The Whitehead Link Complement and the Borromean Rings Complement are Torus Bundles*

Christof Menzel

June 11, 1999

It is well known that hyperbolic manifolds constitute the largest class (in a certain sense the genuine class) of geometrizable 3-manifolds. Among the hyperbolic manifolds, the surface bundles (i.e. manifolds fibering over the circle with fiber a surface of finite type) play an important rôle. In fact, Thurston [3] conjectured that every hyperbolic 3-manifold of finite volume is finitely covered by a surface bundle.

The figure 8 knot complement is long since known to be hyperbolic and to be a once punctured torus bundle. The Whitehead link complement and the Borromean rings complement are known to be hyperbolic as well [4]. Up to my knowledge, however, no one has recognized them as surface bundles.

In this short note I will show that the Whitehead link complement is a twice punctured torus bundle and the Borromean rings complement is a thrice punctured torus bundle by specifying automorphism of the surfaces in question. In order to do this, it is enough to specify automorphisms of the fundamental groups of the surfaces and to show isomorphy of the fundamental groups of the 3-manifolds in question [2].

Let $S_g^{(r)}$ be the orientable surface of genus g with r punctures. Its fundamental group is given by

$$\begin{aligned}\pi_1(S_g^{(r)}) &= \langle\, a_1, a_2, \ldots, a_{2g-1}, a_{2g}, \kappa_1, \ldots, \kappa_r \mid \Pi_{i=1}^{g}[a_{2i}, a_{2i-1}]\,\Pi_{j=1}^{r}\kappa_j \,\rangle \\ &\cong \Phi_{2g+r-1}.\end{aligned}$$

A surface bundle with fiber $S_g^{(r)}$ is a manifold

$$M_f(S_g^{(r)}) = S_g^{(r)} \times [0,1]/(x,0) \sim (f(x),1)\,\forall x \in S_g^{(r)},$$

*This work has been supported by the Deutsche Forschungsgemeinschaft – Sonderforschungsbereich 343 Diskrete Strukturen in der Mathematik

where f is a homeomorphism of the surface. The fundamental group of this surface bundle is given by

$$\pi_1(S_g^{(r)}) \rtimes \mathbf{Z} = \langle\, \Phi_{2g+r-1}, \mu \,|\, \mu\omega\mu^{-1} = f_*(\omega)\, \forall \omega \in \Phi_{2g+r-1} \,\rangle.$$

Here, f_* is the automorphism of $\pi_1(S_g^{(r)})$ induced by f. Since the mapping class of f is completely determined by f_*, I omit the $*$.

Let $g = 1, r = 2$, so the fiber is the twice punctured torus. Its fundamental group is given by

$$\pi_1(S_1^{(2)}) = \langle a, b, \kappa_1, \kappa_2 \,|\, \kappa_2^{-1} = b^{-1}a^{-1}ba\kappa_1 \rangle \cong \Phi_3.$$

According to Magnus [1], the group of automorphisms of $\pi_1(S_1^{(2)})$ leaving κ_2 invariant is generated by

$$r : (a, b, \kappa_1) \mapsto (a, a^{-1}b, \kappa_1),$$
$$s : (a, b, \kappa_1) \mapsto (a^{-1}ba, a^{-1}, \kappa_1),$$

and

$$\tau : (a, b, \kappa_1) \mapsto (a\kappa_1 a\kappa_1^{-1}a^{-1}, a\kappa_1 a^{-1}b, a\kappa_1 a^{-1}).$$

(Here, my κ_1 is Magnus' u^{-1}, my b is Magnus' b^{-1}.) Define $f := r^{-1}sr^{-1} : (a, b, \kappa_1) \mapsto (ba, b, \kappa_1)$.

Then $f\tau : (a, b, \kappa_1) \mapsto (ba\kappa_1 ba\kappa_1^{-1}a^{-1}b^{-1}, ba\kappa_1 a^{-1}, ba\kappa_1 a^{-1}b^{-1})$ is the desired automorphism, as I will show now.

