Estimates of variable stepsize Runge–Kutta methods for sectorial evolution equations with nonsmooth data

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1 Introduction

In a recent series of papers, Lubich and Ostermann [16], [17], [18], [19], [20] have established the basic properties of Runge–Kutta dicretizations of sectorial evolution equations and have created the basis for a qualitative theory to follow. The main features of their results may be summarized as follows:

- The stability theory of Runge–Kutta methods for sectorial evolution equations can be drawn from a general approach to Runge Kutta methods for integral equations of convolution type [16],
- the basic assumption on the Runge–Kutta method is the so-called $A(\theta)$-stability. It requires a bound of the stability function evaluated on the boundary of a sector that is slightly bigger than the sector which contains the spectrum of the unbounded operator [16],
- due to incompatibilites of initial boundary conditions the order of convergence is generally less than the classical order for ODE's, it may however be conserved in the interior of the domain [16],[17],

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- for nonsmooth initial data uniform estimates of the first order can be obtained for the solutions as well as for derivatives with respect to intial values. These can then be used to study the error of longtime dynamics, such as periodic solutions [19] or transitions to periodic solutions at a Hopf bifurcation [20].

In this paper we develop a unified approach to some of their results in the nonsmooth data case and, based on this, we present extensions to the case of variable stepsizes. For discretizations in space of semilinear evolution equations with nonsmooth data we refer for completeness to [12] as the starting point of the general theory and to [23] where more recent references can be found.

There are various approaches in the literature which deal with Runge-Kutta estimates for sectorial evolution equations [14],[3], [1],[21],[7],[2]. We mention in particular the work of Bakaev [1],[2] and Gonzalez, Palencia [21],[7] who both treat variable stepsizes and (more generally than in this paper) sectorial operators that vary in time. However, they restrict to linear equations in the homogenous [21],[7] or inhomogenous case [2] and – more importantly – they concentrate on stability estimates with L_1-norms in time. In accordance with [19] we think that such estimates are not sufficient to yield sharp error bounds in the nonsmooth data case. Rather, one has to make use of the smoothing effect of the discrete equations by applying in some way or another Abels summation trick (summation by parts). This will be worked out in detail in sections 5 and 6.

In a first step in section 3 we set up a refined operational calculus for unbounded operators and for functions which are analytic in a neighbourhood of a sector in the negative half plane and which vanish at infinity to some algebraic (in general noninteger) order. This function class extends the Dunford-Taylor class which requires analyticity at infinity and it can be considered as an extension of the classical Dunford-Gelfand class to functions with certain weak singularities. This operational calculus turns out to be the appropriate tool for the basic stability estimates and it allows us to restructure and unify the Lubich, Ostermann results. Moreover, we use it later on to derive estimates for the case of variable stepsizes.

These results will be prepared in sections 4 and 5 by some stability estimates for variable stepsizes and a convergence result for constant stepsize. Though parts of these results are well-known we provide full proofs here because this will motivate the estimates for variable stepsizes in section 6 and because our approach sometimes differs from standard proofs, e.g. for the classical convergence result of LeRoux [14].

In section 6 we then deal with the general case of variable stepsize and nonsmooth data. It turns out that extra assumptions are needed here which either require that the stability function vanishes at infinity or which impose some condition of quasi-uniformity on the underlying grid. It remains as an open problem whether such additional conditions can be avoided altogether. If this is not the case it would be highly desirable to formulate a simple grid condition which is sufficient for convergence and which subsumes all of the above-mentioned additional requirements.

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2 Sectorial evolution equations

The equation we deal with is of the form

$$\dot{u} = Au + f(u), \quad t > 0 \tag{2.1}$$

where A is the infinitesimal generator of an analytic semigroup $\{e^{At}\}_{t\geq 0}$ in some Banach space $(X, |\cdot|)$. In other words our assumption is

(A0) A is a densely defined closed linear operator on X satisfying the spectral condition

$$\sigma(A) \subset \Sigma_{a,\theta} = \{z \in \mathbb{C} \setminus \{a\} : \theta < |\arg(z-a)| \leq \pi\}$$

for some real a, θ with $\theta \in (\frac{\pi}{2}, \pi)$ and the resolvent estimate

$$|(zI - A)^{-1}| \leq \frac{M}{1 + |z-a|} \quad \text{whenever} \quad z \notin \Sigma_{a,\theta} \tag{2.2}$$

for some positive constant M.

Since $a \notin \sigma(A)$ and $\sigma(A)$ is closed, there exists an $\omega_0 > 0$ with the property that

$$\sigma(A) \subset \mathcal{S}_{a,\theta} = \Sigma_{a,\theta} \setminus \{z \in \mathbb{C} : |z-a| \leq \omega_0\}.$$

The reason for introducing the modified sector $\mathcal{S}_{a,\theta}$ is that its boundary curve $\Gamma = \partial\mathcal{S}_{a,\theta}$ will serve as a proper contour for the operational calculus to follow, see (2.4), (2.5) below.

By passing to a somewhat greater M if necessary, we may assume that

$$|(zI - A)^{-1}| \leq \frac{M}{|z-a|} \quad \text{for all} \quad z \notin \mathcal{S}_{a,\theta}. \tag{2.3}$$

We then have the contour integral representation

$$e^{At} = \frac{1}{2\pi i} \int_\Gamma e^{zt}(zI - A)^{-1} dz, \quad t > 0 \tag{2.4}$$

where the boundary curve $\Gamma = \partial\mathcal{S}_{a,\theta}$ is oriented upwards. Note that $\{e^{At}\}_{t\geq 0}$ is the solution semigroup of the homogeneous equation $\dot{u} = Au$. For these and other properties of analytic semigroups, see Hale [9], Henry [11] and Pazy [22]. For a particularly good survey of the essentials, see Appendix A in Stuart [23].

The simplest example in infinite dimensions is

$$X = L_2(0,\pi) \quad \text{equipped with norm} \quad |u| = (\frac{1}{\pi}\int_0^\pi |u(x)|^2 dx)^{\frac{1}{2}}$$

and $A = \Delta_D$, the Laplacian subject to Dirichlet boundary conditions on $[0,\pi]$. Then the spectrum $\sigma(\Delta_D)$ consists of the sequence of eigenvalues $\{-k^2\}_{k=1}^\infty$ and the constants $a \in (-1,\infty)$ resp. $\theta \in (\frac{\pi}{2},\pi)$ can be chosen arbitrarily. Inequality (2.2) may be written more precisely as

$$\max\{|z-a| \cdot |(zI-\Delta_D)^{-1}| : z \notin \Sigma_{a,\theta}\} = \frac{1}{\sin\theta}$$

while the resolvent resp. the contour integral representation simplifies to the eigenfunction expansions

$$(zI-\Delta_D)^{-1}u = \sum_{k=1}^\infty c_k(z+k^2)^{-1}\sin kx, \quad z \notin \{-k^2\}_{k=1}^\infty$$

resp.

$$e^{\Delta_D t}u = \sum_{k=1}^\infty c_k e^{-k^2 t}\sin kx, \quad t \geq 0$$

for all $u = \sum_{k=1}^\infty c_k \sin kx \in L_2(0,\pi)$.

In order to describe our requirements on the nonlinearity f in (2.1), we briefly recall the concept of fractional power spaces. The linear operator A gives rise to a collection of bounded linear operators

$$(aI-A)^{-\alpha} = \frac{1}{2\pi i}\int_\Gamma (a-z)^{-\alpha}(zI-A)^{-1}dz, \quad \alpha > 0. \tag{2.5}$$

Here we let $(a-z)^{-\alpha}$ be the principal branch of the reciprocal root function which is analytic on $\mathbb{C}\backslash\{t \in \mathbb{R} : t \geq a\}$ and positive for $z = t < a$. The operators $(aI-A)^{-\alpha}$, $\alpha > 0$ have densely defined inverses

$$(aI-A)^\alpha = ((aI-A)^{-\alpha})^{-1}, \quad \alpha > 0$$

which are unbounded in general and they give rise to a nested collection of fractional power spaces

$$X^\alpha = \text{range}((aI-A)^{-\alpha}), \quad \alpha > 0$$

equipped with the norm

$$|u|_\alpha = |(aI-A)^\alpha u| \quad \text{for all} \quad u \in X^\alpha, \alpha > 0.$$

It is important to note that $\{X^\alpha\}_{\alpha>0}$ does not depend on the particular choice of the constant a and that the norms corresponding to different a's are equivalent.

Note also that $X^1 = \mathcal{D}(A)$, the domain of A and $(aI - A)^1 = aI - A$. Finally, we set $X^0 = X$ and $(aI - A)^0 = I$.

Returning to the example $(X, A) = (L_2(0,\pi), \Delta_D)$, it is easily checked that with the choice $a = 0$ we have

$$(L_2(0,\pi))^\alpha = \{u = \sum_{k=1}^{\infty} c_k \sin kx \in L_2(0,\pi) : \quad |u|_\alpha = (\frac{1}{2}\sum_{k=1}^{\infty} k^{4\alpha}|c_k|^2)^{\frac{1}{2}} < \infty\}$$

and

$$(aI - \Delta_D)^\alpha u = \sum_{k=1}^{\infty} c_k (a + k^2)^\alpha \sin kx \quad \text{for all } u \in X^\alpha, a > -1 \text{ and } \alpha \geq 0.$$

Note that $X^1 = D(\Delta_D) = H^2(0,\pi) \cap H_0^1(0,\pi)$, endowed with the H^2–norm from the Sobolev space $H^2(0,\pi)$ and $X^{\frac{1}{2}} = H_0^1(0,\pi)$. For details, we refer again to [9], [11] and [22].

We also recall two inequalities which play a fundamental role in the whole theory of sectorial evolution equations. There exists a constant $\Omega > 0$ such that

$$|e^{At}| \leq \Omega e^{at} \quad \text{for all} \quad t \geq 0 \tag{2.6}$$

and

$$|(aI - A)^\alpha e^{At}| \leq \Omega t^{-\alpha} e^{at} \quad \text{for all } t > 0 \text{ and } \alpha \in [0, 1). \tag{2.7}$$

The $(L_2(0,\pi), \Delta_D)$ example shows that both (2.6) and (2.7) are sharp. With Ω replaced by a continuous function $\Omega(\cdot)$, inequality (2.7) holds true for all $\alpha \in [0, \infty)$. For proofs, we recommend [22], especially p. 30.

Now we are in a position to formulate the assumptions for f. Throughout the paper we make the following assumption

(F0) For a fixed $\alpha \in [0, 1)$ a mapping $f : X^\alpha \to X$ is given which satisfies for some $L \geq 0$ the Lipschitz condition

$$|f(u) - f(\tilde{u})| \leq L|u - \tilde{u}|_\alpha \quad \text{for all} \quad u, \tilde{u} \in X^\alpha. \tag{2.8}$$

It follows that the solutions of equation (2.1) form a nonlinear C^0 semigroup in X^α. In other words, given $u_0 \in X^\alpha$ arbitrarily, there exists a unique continuous function $\Phi(\cdot, u_0) : [0, \infty) \to X^\alpha$ with the properties that $\Phi(0, u_0) = u_0$, $f(\Phi(\cdot, u_0)) : [0, \infty) \to X$ is continuous, $\Phi(t, u_0) \in \mathcal{D}(A)$ for $t > 0$ and $\Phi(\cdot, u_0)$ satisfies (2.1) for $t > 0$ (this is the definition of a solution as in [9]) and the function $\Phi : [0, \infty) \times X^\alpha \to X^\alpha$ is continuous and has the semigroup property.

Recall that the solutions of (2.1) in the above sense coincide with those solutions of the integral equation

$$u(t) = e^{At}u_0 + \int_0^t e^{A(t-s)} f(u(s))ds, \quad t \geq 0 \tag{2.9}$$

for which both $u : [0, \infty) \to X^\alpha$ and $f(u(\cdot)) : [0, \infty) \to X$ are continuous. Note that the function $\Phi(\cdot, u_0) : (0, \infty) \to X^\alpha$ is continuously differentiable but in general differentiability at $t = 0$ only holds under extra conditions on u_0. Further, if $f : X^\alpha \to X$ is of class C^r, $r = 1, 2, \dots, \infty$ or analytic, then also $\Phi : (0,\infty) \times X^\alpha \to X^\alpha$ is of class C^r, $r = 1, 2, \dots, \infty$ or analytic. The underlying fact is that the classical Picard–Lindelöf theory for ODE's extends to sectorial evolution equations when working with a pair of embedded Banach spaces (X^α, X).

For later use, we note there is a continuous function $b : [0,\infty) \times [0,\infty) \to [0,\infty)$ for which

$$|\Phi(t,u_0)|_\alpha \le b(t, |u_0|_\alpha) \quad \text{for all} \quad t \ge 0, u_0 \in X^\alpha. \tag{2.10}$$

The particular form of the function b can be given via a Gronwall inequality with singular kernel function. We refer also to the estimate of $\dot{u}(t) = d\Phi(t,u_0)/dt$ in X^α as given in Lemma 5.1 (see [11],p.71). This is an important regularity result which has played a crucial role in the error analysis with nonsmooth data since the pioneering papers by Crouzeix and Thomée [5] as well as Hale, Lin and Raugel [10]. Finally, we note that assumptions (A0) and (F0) are met by broad classes of reaction–diffusion and Navier–Stokes equations [11], [22], [9].

