L^p uniqueness of non–symmetric diffusion operators with singular drift coefficients I. The finite–dimensional case

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Abstract.
Two uniqueness results for C^0 semigroups on weighted L^p spaces over $\mathbf{R}^n$ generated by operators of type $\Delta + \beta \cdot \nabla$ with singular drift β are proven. A key ingredient in the proofs is the verification of some kind of "weak Kato inequality" which seems to break down exactly for those drift singularities where L^p uniqueness breaks down as well.

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1. Introduction

This article contains two (existence and) uniqueness results for C^0 semigroups generated by operators of type $\Delta + \beta \cdot \nabla$ on weighted L^p spaces over $\mathbf{R}^n$. They partially generalize related results obtained in the special symmetric case in several recent publications, cf. in particular [LiSem 92] and [BogKryRö 96]. The results are part of my Doktor thesis [Eb 98].

The necessity to develop new analytic techniques for proving sufficiently sharp uniqueness results in singular cases is demonstrated by counterexamples like the following, cf. [Eb 98] for details :

Example. Fix a real $d \geq 2$ and let $1 < p < d$. Then the operator $\mathcal{L}$ with domain $C_0^\infty(\mathbf{R})$ given by $(\mathcal{L}f)(x) = (f''(x) + (d-1)\, x^{-1} f'(x))/2$ is a densely defined dissipative linear operator on $L^p(\mathbf{R};\ x^{d-1}dx)$. However, for $p > d/2$, $\mathcal{L}$ is not $L^p(\mathbf{R};\ x^{d-1}dx)$ *unique*, i.e., there exists more than one C^0 semigroup $(T_t)_{t \geq 0}$ of bounded linear operators on the L^p space such that its generator extends $\mathcal{L}$.

The non–uniqueness in the example is not caused by any bad behaviour at infinity, but solely by the singularity of the drift coefficient $(d-1)x^{-1}$ at zero. That uniqueness does not hold can be proven easily analytically, cf. [Eb 98, Ch. 2, Sect. d)], but it is worth noting as well that several transition semigroups generated by extensions of ($\mathcal{L}$, $C_0^\infty(\mathbf{R})$) can be constructed explicitly probabilistically by using transition functions of certain systems of charged particles. This is true although only one of these semigroups is the transition semigroup of an ordinary Markov process. The reader is referred to [Eb 98, Ch. 4, Sect. c)] and a forthcoming publication for the latter point of view which provides an intuitive probabilistic explanation for certain non–uniqueness results on L^p spaces. Similar counterexamples on $\mathbf{R}^n$, $n \geq 2$, can be easily constructed from the one above, cf. e.g. the remarks below Theorem 2.6 in [Eb 98].

After being convinced by examples that singularities can destroy L^p uniqueness, one might be tempted to prove uniqueness results for singular operators by replacing the state space $\mathbf{R}^n$ by $\mathbf{R}^n \setminus S$, where S is the set of singularities of the drift coefficients (– defined in some appropriate sense). Examples and one–dimensional results indicate, however, that this approach, which consists in viewing singularities as some kind of boundary points, in general does not produce sufficiently strong uniqueness results, cf. e.g. Theorem 2.4 and the comments above in [Eb 98]. The techniques used here are quite different, and do not depend on the form of the singularity set.

To state our two main results, let m be a positive Radon measure on $\mathbf{R}^n$, $n \geq 2$. The one–dimensional case is studied separately in [Eb 98], because different techniques enable us to obtain a much more complete picture of L^p uniqueness for diffusion operators with one–dimensional state space. We fix $p \in [1, \infty)$, and let β be function in $L^p_{\mathrm{loc}}(\mathbf{R}^n \to \mathbf{R}^n\,;\, m)$. We assume that the support of m is $\mathbf{R}^n$, or, more generally, that the boundary of the support of m has measure zero. This condition ensures that the derivation operators ∇ and Δ on $C_0^\infty(\mathbf{R}^n)$ respect m–classes. Let $\mathcal{L}$ denote the densely defined linear operator $\Delta + \beta \cdot \nabla$ with domain $C_0^\infty(\mathbf{R}^n)$ on $L^p(\mathbf{R}^n\,;\, m)$. Throughout this article we assume :

(A 1) There exists a finite constant $\alpha \geq 0$ such that

$$\int \mathcal{L} f \; dm \;\leq\; \alpha \int f \; dm \quad \text{for all positive functions } f \in C_0^\infty(\mathbf{R}^n).$$

Note that the condition is *automatically satisfied* if there exists a C^0 semigroup $(T_t)_{t\geq 0}$ on $L^p(\mathbf{R}^n\,;\, m)$ such that its generator extends $(\,\mathcal{L}\,,\, C_0^\infty(\mathbf{R}^n)\,)$, and m is a sub–invariant measure for $(e^{-\alpha t} T_t)_{t\geq 0}$ for some $\alpha \geq 0$. If m has a density $\rho \in H^{1,1}_{\mathrm{loc}}(\mathbf{R}^n;\, dx)\,,\ \rho > 0\ dx$–a.e., w.r.t. Lebesgue measure, then (A 1) holds if and only if the m–divergence $\frac{1}{\rho}\,\mathrm{div}\,(\rho\,\cdot\,)$ of the anti–symmetric part $\beta - \frac{\nabla \rho}{\rho}$ of the operator drift is bounded from below in the distributional sense, i.e.,

(A 1′) There exists $\alpha \geq 0$ such that

$$-\int (\beta - \frac{\nabla \rho}{\rho}) \cdot \nabla f \; dm \;\geq\; -\alpha \int f \; dm \quad \text{for all positive functions } f \in C_0^\infty(\mathbf{R}^n).$$

The latter condition can be easily checked in concrete cases.