The fundamental group of $M_{f\tau}(S_1^{(2)})$ can be represented as a subgroup of $\mathrm{SL}_2(\mathbf{Z}[i])$ generated by the matrices

$$G = \begin{pmatrix} i & 1 \\ 1-i & -1-2i \end{pmatrix}, \; M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(M represents μ.) These two define the other matrices A, B, K_1, K_2 representing a, b, κ_1, κ_2 via

$$\begin{aligned} K_1 &= A^{-1}B^{-1}MBM^{-1}A, \\ A &= GM^2B^{-1}, \quad \text{(i.e. } G = ABM^{-2}\text{)} \\ B &= GM^{-1}G^{-1}M, \\ K_2^{-1} &= M^{-2}G^{-1}MG^2MG^{-1}. \end{aligned}$$

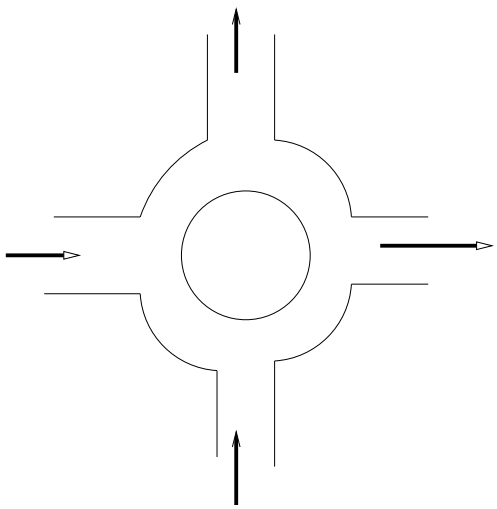

Figure 1: *Twice punctured torus: identify according to arrows.*

It is easy to check that all relations except one become trivial. The only remaining relation is

$$MG^{-1}MG^2MG^{-1}M^{-1}GM^{-1}G^{-2}M^{-1}G,$$

which is the relation of the Whitehead link, as I will show below.

From figure 1 and 2 one can see a possible twice punctured torus Seifert surface for the Whitehead link. This object may as well be viewed as the projection of a handlebody, obtained by thickening the surface $S_1^{(2)}$, i.e. by embedding

$$S_1^{(2)} \times \left[0, \frac{1}{2}\right] / \left\{(x,0) \sim (y,0), \left(x, \frac{1}{2}\right) \sim \left(y, \frac{1}{2}\right) \forall x, y \in \partial S_1^{(2)}\right\}$$

into $\mathbf{R}^3$. This handlebody is obviously unknotted in the sense, that its complement has the same fundamental group. Note the following obvious necessary condition for a link complement to be a fiber bundle over the circle with fiber a surface $S_g^{(r)}$.

Lemma. *If an r-component link complement in the three-sphere is an $S_g^{(r)}$-bundle, then the link admits a Seifert surface homeomorphic to $S_g^{(r)}$ such that the handlebody of genus $2g + r - 1$ obtained by thickening the surface*