All the previous results on existence, uniqueness and continuous dependence remain valid if (2.1) is replaced by

$$\dot{u} = Au + g(t,u), \quad t > 0 \tag{2.11}$$

where A is as above and $g : \mathbb{R} \times X^\alpha \to X$ ($\alpha \in [0,1)$) is a function satisfying for some $\mu \in (0,1]$ the Hölder–Lipschitz condition

$$|g(t,u) - g(\tilde{t},\tilde{u})| \le L(|t-\tilde{t}|^\mu + |u-\tilde{u}|_\alpha) \quad \text{for all} \quad (t,u),(\tilde{t},\tilde{u}) \in \mathbb{R}\times X^\alpha. \tag{2.12}$$

Also Lubich and Ostermann [16], [17], [18], [19], [20] work in this nonautonomous setting. Moreover, in the special case when X is a Hilbert space and $\alpha = \frac{1}{2}$, they allow A to depend on t (or even on u, [18]) in a particular way. We consider only autonomous equations in this paper. However, it is a straightforward but sometimes lengthy technical task to show that all results we present in the sequel remain valid if (2.1) and (2.8) are replaced by (2.11) and case $\mu = 1$ of (2.12), respectively. If our interest is focused on a bounded subset of X^α as well as on short intervals of existence of the solutions, then the global Lipschitz estimate (2.8) can be weakened to a local one.

Consider again the $(X,A) = (L_2(0,\pi), \Delta_D)$ example with $f : X^{\frac{1}{2}} \to X$ defined by $(f(u))(x) = (1+x)u^2(x)$, $x \in (0,\pi)$. Since $X^{\frac{1}{2}} = H_0^1(0,\pi)$ consists of absolutely continuous functions and we have the Sobolev imbedding estimate

$$\begin{aligned} \sup_{0<x<\pi} |u(x)| &= \sup_{0<x<\pi} \left| \int_0^x u'(\xi)d\xi \right| \\ \le \left(\pi \int_0^\pi |u'(\xi)|^2 d\xi\right)^{\frac{1}{2}} &= \pi|u|_{\frac{1}{2}} \quad \text{for all} \quad u \in X^{\frac{1}{2}}, \end{aligned}$$

it is easy to modify f outside an arbitrary ball $\{u \in X^{\frac{1}{2}} : |u|_{\frac{1}{2}} \le \rho\}$ so that the new function f_ρ satisfies inequality (2.8) with some constant $L = L(\rho)$

and $\alpha = \frac{1}{2}$. For example, let $\mu_\rho : [0, \infty) \to [0, 1]$ be a C^∞ function with the properties that $\mu_\rho(r) = 1$ whenever $r \le \rho$ and $\mu_\rho(r) = 0$ whenever $r \ge \rho + 1$ and take $f_\rho(u) = \mu_\rho(|u|_{\frac{1}{2}})f(u)$, $u \in X^{\frac{1}{2}}$.

3 Some operational calculus and its application to Runge–Kutta schemes

Consider a general m-stage Runge–Kutta method with parameters $\mathcal{A} = \{a_{ij}\}_{i,j=1}^m$, $b = \{b_j\}_{j=1}^m$ and $\{c_i\}_{i=1}^m$. Applied to equation (2.1), the Runge–Kutta method takes the form

$$U_{n+1} = U_n + h \sum_{j=1}^m b_j(AU_{nj} + f(U_{nj})), \quad n = 0, 1, \ldots \tag{3.1}$$

where $U_0 = u_0 \in X^\alpha$ and the internal stage values $\{U_{ni}\}_{i=1}^m$ are defined by the system of equations

$$U_{ni} = U_n + h \sum_{j=1}^m a_{ij}(AU_{nj} + f(U_{nj})), \quad i = 1, 2, \ldots, m, \tag{3.2}$$

$n = 0, 1, \ldots$. The underlying idea is of course to approximate $\Phi(nh, u_0)$ by U_n and $\Phi(nh + c_i h, u_0)$, $i = 1, 2, \ldots, m$ by U_{ni} with the accuracy depending on the choice of parameters. We assume that $\sum_{j=1}^m b_j = 1$.

In what follows we investigate if, for stepsize h sufficiently small and $U_n \in X^\alpha$, a transformed version of equation (3.2) can be solved for the vector

$$(U_{n1}, \ldots, U_{nm})^T \in \mathcal{X}^\alpha := (X^\alpha)^m = X^\alpha \times \ldots \times X^\alpha (m \text{ times}).$$

Rewrite (3.2) as

$$U_{ni} - h \sum_{j=1}^m a_{ij} AU_{nj} = U_n + h \sum_{j=1}^m a_{ij} f(U_{nj}), \quad i = 1, 2, \ldots, m \tag{3.3}$$

and, with $h > 0$ as a parameter, consider the matrix function

$$\mathcal{M}_h(z) = I_m - h\mathcal{A}z, \quad z \in \mathbb{C}.$$

Assume that

(A1) $\mathcal{A}$ is invertible

(A2) the eigenvalues of $\mathcal{A}$ satisfy $|\arg(z)| < \theta$.

By Cramer's rule, there exist rational functions $\mathcal{N}_{h,ik}$ with poles at $\{z \in \mathbb{C} : \det(\mathcal{M}_h(z)) = 0\}$ such that

$$\sum_{k=1}^m \mathcal{N}_{h,ik}(z) \cdot \mathcal{M}_{h,kj}(z) = \sum_{k=1}^m \mathcal{M}_{h,ik}(z) \cdot \mathcal{N}_{h,kj}(z) = \delta_{ij}, \quad i, j = 1, \ldots, m \tag{3.4}$$

whenever $\det(\mathcal{M}_h(z)) \neq 0$. In virtue of (A1), $\mathcal{N}_{h,ij}$ is a proper rational function in the sense that the degree of its denominator is greater than the degree of its nominator. Since $\det(\mathcal{M}_h(z)) = 0$ is equivalent to $(zh)^{-1} \in \sigma(\mathcal{A})$, the assumption (A2) implies that, for h sufficiently small, say $h \in (0, h_0]$, no pole of $\mathcal{N}_{h,ij}$ is contained in $cl(\Sigma_{a,\theta})$. Actually, one can take $h_0 = \infty$ if $a \leq 0$ and, $h_0 = \Delta/a$ with some positive constant Δ if $a > 0$.

In particular, $\mathcal{N}_{h,ij}$ $(i, j = 1, 2, \ldots, m)$ belongs to the function class

> $\mathcal{C}_{a,\theta} = \{\eta : \mathbb{C} \hookrightarrow \mathbb{C} \mid \eta$ is defined and is analytic on an open neighborhood of $\Gamma \cup \mathcal{S}_{a,\theta}$ in $\mathbb{C}$ and there exists a $\kappa > 0$ such that $\limsup\{|z|^\kappa \cdot |\eta(z)| : z \in \Sigma_{a,\theta}, |z| \to \infty\} < \infty\}$.

In view of inequality (2.3), we may define

$$\mathcal{N}_{h,ij}(A) = \frac{1}{2\pi i} \int_\Gamma \mathcal{N}_{h,ij}(z) \cdot (zI - A)^{-1} dz, \quad i, j = 1, \ldots, m. \tag{3.5}$$

Lemma 3.1 *Formula* (3.5) *defines a collection of bounded linear operators on* X *satisfying the identities* $(i, j = 1, \ldots, m)$

$$\sum_{k=1}^{m} \mathcal{N}_{h,ik}(A) \cdot \mathcal{M}_{h,kj}(A) u = \delta_{ij} u \quad \textit{whenever} \quad u \in \mathcal{D}(A) \tag{3.6}$$

and

$$\sum_{k=1}^{m} \mathcal{M}_{h,ik}(A) \cdot \mathcal{N}_{h,kj}(A) u = \delta_{ij} u \quad \textit{whenever} \quad u \in X. \tag{3.7}$$

Proof: This is an application of various results from A. E. Taylor's operational calculus for densely defined closed linear operators [24]. He considered the function class

> $\mathcal{T} = \{\tau : \mathbb{C} \hookrightarrow \mathbb{C} \mid \tau$ is analytic at ∞ and on some neighborhood U^τ of $\sigma(A) \cup \{\infty\}$ in $\mathbb{C}\}$

and defined a linear multiplicative mapping of $\mathcal{T}$ into $L(X, X)$, the Banach space of bounded linear operators on X. This map $\tau \to \tau(A)$ is defined as follows. Let D^τ be a closed neighborhood of $\sigma(A) \cup \{\infty\}$ in U^τ with the property that its boundary $\gamma^\tau = \partial D^\tau$ consists of a finite number of positively oriented (pairwise nonintersecting) smooth Jordan curves, then set

$$\tau(A) = \tau(\infty) I + \frac{1}{2\pi i} \int_{\gamma^\tau} \tau(z) \cdot (zI - A)^{-1} dz, \quad \tau \in \mathcal{T}. \tag{3.8}$$

In case of $\tau \in \mathcal{C}_{a,\theta} \cap \mathcal{T}$ (because the integral does not depend on the particular choice of D^τ), we may take $\gamma^\tau = \Gamma_R \cup c_0^R$ where $R > 0$ is sufficiently large and

$$\Gamma_R = \{z \in \Gamma : |z| \leq R\} \quad \text{and} \quad c_0^R = \{Re^{i\vartheta} \in \mathbb{C} : |\vartheta| \leq \theta\}.$$

Inequality (2.3) and the growth order condition on $\tau \in \mathcal{C}_{a,\theta} \cap \mathcal{T}$ imply $\tau(\infty) = 0$ and that the integral over c_0^R tends to zero as $R \to \infty$. Formula (3.8) then simplifies to

$$\tau(A) = \frac{1}{2\pi i} \int_\Gamma \tau(z) \cdot (zI - A)^{-1} dz, \quad \tau \in \mathcal{C}_{a,\theta} \cap \mathcal{T}. \tag{3.9}$$

Since $\mathcal{N}_{h,ij} \in \mathcal{C}_{a,\theta} \cap \mathcal{T}$ $(i,j = 1, \ldots, m)$, we may apply A. E. Taylor's operational calculus to (3.4): our lemma follows from Lemma V.8.6 of [24] or from Theorem VII.9.8 of [6]. ■

In what follows we shall frequently work with functions of the form $(a - z)^\alpha \xi(z)$ where $\alpha \in [0, 1)$ and ξ is a proper rational function with no poles on $\Gamma \cup \mathcal{S}_{a,\theta}$. Such functions belong to $\mathcal{C}_{a,\theta} \setminus \mathcal{T}$ (if $\alpha \neq 0$). Note also that (as a function of z) $e^{zt} \in \mathcal{C}_{a,\theta} \cap \mathcal{T}$ for all $t > 0$. Hence we find it more comfortable to work with the function class $\mathcal{C}_{a,\theta}$ and to define

$$\eta(A) = \frac{1}{2\pi i} \int_\Gamma \eta(z) \cdot (zI - A)^{-1} dz, \quad \eta \in \mathcal{C}_{a,\theta}. \tag{3.10}$$

Lemma 3.2 *The mapping $\mathcal{C}_{a,\theta} \to L(X, X)$, $\eta \mapsto \eta(A)$ is linear and multiplicative.*

Proof: Since $\eta(A) \in L(X, X)$ and linearity are trivial and $\mathcal{C}_{a,\theta}$ is closed under multiplication, we only need to show that $\eta_1(A)\eta_2(A) = \eta_1\eta_2(A)$ for all $\eta_1, \eta_2 \in \mathcal{C}_{a,\theta}$. We mimic the proof of the semigroup property $e^{At}e^{As} = e^{A(t+s)}$, $t, s > 0$ from p. 21 of [11]. Defining $\eta_2(A)$, the integration contour can be taken as an unbounded Jordan arc Δ being a slightly deformed copy of Γ shifted into $\mathcal{S}_{a,\theta} \setminus \sigma(A)$. Using the resolvent identity and Fubini's theorem, we obtain that

$$\begin{aligned}
&\eta_1(A)\eta_2(A) \\
&= \frac{1}{2\pi i} \int_\Gamma \eta_1(z)(zI - A)^{-1} dz \cdot \frac{1}{2\pi i} \int_\Delta \eta_2(w)(wI - A)^{-1} dw \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_\Gamma \int_\Delta \eta_1(z)\eta_2(w) \frac{1}{z - w} \left\{(wI - A)^{-1} - (zI - A)^{-1}\right\} dw\, dz \\
&= \frac{1}{2\pi i} \int_\Delta \left(\frac{1}{2\pi i} \int_\Gamma \frac{\eta_1(z)}{z - w} dz\right) \eta_2(w)(wI - A)^{-1} dw \\
&\quad + \frac{1}{2\pi i} \int_\Gamma \left(\frac{1}{2\pi i} \int_\Delta \frac{\eta_2(w)}{w - z} dw\right) \eta_1(z)(zI - A)^{-1} dz.
\end{aligned}$$

Next we apply the Cauchy formula to each inner integral. In view of the growth condition imposed on our function class $\mathcal{C}_{a,\theta}$, standard manipulations lead to closed Jordan curves as integral contours. We conclude that the first inner integral equals $\eta_1(w)$ whereas the second inner integral is equal to zero. Hence

$$\eta_1(A)\eta_2(A) = \frac{1}{2\pi i} \int_\Delta \eta_1(w)\eta_2(w)(wI - A)^{-1} dw = \eta_1\eta_2(A).$$

■

Remark 3.3 It is important to note that (3.10) implies

$$\eta(A) = (z_0 I - A)^{-k} \quad \text{whenever} \quad \eta(z) = (z_0 - z)^{-k} \quad \text{for} \quad z \in \Gamma \cup \mathcal{S}_{a,\theta}, z_0 \notin \Gamma \cup \mathcal{S}_{a,\theta}$$

and $k = 1, 2, \ldots$. In fact, the integration contour can be transformed to $\Gamma_R \cup c_0^R$ with R sufficiently large and then to a small circle (with negative orientation) around z_0 and then the classical formula

$$\frac{(k-1)!}{2\pi i} \oint \frac{(zI - A)^{-1}}{(z - z_0)^k} dz = \frac{d^{k-1}}{dz^{k-1}} (zI - A)^{-1} \Big|_{z=z_0} = (-1)^{k-1} (k-1)! (z_0 I - A)^{-k}$$

applies. This observation is not only a consistency–type property but paves the way for extending the spectral calculus to polynomials of A, leads easily to a direct proof of Lemma 3.1 above and serves also a basis for the proof of Lemma V.8.6 of [24] and of Theorem VII.9.8 of [6] our Lemma 3.1 is based on. Since $\mathcal{M}_h(A) = I - h\mathcal{A}A$ and neither $\mathcal{T}$ nor $\mathcal{C}_{a,\theta}$ contain polynomials of degree ≥ 1, it is absolutely essential that the operational calculus extends to polynomials. For the $\mathcal{T}$–based calculus, this is done in a series of lemmas, theorems and exercises in Section V.8 of [24] and in Section VII.9 of [6]. (See also the more detailed presentation in Chapter X of [4].) It is not a hard but rather lengthy task to check that the same results hold true for the $\mathcal{C}$–based operational calculus, too. The results are listed in Appendix B. We refer also to the Appendix A where we point out that the operational calculus in $\mathcal{C}_{a,\theta}$ based on (3.10) is equivalent to extending the standard Dunford–Gelfand calculus for bounded operators to integrals with certain weak singularities.