Even if β is smooth and m has a smooth strictly positive density ρ, L^p uniqueness in the sense claimed below can not be expected in the non–symmetric case (– in contrast to the symmetric case –) without any growth restrictions at infinity. In fact, for smooth coefficients it is known that a growth of $|\beta|$ of order $|x|\,(\log|x|)^{1+\varepsilon}$ is sufficient to guarantee L^p uniqueness for $1 < p \leq 2$ if $\varepsilon = 0$, but not for any $\varepsilon > 0$, cf. the results and counterexamples in Chapter 2, Sections b) and c) of [Eb 98]. For the singular drifts considered here, it would be too restrictive to assume that $\beta(x)$ is dominated by a locally bounded function for large $|x|$. Instead, we make the following assumption :

(A 2) There exists a decomposition $\beta = \beta^{\text{sing}} + \beta^{\text{reg}}$, $\beta^{\text{sing}}, \beta^{\text{reg}} : \mathbf{R}^n \to \mathbf{R}^n$, such that the "singular" part β^{sing} satisfies

$$(1) \qquad \lim_{r\to\infty} \frac{1}{r} \, \|(\beta_r^{\text{sing}})^-\|_{L^p(B_r;m)} \;=\; 0,$$

and the "regular" part β^{reg} is locally bounded, and satisfies

$$(2) \qquad \limsup_{r\to\infty} \frac{m\,(B_r)}{r^k} \;<\; \infty \quad \text{for some } k > 0, \text{ and}$$

$$\limsup_{|x|\to\infty} \frac{(\beta_r^{\text{reg}})^-(x)}{|x|} \;<\; \infty,$$

or

(3) The measure m is finite, and

$$\limsup_{|x|\to\infty} \frac{(\beta_r^{\text{reg}})^-(x)}{|x|\,\log|x|} \;<\; \infty.$$

Here $\beta_r := e_r \cdot \beta$ with $e_r(x) = x/|x|$, B_r denotes the open ball of radius r around 0, and $f^- := -\min(f, 0)$ is the negative part of a function f. In the sequel, we set $p/(2-p) := \infty$ if $p = 2$. We now state the results, and then comment once more on the conditions imposed :

Theorem 1 *Let $p \in [1, 2]$. Suppose that $m = \rho\, dx$ with $\rho \in L^{p/(2-p)}_{\text{loc}}(\mathbf{R}^n, dx)$, β is in $L^{2p}_{\text{loc}}(\mathbf{R}^n \to \mathbf{R}^n\,;\, m)$, $\beta\rho^{1/2}$ is in $L^{2p/(2-p)}_{\text{loc}}(\mathbf{R}^n \to \mathbf{R}^n\,;\, m)$, and (A 1) and (A 2) hold.
Then the closure of $(\,\mathcal{L}\,,\, C_0^\infty(\mathbf{R}^n)\,)$ generates a C^0 semigroup on $L^p(\mathbf{R}^n\,;\, m)$.*

Theorem 2 *Let $p \in [1, \infty)$. Suppose that β is in $L^{(1+\frac{n}{2})p+\varepsilon}_{\text{loc}}(\mathbf{R}^n \to \mathbf{R}^n\,;\, m)$ for some $\varepsilon > 0$, and (A 1) and (A 2) hold. Then the closure of $(\,\mathcal{L}\,,\, C_0^\infty(\mathbf{R}^n)\,)$ generates a C^0 semigroup on $L^p(\mathbf{R}^n\,;\, m)$.*

Corollary 1 *If the conditions imposed in Theorem 1 or in Theorem 2 hold, then the operator* $(\mathcal{L}, C_0^\infty(\mathbf{R}^n))$ *is* $L^p(\mathbf{R}^n ; m)$ unique, *i.e., there exists only one* C^0 *semigroup on* $L^p(\mathbf{R}^n ; m)$ *such that its generator extends* $(\mathcal{L}, C_0^\infty(\mathbf{R}^n))$.

The example given in the beginning can be used to demonstrate that for $\varepsilon > 0$ the assumption $\beta \in L^{2p}_{\text{loc}}(\mathbf{R}^n \to \mathbf{R}^n ; m)$ in Theorem 1 can not be replaced by $\beta \in L^{2p-\varepsilon}_{\text{loc}}(\mathbf{R}^n \to \mathbf{R}^n ; m)$. The additional assumptions in Theorem 1 seem to be less optimal, but it is not clear to me how to drop them. The study of concrete examples shows that they are not very restrictive in low dimensions, but problematic in certain higher dimensional cases. Similarly, the assumption $\beta \in L^{(1+\frac{n}{2})p+\varepsilon}_{\text{loc}}(\mathbf{R}^n \to \mathbf{R}^n ; m)$ is almost sharp for $n = 2$, but restrictive for large n.

We will give all the proofs in Section 2 below. The proof of Theorem 1 uses only rather standard analytic techniques, and therefore seems to be conceptually more simple than previous proofs of related results. The new key ingredient is the verification of some kind of "*weak Kato inequality*" in Step 2. It is remarkable that this inequality seems to break down exactly for those drift singularities where L^p uniqueness breaks down as well. The proof of Theorem 2 is more involved because it is based on a deeep regularity result for invariant measures obtained in [BogKryRö 96], cf. the second lemma in Section 2 below.

We finally comment briefly on the conditions imposed, and the relations to other results. For a detailled discussion of the L^p and related uniqueness problems for C^0 semigroups the reader is once more referred to [Eb 98].