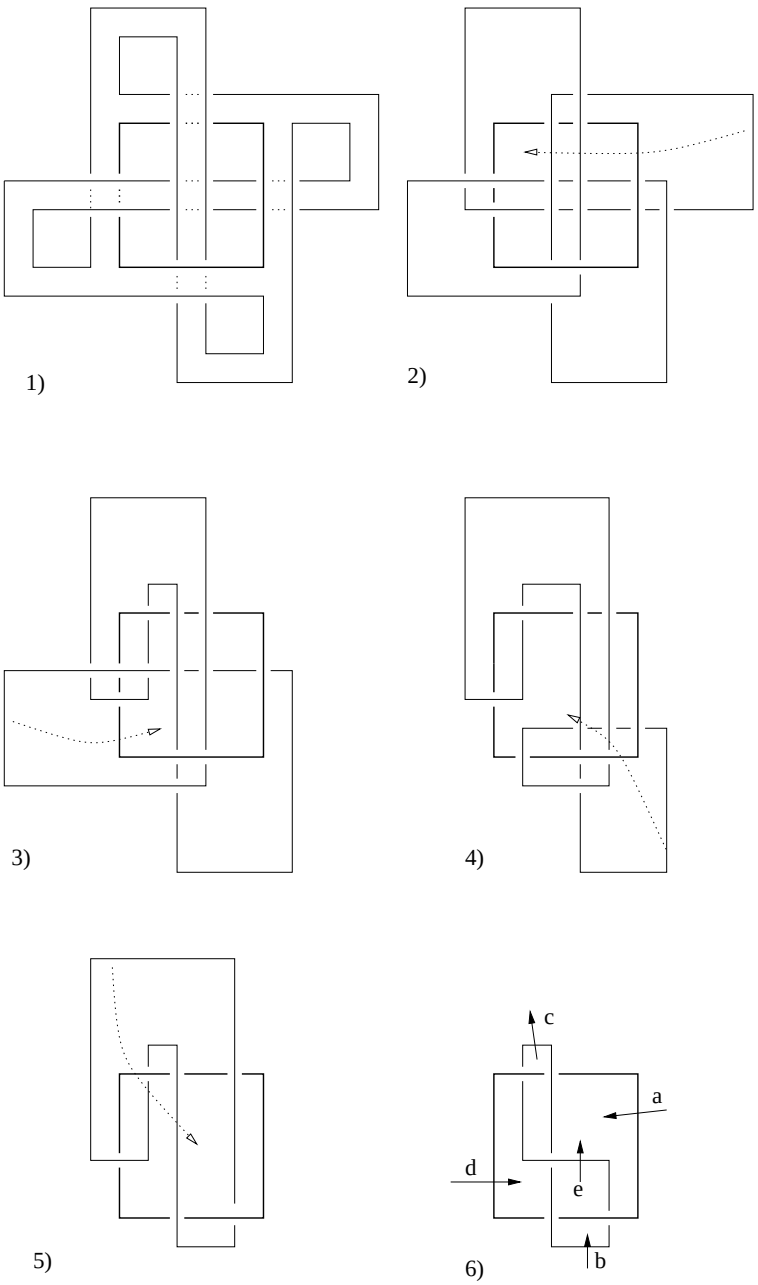

Figure 2: *The twice punctured torus as possible Seifert surface for the Whitehead link. The last picture shows a Wirtinger presentation for the fundamental group of the complement.*

(without self intersection)

$$S_g^{(r)} \times \left[0, \frac{1}{2}\right] / \left\{(x,0) \sim (y,0), \left(x, \frac{1}{2}\right) \sim \left(y, \frac{1}{2}\right) \forall x, y \in \partial S_g^{(r)}\right\}$$

is unknotted.

From the last picture in figure 2 the following system of relators for the complement can be read off:

$$\begin{aligned} R_1 &= acd^{-1}c^{-1}, \\ R_2 &= ba^{-1}ea, \\ R_3 &= a^{-1}bdb^{-1}, \\ R_4 &= e^{-1}dc^{-1}d^{-1}, \\ R_5 &= ceb^{-1}e^{-1}. \end{aligned}$$

Replace $b = a^{-1}e^{-1}a$, $a = cdc^{-1}$, $e = dc^{-1}d^{-1}$. Then:

$$\begin{aligned} R_3 &= cd^{-1}c^{-1}dcd^{-1}cdc^{-1}dcd^{-1}c^{-1}dc^{-1}d^{-1}, \\ R_5 &= cdc^{-1}d^{-1}cd^{-1}c^{-1}dc^{-1}d^{-1}cdc^{-1}dcd^{-1}. \end{aligned}$$

Now R_5^{-1} is cyclically equivalent to R_3. Replacing $c = GM$, $d = M$ and inverting R_3 leads to the desired relator. Therefore the Whitehead link complement is a twice punctured torus bundle.

From figure 3 it can be seen that the thrice punctured torus is a Seifert surface for the Borromean rings, the thickening of which is an unknotted handlebody of genus 4.

I will now present an automorphism g of the fundamental group of $S_1^{(3)}$, such that $\pi_1(M_g(S_1^{(3)}))$ is isomorphic to the fundamental group of the Borromean rings complement. This automorphism has been found in the following way (cf. figure 4): Draw the generators a, b, κ_1, κ_2 of $\pi_1(S_1^{(3)})$ on the bottom of the handlebody = thickened surface. Deform them homotopically through the complement of the handlebody onto its top. Then the presentation of these deformed curves as words in a, b, κ_1, κ_2 are the images of a, b, κ_1, κ_2 under g. The curve κ_3, on which the base point lies, is of course left invariant.