Our next lemma is a consequence of the $\mathcal{C}$–based operational calculus and its extension to polynomials.

Lemma 3.4 *Let η be a function of the form $\eta(z) = (a-z)^\alpha \xi(z)$ where $\alpha \in [0, 1)$ and ξ is a proper rational function with no poles on $\Gamma \cup \mathcal{S}_{a,\theta}$. Then $\eta, \xi \in \mathcal{C}_{a,\theta}$ and $\eta(A) = (aI - A)^\alpha \xi(A)$.*

Remark 3.5 Since $\eta(z) = (a - z)(a - z)^{\alpha-1}\xi(z)$, this Lemma can be derived from Proposition B1 in Appendix B. But we prefer to give a direct proof which is shorter than the presentation of the extended operational calculus. We shall make use of Theorem 12.1 of Komatsu [13] stating that $range(e^{At}) \subset X^\alpha$ and

$$(aI - A)^\alpha e^{At} = \frac{1}{2\pi i} \int_\Gamma (a - z)^\alpha e^{zt} (zI - A)^{-1} dz, \quad t > 0 \text{ and } \alpha \in (0, 1). \quad (3.11)$$

Notice that (3.11) also holds for $\alpha = 0$ due to (2.4).

Proof: Decomposing ξ into partial fractions we obtain $range(\xi(A)) \subset X^1$ by the previous remarks. Hence $(\alpha I - A)^\alpha \xi(A)$ is well defined and it is sufficient to prove that

$$(aI - A)^\alpha (wI - A)^{-k} = \frac{1}{2\pi i} \int_\Gamma (a - z)^\alpha \frac{1}{(z - w)^k} (zI - A)^{-1} dz \qquad (3.12)$$

for each $w \notin \Gamma \cup \mathcal{S}_{a,\theta}$ and $k = 1, 2, \ldots$. Since both sides are analytic in w and analytic continuations from $\{w \in \mathbb{C} : \mathrm{Re}\, w > a\}$ to $\mathbb{C} \setminus (\Gamma \cup \mathcal{S}_{a,\theta})$ are unique, we may assume that $\mathrm{Re}\, w > a$. The starting point for the computation to follow is the well–known Laplace transform formula (see e.g. [22], p. 8)

$$(wI - A)^{-1} = \int_0^\infty e^{-wt} e^{At} dt, \quad \mathrm{Re}\, w > a.$$

Differentiating both sides $(k-1)$ times with respect to w and multiplying by $(aI - A)^\alpha$ yields

$$(aI - A)^\alpha (wI - A)^{-k} = \frac{1}{(k-1)!} \int_0^\infty t^{k-1} e^{-wt} (aI - A)^\alpha e^{At} dt.$$

This is possible because $(aI - A)^\alpha$ is closed as the inverse of a bounded linear operator. The remaining steps of the computation are justified by inequalities (2.6) and (2.7). Combining (3.11) with Fubini's theorem, (3.12) follows immediately. In fact,

$$\begin{aligned}
&\frac{1}{(k-1)!} \int_0^\infty t^{k-1} e^{-wt} \left(\frac{1}{2\pi i} \int_\Gamma (a - z)^\alpha e^{zt} (zI - A)^{-1} dz \right) dt \\
&= \frac{1}{2\pi i} \int_\Gamma (a - z)^\alpha \left(\frac{1}{(k-1)!} \int_0^\infty t^{k-1} e^{(z-w)t} dt \right) (zI - A)^{-1} dz \\
&= \frac{1}{2\pi i} \int_\Gamma (a - z)^\alpha \frac{1}{(w - z)^k} (zI - A)^{-1} dz.
\end{aligned}$$

■

Returning to (3.3), consider the related system of equations

$$U_{ni} = \sum_{j=1}^m \mathcal{N}_{h,ij}(A)(U_n + h \sum_{k=1}^m a_{jk} f(U_{nk})), \quad i = 1, \ldots, m. \tag{3.13}$$

Recall that $X \supset X^\alpha \supset X^1 = \mathcal{D}(A) = \mathcal{D}(\mathcal{M}_h(A))$, $U_n \in X^\alpha$ and $f : X^\alpha \to X$. As a by–product of Lemma 3.1, equation (3.3) is equivalent to (3.13) and any solution $(U_{n1}, \ldots, U_{nm})^T \in \mathcal{X}^\alpha = (X^\alpha)^m$ of (3.13) is automatically in $\mathcal{X}^1 = (X^1)^m$. Equipped with the norm $||\mathcal{U}||_\alpha = \max\{|U^i|_\alpha : i = 1, \ldots, m\}$, $\mathcal{X}^\alpha$ is a Banach space.

Lemma 3.6 *For any fixed $V \in X^\alpha$ and for sufficiently small $h \in (0, h_0]$, the operator $\mathcal{G}_V : \mathcal{X}^\alpha \to \mathcal{X}^\alpha$ defined by*

$$(\mathcal{G}_V(\mathcal{U}))^i = \sum_{j=1}^m \mathcal{N}_{h,ij}(A)(V + h \sum_{k=1}^m a_{jk} f(U^k)), \quad i = 1, \ldots, m$$

for $\mathcal{U} = (U^1, \ldots, U^m)^T \in \mathcal{X}^\alpha$, is a contraction on $\mathcal{X}^\alpha$. In particular, the system (3.13) has a unique solution in $\mathcal{X}^\alpha$. Moreover, we have the estimate

$$||\mathcal{G}_V(\cdot) - \mathcal{G}_{\tilde{V}}(\cdot)||_\alpha \leq \mathit{const} \cdot |V - \tilde{V}|_\alpha \quad \textit{for any} \quad V, \tilde{V} \in X^\alpha. \tag{3.14}$$

Proof: By (2.8) and the definition of the various norms, it is obvious that

$$\begin{aligned}
&||\mathcal{G}_V(\mathcal{U}) - \mathcal{G}_V(\tilde{\mathcal{U}})||_\alpha \\
&\leq \quad hL \cdot const(\mathcal{A}) \cdot \max_{1\leq i\leq m} \left\{ \sum_{j=1}^m |(aI-A)^\alpha \cdot \mathcal{N}_{h,ij}(A)| \right\} \cdot ||\mathcal{U} - \tilde{\mathcal{U}}||_\alpha
\end{aligned}$$

for all $\mathcal{U},\tilde{\mathcal{U}} \in \mathcal{X}^\alpha$. In order to estimate $|(aI-A)^\alpha \cdot \mathcal{N}_{h,ij}(A)|$, note that $\mathcal{M}_h(z) = I_m - h\mathcal{A}z = \mathcal{M}_1(hz)$ and correspondingly, $\mathcal{N}_h(z) = \mathcal{N}_1(hz)$ for all $(h,z) \in \mathbb{R}^+ \times \mathbb{C}$ with $(hz)^{-1} \notin \sigma(\mathcal{A})$. For h sufficiently small, it follows from (A1), (A2) that $\mathcal{N}_{h,ij}$ satisfies the conditions imposed on ξ in Lemma 3.4. Moreover, no pole of $\mathcal{N}_{h,ij}$ is contained in $cl(\Sigma_{a,\theta})$, $i,j = 1,\ldots,m$. Hence

$$(aI-A)^\alpha \cdot \mathcal{N}_{h,ij}(A) = \frac{1}{2\pi i}\int_\Gamma (a-z)^\alpha \cdot \mathcal{N}_{1,ij}(hz)(zI-A)^{-1}dz$$

and, though $(a-z)^\alpha$ is not analytic at $z=a$ for $\alpha \in (0,1)$, we may replace Γ by the curve $\gamma_a = \partial(\Sigma_{a,\theta})$, oriented upwards. Since

$$|\mathcal{N}_{1,ij}(hz)| \leq const \cdot \min\{1,(h\rho)^{-1}\} \quad \text{for} \quad a \pm \rho e^{i\theta} = z \in \gamma_a, \rho > 0, \tag{3.15}$$

it follows from (2.2) that for $i,j = 1,\ldots,m$ and $0 < h \leq h_0 < 1$

$$\begin{aligned}
&|(aI-A)^\alpha \cdot \mathcal{N}_{h,ij}(A)| \leq \text{const}\int_0^\infty \rho^\alpha \cdot \min\{1,(h\rho)^{-1}\}\frac{1}{1+\rho}d\rho \\
&\leq \quad \text{const} \cdot \int_0^{1/h} \frac{\rho^\alpha}{1+\rho}d\rho + \text{const} \cdot \int_{1/h}^\infty h^{-1}\rho^{\alpha-2}d\rho \\
&\leq \quad \text{const}(\alpha)\cdot h^{-\alpha} \quad \text{if} \quad \alpha \in (0,1) \quad \text{and} \quad \text{const}\cdot \log h^{-1} \quad \text{if} \quad \alpha = 0.
\end{aligned}$$

In particular, by passing to a smaller h_0 if necessary, we have for $\alpha \in (0,1)$ and $h \in (0,h_0]$ that

$$||\mathcal{G}_V(\mathcal{U}) - \mathcal{G}_V(\tilde{\mathcal{U}})||_\alpha \leq \text{const}\cdot h^{1-\alpha}||\mathcal{U}-\tilde{\mathcal{U}}||_\alpha \leq \frac{1}{2}||\mathcal{U}-\tilde{\mathcal{U}}||_\alpha, \tag{3.16}$$

the desired contraction estimate.

For the proof of (3.14) we use the fact that the operators $\mathcal{N}_{1,ij}(hA) = \mathcal{N}_{ij}(hA)$, $i,j = 1,\ldots,m$ have the same norm considered as operators in X or in X^α (see equation (3.22) below). We obtain

$$\begin{aligned}
&||\mathcal{G}_V(\mathcal{U}) - \mathcal{G}_{\tilde{V}}(\mathcal{U})||_\alpha = \max_{1\leq i\leq m}\Big|\sum_{j=1}^m \mathcal{N}_{1,ij}(hA)\cdot(V-\tilde{V})\Big|_\alpha \\
&\leq \quad \left(\max_{1\leq i\leq m}\sum_{j=1}^m |\mathcal{N}_{h,ij}(A)|\right)\cdot |V-\tilde{V}|_\alpha \quad \text{for any} \quad V,\tilde{V} \in X^\alpha, \mathcal{U} \in \mathcal{X}^\alpha.
\end{aligned}$$

We know already that $|\mathcal{N}_{h,ij}(A)| \leq \text{const} \cdot \log h^{-1}$. By (A1) there are no poles of $\mathcal{N}_{h,ij}$ in the complex disc $\{z \in \mathbb{C} : |z| \leq \delta/h\}$ and hence we can shift the integration from γ_a to the keyhole contour

$$\gamma_{KH} = \{a + \delta e^{i\vartheta}/h : \vartheta \in [-\theta, \theta]\} \cup \{a \pm \rho e^{i\theta} : \rho \geq \delta/h\}$$

Then we obtain using (2.2) that

$$\begin{aligned} |\mathcal{N}_{h,ij}(A)| &\leq \Big|\frac{1}{2\pi i}\int_{\gamma_k} \mathcal{N}_{1,ij}(hz)(zI - A)^{-1}dz\Big| \\ &\leq \text{const} \cdot \int_{-\theta}^{\theta} \frac{M}{1+\delta/h} \cdot \frac{\delta}{h} d\vartheta + \text{const} \cdot \int_{\delta/h}^{\infty} \frac{1}{h\rho} \cdot \frac{M}{1+\rho} d\rho \\ &\leq \text{const} \quad \text{for all} \quad i,j = 1,\ldots,m, h \in (0, h_0]. \end{aligned}$$

This proves (3.14) and the very same integration trick shows that (3.16) is valid for $\alpha = 0$, too. ■

For the following theorem it is useful to introduce for $i,j = 1,\ldots,m$ the rational functions

$$r, q_j, s_i, s_{ij} : \mathbb{C} \hookrightarrow \mathbb{C}, \quad i,j = 1\ldots,m$$

in matrix notation by

$$r(z) = 1 + zb^T\mathcal{N}(z)\mathbf{1}, \tag{3.17}$$

$$q^T(z) = \{q_j(z)\} = b^T\mathcal{N}(z), \quad s(z) = \{s_i(z)\} = \mathcal{N}(z)\mathbf{1}, \quad \{s_{ij}(z)\} = \mathcal{N}(z)\mathcal{A}, \tag{3.18}$$

where $\mathcal{N}(z) = (I_m - \mathcal{A}z)^{-1}$ and $\mathbf{1} = (1,1\ldots,1)^T \in \mathbb{R}^m$. As is well known the stability function $r(z)$ plays a crucial role in the whole theory of discretization.