Remarks. (i) (*Symmetric case*) Symmetric diffusion operators of the type considered here are sometimes called *generalized Schrödinger operators* because they appear as ground state transformations of ordinary Schrödinger operators. To my best knowledge, V. Liskevich and Y. Semenov were the first to prove a general L^p uniqueness result for generalized Schrödinger operators with very singular drifts. In [LiSem 92], they used an approximative criterion and an L^4 gradient estimate for solutions of parabolic PDE to show essential self–adjointness of $(\Delta + \beta \cdot \nabla, C_0^\infty(\mathbf{R}^n))$ on $L^2(\mathbf{R}^n; \rho\, dx)$ for $\rho \in H^{1,2}_{\text{loc}}(\mathbf{R}^n ; dx)$, $\rho > 0$ a.e., and $\beta = \frac{\nabla\rho}{\rho}$, provided $|\beta|$ is in $L^4(\mathbf{R}^n; \rho\, dx)$. It is well–known that in the symmetric case, essential self–adjointness on $L^2(\mathbf{R}^n; \rho\, dx)$ is equivalent to $L^2(\mathbf{R}^n; \rho\, dx)$ uniqueness, cf. [Ar 86]. The method used in [LiSem 92] has been extended to prove $L^p(\mathbf{R}^n; \rho\, dx)$ uniqueness of generalized Schrödinger operators provided $p > 3/2$ and $|\beta| \in L^{2p}(\mathbf{R}^n; \rho\, dx)$, cf. [Li 94].

In the symmetric case, the assumptions in [LiSem 92] and [Li 94] are locally weaker than the conditions from Theorem 1 and 2 above. However, the assumed global integrability of $|\beta|^{2p}$ is restrictive. In fact, in the non–singular symmetric case, essential self–adjointness is known to hold under purely local assumptions on ρ, cf. [Wie 85]. V. Bogachev, N. Krylov and M. Röckner [BogKryRö 96] as well as, recently, V. Liskevich [Li 99] were able to drop the global assumptions also in the singular symmetric case, but instead they had to add more restrictive local assumptions. It is still an *open problem* whether L^p uniqueness in the symmetric case holds under the assumption $|\beta| \in L^{2p}_{\mathrm{loc}}(\mathbf{R}^n; m)$ only.

(ii) (*Markov uniqueness*) In the symmetric case, much stronger uniqueness results are known to hold if one only looks at extensions of the operator $(\mathcal{L}, C_0^\infty(\mathbf{R}^n))$ which generate symmetric C^0 semigroups $(T_t)_{t\geq 0}$ on $L^2(E; m)$ that are in addition *sub–Markovian*, i.e., $0 \leq T_t f \leq 1$ m–a.e. holds for all $t \geq 0$ and all positive functions $f \in C_0^\infty(\mathbf{R}^n)$. Uniqueness among all such extensions is called *Markov uniqueness.* It is known that in contrast to L^p uniqueness, Markov uniqueness of finite–dimensional diffusion operators is not destroyed by singularities of the drift coefficients, cf. [RöZha 92, 94], [Eb 95], and Chapter 3 in [Eb 98]. This is even true for locally strictly elliptic diffusion operators with non–constant second order coefficients. However, degeneracy of the diffusion matrix can destroy Markov uniqueness as well, cf. [Eb 98] for details.

(iii) (*L^1 uniqueness*) In the case $p = 1$, W. Stannat [St 99] has recently proven much more precise criteria for uniqueness than stated here. This considerably extends previous results by V. Liskevich and Y. Semenov [LiSem 96]. The phenomena causing non–uniqueness turn out to be quite different for $p = 1$ and for $p > 1$, cf. the discussion in Chapter 4 of [Eb 98].

(iv) (Non–constant second order coefficients) Results about L^p uniqueness of diffusion operators with non–constant second order coefficients exist in the strictly elliptic symmetric case [LiTuv 93] and for $p = 1$ [St 99]. For $p > 1$, the existing results still provide a rather incomplete picture. As already mentioned above, much more can be said about Markov uniqueness of symmetric diffusion operators with non–constant diffusion matrices.

2. Proofs

Let $C^1([0,\infty))$ denote the space of all continuously differentiable functions on $[0,\infty)$, where the derivative is taken to the right at 0. For the proof of the theorems, we need the following comparison lemma:

Lemma 1 *Let $A \in C([0,\infty))$, $B \in C^1([0,\infty))$, and $r_1 \in (0,\infty)$, such that $A + B' > 0$ on (r_1,∞). Suppose G and K are functions in $C^1([0,\infty))$ such* *that $G(0) = K(0) = 0$, and the following inequalities hold:*

$$(4)\qquad -G'(r) + \int_0^r A(s)\,G(s)\,ds \;\le\; \int_0^r B(s)\,G'(s)\,ds \qquad \textit{and}$$
$$-K'(r) + \int_0^r A(s)\,K(s)\,ds \;\ge\; \int_0^r B(s)\,K'(s)\,ds \quad \textit{for all } r \ge r_1.$$

$$(5)\qquad G(r_1) \;>\; K(r_1) \quad \textit{and}$$
$$\int_0^{r_1} (\,A(s) + B'(s)\,)\,G(s)\,ds \;\ge\; \int_0^{r_1} (\,A(s) + B'(s)\,)\,K(s)\,ds.$$

Then $G(r) > K(r)$ for all $r \in [r_1, \infty)$.