The automorphism g is determined by:

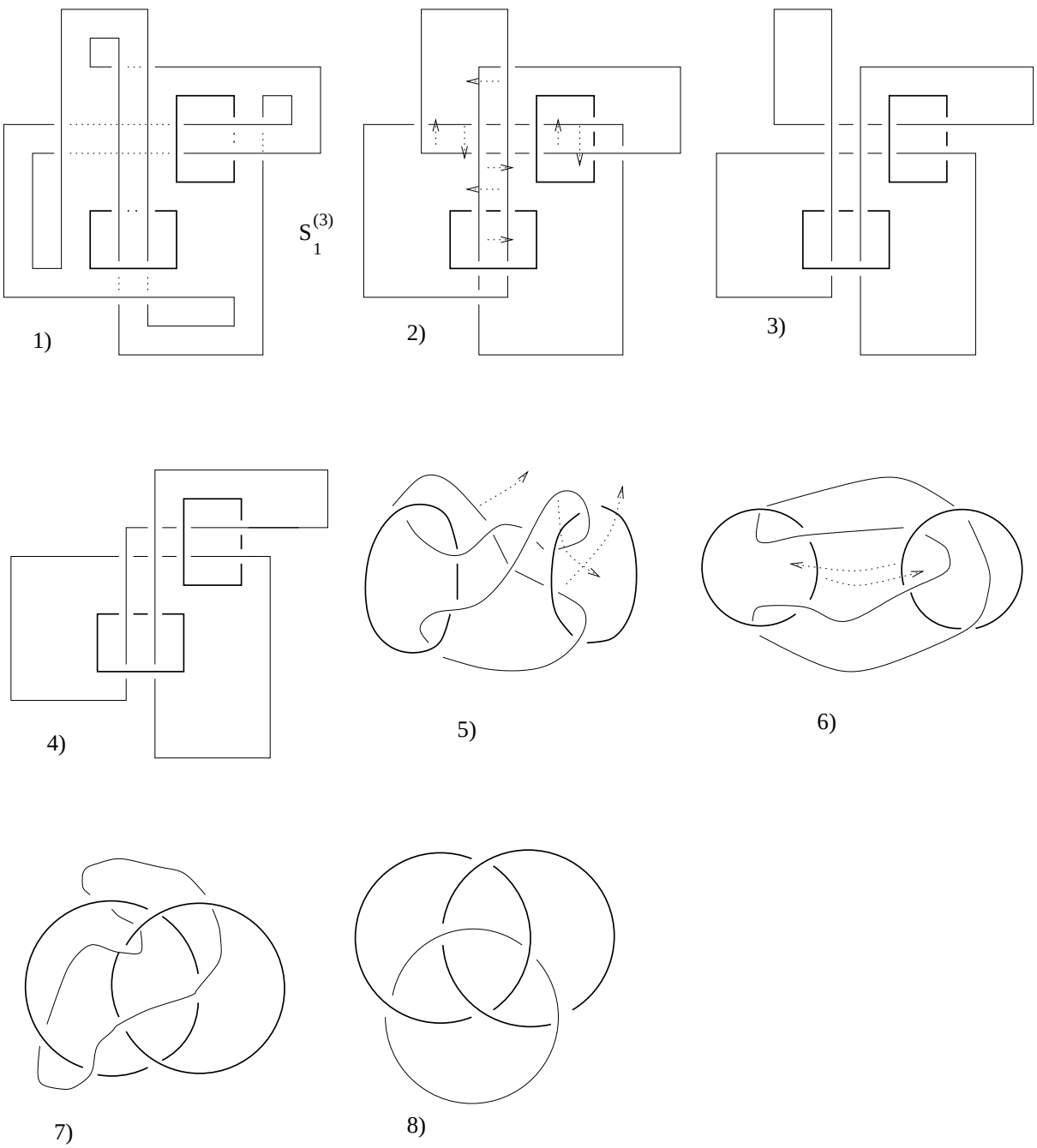

Figure 3: *A thrice punctured torus as Seifert surface for the Borromean rings.*

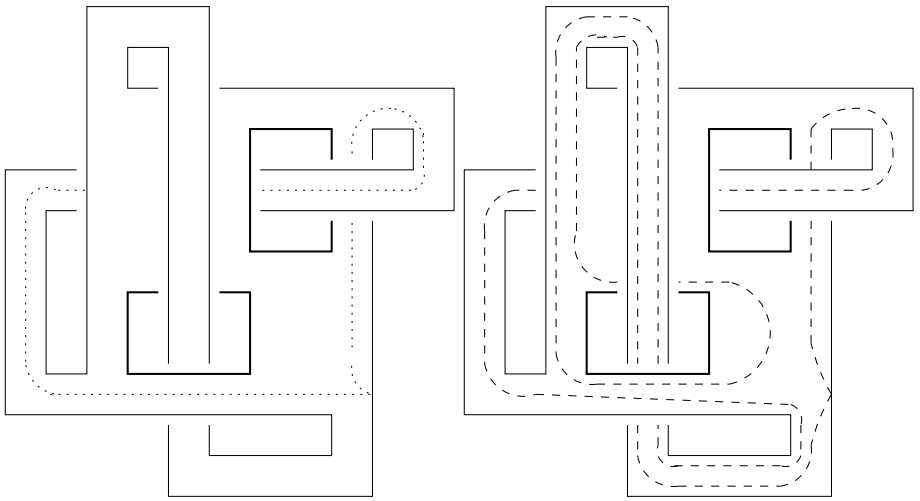

Figure 4: *One of the generators of the fundamental group of $S_1^{(3)}$: dotted line, left. Deform this curve from the bottom of the thickened surface through the complement onto its top (dashed line, right) and read off its presentation, which is its image under the automorphism g that one is looking for.*

$$\begin{array}{rlll}
a & \mapsto \kappa_2^{-1}a\kappa_1 a^{-1} & a\,b\,\kappa_2^{-1}\,b^{-1} & a\kappa_1^{-1}a^{-1}\kappa_2, \\
b & \mapsto \kappa_2^{-1}a\kappa_1 a^{-1} & b\,a\,\kappa_1\,a^{-1} & a\kappa_1^{-1}a^{-1}\kappa_2, \\
\kappa_1 & \mapsto \kappa_2^{-1}a\kappa_1 a^{-1} & b\kappa_2 b^{-1}a\,\kappa_1\,a^{-1}b\kappa_2^{-1}b^{-1} & a\kappa_1^{-1}a^{-1}\kappa_2, \\
\kappa_2 & \mapsto \kappa_2^{-1}a\kappa_1 a^{-1} & b\,\kappa_2\,b^{-1} & a\kappa_1^{-1}a^{-1}\kappa_2.