Theorem 3.7 *Starting from an arbitrary $U_0 = u_0 \in X^\alpha$, the Runge–Kutta method* (3.1)–(3.2) *gives rise to a sequence $\{U_n\}_{n=0}^\infty \subset X^\alpha$ via the recursion*

$$\left.\begin{aligned} U_{n+1} &= r(hA)U_n + h \cdot F_h(U_n), \quad n = 0,1,\ldots \\ F_h(U_n) &= \sum_{j=1}^m q_j(hA)f(U_{nj}), \quad h \in (0,h_0] \end{aligned}\right\} \tag{3.19}$$

where $(U_{n1},\ldots,U_{nm})^T \in \mathcal{X}^\alpha$ is uniquely determined by the system

$$U_{ni} = s_i(hA)U_n + h\sum_{j=1}^m s_{ij}(hA)f(U_{nj}), \quad i = 1,2,\ldots,m. \tag{3.20}$$

The numerical solution can be represented as

$$U_n = r^n(hA)U_0 + h\sum_{k=0}^{n-1} r^{n-k-1}(hA)\sum_{j=1}^m q_j(hA)f(U_{kj}), \quad n = 1,2,\ldots. \tag{3.21}$$

where the internal stage values $\{\{U_{kj}\}_{j=1}^m\}_{k=0}^\infty$ are determined by (3.20).

In these expressions the operator functions $r(hA)$, $q_j(hA)$, $s_i(hA)$, $s_{ij}(hA)$ all belong to $L(X,X)$ and their restrictions to X^α belong to $L(X^\alpha, X^\alpha)$, $i,j = 1,2,\ldots,m$.

Proof: This is simply an amalgamation of the previous results. With the notations used in Lemmas 3.1 and 3.6, $\mathcal{N}(hz) = \mathcal{N}_h(z)$ whenever $(zh)^{-1} \notin \sigma(\mathcal{A})$. In particular, for $h \in (0, h_0]$, the rational functions r_h, $q_{h,j}$, $s_{h,i}$, $s_{h,ij} : \mathbb{C} \hookrightarrow \mathbb{C}$ defined by $r_h(z) = r(hz)$ etc. have no poles on $cl(\Sigma_{a,\theta})$ and $r_h - r_h(\infty)$, $q_{h,j}$, $s_{h,i}$, $s_{h,ij}$ are proper. Thus Lemma 3.2 applies.

It is left to prove that $r(hA)_{|X^\alpha}$ etc. belong to $L(X^\alpha, X^\alpha)$. Using Lemma 3.2 again,

$$(r(hA) - r(\infty)I)(aI - A)^{-\alpha}u = (aI - A)^{-\alpha}(r(hA) - r(\infty)I)u \in X^\alpha$$

for all $u \in X$. Since $r(\infty)I(aI - A)^{-\alpha}u = (aI - A)^{-\alpha}r(\infty)Iu \in X^\alpha$, it follows that $r(hA)(aI - A)^{-\alpha}u \in X^\alpha$ for all $u \in X$. But $v \in X^\alpha$ if and only if $v = (aI - A)^{-\alpha}u$ for some $u \in X$. Hence $r(hA)v \in X^\alpha$ and $(aI - A)^\alpha r(hA)v = r(hA)(aI - A)^\alpha v$ for all $v \in X^\alpha$. By the definition of the norm $|\cdot|_\alpha$ and the density of X^α in X, we conclude that

$$r(hA)_{|X^\alpha} \in L(X^\alpha, X^\alpha) \quad \text{and} \quad |r(hA)_{|X^\alpha}|_\alpha = |r(hA)|. \tag{3.22}$$

■

In the following definition we use the curve

$$\gamma_0 = \{z \in \mathbb{C} : |\arg(z)| = \theta \text{ or } z = 0\}$$

and notice that $r(0) = 1$. Moreover,we introduce the acute angle $\theta^c = \pi - \theta$ between γ_0 and the negative half-axis. This angle is in common use with Runge–Kutta methods while it is customary to work with the obtuse angle θ for sectorial operators.

Definition 3.8 We say that the Runge–Kutta method (3.1)–(3.2) is **strictly $A(\theta^c)$–stable** if it satisfies (A1), (A2) as well as

(A3) $|r(\infty)| < 1$

(A4) $|r(z)| < 1$ whenever $z \in \gamma_0 \setminus \{0\}$.

This is a stricter version of what is called strong $A(\theta^c)$–stability by Lubich and Ostermann in [16],[19]. They use the term strongly $A(\theta^c)$–stable if (A1), (A2), (A3) and

(A4)' $|r(z)| \leq 1$ whenever $z \in \gamma_0$.

are satisfied. Then they usually assume strong $A(\theta^c + \varepsilon)$-stability for some $\varepsilon > 0$ which implies strict $A(\theta^c)$–stability by the maximum principle. It seems to us that the above notion is more convenient to work with when handling contour integrals and deriving basic estimates (see Section 4) and this pays off in the case of variable stepsizes (see Section 6).

Concluding this section, we recall some elementary consequences of assumptions (A1)-(A4). As a direct corollary of the standing assumption $\sum_{j=1}^m b_j = 1$, met by all consistent Runge–Kutta methods, we have that $r'(0) = 1$ and further, for $|ha|$ small enough,

$$e^{ha-Q|ha|^2} \leq r(ha) \leq e^{ha+Q|ha|^2} \tag{3.23}$$

with some positive constant Q.

Lemma 3.9 *Assume that the Runge-Kutta method is strictly $A(\theta^c)$–stable. Then there are positive constants h_0, p, q and δ such that for $h \in (0, h_0]$*

$$|r(hz)| \leq r(ha)e^{2\rho h} \quad \textit{whenever} \quad z = a + \rho e^{i\vartheta}, 0 \leq \rho h \leq \delta, \vartheta \in [-\pi, \pi] \tag{3.24}$$

$$|r(hz)| \leq r(ha)e^{-p\rho h} \quad \textit{whenever} \quad z = a \pm \rho e^{i\theta}, 0 \leq \rho h \leq \delta \tag{3.25}$$

$$|r(hz)| \leq r(ha)(1-q) \quad \textit{whenever} \quad z = a \pm \rho e^{i\theta}, \delta \leq \rho h \tag{3.26}$$

$$|r(\infty)| \leq r(ha)(1-q) \tag{3.27}$$

Proof: In proving (3.24) and (3.25), assume that $|ha| \leq \Delta$ and $0 \leq \rho h \leq \delta$ where Δ and δ are positive constants we specify later. Here we require only that the poles of r are exterior to the complex disc $\{z \in \mathbb{C} : |z| \leq \Delta + \delta\}$. Our starting point is the integral representation

$$r(hz) = r(ha) + r'(0)(hz - ha) + \int_0^1 [r'(ha + \tau(hz - ha)) - r'(0)](hz - ha)d\tau.$$

With $z = a + \rho e^{i\vartheta}$, property $r'(0) = 1$ implies that

$$|r(hz)| \leq \left(r^2(ha) + 2r(ha)\rho h \cdot \cos\vartheta + \rho^2 h^2\right)^{\frac{1}{2}} + \varepsilon(\Delta + \delta)\rho h$$

with some positive $\varepsilon(\Delta + \delta)$ satisfying $\varepsilon(\Delta + \delta) \to 0$ as $\Delta + \delta \to 0$. There is no loss of generality in assuming that $7/8 \leq r(ha) \leq 9/8$ and $\delta \leq 1/2$. It is readily checked that

$$\left(r^2(ha) + 2r(ha)\rho h \cdot \cos\vartheta + \rho^2 h^2\right)^{\frac{1}{2}} \leq r(ha) + (\cos\vartheta + \delta)\rho h$$

for all ϑ. Since $\cos\theta < 0$ we can choose $0 < p < \frac{4}{5}$ and $\delta > 0$ such that

$$\cos\theta + \delta \leq -\frac{5}{4}p \quad \text{and} \quad \sup\{\varepsilon(\tau) : 0 < \tau \leq 2\delta\} \leq p/8$$

By passing to a new Δ if necessary, we may assume that $\Delta \leq \delta$. Inequalities (3.24) and (3.25) then follow from

$$\begin{aligned} r(hz) &\leq r(ha) + (1 + 1/2)\rho h + \varepsilon(\Delta + \delta)\rho h \leq r(ha) + 7\rho h/4 \\ &\leq r(ha)(1 + 2\rho h) \leq r(ha)e^{2\rho h} \end{aligned}$$

whenever $|\vartheta| \leq \pi$ and

$$r(hz) \leq r(ha) - 5p\rho h/4 + p\rho h/8 = r(ha) - 9p\rho h/8 \leq r(ha)(1 - p\rho h) \leq r(ha)e^{-p\rho h}$$

whenever $\vartheta = \pm\theta$. Inequalities (3.26) and (3.27) are obtained from (A3) and (A4) via a simple compactness argument. This step might require a further modification of Δ. ■

4 Estimates for strictly $A(\theta^c)$-stable Runge–Kutta methods

In order to prove approximation results for the Runge–Kutta method (3.1)–(3.2), some preparatory inequalities are needed. The most important ones are discrete analogs of the inequalities (2.6) and(2.7), respectively.

Lemma 4.1 *Assume that the Runge–Kutta method is strictly $A(\theta^c)$–stable. Then there exist positive constants Ω and h_0 such that*

$$|r^k(hA)| \leq \Omega \cdot r^k(ha), \quad h \in (0, h_0], k \in \mathbb{N}. \tag{4.1}$$

Lemma 4.2 *Let the Runge–Kutta method be strictly $A(\theta^c)$–stable and let $\alpha \in [0,1)$ be arbitrary. Then there exist positive constants Ω and h_0 such that for $j = 1, \dots, m$*

$$|(aI - A)^\alpha \cdot r^k(hA) \cdot q_j(hA)| \leq \Omega(k+1)^{-\alpha} h^{-\alpha} r^k(ha), \quad h \in (0, h_0], k \in \mathbb{N}. \tag{4.2}$$

Remark 4.3 Lemma 4.1 is a restatement of Lemma 3.1(ii) in Lubich and Ostermann [20]. Actually, they state (4.1) with $\Omega r^k(ha)$ replaced by $\Omega(\varepsilon)e^{(a+\varepsilon)kh}$, $\varepsilon > 0$, but in view of (3.23), this is equivalent to the formulation above. Lemma 4.2 is a slight modification of Lemma 3.1(i) of [20]. In order to handle internal stage values directly, Lubich and Ostermann estimate $|(aI - A)^\alpha r^k(hA)s(hA)q_j(hA)|$ instead. The constants Ω and h_0 depend only on α, the Runge-Kutta method itself and the three constants in (2.2).

Assuming strong $A(\theta^c)$-stability, case $a = 0$ of (4.1) is proved in Lubich and Nevanlinna [15]. However, in case $a \neq 0$ the inequalities (4.1) and (4.2) seemingly do not remain valid if strict $A(\theta^c)$-stability is weakened to strong $A(\theta^c)$-stability. The case $a = 0$ in equation (4.2) is more delicate. If (A4) is weakened to (A4)', then the equation $|r(\pm\rho e^{i\theta})| = 1$, $\rho \neq 0$ has finitely many solutions, say $\rho_1, \rho_2, \dots, \rho_N$ and, for these we have

$$|r(\pm\rho e^{i\theta})| = 1 - c_k(\rho - \rho_k)^{2(1+\mu_k)} + \dots \quad \text{for } \rho - \rho_k \text{ small and } k = 1, 2, \dots, N$$

with suitable $c_k > 0$ and $\mu_k \in \mathbb{N}$. Then a subtle integration estimate near ρ_k (similar to the one used in proving Theorem 3.3 of [15]) shows that inequality (4.2) still holds for $\alpha \in [0, \alpha_0]$ where $\alpha_0 = \min\{(2(1 + \mu_k))^{-1} : k = 1, 2, \dots, N\}$.

The estimate (4.1) is the genuine counterpart of (2.6) and has a long history in numerical analysis , see e.g. [3]. For an arbitrary $\alpha \geq 0$ it is not hard to show that strict $A(\theta^c)$-stability (or strong $A(\theta^c)$-stability if $a = 0$ and $\alpha \in [0, \alpha_0]$) plus the extra assumption $r(\infty) = 0$ imply the estimate

$$|(aI - A)^\alpha \cdot r^k(hA)| \leq \Omega k^{-\alpha} h^{-\alpha} r^k(ha), \quad h \in (0, h_0], k \in \mathbb{N}, k > \alpha. \tag{4.3}$$

This latter inequality has certainly more resemblance to (2.7) than (4.2). But, as the solution formula (3.21) suggests, the inequality (4.3) fits better to nonlinear problems.

In what follows we will prove a generalization of Lemma 4.1 for variable stepsizes. As a by-product we obtain that by a suitable choice of norms one may take $\Omega = 1$ in (4.1).

Lemma 4.4 *Assume that the Runge–Kutta method is strictly $A(\theta^c)$–stable. Then there exist positive constants h_0 and Ω such that*

$$|r(h_1 A) r(h_2 A) \ldots r(h_N A)| \leq \Omega \cdot r(h_1 a) r(h_2 a) \ldots r(h_N a) \tag{4.4}$$

for all finite sequences $(h_1, h_2, \ldots, h_N), h_n \in (0, h_0], N \in \mathbb{N}$. Moreover, there is an equivalent norm $||\cdot||$ on X such that

$$|r^k(hA)| \leq r^k(ha), \quad h \in (0, h_0], k \in \mathbb{N}. \tag{4.5}$$

Remark 4.5 Note that both Bakaev [1] and Palencia [21] prove case a=0 of Lemma 4.4 assuming only strong $A(\theta^c)$-stability. For evolution operators that vary in time this latter result is generalized by Gonzalez and Palencia [7]. In case $a < 0$ the inequality (4.5) can be interpreted as a strong form of exponential stability for the linear recursion $U_{n+1} = r(hA)U_n$, the discretized version of the linear equation $\dot{u} = Au$. The variable stepsize estimate (4.4) has a similar interpretation since we have

$$r(h_1 a) r(h_2 a) \ldots r(h_N a) \leq e^{a(1+Qah_0)(h_1+h_2+\ldots+h_N)}$$

as a simple consequence of (3.23).