Proof of the lemma. Partial integration yields

$$-G'(r) + \int_0^r (A + B')\,G\,ds \;\le\; B(r)\,G(r), \quad \text{and}$$
$$-K'(r) + \int_0^r (A + B')\,K\,ds \;\ge\; B(r)\,K(r) \quad \text{for all } r \ge r_1.$$

Suppose that $G(r) \le K(r)$ for some $r \ge r_1$, and let $u := \inf\{r \ge r_1\,;\, G(r) \le K(r)\}$. Since $G(r_1) > K(r_1)$, u is in (r_1, ∞). Obviously, $G(u) = K(u)$ and $G'(u) \le K'(u)$. Hence

$$\int_0^u (A+B')\,(G-K)\,ds \;\le\; G'(u) + B(u)\,G(u) - K'(u) - B(u)\,K(u) \;\le\; 0.$$

This is a contradiction, because, on the other hand,

$$\int_0^u (A+B')\,(G-K)\,ds \;=\; \int_0^{r_1} (A+B')\,(G-K)\,ds + \int_{r_1}^u (A+B')\,(G-K)\,ds,$$

which is strictly positive, since $G > K$ and $A + B' > 0$ on (r_1, u), and (5) holds. ■

Proof of Theorem 1. Let $q \in [2, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$. We choose $\lambda \ge 0$ such that $(\mathcal{L} - \lambda\,,\, C_0^\infty(\mathbf{R}^n)\,)$ is dissipative on $L^p(\mathbf{R}^n\,;\, m)$, and we fix a large constant $\gamma > \lambda$ to be specified below. We show in several steps that the existence of a non–trivial solution $u \in L^q(\mathbf{R}^n\,;\, m)$ of the equation $\mathcal{L}^* u = \gamma u$ leads to a contradiction, if γ is chosen large enough. This implies that the range of $(\,\mathcal{L} - \lambda\,,\, C_0^\infty(\mathbf{R}^n)\,)$ is dense in $L^p(\mathbf{R}^n\,;\, m)$, and the assertion now follows by the Lumer–Phillips theorem, cf. e.g. [Pa 85, Ch. I, Thm. 4.3]. Thus suppose u is a non–trivial solution of $\mathcal{L}^* u = \gamma u$.

Step 1: Regularity. We will show that ρu is in $H^{1,2}_{\mathrm{loc}}(\mathbf{R}^n\,;\, dx)$, and

$$(6)\qquad \int \nabla f \cdot \nabla(\rho u)\,dx \;+\; \gamma \int f\,\rho\,u\,dx \;=\; \int \beta \cdot \nabla f\,\rho\,u\,dx$$

for all compactly supported functions $f \in H^{1,2}(\mathbf{R}^n\,;\,dx)$.
Note that the equation $\mathcal{L}^* u = \gamma u$ can be rewritten as

$$(7)\qquad \int_{\mathbf{R}^n} (\gamma - \Delta)\, f\, \rho\, u\, dx \;=\; \int_{\mathbf{R}^n} \beta \cdot \nabla f\, \rho\, u\, dx \quad \text{for all } f \in C_0^\infty(\mathbf{R}^n).$$

Let $\varphi \in C_0^\infty(\mathbf{R}^n)$ be a positive function such that $\int_{\mathbf{R}^n} \varphi\, dx = 1$ and $\varphi(x) = \varphi(-x)$ for all x, and let φ_ε, $\varepsilon > 0$, $\varphi_\varepsilon(x) = \varepsilon^{-n} \cdot \varphi(x/\varepsilon)$, denote the corresponding dirac sequence. Let $\mathcal{E}_\gamma$ be the bilinear form

$$\mathcal{E}_\gamma\,(f, g) \;=\; \int \nabla f \cdot \nabla g\; dx \;+\; \gamma \int f \cdot g\; dx$$

on $H^{1,2}(\mathbf{R}^n\,;\,dx)$. Fix a function η in $C_0^\infty(\mathbf{R}^n)$ such that $0 \le \eta \le 1$. Then for all $f \in C_0^\infty(\mathbf{R}^n)$, and $0 < \varepsilon \le 1$,

$$\begin{aligned}
\mathcal{E}_\gamma\,(f\,,\,(\eta\rho u) * \varphi_\varepsilon\,) \;&=\; \int_{\mathbf{R}^n} (\gamma - \Delta)\, f\;\; (\eta\rho u) * \varphi_\varepsilon\; dx \\
=\;& \int_{\mathbf{R}^n} (\gamma - \Delta)\, (f * \varphi_\varepsilon)\; \eta\, \rho\, u\; dx \\
=\;& \int (\gamma - \Delta)\, ((f * \varphi_\varepsilon)\, \eta)\; \rho\, u\; dx \;+\; \int (f * \varphi_\varepsilon\, \Delta\eta + 2\, \nabla(f * \varphi_\varepsilon) \cdot \nabla\eta)\; \rho\, u\; dx \\
=\;& \int \beta \cdot \nabla(f * \varphi_\varepsilon)\; \eta\, \rho\, u\; dx \;+\; \int (f * \varphi_\varepsilon\, (\Delta\eta + \beta \cdot \nabla\eta) + 2\, \nabla(f * \varphi_\varepsilon) \cdot \nabla\eta)\; \rho\, u\; dx \\
\le\;& C \cdot \mathcal{E}_\gamma\,(f\,,\,f)^{1/2} \cdot \|\,(1 + |\beta|)\, \rho u\|_{L^2(\mathrm{supp}\,\eta\,;\,dx)}
\end{aligned}$$

for some finite constant C depending only on η. Choosing $f = (\eta\rho u) * \varphi_\varepsilon$, we obtain

$$\begin{aligned}
(8)\qquad & \mathcal{E}_\gamma\,(\,(\eta\rho u) * \varphi_\varepsilon\,,\,(\eta\rho u) * \varphi_\varepsilon\,)^{1/2} \\
\le\;& C \cdot \|\,(1 + |\beta|) \cdot \rho u\|_{L^2(\mathrm{supp}\,\eta\,;\,dx)} \;=\; C \cdot \|\,(1 + |\beta|)\rho^{1/2} u\|_{L^2(\mathrm{supp}\,\eta\,;\,m)} \\
\le\;& C \cdot \|\,(1 + |\beta|)\rho^{1/2}\|_{L^s(\mathrm{supp}\,\eta\,;\,m)} \cdot \|u\|_{L^q(\mathbf{R}^n\,;\,m)},
\end{aligned}$$