\end{array}$$

The inverse is determined by

$$\begin{array}{rlll}
a & \mapsto b^{-1}\kappa_2 b\kappa_2^{-1}\kappa_1^{-1}\kappa_2 & a\,\kappa_2 & \kappa_2^{-1}\kappa_1\kappa_2 b^{-1}\kappa_2^{-1}b, \\
b & \mapsto b^{-1}\kappa_2 b\kappa_2^{-1}\kappa_1^{-1}\kappa_2 & b\,\kappa_2^{-1}\,\kappa_1^{-1}\,\kappa_2 & \kappa_2^{-1}\kappa_1\kappa_2 b^{-1}\kappa_2^{-1}b, \\
\kappa_1 & \mapsto b^{-1}\kappa_2 b\kappa_2^{-1}\kappa_1^{-1}\kappa_2 & \kappa_2^{-1}\,a^{-1}\,\kappa_2^{-1}\,\kappa_1\,\kappa_2\,a\,\kappa_2 & \kappa_2^{-1}\kappa_1\kappa_2 b^{-1}\kappa_2^{-1}b, \\
\kappa_2 & \mapsto b^{-1}\kappa_2 b\kappa_2^{-1}\kappa_1^{-1}\kappa_2 & \kappa_2^{-1}\,\kappa_1\,\kappa_2\,b^{-1}\,\kappa_2\,b\,\kappa_2^{-1}\,\kappa_1^{-1}\,\kappa_2 & \kappa_2^{-1}\kappa_1\kappa_2 b^{-1}\kappa_2^{-1}b.
\end{array}$$

Lemma: *The fundamental group of $M_g(S_1^{(3)})$ is isomorphic to the fundamental group of the Borromean rings complement.*

Proof.