Proof: First we repeat the renorming argument from p.18 of [22] to show that (4.5) is a consequence of (4.4). For brevity, we set

$$\mathcal{H} = \{\chi = (h_1, \ldots, h_N) \in \mathbb{R}^N : h_n \in (0, h_0], n = 1, 2, \ldots, N; N \in \mathbb{N}\}.$$

Assuming (4.4), it is readily checked that the formula

$$||u|| = \sup \left\{ \frac{|r(h_1 A) r(h_2 A) \ldots r(h_N A) u|}{r(h_1 a) r(h_2 a) \ldots r(h_N a)} \; : \; \chi \in \mathcal{H} \right\}, \quad u \in X \tag{4.6}$$

defines a norm with all the required properties. Notice that in case $N = 0$ the fraction in (4.6) is set to $|u|$ by definition.

Consider now a $\chi \in \mathcal{H}$ and assume without loss of generality that

$$h_1 \leq h_2 \leq \ldots \leq h_N \leq h_0 \tag{4.7}$$

where h_0 and the constants p, q, δ below are taken from Lemma 3.9. Since $r - r(\infty) \in \mathcal{C}_{a,\theta}$, we can use the $\mathcal{C}$-based operational calculus and the integral representation

$$r(h_1 A) r(h_2 A) \cdots r(h_N A) = r^N(\infty) \cdot I$$

$$+ \sum_{k=1}^{N} \frac{1}{2\pi i} \int_\Gamma r(h_1 z) r(h_2 z) \ldots r(h_{k-1} z) \left(r(h_k z) - r(\infty) \right) r^{N-k}(\infty) (zI - A)^{-1} dz.$$

To evaluate the k-th integral, we shift the integration contour to $\Gamma_1^k \cup \Gamma_2^k \cup \Gamma_3^k$, $k = 1, 2, \ldots, N$ where

$$\Gamma_1^k = \{z = a + (h_1 + h_2 + \ldots + h_k)^{-1}\delta e^{i\vartheta} \in \mathbb{C} : \vartheta \in [-\theta, \theta]\},$$
$$\Gamma_2^k = \{z = a \pm \rho e^{i\theta} \in \mathbb{C} : (h_1 + h_2 + \ldots + h_k)^{-1}\delta \leq \rho \leq \delta/h_k\},$$
$$\Gamma_3^k = \{z = a \pm \rho e^{i\theta} \in \mathbb{C} : \rho \geq \delta/h_k\}$$

with orientations inherited from the orientation of Γ. Since $r - r(\infty)$ is a proper rational function, we have that

$$|r(hz) - r(\infty)| \leq \text{const} \cdot \min\{\delta^{-1}, (\rho h)^{-1}\} \quad \text{for} \quad h \in (0, h_0], z = a \pm \rho e^{i\theta} \in \mathbb{C}. \tag{4.8}$$

For brevity, we write

$$\Pi = r(h_1 a) r(h_2 a) \ldots r(h_N a), \quad H_k = h_1 + \ldots + h_k$$

and note that $r^N(\infty) \leq \Pi \cdot (1-q)^N$ by (3.27). In view of (3.24), (3.27) and (2.2), the integral over Γ_1^k is bounded by

$$\int_{-\theta}^{\theta} r(h_1 a) e^{2\delta \frac{h_1}{H_k}} \ldots r(h_{k-1} a) e^{2\delta \frac{h_{k-1}}{H_k}} \cdot \text{const}$$
$$\cdot r(h_{k+1} a)(1-q) \ldots r(h_N a)(1-q) \cdot \frac{M}{1 + \frac{\delta}{H_k}} \cdot \frac{\delta}{H_k} d\vartheta$$
$$\leq \text{const} \cdot \Pi (1-q)^{N-k} \cdot \int_{-\pi}^{\pi} e^{2\delta \frac{H_{k-1}}{H_k}} d\vartheta \leq \text{const} \cdot \Pi (1-q)^{N-k}.$$

In view of (3.25), (4.7),(4.8), (3.27) and (2.2), the integral over Γ_2^k is bounded by

$$\text{const} \int_{\delta/H_k}^{\delta/h_k} r(h_1 a) e^{-h_1 \rho p} \ldots r(h_{k-1} a) e^{-h_{k-1} \rho p} r(h_{k+1} a)(1-q) \ldots r(h_N a)(1-q) \cdot \frac{M}{1+\rho} d\rho$$

$$\begin{aligned}
&\leq \quad \text{const} \cdot \Pi (1-q)^{N-k} \cdot \int_{\delta/H_k}^{\delta/h_k} e^{-H_{k-1}\rho p} \left(e^{-h_k \rho p} \cdot e^{\delta p}\right) \frac{1}{\rho} d\rho \\
&\leq \quad \text{const} \cdot \Pi (1-q)^{N-k} \cdot \int_{\delta/H_k}^{\infty} e^{-H_k \rho p} \cdot \frac{1}{\rho} d\rho \\
&= \quad \text{const} \cdot \Pi (1-q)^{N-k} \cdot \int_{\delta}^{\infty} e^{-\tau p} \cdot \frac{1}{\tau} d\tau.
\end{aligned}$$

Finally, the combination of (3.24) and (3.26), (4.8), (3.27) and (2.2) yield that the integral over Γ_3^k is not greater than

$$\int_{\delta/h_k}^{\infty} r(h_1 a) \ldots r(h_{k-1} a) \cdot \frac{\text{const}}{h_k \rho} \cdot r(h_{k+1} a)(1-q) \ldots r(h_N a)(1-q) \cdot \frac{M}{1+\rho} d\rho$$
$$\leq \quad \text{const} \cdot \Pi (1-q)^{N-k} \cdot \int_{\delta/h_k}^{\infty} \frac{1}{h_k \rho^2} d\rho = \text{const} \cdot \Pi (1-q)^{N-k} \cdot \int_{\delta}^{\infty} \frac{1}{\tau^2} d\tau.$$

Summing all contributions, we see that

$$|r(h_1A)r(h_2A)\cdots r(h_NA)| \le \Pi(1-q)^N + \sum_{k=1}^{N} \text{const}\cdot\Pi(1-q)^{N-k}$$

$$\le \text{const}\cdot\Pi\cdot\sum_{k=0}^{N}(1-q)^{N-k} \le \text{const}\cdot\Pi$$

and this ends the proof of Lemma 4.4. ■

It is a natural question if and how Lemma 4.2 can be generalized for methods with variable stepsize. In Section 5 we will show that under the additional assumption $r(\infty)=0$ we have an estimate

$$|(aI-A)^\alpha r(h_1A)\cdots r(h_NA)q_j(h_{N+1}A)| \le \Omega(\sum_{n=1}^{N+1} h_n)^{-\alpha} r(h_1a)\cdots r(h_Na) \tag{4.9}$$

for all finite sequences $(h_1,\ldots,h_{N+1}) \in \mathbb{R}^{N+1}, h_n \in (0,h_0], n=1,\ldots,N+1; N\in\mathbb{N}$ and for $j=1,\ldots,m$. The conditions of Lemma 4.2 alone do not seem to imply inequality (4.9). In fact, consider the simple special case $a=0, \alpha=1/2, h_1=h_2=\ldots=h_N=h, Nh=1$ and $h_{N+1}\le h/2$. Then $|(-A)^{1/2}r^N(hA)q_j(h_{N+1}A)|$ does not seem to be bounded as $h/h_{N+1}\to\infty$. The difficulty appears when evaluating the integral representation over $\tilde{\Gamma} = \{\rho e^{i\theta}=z\in\mathbb{C} : \delta/h \le \rho \le \delta/h_{N+1}\}$ — this part of the integration contour Γ can not be shifted too much. One can hardly obtain a better bound for the integral over $\tilde{\Gamma}$ than $\text{const}\cdot(1-q)^N h_{N+1}^{-1/2}$.

The extra assumption $r(\infty)=0$ can be replaced by other requirements. For example, (4.9) is valid if $\alpha=0$. This follows immediately from (4.4) and the uniform boundedness of $\{|q_j(hA)|\}_{h\in(0,h_0]}$. An alternative extra assumption for (4.9) is the existence of positive constants c and C for which $c\le h_i/h_k\le C, i,k=1,2,\ldots,N; N\in\mathbb{N}$. The proof of this latter result is just a little harder than the proof of Lemma 4.2.

Next, we outline for completeness the proof of Lemma 4.2.

Proof: The operator $(aI-A)^\alpha r^k(hA)q_j(hA)$ may be represented as

$$\frac{1}{2\pi i}\left(\int_{\Gamma_1}+\int_{\Gamma_2}\right)(a-z)^\alpha r^k(hz)q_j(hz)(zI-A)^{-1}dz$$

where $\Gamma_1=\{a\pm\rho e^{i\theta} : 0\le\rho h\le\delta\}$ and $\Gamma_2=\{a\pm\rho e^{i\theta} : \delta\le\rho h\}$. Combining Lemma 3.9, the analogue of (4.8) for q_j and (2.2), it is easily seen that $|(aI-A)^\alpha r^k(hA)q_j(hA)|$ is bounded by

$$\text{const}\cdot\int_0^{\delta/h}\rho^\alpha r^k(ha)e^{-kp\rho h}\cdot\frac{M}{1+\rho}d\rho$$
$$+\quad\text{const}\cdot\int_{\delta/h}^{\infty}\rho^\alpha r^k(ha)(1-q)^k\cdot\frac{1}{h\rho}\cdot\frac{M}{1+\rho}d\rho$$

$$
\begin{aligned}
&\leq \text{const}\cdot r^k(ha)\left(\int_0^\delta\left(\frac{\tau}{h}\right)^\alpha e^{-(k+1)p\tau}\cdot\frac{1}{\tau}d\tau+\int_\delta^\infty\left(\frac{\tau}{h}\right)^\alpha(1-q)^k\cdot\frac{1}{\tau^2}d\tau\right)\\
&\leq \text{const}\cdot r^k(ha)h^{-\alpha}\left(\left(\frac{1}{k+1}\right)^\alpha\cdot\int_0^\infty\tau^{\alpha-1}e^{-\tau}d\tau+(1-q)^k\cdot\int_\delta^\infty\tau^{\alpha-2}d\tau\right)\\
&\leq \text{const}\cdot r^k(ha)h^{-\alpha}(k+1)^{-\alpha}\quad\text{provided that}\quad\alpha\in(0,1).
\end{aligned}
$$

In view of (4.1), the remaining case $\alpha=0$ is equivalent to subcase $\alpha=0,k=0$. Inequality $|q_j(hA)|\leq\Omega$ follows from integrating over the keyhole contour $\{a+\delta e^{i\vartheta}/h:\vartheta\in[-\theta,\theta]\}\cup\Gamma_2$. ■

In what follows, we discuss Runge–Kutta estimates for the homogenous equation $\dot{u}=Au$. We present a truly fundamental theorem of Le Roux [14] first (see Lemma 4.7 below) and then turn to various generalizations.

Definition 4.6 For $\mathcal{P}=1,2,\ldots$ we say that a Runge–Kutta method is of order $\mathcal{P}$ if its stability function satisfies $r(0)=r'(0)=\ldots=r^{(\mathcal{P})}(0)=1$ but $r^{(\mathcal{P}+1)}(0)\neq 1$.

The fundamental estimate is the following.

Lemma 4.7 *[14]. Let the operator A satisfy (A0) with $a=0$ and assume that the Runge–Kutta method is of order $\mathcal{P}$ and strictly $A(\theta^c)$-stable. Then there exists positive constants Ω and h_0 such that*

$$|r^N(hA)-e^{ANh}|\leq\Omega N^{-\mathcal{P}}\qquad\textit{whenever}\qquad h\in(0,h_0],N=1,2,\ldots.\tag{4.10}$$

For illustration we treat an example.

Example 4.8 Reconsider the model case $(X,A)=(L_2(0,\pi),\Delta_D)$ from Section 1 with $r(z)=(1-z)^{-1}$, the stability function of the backward Euler method. For $u=\sum_{k=1}^\infty c_k\sin kx\in L_2(0,\pi)$ we have

$$|\left(r^N(h\Delta_D)-e^{\Delta_D Nh}\right)u|^2=\frac{1}{2}\sum_{k=1}^\infty\left((1+k^2h)^{-N}-e^{-k^2hN}\right)^2c_k^2$$

for each $N\in\mathbb{N},h\geq 0$. Hence

$$\lim_{h\to 0^+}|r^N(h\Delta_D)-e^{\Delta_D Nh}|=M_N\quad\text{where}\quad M_N=\max_{x\leq 0}\left((1-x)^{-N}-e^{xN}\right)$$

for each $N\in\mathbb{N}$. By an elementary Taylor expansion argument,

$$\lim_{N\to\infty}N\cdot M_N=2/e^2.$$

There are two important consequences. In contrast to ordinary equations, local error terms do not tend (uniformly) to zero with decreasing stepsize. In fact, $|r(h\Delta_D)-e^{\Delta_D h}|\to M_1=0.2036\ldots$ as $h\to 0^+$. The second consequence is that, at least for the $\mathcal{P}=1$ case, Lemma 4.7 is sharp. If r is the stability function of the $\mathcal{P}$-th order RADAU IIA method (a strictly $A(\theta^c)$-stable Runge–Kutta method for all $\theta^c \in (\frac{\pi}{2}, \pi)$ with $r(\infty) = 0$ and odd $\mathcal{P}$, see [8]), then — as before

$$\lim_{h \to 0^+} |r^N(h\Delta_D) - e^{\Delta_D N h}| = \max_{x \le 0} |r^N(x) - e^{xN}|.$$

A direct computation shows that

$$\begin{aligned} N^{\mathcal{P}} \left(r^N(-\frac{1}{N}) - \frac{1}{e} \right) &= N^{\mathcal{P}} \cdot \frac{1}{e} \left\{ e^{N \log(1 + [r(-1/N) - e^{-1/N}]/e^{-1/N})} - 1 \right\} \\ \to \quad (-1)^{\mathcal{P}} \frac{1 - r^{(\mathcal{P}+1)}(0)}{e(\mathcal{P}+1)!} & \quad \text{as} \quad N \to \infty. \end{aligned}$$

In particular, Lemma 4.7 is sharp for any odd integer $\mathcal{P}$.