where $s = \frac{2p}{2-p}$, i.e., $\frac{1}{s} = \frac{1}{p} - \frac{1}{2} = \frac{1}{2} - \frac{1}{q}$. By assumption, $(1 + |\beta|)\rho^{1/2}$ is in $L^s_{\mathrm{loc}}(\mathbf{R}^n\,;\,m)$, whence the right hand side is finite. Hence the functions $(\eta\rho u) * \varphi_\varepsilon$, $0 < \varepsilon \le 1$, are uniformly bounded elements in $H^{1,2}(\mathbf{R}^n\,;\,dx)$. Therefore, $\eta\rho u$ is in $H^{1,2}(\mathbf{R}^n\,;\,dx)$ as well. Since η is an arbitrary function in $C_0^\infty(\mathbf{R}^n)$ such that $0 \le \eta \le 1$, we obtain $\rho u \in H^{1,2}_{\mathrm{loc}}(\mathbf{R}^n\,;\,dx)$. Integration by parts in (7) now yields (6) for all $f \in C_0^\infty(\mathbf{R}^n)$. Since, by the last estimate in (8), $\beta\rho u$ is in $L^2_{\mathrm{loc}}(\mathbf{R}^n \to \mathbf{R}^n\,;\,dx)$, (6) even holds for all compactly supported functions $f \in H^{1,2}(\mathbf{R}^n\,;\,dx)$.

Step 2: Inequality for $\boldsymbol{\rho\,|u|}$. Since $\rho\, u$ is in $H^{1,2}_{\mathrm{loc}}(\mathbf{R}^n\,;\,dx)$, $\rho\,|u|$ is in

$H^{1,2}_{\rm loc}(\mathbf{R}^n\,;\,dx)$ as well. We will derive the following inequality for $\rho\,|u|$ from (6):

$$\int \nabla\xi\cdot\nabla\,(\rho\,|u|)\;dx \;+\;\gamma\int \xi\;\rho\,|u|\;dx \;\le\; \int \beta\cdot\nabla\xi\;\rho\,|u|\;dx \tag{9}$$

for all positive, compactly supported functions $\xi\in H^{1,2}(\mathbf{R}^n\,;\,dx)$.

For $\varepsilon>0$, let $\psi_\varepsilon:\mathbf{R}\to\mathbf{R}$ be given by $\psi_\varepsilon(x)=\operatorname{sgn}(x)$ if $|x|\ge\varepsilon$, and $\psi_\varepsilon(x)=x/\varepsilon$ if $|x|\le\varepsilon$. Obviously, ψ_ε is Lipschitz continuous, whence $\psi_\varepsilon(\rho u)$ is in $H^{1,2}_{\rm loc}(\mathbf{R}^n\,;\,dx)$, and

$$\nabla\,\psi_\varepsilon(\rho u)\;=\;\psi_\varepsilon'(\rho u)\cdot\nabla(\rho u)\;=\;\varepsilon^{-1}\cdot\chi_{\{\rho|u|\le\varepsilon\}}\cdot\nabla(\rho u)\qquad dx\text{–a.e.}$$

Fix a positive function $\xi\in C_0^\infty(\mathbf{R}^n)$. Setting $f:=\xi\cdot\psi_\varepsilon(\rho u)$ in (6), we obtain

$$\begin{aligned}&\int\nabla\xi\cdot\nabla(\rho u)\;\psi_\varepsilon(\rho u)\;dx\;+\;\gamma\int\xi\,\rho\,u\;\psi_\varepsilon(\rho u)\;dx\\ &=\;\int\beta\cdot\nabla\xi\;\rho\,u\;\psi_\varepsilon(\rho u)\;dx\;+\;\varepsilon^{-1}\int_{\{\rho|u|\le\varepsilon\}}\xi\;\beta\cdot\nabla(\rho u)\;\rho\,u\;dx\\ &\quad-\;\varepsilon^{-1}\int_{\{\rho|u|\le\varepsilon\}}\xi\;|\nabla(\rho u)|^2\;dx.\end{aligned}\tag{10}$$

To estimate the right–hand side of (10) note that

$$\int_{\{\rho|u|\le\varepsilon\}}\xi\;\beta\cdot\nabla(\rho u)\;\rho\,u\;dx\;\le\;\left(\int_{\{\rho|u|\le\varepsilon\}}\xi\;|\nabla(\rho u)|^2\;dx\right)^{1/2}\cdot C_\varepsilon^{1/2},$$

where

$$C_\varepsilon\;:=\;\int_{\{\rho|u|\le\varepsilon\}}\xi\;|\beta|^2\,\rho^2\,u^2\;dx\;\le\;\varepsilon\cdot\|u\|_{L^q(\mathbf{R}^n\,;\,m)}\cdot\|\chi_{\{0<\rho|u|\le\varepsilon\}}\xi|\beta|^2\|_{L^p(\mathbf{R}^n\,;\,m)}.$$

Since β is in $L^{2p}_{\rm loc}(\mathbf{R}^n\to\mathbf{R}^n\,;\,m)$, $\varepsilon^{-1}C_\varepsilon$ converges to 0 as ε tends to 0. We obtain

$$\begin{aligned}&\varepsilon^{-1}\int_{\{\rho|u|\le\varepsilon\}}\xi\;\beta\cdot\nabla(\rho u)\;\rho\,u\;dx\;-\;\varepsilon^{-1}\int_{\{\rho|u|\le\varepsilon\}}\xi\;|\nabla(\rho u)|^2\;dx\\ &\le\;\varepsilon^{-1}\left(\left(\int_{\{\rho|u|\le\varepsilon\}}\xi\;|\nabla(\rho u)|^2\;dx\right)^{1/2}\cdot C_\varepsilon^{1/2}\;-\;\int_{\{\rho|u|\le\varepsilon\}}\xi\;|\nabla(\rho u)|^2\;dx\right)\\ &\le\;C_\varepsilon/4\varepsilon\;\overset{\varepsilon\downarrow0}{\longrightarrow}\;0.\end{aligned}$$