The group $\pi_1(M_g(S_1^{(3)}))$ is generated by a, b, κ_1, κ_2 and μ and has the relators $S_1, \ldots, S_4$:

$$\begin{aligned}
S_1 &= \mu\, a^{-1}\, \mu^{-1}\, \kappa_2^{-1}\, a\, \kappa_1\, b\, \kappa_2^{-1}\, b^{-1}\, a\, \kappa_1^{-1}\, a^{-1}\, \kappa_2, \\
S_2 &= \mu\, b^{-1}\, \mu^{-1}\, \kappa_2^{-1}\, a\, \kappa_1\, a^{-1}\, b\, \kappa_2, \\
S_3 &= \mu\, \kappa_1^{-1}\, \mu^{-1}\, \kappa_2^{-1}\, a\, \kappa_1\, a^{-1}\, b\, \kappa_2\, b^{-1}\, a\, \kappa_1\, a^{-1}\, b\, \kappa_2^{-1}\, b^{-1}\, a\, \kappa_1^{-1}\, a^{-1}\, \kappa_2, \\
S_4 &= \mu\, \kappa_2^{-1}\, \mu^{-1}\, \kappa_2^{-1}\, a\, \kappa_1\, a^{-1}\, b\, \kappa_2\, b^{-1}\, a\, \kappa_1^{-1}\, a^{-1}\, \kappa_2.
\end{aligned}$$

Replace κ_1 with $a^{-1}\, \kappa_2\, \mu\, b\, \mu^{-1}\, \kappa_2^{-1}\, b^{-1}\, a$ according to S_2. Then we have (up to cyclic rotations and inversions):

$$\begin{aligned}
S_1 &= a^{-1}\, b\, \mu^{-1}\, \kappa_2^{-1}\, b^{-1}\, a\, b\, \mu\, b^{-1}, \\
S_3 &= a^{-1}\, b\, \kappa_2\, \mu\, b^{-1}\, \mu^{-1}\, \kappa_2^{-1}\, a\, b\, \mu^{-1}\, b^{-1}\, \kappa_2\, \mu\, b\, \mu^{-1}\, \kappa_2^{-1}\, \mu\, b^{-1}, \\
S_4 &= \kappa_2^{-1}\, b\, \mu^{-1}\, \kappa_2\, \mu\, b^{-1}.
\end{aligned}$$

Replace κ_2 by $b^{-1}\, a\, b\, \mu\, b^{-1}\, a^{-1}\, b\, \mu^{-1}$ according to S_1:

$$\begin{aligned}
S_3 &= \mu\, b^{-1}\, a^{-1}\, b^{-1}\, a\, b\, \mu^{-1}\, b^{-1}\, a^{-1}\, b\, a\, b\, \mu^{-1}\, b^{-1}\, b^{-1}\, a\, b\, \mu\, b^{-1}\, a^{-1}\, b\, a\, b\, \mu^{-1}\, b^{-1}\, a^{-1}\, b\, \mu, \\
S_4 &= \mu\, b^{-1}\, a\, b\, \mu^{-1}\, b^{-1}\, a^{-1}\, b^2\, \mu^{-1}\, b^{-1}\, a\, b\, \mu\, b^{-1}\, a^{-1}.
\end{aligned}$$

Introduce s and replace $\mu = b^{-1}\, a^{-1}\, s^{-1}\, a\, b$:

$$\begin{aligned}
S_3 &= s^{-1}\, b^{-1}\, s\, b\, s\, a\, b^{-1}\, s^{-1}\, b\, s\, a^{-1}\, s^{-1}, \\
S_4 &= b^{-1}\, a^{-1}\, s^{-1}\, a\, s\, b\, a^{-1}\, s\, a\, s^{-1}.
\end{aligned}$$