In the next step we generalize Lemma 4.7 to the case $a \neq 0$. As a motivation we consider the case $a \neq 0$ under the additional assumption

$$\sigma(A) \cap \{z \in \mathbb{C} : \operatorname{Re} z = b\} = \emptyset \quad \text{for some } b \le 0$$

By a standard Dunford-Gelfand projection we may then decompose the space $X = X_{cu} \oplus X_{ss}$ and the operator $A = A_{cu} + A_{ss}$ into a center-unstable and a strongly stable part (cf. [20] for a similar reasoning), where $A_{cu} : X_{cu} \subset D(A) \to X_{cu}$ is bounded with spectrum $Rez > b$ and $A_{ss} : D(A) \cap X_{ss} \to X_{ss}$ is sectorial with constant $a = 0$. Since Runge-Kutta methods inherit this decomposition we may then apply Lemma 4.4 to A_{ss} and use standard $O(h^{\mathcal{P}})$ estimates for the A_{cu}-part.

Lemma 4.9 *Assume that the Runge–Kutta method is of order $\mathcal{P}$ and strictly $A(\theta^c)$-stable. Then there exists positive constants Ω and h_0 such that*

$$|r^N(hA) - e^{ANh}| \le \Omega \mathcal{M}^N (|a| h^{\mathcal{P}} + N^{-\mathcal{P}}) \quad \textit{for} \quad 0 < h \le h_0, N = 1, 2, \dots \tag{4.11}$$

holds with $\mathcal{M} = \max\{r(ha), e^{ah}\}$. The constants Ω and h_0 depend only on $\mathcal{P}$, the three constants in (2.2) as well as on the stability function r i.e. on the Runge–Kutta method itself.

Proof: We use Lemma 3.9 in estimating each term in the integral representation

$$\begin{aligned} r^N(hA) - e^{ANh} &= r^N(\infty) \cdot I \\ + \frac{1}{2\pi i} \int_{\Gamma_1} \left(r^N(hz) - e^{zNh} \right) (zI - A)^{-1} dz &- \frac{1}{2\pi i} \int_{\Gamma_3} r^N(\infty) (zI - A)^{-1} dz \\ + \frac{1}{2\pi i} \int_{\Gamma_2} \left(r^N(hz) - r^N(\infty) \right) (zI - A)^{-1} dz &- \frac{1}{2\pi i} \int_{\Gamma_2} e^{zNh} (zI - A)^{-1} dz \end{aligned}$$

where, with the usual orientation,

$$\Gamma_1 = \{a \pm \rho e^{i\theta} : 0 \le \rho h \le \delta\},\ \Gamma_2 = \{a \pm \rho e^{i\theta} : \delta \le \rho h\},\ \Gamma_3 = \{a + \delta e^{i\vartheta}/h : \vartheta \in [-\theta, \theta]\}.$$

Note that $|r^N(\infty)| \leq r^N(ha)(1-q)^N \leq \mathcal{M}^N N^{-\mathcal{P}}$, $N = 1,2,\ldots$. The main term is the first integral. Observe that

$$|e^{zh}| = e^{ah}e^{\rho h\cos\theta} \leq e^{ah}e^{-p\rho h} \quad \text{for} \quad z \in \Gamma_1 \cup \Gamma_2. \tag{4.12}$$

Moreover, for $z \in \Gamma_1, h \in (0, h_0]$ and $N = 1,2,\ldots$ we have

$$\begin{aligned} |r^N(hz) - e^{zNh}| &= |r(hz) - e^{zh}| \textstyle\sum_{k=0}^{N-1} |r^k(hz)| \cdot |e^{z(N-1-k)h}| \\ &\leq \text{const} \cdot |hz|^{\mathcal{P}+1} N e^{-(N-1)p\rho h} \mathcal{M}^{N-1} \end{aligned} \tag{4.13}$$

and

$$|z|^{\mathcal{P}+1} = (a^2 + 2a\rho\cos\theta + \rho^2)^{(\mathcal{P}+1)/2} \leq \text{const}(a,\mathcal{P}) \cdot (|a| + \rho^{\mathcal{P}+1}). \tag{4.14}$$

Thus the first integral is bounded by

$$\begin{aligned} &\text{const} \cdot N\mathcal{M}^N \left(\textstyle\int_0^\infty |a| h^{(\mathcal{P}+1)} e^{-p\rho hN} d\rho + \int_0^\infty (h\rho)^{(\mathcal{P}+1)} e^{-p\rho hN} \cdot \frac{d\rho}{\rho} \right) \\ &\leq \text{const} \cdot \mathcal{M}^N (|a| h^{\mathcal{P}} + N^{-\mathcal{P}}). \end{aligned} \tag{4.15}$$

The integral over Γ_3 can be easily estimated by

$$|r^N(\infty)| \left(\int_{-\theta}^{\theta} \frac{M}{1+\delta/h} \cdot \frac{\delta}{h} d\vartheta \right) \leq \text{const} \cdot (1-q)^N r^N(ha).$$

Using (4.8), we find for the third integral the bound

$$\text{const} \int_{\delta/h}^{\infty} |r(hz) - r(\infty)| N(1-q)^{N-1} r^{N-1}(ha) \frac{M}{1+\rho} d\rho$$

$$\leq \text{const} \cdot N(1-q)^N r^N(ha) \cdot \int_{\delta/h}^{\infty} \frac{1}{\rho h} \cdot \frac{1}{\rho} d\rho \leq \text{const} \cdot N(1-q)^N r^N(ha).$$

Finally, the last integral is bounded by

$$\text{const} \cdot e^{aNh} \int_{\delta/h}^{\infty} e^{-p\rho hN} \cdot \frac{1}{\rho} d\rho \leq \text{const} \cdot e^{aNh} \frac{1}{N} e^{-p\delta N}.$$

Summing all contributions, (4.11) follows. ■

Remark 4.10 Replacing N by $N+1$ as in (4.9) the variable stepsize analogue of (4.11) is

$$|r(h_1 A) r(h_2 A) \cdots r(h_{N+1} A) - e^{AH}| \leq \Omega \mathcal{N} \left(\max_{1 \leq i \leq N+1} h_i \right)^{\mathcal{P}} (|a| + H^{-\mathcal{P}}) \tag{4.16}$$

for all finite sequences $(h_1, h_2, \ldots, h_{N+1}) \in \mathbb{R}^{N+1}$ with $h_n \in (0, h_0]$,$n = 1, 2, \ldots, N+1$ where

$$H = \sum_{j=1}^{N+1} h_j \quad \text{and} \quad \mathcal{N} = \prod_{j=1}^{N+1} \max\{r(h_j a); e^{h_j a}\}.$$

Inequality (4.16) is proved in Section 5 as Lemma 6.2 under the conditions of Lemma 4.9 plus the extra assumption $r(\infty) = 0$. As with (4.9) it seems that (4.16) is no longer valid if this extra assumption is dropped. Consider again the special case $\mathcal{P} = 1, a = 0, h_1 = h_2 = \ldots = h_N = h, h_{N+1} \leq h/2$, $Nh = 1$ and $h/h_{N+1} \to \infty$ as $N \to \infty$. Arguing as in the proof of Lemma 4.9, a straightforward but rather lengthy computation yields that

$$|r^N(hA)r(h_{N+1}A) - e^{A(Nh+h_{N+1})}| \leq \text{const} \cdot \frac{1}{N}$$

$$+\frac{1}{2\pi}\left|\int_{\Gamma_\star} r^N(hz)(r(h_{N+1}z) - r(\infty))(zI - A)^{-1}dz\right|$$

where $\Gamma_\star = \{\pm\rho e^{i\theta} : \delta/h \leq \rho \leq \delta/h_{N+1}\}$. There is no possibility to change the $\Gamma_\star$-part of the original contour $\Gamma_1 \cup \Gamma_2$ too much. Integrating over $\Gamma_\star$, inequalities (3.26), (4.8) and (2.2) imply only that

$$|r^N(hA)r(h_{N+1}A) - e^{A(1+h_{N+1})}| \leq \text{const}\left(\frac{1}{N} + (1-q)^N \log\frac{h}{h_{N+1}}\right)$$

and this is not of order $\mathcal{O}(N^{-1})$ if $h/h_{N+1} \to \infty$ as $N \to \infty$. This makes plausible that (4.16) does not follow from the conditions of Lemma 4.9 alone. Besides $r(\infty) = 0$, an alternative extra assumption for (4.16) is the existence of positive constants c and C for which $c \leq h_i/h_k \leq C, i, k = 1, 2, \ldots, N+1; N \in \mathbb{N}$. The proof of this latter result is just a little harder than the proof of Lemma 4.9.

5 Convergence in case of constant stepsizes

In this section we prove the fundamental convergence theorem for the case of constant stepsizes. Neglecting notational discrepancies Theorem 5.3 is identical to Theorem 2.1 in [19]. We present a much simpler proof than the original. It follows the same pattern as standard Runge–Kutta proofs for the global error in ordinary differential equations [8] :

1. Derive a linear recurrence for the sequence of consecutive errors and
2. apply a discrete Gronwall lemma.

When deriving this linear recurrence, we apply Abel's rearrangement trick (summation by parts) enabling us to prove sharp upper bounds for certain contour integrals. Abel's rearrangement trick (5.4) complies with the smoothing effect [11] of sectorial evolution equations and its application constitutes the main technical novelty of the present paper. Besides, as we shall see in Section 6 below, our proof works also in the variable stepsize case.

We start again with equation (2.1) and assume that the linear operator A satisfies (A0) and the nonlinear operator $f : X^\alpha \to X$ satisfies the condition (F0). For a fixed $u_0 \in X^\alpha$ let $\bar{u}$ denote the solution of (2.1) with $\bar{u}(0) = u_0$. We know already from Section 2 that $\bar{u}$ is defined on $[0, \infty)$, $\bar{u} : [0, \infty) \to X^\alpha$ is continuous and $\bar{u} : (0, \infty) \to X^\alpha$ is continuously differentiable. As in Section 3, the corresponding Runge-Kutta solutions are denoted by $\{U_n\}_{n=0}^\infty$ and the stage values are $\{U_{ni}\}_{i=1}^m$, $n = 0, 1, 2, \ldots$. We assume that the Runge-Kutta method is of order $\mathcal{P}$ and is $A(\theta^c)$-stable. For the stepsize we require that $h \in (0, h_0]$ where h_0 is chosen in such a way that the statements of Lemmas 4.1, 4.2 and 4.9 hold.

We recall a fundamental regularity estimate from Henry [11] p.71 which will be used repeatedly in the proof of Theorem 5.3 below.

Lemma 5.1 *Let $T > 0$ be arbitrary. Then there exists a constant $K > 0$ with the property that*

$$|\dot{\bar{u}}(t)|_\alpha \leq K/t \quad \textit{for all} \quad t \in (0, T]. \tag{5.1}$$

The constant K depends on T as well as on $\alpha, L, |u_0|_\alpha, |f(u_0)|$ and on the three constants M, θ, a from (2.2) .

Remark 5.2 Scrutinizing the proof in Henry [11], a sharper estimate can be derived. For example, assume that $\{\bar{u}(t) : t \geq 0\}$ is bounded in X^α. Then

$$|\dot{\bar{u}}(t)|_\alpha \leq K(t^{-1}e^{at} + t^{-\alpha}e^{Qt}) \quad \text{for all} \quad t > 0$$

where

$$Q = 2^{-1}(2a + |a| + (\Omega L \Gamma(1-\alpha))^{1/(1-\alpha)}).$$

Here, Γ stands for the Gamma-function, and the constant K depends only on a, α, Ω, L (the constants in (2.7) and (2.8), respectively) as well as on $|f(0)|_\alpha$ and $\sup\{|\bar{u}(t)|_\alpha : t \geq 0\}$. This suggests we cannot expect nice formulae derived from this and it explains why, in contrast to our previous estimates, we will use the finite time T in the sequel.

Theorem 5.3 *[19]. Assume that the conditions listed prior to Lemma 5.1 are all satisfied and let $T > 0$ be arbitrary. Then there exists a constant $\mathcal{K} > 0$ such that, for all $h \in (0, h_0]$ and $n = 1, 2, \ldots$ with $nh \leq T$,*

$$|U_n - \bar{u}(nh)|_\alpha \leq \mathcal{K}(n^{-\mathcal{P}} + h^{1-\alpha}n^{-\alpha}\log n). \tag{5.2}$$

The constant $\mathcal{K}$ depends on $T, \alpha, L, |u_0|_\alpha, |f(u_0)|, M, \theta, a$ as well as on the stability function of the Runge-Kutta method.