Now, by letting ε tend to 0 in (10), we obtain (9), because

$$\begin{aligned}\lim_{\varepsilon\downarrow0}\int\nabla\xi\cdot\nabla(\rho u)\;\psi_\varepsilon(\rho u)\;dx\;&=\;\int\nabla\xi\cdot\nabla(\rho u)\operatorname{sgn}(\rho u)\;dx\\ &=\;\int\nabla\xi\cdot\nabla(\rho\,|u|)\;dx\end{aligned}$$

by dominated convergence, and the other integrals in (10) converge similarly. Hence (9) holds for all positive functions $\xi \in C_0^\infty(\mathbf{R}^n)$. Since $|\beta|\rho u$ is in $L^2_{\mathrm{loc}}(\mathbf{R}^n\,;\,dx)$ by (8), (9) is true for all positive compactly supported functions $\xi \in H^{1,2}(\mathbf{R}^n\,;\,dx)$ as well.

Step 3: Localization. We finally show that by (9) and Assumption (A 2), u cannot be globally in $L^q(\mathbf{R}^n\,;\,m)$ if γ has been chosen large enough, i.e., there is no non–trivial solution of $\mathcal{L}^* u = \gamma u$ in this case.

Fix $r > 0$, and let $\xi(x) := (r - |x|)^+$. By (9), we obtain

$$
\begin{aligned}
-\int_{B_r} e_r \cdot \nabla\,(\rho\,|u|)\,dx \;\;+\;\; \gamma \int_{B_r} (r - |x|)\,\rho\,|u|\,dx \\
\leq\;\; -\int_{B_r} \beta_r\,\rho\,|u|\,dx.
\end{aligned}
\tag{11}
$$

Note that $\operatorname{div} e_r\,(x) = (n-1)/|x| \geq 0\;\; dx$–a.e., whence $\operatorname{div} e_r$ is in $L^{2-\varepsilon}_{\mathrm{loc}}(\mathbf{R}^n\,;\,dx)$ for every $\varepsilon > 0$. The function $\rho|u|$ is in $H^{1,2}_{\mathrm{loc}}(\mathbf{R}^n;\,dx)$, and hence, by the Sobolev embedding theorem, in $L^{2+\delta}_{\mathrm{loc}}(\mathbf{R}^n;\,dx)$ for some $\delta > 0$. Thus, $\operatorname{div}\,(\rho|u|e_r)$ is in $L^1_{\mathrm{loc}}(\mathbf{R}^n\,;\,dx)$. We have

$$
-e_r \cdot \nabla(\rho\,|u|) \;=\; -\operatorname{div}\,(\rho|u|e_r) \;+\; \rho|u|\,\operatorname{div}\,(e_r) \;\geq\; -\operatorname{div}\,(\rho|u|e_r)
\tag{12}
$$

dx–a.e. For $s \geq 0$ let

$$
g(s) \;:=\; \int_{B_s} \operatorname{div}\,(\rho|u|e_r)\,dx.
$$

By multiplying with a test–function and integrating, one easily verifies that

$$
g(r) \;=\; \int_{\partial B_r} \rho\,|u|\,dy \qquad \text{for a.e. } r \geq 0,
$$

i.e., g is a continuous modification of the function $r \mapsto \int_{\partial B_r} \rho\,|u|\,dy$. Moreover,

$$
\int_{B_r} (r - |x|)^+\,\rho\,|u|\,dx \;=\; \int_0^r \int_{B_s} \rho\,|u|\,dx\,ds \;=\; \int_0^r \int_0^s g\,(t)\,dt\,ds.
$$

Hence, by (11) and (12),

$$
\begin{aligned}
&-g(r) \;+\; \gamma \int_0^r \int_0^s g\,(t)\,dt\,ds \\
&\leq\; -\int_{B_r} \beta_r\,\rho\,|u|\,dx \;\leq\; \int_{B_r} \beta_r^-\;\rho\,|u|\,dx \\
&\leq\; \|(\beta_r^{\mathrm{sing}})^-\|_{L^p(B_r;m)} \cdot \|u\|_{L^q(\mathbf{R}^n\,;\,m)} \;+\; \int_{B_r} (\beta_r^{\mathrm{reg}})^-\;\rho\,|u|\,dx,
\end{aligned}
\tag{13}
$$

where $\beta = \beta^{\text{sing}} + \beta^{\text{reg}}$ is a decomposition as in (A 2). We first consider the case where Assumption (2) holds. Then there exist finite positive constants α and C such that

$$
\begin{aligned}
&(14) \qquad \lim_{r\to\infty} r^{-\alpha p}\, m(B_r) \;=\; 0\,, \qquad \text{and} \\
&(15) \qquad (\beta_r^{\text{reg}}(x))^- \;\le\; C\cdot|x| \quad \text{for all } x \text{ outside some ball around } 0.
\end{aligned}
$$

By changing the decomposition $\beta = \beta^{\text{sing}} + \beta^{\text{reg}}$, we may even assume that (15) holds for all $x \in \mathbf{R}^n$. Since u does not vanish, we have

$$
\begin{aligned}
\liminf_{r\to\infty}\; r^{-1} \int_0^r \int_0^s g(t)\, dt\, ds \;&>\; 0, \qquad \text{whereas} \\
\lim_{r\to\infty}\; r^{-1} \left\| (\beta_r^{\text{sing}})^- \right\|_{L^p(B_r;m)} \;&=\; 0
\end{aligned}
$$

by Assumption (1). Hence there exists $r_0 > 0$ such that $\int_0^{r_0} g(t)\, dt > 0$, and

$$
\begin{aligned}
(16) \qquad -g(r) \;+\; \frac{\gamma}{2} \int_0^r \int_0^s g(t)\, dt\, ds \;&\le\; \int_{B_r} (\beta_r^{\text{reg}})^-\; \rho\, |u|\, dx \\
\le\; C \int_{B_r} |x|\; \rho\, |u|\, dx \;&=\; C \int_0^r s\, g(s)\, ds
\end{aligned}
$$

for all $r \ge r_0$.