Introduce w and replace $a = s^{-1} w^{-1}$:

$$\begin{aligned}
S_3 &= s^{-1}\, b^{-1}\, s\, b\, w^{-1}\, b^{-1}\, s^{-1}\, b\, s\, w, \\
S_4 &= b^{-1}\, w\, s^{-1}\, w^{-1}\, s\, b\, w\, s\, w^{-1}\, s^{-1}.
\end{aligned}$$

Introduce u and replace $b = s u^{-1}$:

$$\begin{aligned}
S_3 &= s^{-1}\, u^{-1}\, s\, u\, w^{-1}\, u^{-1}\, s^{-1}\, u\, s\, w, \\
S_4 &= u^{-1}\, s\, w\, s^{-1}\, w^{-1}\, u\, w\, s\, w^{-1}\, s^{-1},
\end{aligned}$$

which can be written as

$$\begin{aligned}
S_3 &= [w, [u, s]], \\
S_4 &= [u^{-1}, [w^{-1}, s^{-1}]].
\end{aligned}$$

From figure 5 the following relators for the complement of the Borromean

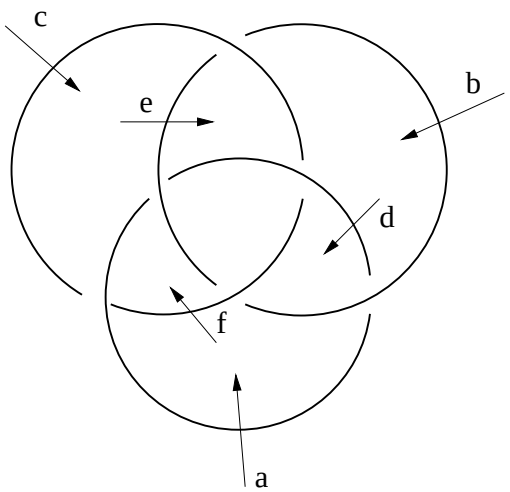

Figure 5: *Wirtinger presentation for the fundamental group of the complement of the Borromean rings.*

rings can be read off:

$$\begin{aligned} R_1 &= d^{-1}bab^{-1}, \\ R_2 &= fdc^{-1}d^{-1}, \\ R_3 &= efb^{-1}f^{-1}, \\ R_4 &= fac^{-1}a^{-1}, \\ R_5 &= ecb^{-1}c^{-1}, \\ R_6 &= dea^{-1}e^{-1}. \end{aligned}$$

Replace $d = bab^{-1}$, $f = aca^{-1}$, $e = cbc^{-1}$. Then the remaining relators are

$$\begin{aligned} R_2 &= a\,c\,a^{-1}bab^{-1}\,c^{-1}\,ba^{-1}b^{-1}, \\ R_3 &= c\,b\,c^{-1}aca^{-1}\,b^{-1}\,ac^{-1}a^{-1}, \\ R_6 &= bab^{-1}cbc^{-1}a^{-1}cb^{-1}c^{-1}. \end{aligned}$$

Now $R_6 = (R_3R_2)^{-1}$. Replace $c = u, b = w, a^{-1} = s$. Then

$$\begin{aligned} R_2 &= [u^{-1}, [w^{-1}, s^{-1}]], \\ R_3 &= [w, [u, s]]. \end{aligned}$$

This completes the proof. □

References

[1] Magnus, Wilhelm: "Über Automorphismen von Fundamentalgruppen berandeter Flächen." Math. Annalen **109** (1934), 617-646.

[2] Stallings, John: "On fibering certain 3-manifolds." In: *Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961).* Englewood Cliffs, N.J.: Prentice-Hall, 1962, 95-100.

[3] Thurston, William P.: "Three Dimensional Manifolds, Kleinian Groups and Hyperbolic Geometry." Bull. Amer. Math. Soc. **6** (1982), 357-382.

[4] Thurston, William P.: *The Geometry and Topology of Three-Manifolds. Electronic version 1.0 - October 1997.*

Dr. Christof Menzel
SFB 343/Fakultät für Mathematik
Universität Bielefeld
P.O.-Box 100131
D-33501 Bielefeld
e-mail: menzel@mathematik.uni-bielefeld.de