Proof: Starting from (3.19)–(3.21) and the integral representation

$$\bar{u}(nh) = e^{Anh}u_0 + \int_0^{nh} e^{A(nh-s)}f(\bar{u}(s))ds,$$

we derive a recursive chain of inequalities for the sequence of errors $\{|e_n|_\alpha\}$ given by

$$e_n = U_n - \bar{u}(nh) \quad , \quad n = 0, 1, 2, \ldots .$$

We write e_n as a sum of five terms as follows

$$
\begin{aligned}
e_n &= I_n + II_n + III_n + IV_n + V_n \\
&= (r^n(hA) - e^{Anh})u_0 \\
&+ h\sum_{k=0}^{n-1} r^{n-1-k}(hA)\sum_{j=1}^{m} q_j(hA)(f(U_{kj}) - f(\bar{u}_{kj})) \\
&+ h\sum_{k=0}^{n-1} r^{n-1-k}(hA)\sum_{j=1}^{m} q_j(hA)(f(\bar{u}_{kj}) - f(\bar{u}(kh+h))) \\
&+ \sum_{k=0}^{n-1}\int_{kh}^{kh+h}\left(r^{n-1-k}(hA)\sum_{j=1}^{m} q_j(hA) - e^{A(nh-s)}\right) f(\bar{u}(kh+h))ds \\
&+ \sum_{k=0}^{n-1}\int_{kh}^{kh+h} e^{A(nh-s)}(f(\bar{u}(kh+h)) - f(\bar{u}(s)))ds,
\end{aligned}
$$

$n = 1, 2, \ldots$. Here the auxiliary stage values $(\bar{u}_{k1}, \ldots, \bar{u}_{km})^T \in \mathcal{X}^\alpha$; $k = 0, 1, 2, \ldots$ are uniquely determined by the system

$$
\bar{u}_{ki} = s_i(hA)\bar{u}(kh) + h\sum_{j=1}^{m} s_{ij}(hA)f(\bar{u}_{kj}), \quad i = 1, 2, \ldots, m. \tag{5.3}
$$

By (3.22) and Lemma 4.9,

$$
|I_n|_\alpha \le |r^n(hA) - e^{Anh}|_\alpha \cdot |u_0|_\alpha = |r^n(hA) - e^{Anh}| \cdot |u_0|_\alpha \le \Omega\mathcal{M}^n(|a|h^{\mathcal{P}} + n^{-\mathcal{P}})
$$

and therefore, by using $h \le T/n \le \text{const}/n$,

$$
|I_n|_\alpha \le \text{const} \cdot n^{-\mathcal{P}} \quad \text{whenever} \quad n = 1, 2, \ldots \quad \text{and} \quad nh \le T.
$$

Applying Lemma 4.2, inequality (2.8) and the combination of (3.16) and (3.14) with $V = U_k, \tilde{V} = \bar{u}(kh)$, we obtain that

$$
\begin{aligned}
|II_n|_\alpha &\le h\sum_{k=0}^{n-1}\sum_{j=1}^{m}|(aI - A)^\alpha r^{n-k-1}(hA)q_j(hA)| \cdot |f(U_{kj}) - f(\bar{u}_{kj})| \\
&\le h\sum_{k=0}^{n-1}\Omega((n-k)h)^{-\alpha} r^{n-1-k}(ha) \cdot L \cdot \max_{1\le j\le m}|U_{kj} - \bar{u}_{kj}|_\alpha \\
&\le h \cdot \text{const}\sum_{k=0}^{n-1}((n-k)h)^{-\alpha}|e_k|_\alpha \quad \text{whenever} \quad n = 1, 2, \ldots; nh \le T.
\end{aligned}
$$

The very same reasoning yields that

$$
|III_n| \le h \cdot \text{const}\sum_{k=0}^{n-1}\Omega((n-k)h)^{-\alpha} \cdot \max_{1\le j\le m}|\bar{u}_{kj} - \bar{u}(kh+h)|_\alpha
$$

whenever $n = 1, 2, \ldots$ and $nh \le T$. In view of (5.3), inequality (3.16) with $V = \bar{u}(kh)$ and $\mathcal{U} = (\bar{u}(kh+h), \ldots, \bar{u}(kh+h))^T \in \mathcal{X}^\alpha$ as starting point of the iteration to the fixed point $\mathcal{U}^* = (\bar{u}_{k1}, \ldots, \bar{u}_{km})^T \in \mathcal{X}^\alpha$ implies that

$$
\max_{1\le j\le m}|\bar{u}_{kj} - \bar{u}(kh+h)|_\alpha = ||\mathcal{U}^* - \mathcal{U}||_\alpha \le 2||\mathcal{G}_{\bar{u}(kh)}(\mathcal{U}) - \mathcal{U}||_\alpha =
$$

$$2\max_{1\le i\le m}|s_i(hA)\bar{u}(kh)+h\sum_{j=1}^{m}s_{ij}(hA)f(\bar{u}(kh+h))-\bar{u}(kh+h)|_\alpha$$

and further, replacing $f(\bar{u}(kh+h))$ by $-A\bar{u}(kh+h)+\dot{\bar{u}}(kh+h)$,

$$=2\cdot\max_{1\le i\le m}|III_{ik}^1+III_{ik}^2+III_{ik}^3|_\alpha$$

where, for each $i=1,2,\ldots,m$ and $k=0,1,2,\ldots$,

$$III_{ik}^1=s_i(hA)(\bar{u}(kh)-\bar{u}(kh+h)),$$

$$III_{ik}^2=[s_i(hA)-I-h\sum_{j=1}^{m}s_{ij}(hA)A]\bar{u}(kh+h),$$

$$III_{ik}^3=h\sum_{j=1}^{m}s_{ij}(hA)\dot{\bar{u}}(kh+h).$$

Observe first that $III_{ik}^2=0\in\mathcal{X}^\alpha$. This follows immediately from the $\mathcal{T}$-based operational calculus. In fact, the rational function

$$\mathbb{C}\setminus\sigma(\mathcal{A}^{-1})\to\mathbb{C},z\to s_i(z)-1-z\sum_{j=1}^{m}s_{ij}(z)$$

is just the i-th coordinate of the function

$$\begin{aligned}\mathbb{C}\setminus\sigma(\mathcal{A}^{-1})\to\mathbb{C}^m,z\to s(z)-\mathbf{1}-zS(z)\mathbf{1}\\ =(I_m-\mathcal{A}z)^{-1}\mathbf{1}-\mathbf{1}-z(I_m-\mathcal{A}z)^{-1}\mathcal{A}\mathbf{1}\\ =(I_m-\mathcal{A}z)^{-1}\{I_m-(I_m-\mathcal{A}z)-\mathcal{A}z\}\mathbf{1}=0\end{aligned}$$

and thus the expression in the square bracket vanishes.

The remaining two terms can be estimated by using Lemma 5.1. The very same integration trick we used in deriving the boundedness of $\{|\mathcal{N}_{h,ij}(A)|:h\in(0,h_0]\}$ (see Lemma 3.6) applies to $\{s_i(hA)\}_{h\in(0,h_0]}$ and $\{s_{ij}(hA)\}_{h\in(0,h_0]}$. Hence

$$\begin{aligned}|III_{ik}^1|_\alpha+|III_{ik}^3|_\alpha&\le\text{const}\cdot\left|\int_{kh}^{kh+h}\dot{\bar{u}}(\tau)d\tau\right|_\alpha+h|\dot{\bar{u}}(kh+h)|_\alpha\\ &\le\ \text{const}\int_{kh}^{kh+h}\frac{K}{\tau}d\tau+h\frac{K}{kh+h}\le\text{const}\cdot\frac{1}{k+1}\quad\text{for}\quad k=1,2,\ldots\end{aligned}$$

and, using the boundedness of $\{|\bar{u}(\tau)|_\alpha:0\le\tau\le T\}$ guaranteed by (2.10),

$$|III_{ik}^1|_\alpha+|III_{ik}^3|_\alpha\le\text{const}\quad\text{for}\quad k=0,i=1,2,\ldots,m.$$

The conclusion is that

$$|III_n|_\alpha\le h\cdot\text{const}\sum_{k=0}^{n-1}((n-k)h)^{-\alpha}(k+1)^{-1}\quad\text{whenever}\quad nh\le T.$$

A similar application of (5.1) plus (2.7) shows that

$$
\begin{aligned}
|V_n|_\alpha &\le \sum_{k=0}^{n-1} \int_{kh}^{kh+h} |(aI-A)^\alpha e^{A(nh-s)}| \cdot |(f(\bar{u}(kh+h)) - f(\bar{u}(s))| ds \\
&\le \left(\int_0^h + \sum_{k=1}^{n-2} \int_{kh}^{kh+h} + \int_{nh-h}^{nh} \right) \Omega \frac{e^{a(nh-s)}}{(nh-s)^\alpha} \cdot L \cdot |\bar{u}(kh+h) - \bar{u}(s)|_\alpha ds \\
&\le h \cdot \mathrm{const} \sum_{k=0}^{n-1} ((n-k)h)^{-\alpha} (k+1)^{-1} \quad \text{whenever} \quad nh \le T.
\end{aligned}
$$

The remaining term $|IV_n|_\alpha$ needs considerably more care. We apply Abel's rearrangement trick and rewrite $IV_n = \sum_{k=1}^n \omega_{n-k} g_k$, $n = 1, 2, \dots$ as

$$
IV_n = \sum_{k=1}^{n-1} \left(\sum_{\ell=n-k}^{n-1} \omega_\ell \right) (g_k - g_{k+1}) + \left(\sum_{\ell=0}^{n-1} \omega_\ell \right) \cdot g_n \tag{5.4}
$$

where

$$
\omega_\ell = h r^\ell (hA) \sum_{j=1}^m q_j(hA) - \int_{(n-\ell-1)h}^{(n-\ell)h} e^{A(nh-s)} ds, \quad \ell = 0, 1, \dots, n-1;
$$

$$
g_k = f(\bar{u}(kh)), \quad k = 1, 2, \dots, n.
$$

We claim that there exists a positive constant Ω (depending on $T, \alpha, \mathcal{P}$, the three constants in (2.2) and on the Runge-Kutta method itself) such that

$$
|\omega_\ell|_{0\to\alpha} \le \Omega h^{1-\alpha} (\ell+1)^{-\alpha-\mathcal{P}}, \quad \ell = 0, 1, \dots, n-1 \tag{5.5}
$$

and

$$
|\sum_{\ell=0}^{n-1} \omega_\ell|_{0\to\alpha} \le \Omega h^{1-\alpha} n^{-\alpha-\mathcal{P}+1} \tag{5.6}
$$

whenever $h \in (0, h_0]$, $n = 1, 2, \dots$ and $nh \le T$. Here $|\cdot|_{0\to\alpha}$ denotes the norm of an operator from X into X^α.

To prove (5.5) and (5.6), we need a suitable integral representation for $\omega_\ell, \ell = 0, 1, \dots, n-1$. Combining (2.4) with Fubini theorem, we obtain that

$$
\begin{aligned}
2\pi i \int_{(n-\ell-1)h}^{(n-\ell)h} e^{A(nh-s)} ds &= \int_{(n-\ell-1)h}^{(n-\ell)h} \int_\Gamma e^{z(nh-s)} (zI-A)^{-1} dz ds \\
&= \int_\Gamma \int_{(n-\ell-1)h}^{(n-\ell)h} e^{z(nh-s)} ds (zI-A)^{-1} dz = \int_\Gamma e^{\ell hz} \frac{e^{zh}-1}{z} (zI-A)^{-1} dz.
\end{aligned}
$$

Together with the simple identity

$$
\sum_{j=1}^m q_j(hz) = b^T (I_m - \mathcal{A}hz)^{-1} \mathbf{1} = (r(hz)-1)/(hz),
$$

this implies that

$$
\omega_\ell = \frac{1}{2\pi i} \int_\Gamma \left(r^\ell(hz) \frac{r(hz)-1}{z} - e^{\ell hz} \frac{e^{zh}-1}{z} \right) (zI-A)^{-1} dz
$$

or, equivalently

$$\omega_\ell = \frac{h}{2\pi i}\int_{\Gamma_1}\left((r^\ell(hz) - e^{\ell hz})\frac{e^{zh}-1}{zh} + r^\ell(hz)\frac{r(hz)-e^{zh}}{hz}\right)(zI-A)^{-1}dz$$
$$+ \quad \frac{h}{2\pi i}\int_{\Gamma_2}\left(r^\ell(hz)\frac{r(hz)-1}{hz} - e^{\ell hz}\frac{e^{zh}-1}{zh}\right)(zI-A)^{-1}dz$$

where $\Gamma_1 = \{a \pm \rho e^{i\theta} : 0 \le \rho h \le \delta\}$ and $\Gamma_2 = \{a \pm \rho e^{i\theta} : \delta \le \rho h\}$.

Using (4.12) the integral over Γ_2 can be estimated by

$$\begin{aligned}
&\text{const}\cdot h\int_{\delta/h}^{\infty}\rho^\alpha\left((1-q)^\ell r^\ell(ha)\frac{1}{h\rho} + e^{\ell ha}e^{-\ell p\rho h}\frac{1}{h\rho}\right)\frac{M}{1+\rho}d\rho \\
&\le \quad \text{const}\cdot\mathcal{M}^\ell\int_{\delta/h}^{\infty}\rho^{\alpha-1}((1-q)^\ell + e^{-\ell p\delta})\frac{d\rho}{\rho} \\
&= \quad \text{const}\mathcal{M}^\ell((1-q)^\ell + e^{-\ell p\delta})\int_\delta^\infty\left(\frac{\tau}{h}\right)^{\alpha-1}\frac{d\tau}{\tau} \le \text{const}\cdot h^{1-\alpha}\mathcal{M}^{\ell+1}(\ell+1)^{-\mathcal{P}-1}.
\end{aligned}$$

On the other hand, case $N=\ell$ (and $\alpha \ne 0$) of the estimates (4.12),(4.15) and of (4.11) imply that the integral over Γ_1 is bounded by

$$\text{const}\cdot h\int_0^{\delta/h}\rho^\alpha(\ell\mathcal{M}^\ell e^{-\ell p\rho h}|zh|^{\mathcal{P}+1} + r^\ell(ha)e^{-\ell p\rho h}|zh|^{\mathcal{P}})e^{-p\rho h}e^{p\delta}\frac{M}{1+\rho}d\rho$$
$$\le \text{const}\cdot h\mathcal{M}^{\ell+1}(h(\ell+1))^{-\alpha}(|a|h^{\mathcal{P}} + (\ell+1)^{-\mathcal{P}} + (\ell+1)^{-1}(|a|h^{\mathcal{P}-1} + (\ell+1)^{-(\mathcal{P}-1)}))$$

and (5.5) follows from the simple inequality $h \le T/(\ell+1) = \text{const}/(\ell+1), \ell = 0,1,\dots,n-1$.