We can now apply the comparison lemma (Lemma 1). Let $G(r) := \int_0^r g(t)\, dt$, and $K(r) := \varepsilon \cdot r^\alpha$, where $\varepsilon > 0$ is a fixed constant. If γ has been chosen sufficiently large (i.e., $\gamma > 2C\alpha$) in the beginning, and $r_1 \in [r_0, \infty)$ is a sufficiently large constant (which does not depend on ε), then

$$
\begin{aligned}
-K'(r) \;+\; \frac{\gamma}{2} \int_0^r K(s)\, ds \;&=\; \varepsilon \left(-\alpha\, r^{\alpha-1} \;+\; \frac{\gamma}{2(\alpha+1)}\, r^{\alpha+1} \right) \\
\ge\; \varepsilon \cdot C \cdot \frac{\alpha}{\alpha+1}\, r^{\alpha+1} \;&=\; C \cdot \int_0^r s\, K'(s)\, ds
\end{aligned}
$$

for all $r \ge r_1$, whereas G satisfies the opposite inequality by (16). Thus (4) holds with $A(s) = \gamma/2$ and $B(s) = Cs$. Since $r_1 \ge r_0$, we have $G(r_1) = \int_0^{r_1} g(t)\, dt > 0$, and $\int_0^{r_1} (A(s) + B'(s))\, G(s)\, ds > 0$, whence (5) holds if ε is chosen small enough. By Lemma 1, we then obtain

$$
\begin{aligned}
\varepsilon\, r^\alpha \;=\; K(r) \;<\; G(r) \;&=\; \int_0^r g(s)\, ds \;=\; \int_{B_r} \rho\, |u|\, dx \\
&\le\; (m(B_r))^{1/p}\, \|u\|_{L^q(\mathbf{R}^n;m)} \qquad \text{for all } r \ge r_1.
\end{aligned}
$$

This is a contradiction to the volume growth restriction (14), so there is no non–trivial solution u of $\mathcal{L}^* u = \gamma u$ if (2) holds, and γ is large enough.

If (3) holds then we can argue similarly. We have

$$(17)\qquad (\beta_r^{\mathrm{reg}}(x))^- \;\leq\; C\,\left(1+|x|\log^+|x|\right) \;\leq\; C\,(|x|+e)\log(|x|+e)$$

for all $x\in\mathbf{R}^n$, where C is a finite constant. Hence, by (13) and the assumption on β^{sing}, there exists $r_0>0$ such that $\int_0^{r_0} g(t)\,dt>0$, and

$$(18)\qquad \begin{aligned} -g(r) \;+\; \frac{\gamma}{2}\int_0^r\int_0^s g(t)\,dt\,ds \;&\leq\; \int_{B_r}(\beta_r^{\mathrm{reg}})^-\;\rho\,|u|\;dx \\ \leq\; C\cdot\int_0^r (s+e)\;\log\,(s+e)\;g(s)\;ds &\qquad \text{for all } r\geq r_0. \end{aligned}$$

On the other hand, for $\varepsilon>0$, the function $K(r):=\varepsilon\cdot\log\log(r+e)$ satisfies

$$\begin{aligned} &-K'(r) \;+\; \frac{\gamma}{2}\int_0^r K(s)\;ds \\ &=\; \varepsilon\,\left(-((r+e)\;\log\,(r+e))^{-1} \;+\; \frac{\gamma}{2}\int_0^r \log\log\,(s+e)\,ds\right) \\ &\geq\; \varepsilon\cdot C\cdot r \;=\; C\cdot\int_0^r (s+e)\;\log\,(s+e)\;K'(s)\;ds \end{aligned}$$

for all $r\geq r_1$, provided $r_1\in[r_0,\infty)$ is a sufficiently large constant. Hence (4) holds with $G(r)=\int_0^r g(t)\,dt$, $A(r):=\gamma/2$, and $B(r):=C\cdot(r+e)\;\log\,(r+e)$. As above, we see that (5) also holds, if ε is chosen small enough. By Lemma 1 we then obtain

$$\begin{aligned} \varepsilon\;\log\log\,(r+e) \;&=\; K(r) \;<\; G(r) \;=\; \int_0^r g(s)\;ds \;=\; \int_{B_r}\rho\,|u|\;dx \\ &\leq\; (m(B_r))^{1/p}\;\|u\|_{L^q(\mathbf{R}^n\,;m)} \qquad \text{for all } r\geq r_1. \end{aligned}$$

This is a contradiction to the assumed finiteness of the measure m, so there is no non–trivial solution u of $\mathcal{L}^*u=\gamma u$ if (3) holds either. ■

To prove the second theorem, we need the following special case of a regularity result by Bogachev, Krylov and Röckner [BoKryRö 96, Thm. 1, (ii) and (iii)]:

Lemma 2 *Suppose $n\geq 2$. Let $\kappa\in[1,n]$ and $c\in\mathbf{R}$. Suppose μ is a signed Radon measure on $\mathbf{R}^n$ such that β is in $L^{\kappa+\varepsilon}_{\mathrm{loc}}(\mathbf{R}^n\to\mathbf{R}^n\,;\,\mu)$ for some $\varepsilon>0$, and*

$$\int(\Delta f \;+\; \beta\cdot\nabla f \;+\; cf)\;d\mu \;=\; 0 \qquad \textit{for all } f\in C_0^\infty(\mathbf{R}^n).$$

Then μ is absolutely continuous w.r.t. Lebesgue measure, and $\dfrac{d\mu}{dx}$ is in $H^{1,n/(n-\kappa+1)}_{\mathrm{loc}}(\mathbf{R}^n\,;\,dx)\cap L^{n/(n-\kappa)}_{\mathrm{loc}}(\mathbf{R}^n\,;\,dx)$.

