{g'} \Sigma X'$$

where H_J is injective by our assumption on X. Using the injectivity of H_I, the map $h\colon H_{X'} \to H_X$ induces the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & H_{X'} \arrow[r, "H_{e'}"] \arrow[d, "h"] & H_{I'} \arrow[r, "H_{f'}"] \arrow[d] & H_{J'} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H_X \arrow[r, "H_e"] & H_I \arrow[r, "H_f"] & H_J \arrow[r] & 0 \end{tikzcd}$$

An application of Lemma 1 gives the following commutative diagram

$$\begin{tikzcd} \Sigma^{-1}J' \arrow[r] \arrow[d, "\Sigma^{-1}w"] & X' \arrow[r, "e'"] & I' \arrow[r, "f'"] \arrow[d, "v"] & J' \arrow[d, "w"] \\ \Sigma^{-1}J \arrow[r] & X \arrow[r, "e"] & I \arrow[r, "f"] & J \end{tikzcd}$$

since H_I and H_J are injective, and the fill-in map $u\colon X' \to X$ induces the map $h\colon H_{X'} \to H_X$ by construction.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) Use Lemma 4. ☐

Proof of the Theorem. Throughout the proof we shall use freely the fact that flat and exact functors in $(\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$ coincide (see Lemma 2.7 in [7]).

(1) $\Rightarrow$ (2) Clear.

(2) $\Rightarrow$ (3) If $\operatorname{Ext}^2(-, F) = 0$, then F is exact by Lemma 2 and therefore flat. Conversely, suppose that F is flat and therefore exact. Then $F \cong H_Y$ for some $Y \in \mathcal{T}$ by (2), and the second condition in (2) implies $\operatorname{Ext}^2(-, F) = 0$ by Lemma 4.

(3) $\Rightarrow$ (4) Let $F \in (\mathcal{F}^{\mathrm{op}}, \mathrm{Ab})$, and suppose that $E \to F$ is any map such that E is flat. We need to show that $E \to F$ factors through p_F. We take the composition

$$H_Y \xrightarrow{p_E} E \longrightarrow F \longrightarrow H_I$$

which is of the form H_c for some map $c\colon Y \to I$ by Lemma 1. Clearly, $H_f H_c = H_{fc} = 0$, and therefore $fc = 0$, again by Lemma 1. Therefore c factors through $e\colon X \to I$ via some map $u\colon Y \to X$. Condition (3) and Lemma 2 imply that p_E is an isomorphism, and the composition $H_u p_E^{-1}\colon E \to H_X$ gives the desired map.

Now let $v\colon H_X \to H_X$ be any endomorphism satisfying $p_F v = p_F$. We need to show that v is an isomorphism. To this end observe that $v = H_u$ for some $u\colon X \to X$ by the Proposition. Moreover, $p_F v = p_F$ implies $H_{eu} = H_e$, and therefore $eu = e$ by Lemma 1. An application of Lemma 3 shows that v is an isomorphism, and this completes the proof of the fact that p_F is a flat cover of F.

(4) $\Rightarrow$ (1) Clearly, a flat cover $E \to F$ is an isomorphism if F is flat. Therefore (4) implies that $P_F \colon H_X \to F$ is an isomorphism for every exact functor F. A combination of Lemma 2 and the Proposition shows that every map $H_X \to H_Y$ is induced by a map $X \to Y$ in $\mathcal{T}$. $\square$

ACKNOWLEDGEMENT. This paper was written in Spring 1999 while I visited Santa Barbara. I would like to thank Birge Huisgen-Zimmermann and Ken Goodearl for their warm hospitality.

REFERENCES

[1] J.F. ADAMS, A variant of E.H. Brown's representability theorem, Topology **10** (1971), 185–198.
[2] E.H. BROWN, Cohomology theories, Ann. of Math. **75** (1962), 467–484.
[3] D.J. BENSON AND G.P. GNACADJA, Phantom maps and purity in modular representation theory I, Fund. Math., to appear.
[4] J.D. CHRISTENSEN AND N.P. STRICKLAND, Phantom maps and homology theories, Topology **37** (1998), 339–364.
[5] E.E. ENOCHS, Injective and flat covers, envelopes and resolvents, Israel J. Math. **39** (1981), 189–209.
[6] P. GABRIEL, Des catégories abéliennes, Bull. Soc. Math. France **90** (1962), 323–448.
[7] H. KRAUSE, Smashing subcategories and the telescope conjecture - an algebraic approach, Invent. Math., to appear.
[8] A. NEEMAN, On a theorem of Brown and Adams, Topology **36** (1997), 619–645.
[9] J. XU, Flat covers of modules, Springer Lec. Notes **1634** (1996).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CA 93106, U.S.A.
E-mail address: `krause@math.ucsb.edu`

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT BIELEFELD, 33501 BIELEFELD, GERMANY
E-mail address: `henning@mathematik.uni-bielefeld.de`