The highly non–trivial proof of the lemma is based on regularity results in fractional Sobolev spaces, cf. [BoKryRö 96].

Proof of Theorem 2. Let $u \in L^q(\mathbf{R}^n\,;\,m)$, $\frac{1}{q}+\frac{1}{p}=1$, be a solution of $\mathcal{L}^*u=\gamma u$ for some $\gamma \geq \lambda$. Note that, since $n \geq 2$, the assumption implies $\beta \in L^{2p}_{\rm loc}(\mathbf{R}^n \to \mathbf{R}^n\,;\,m)$, which was one of the assumptions in Theorem 1. We will show moreover, that the signed measure $u\cdot m$ is absolutely continuous w.r.t. Lebesgue measure, the density $\frac{d(u\cdot m)}{dx}$ is in $H^{1,2}_{\rm loc}(\mathbf{R}^n\,;\,dx)$, and $|\beta|\cdot\frac{d(u\cdot m)}{dx}$ is in $L^2_{\rm loc}(\mathbf{R}^n\,;\,dx)$. Once we have shown this, the proof of Theorem 2 can be carried out in the same way as the proof of Theorem 1, starting from the end of Step 1. In fact, we only have to replace everywhere $\rho\,u$ by $\frac{d(u\cdot m)}{dx}$, and $\rho\,|u|$ by $\left|\frac{d(u\cdot m)}{dx}\right|$. Instead of $|\beta|\,\rho\,u \in L^2_{\rm loc}(\mathbf{R}^n\,;\,dx)$ (for which the assumption $|\beta|\,\rho^{1/2} \in L^{2p/(2-p)}_{\rm loc}(\mathbf{R}^n\,;\,m)$ in Theorem 1 was needed), we can use now that $|\beta|\cdot\frac{d(u\cdot m)}{dx}$ is in $L^2_{\rm loc}(\mathbf{R}^n\,;\,dx)$.
By the assumption, there exists $\varepsilon>0$ such that

$$\int_K |\beta|^{1+\frac{n}{2}+\varepsilon}\,|u|\,dm \;\;\leq\;\; \|u\|_{L^q(\mathbf{R}^n\,;\,m)}\cdot\left(\int_K |\beta|^{(1+\frac{n}{2}+\varepsilon)p}\,dm\right)^{1/p} \;\;<\;\; \infty$$

for any compact subset $K\subset\mathbf{R}^n$. Hence, by the lemma above, the measure $u\cdot m$ is absolutely continuous w.r.t. Lebesgue measure, and $\frac{d(u\cdot m)}{dx} \in H^{1,2}_{\rm loc}(\mathbf{R}^n\,;\,dx)\cap L^{2n/(n-2)}_{\rm loc}(\mathbf{R}^n\,;\,dx)$, where $2n/(n-2):=\infty$ if $n=2$. In particular,

$$\begin{aligned}
&\int_K |\beta|^2\left(\frac{d(u\cdot m)}{dx}\right)^2\,dx\\
&\leq\;\left(\int_K |\beta|^{1+\frac{n}{2}}\left|\frac{d(u\cdot m)}{dx}\right|\,dx\right)^{\frac{4}{n+2}}\cdot\left\|\left|\frac{d(u\cdot m)}{dx}\right|^{\frac{2n}{n+2}}\right\|_{L^{\frac{n+2}{n-2}}(K;\,dx)}\\
&=\;\left(\int_K |\beta|^{1+\frac{n}{2}}\;|u|\;dm\right)^{\frac{4}{n+2}}\cdot\left\|\frac{d(u\cdot m)}{dx}\right\|^{\frac{n+2}{2n}}_{L^{\frac{2n}{n-2}}(K;\,dx)}\;\;<\;\;\infty
\end{aligned}$$

for any compact subset $K\subset\mathbf{R}^n$. Thus $|\beta|\,\dfrac{d(u\cdot m)}{dx}$ is in $L^2_{\rm loc}(\mathbf{R}^n\,;\,dx)$, which completes the proof. ∎

Proof of the corollary. Let $\hat{\mathcal{L}}$ be a generator of a C^0 semigroup on $L^p(\mathbf{R}^n\,;\,m)$ that extends $(\,\mathcal{L}\,,\,C_0^\infty(\mathbf{R}^n)\,)$. Then $\hat{\mathcal{L}}$ is closed, and thus an extension of the closure $\bar{\mathcal{L}}$ of $\mathcal{L}$. By Theorem 1 resp. 2, both $\bar{\mathcal{L}}$ and $\hat{\mathcal{L}}$ generate C^0 semigroups. Hence there exists $\lambda_0\geq 0$ such that for all $\lambda\geq\lambda_0$, the operators $\lambda-\hat{\mathcal{L}}$ and $\lambda-\bar{\mathcal{L}}$ are bijections from their domains onto $L^p(\mathbf{R}^n\,;\,m)$, cf. e.g. [Pa 85]. But $\hat{\mathcal{L}}$ extends $\bar{\mathcal{L}}$, so both generators, and the corresponding semigroups, coincide. ∎

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