The proof of (5.6) is based on the integral representation

$$\sum\nolimits_{\ell=0}^{n-1}\omega_\ell = \frac{h}{2\pi i}\left(\int_{\Gamma_1} + \int_{\Gamma_2}\right)\frac{r^n(hz)-e^{nhz}}{hz}(zI-A)^{-1}dz.$$

The very same argument we used in proving (5.5) gives that

$$\begin{aligned}
|\sum\nolimits_{\ell=0}^{n-1}\omega_\ell|_{0\to\alpha} &\le \text{const}\cdot h\int_{\delta/h}^{\infty}\rho^\alpha((1-q)^n\mathcal{M}^n + e^{-np\rho h}\mathcal{M}^n)\frac{1}{h\rho}\cdot\frac{d\rho}{\rho} \\
&+ \quad \text{const}\cdot h\int_0^{\delta/h}\rho^\alpha(n\mathcal{M}^n e^{-np\rho h}|zh|^{\mathcal{P}})\frac{M}{1+\rho}d\rho \\
&\le \quad \text{const}\cdot h\mathcal{M}^n(hn)^{-\alpha}n^{-(\mathcal{P}-1)} = \Omega h^{1-\alpha}n^{-\alpha-\mathcal{P}+1}.
\end{aligned}$$

Returning to (5.4), it follows by Lemma 5.1 that

$$\begin{aligned}
|IV_n|_\alpha &\le \sum\nolimits_{k=1}^{n-1}\sum\nolimits_{\ell=n-k}^{n-1}\Omega h^{1-\alpha}(\ell+1)^{-\alpha-\mathcal{P}}|g_k - g_{k+1}| + \Omega h^{1-\alpha}n^{-\alpha-\mathcal{P}+1}|g_n| \\
&\le \quad \Omega h^{1-\alpha}\sum\nolimits_{k=1}^{n-1}\sum\nolimits_{\ell=n-k}^{\infty}(\ell+1)^{-\alpha-1}\cdot L|\bar u(kh) - \bar u(kh+h)|_\alpha + \Omega h^{1-\alpha}n^{-\alpha}|g_n| \\
&\le \quad \Omega h^{1-\alpha}\sum\nolimits_{k=1}^{n-1}\int_{n-k}^{\infty}\tau^{-\alpha-1}d\tau\cdot L\frac{K}{k} + \Omega h^{1-\alpha}n^{-\alpha}(|f(u_0)| + L|\bar u(nh) - u_0|_\alpha) \\
&\le \quad \text{const}\cdot h^{1-\alpha}\sum\nolimits_{k=0}^{n-1}(n-k)^{-\alpha}(k+1)^{-1} \quad \text{whenever} \quad n = 1,2,\dots; nh \le T.
\end{aligned}$$

Since

$$\left(\sum_{k=0}^{\left[\frac{n-1}{2}\right]} + \sum_{k=\left[\frac{n+1}{2}\right]}^{n-1}\right) (n-k)^{-\alpha}(k+1)^{-1} \leq \text{const} \cdot n^{-\alpha}(1+\log n),$$

we arrive at the inequality

$$|III_n|_\alpha + |IV_n|_\alpha + |V_n|_\alpha \leq \text{const} \cdot h^{1-\alpha} n^{-\alpha}(1+\log n).$$

Summing all the five contributions, we conclude there is a constant C such that, for all $h \in (0, h_0]$ and $n = 1, 2, \dots$ with $nh \leq T$,

$$|e_n|_\alpha \leq C(n^{-\mathcal{P}} + h(nh)^{-\alpha}\log n + h\sum_{k=0}^{n-1}((n-k)h)^{-\alpha}|e_k|_\alpha), \tag{5.7}$$

the desired recursive chain of inequalities for the finite real sequence $\{|e_n|_\alpha\}$. Note that $|e_0|_\alpha = 0$.

The proof can be finished by showing that (5.2) is a direct consequence of (5.7). This will be the content of the next Lemma. ■

The following is a discrete Gronwall type Lemma with weak singularities. It should be compared with the corresponding continuous version in [11], Lemma 7.1.1.

Lemma 5.4 *To any two positive constants T, C one can associate new constants $\mathcal{C}, \mathcal{D}, E, h_0 > 0$ with the following property. For any nonnegative sequence $\eta_n \in \mathbb{R}$ satisfying $\eta_0 = 0$ and*

$$\eta_n \leq C(n^{-\mathcal{P}} + h(nh)^{-\alpha}\log n + h\sum_{k=0}^{n-1}((n-k)h)^{-\alpha}\eta_k) \tag{5.8}$$

for $n = 1, 2, \dots$ with $nh \leq T$ and $h \in (0, h_0]$, the following estimate holds

$$\eta_n \leq \mathcal{C}(n^{-\mathcal{P}} + \mathcal{D}h(nh)^{-\alpha}\log n)e^{Enh}. \tag{5.9}$$

Proof: In the induction step we have to check that

$$\begin{aligned} &C\left(n^{-\mathcal{P}} + h(nh)^{-\alpha}\log n + h\sum_{k=1}^{n-1}((n-k)h)^{-\alpha}\mathcal{C}(k^{-\mathcal{P}} + \mathcal{D}h(kh)^{-\alpha}\log k)e^{Ekh}\right) \\ &\leq \ \mathcal{C}(n^{-\mathcal{P}} + \mathcal{D}h(nh)^{-\alpha}\log n)e^{Enh} \quad \text{whenever} \quad n = 2, 3, \dots \quad \text{and} \quad nh \leq T. \end{aligned}$$

We do this by showing for $n = 2, 3 \dots$ the estimates

$$h\sum_{k=1}^{n-1}((n-k)h)^{-\alpha}k^{-\mathcal{P}}e^{-E(n-k)h} \leq Bh(nh)^{-\alpha}\log n + \varepsilon n^{-\mathcal{P}} \tag{5.10}$$

and

$$h\sum_{k=1}^{n-1}((n-k)h)^{-\alpha}h(kh)^{-\alpha}\log(k)e^{-E(n-k)h} \leq \varepsilon h(nh)^{-\alpha}\log n \tag{5.11}$$

where B is a fixed positive constant and $\varepsilon = \varepsilon(E) \to 0$ as $E \to \infty$. Since $k^{-\alpha}\log k \le 2(k+1)^{-\alpha}\log n$ for $k = 1, 2, \ldots, n-1$ we obtain (5.11) from the following statement

$$\sup_{n=2,3,\ldots}(nh)^{\alpha}\sum_{k=1}^{n-1}((n-k)h)^{-\alpha}((k+1)h)^{-\alpha}e^{-E(n-k)h} \to 0 \quad \text{as} \quad E \to \infty. \tag{5.12}$$

We can also reduce (5.10) to (5.12) because the following estimate holds for all $E > 0$

$$\begin{aligned} h\sum_{k=1}^{n-1}((n-k)h)^{-\alpha}k^{-\mathcal{P}}e^{-E(n-k)h} &= \left(\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\cdots + \sum_{k=\left[\frac{n+1}{2}\right]}^{n-1}\cdots\right) \\ \le\ & h\cdot\mathrm{const}\cdot(nh)^{-\alpha}\sum_{k=1}^{\left[\frac{n-1}{2}\right]}k^{-\mathcal{P}} \\ +\ & h\cdot\mathrm{const}\cdot n^{-\mathcal{P}}\sum_{k=\left[\frac{n+1}{2}\right]}^{n-1}((n-k)h)^{-\alpha}e^{-E(n-k)h} \\ \le\ & h\cdot\mathrm{const}\cdot(nh)^{-\alpha}\log n \\ +\ & h\cdot\mathrm{const}\cdot n^{-\mathcal{P}}\sum_{k=1}^{n-1}((n-k)h)^{-\alpha}((k+1)h)^{-\alpha}(nh)^{\alpha}e^{-E(n-k)h}. \end{aligned}$$

It is left to prove (5.12). Since the sum in (5.12) is the lower approximating Riemann sum of an integral it suffices to show

$$\sup_{x>0}\int_0^x x^{\alpha}\cdot\frac{e^{-E(x-\tau)}}{(x-\tau)^{\alpha}}\cdot\frac{1}{\tau^{\alpha}}d\tau \to 0 \quad \text{as} \quad E \to \infty. \tag{5.13}$$

Substituting $\tau = x/s$, we find for the integral in (5.13) the estimate

$$\int_1^{\infty} x^{\alpha}\cdot\frac{e^{-Ex(s-1)/s}}{(x(s-1)/s)^{\alpha}}\cdot\frac{1}{(x/s)^{\alpha}}\cdot\frac{x}{s^2}ds$$

$$\begin{aligned} \le\ & \int_1^2 x^{\alpha}\cdot\frac{e^{-Ex(s-1)/2}}{(x(s-1))^{\alpha}}\cdot x\,ds + \int_2^{\infty}\frac{e^{-Ex/2}}{(x/2)^{\alpha}}\cdot xs^{\alpha-2}ds \\ =\ & (2/E)^{1-\alpha}\int_0^{Ex/2}\frac{e^{-u}}{u^{\alpha}}du + x^{1-\alpha}e^{-Ex/2}\int_2^{\infty}2^{\alpha}s^{\alpha-2}ds \\ \le\ & \mathrm{const}\left(E^{\alpha-1} + \max\{x^{1-\alpha}e^{-Ex/2} : 0 \le x < \infty\}\right) \le \mathrm{const}/E^{1-\alpha}. \end{aligned}$$

This hold for every $E > 0$ and (5.13) follows. The rest is easy. In view of (5.10) and (5.11), inequality (5.9) holds true provided that

$$C(1+\mathcal{C}\varepsilon) \le \mathcal{C} \quad \text{and} \quad C(1+\mathcal{C}B+\mathcal{C}\mathcal{D}\varepsilon) \le \mathcal{C}\mathcal{D}. \tag{5.14}$$

It is readily seen that both inequalities are satisfied with $\mathcal{C} = 2C, \mathcal{D} = 2BC+1$ whenever E is chosen so large that $\varepsilon = \varepsilon(E) \le 1/(2C)$. ∎

Remark 5.5 The example $(X, A) = (L_2(0,\pi), \Delta_D)$ of Section 2 shows that, at least for the backward Euler method of order $\mathcal{P} = 1$, inequalities (5.5) and (5.6) are sharp. Arguing as in Example 4.8, it is not hard to derive that

$$h^{\alpha-1} \cdot \lim_{h\to 0^+} |\omega_\ell|_{0\to\alpha} = \max_{x\le 0}(-x)^\alpha |(1-q)^{-(\ell+1)} + (e^x - 1)e^{x(\ell+1)}/x|$$

and further,

$$|\omega_\ell|_{0\to\alpha} \ge c_0 h^{1-\alpha}(\ell+1)^{-\alpha-1} \quad \text{and} \quad |\sum\nolimits_{\ell=0}^{n-1} \omega_\ell|_{0\to\alpha} \ge c_0 h^{1-\alpha} n^{-\alpha}$$

for some constant $c_0 > 0$, and each $h \in (0, h_0], \ell \in \mathbb{N}, n \in \mathbb{N}\setminus\{0\}$. These lower estimates point to the importance of Abel's rearrangement trick we used in handling term IV_n in the proof of Theorem 5.3. A direct application of (5.5) via the triangle inequality leads only to the estimate

$$|IV_n|_\alpha = |\sum\nolimits_{k=0}^{n-1} \omega_{n-k-1} f(\bar{u}(kh+h))|_\alpha \le \text{const} \cdot h^{1-\alpha}$$

and therefore, in the final analysis, it does not imply (5.2) but only the much weaker inequality

$$|e_n|_\alpha \le \mathcal{K}(n^{-\mathcal{P}} + h^{1-\alpha}) \quad \text{whenever} \quad h \in (0, h_0], n = 1, 2, \dots \quad \text{and} \quad nh \le T.$$

6 Convergence for variable stepsizes

We consider equation (2.1) again and assume that all conditions listed in the first paragraph of Section 4 are satisfied. We allow arbitrary sequences of stepsizes of the form

$$(h_1, h_2, \dots, h_N) \in \mathbb{R}^N, \quad h_n \in (0, h_0], \quad n = 1, 2, \dots, N$$

and throughout the rest of the section we use the notation

$$H_k = \sum_{j=1}^{k} h_j, \quad h_{*k} = \max_{1\le j\le k} h_j.$$

If no confusion arises we abbreviate the values for the maximal index $k = N$ as

$$H = H_N, \quad h_* = h_{*N}.$$

We set $h = h_n$ in the Runge-Kutta formulas (3.1),(3.2) and obtain as in Theorem 3.7 an approximating sequence $\{U_n^v\}_{n=0}^\infty \subset X^\alpha$ (here the superscript v stands for variable stepsize). By a simple induction argument we find for $n = 1, 2, \dots$ the following explicit representation

$$U_n^v = r(h_1 A)\dots r(h_n A)U_0^v + \sum_{k=0}^{n-1} h_{k+1} r(h_{k+2}A)\dots r(h_n A) \sum_{j=1}^{m} q_j(h_{k+1}A) f(U_{kj}^v), \tag{6.1}$$

where $U_0^v = u_0$ is an initial value and the internal stage values $\{\{U_{kj}^v\}_{j=1}^m\}_{k=0}^\infty$ are determined by

$$U_{ki}^v = s_i(h_{k+1}A)U_k^v + h_{k+1}\sum_{j=1}^m s_{ij}(h_{k+1}A)f(U_{kj}^v), \quad i = 1,2,\dots,m. \tag{6.2}$$

From now on, we assume that $r(\infty) = 0$. This is the price we have to pay for not imposing any quasi-uniformity conditions on the stepsize sequence. A careful step-by-step reconsideration of the proofs below shows that an alternative extra assumption is the existence of positive constants c and C for which $c \le h_i/h_k \le C$, $i,k = 1,2,\dots,n; n \in \mathbb{N}\setminus\{0\}$.

In a first step we restate and prove the estimates (4.9) and (4.16). The main technique used here is a subtle subdivision of the integration contours depending on the values of the step-sizes. Our main goal will then be to prove a variable stepsize analog of Theorem 5.3.

Lemma 6.1 *Assume that the Runge–Kutta method is strictly $A(\theta^c)$-stable and and its stability function satisfies $r(\infty) = 0$. Then for any $\alpha \in [0,1)$ there exist positive constants Ω and h_0 such that for all $j = 1,\dots,m$*

$$|(aI-A)^\alpha r(h_1A)\cdots r(h_NA)q_j(h_{N+1}A)| \le \Omega H_{N+1}^{-\alpha} r(h_1a)\cdots r(h_Na) \tag{6.3}$$

holds with $H_{N+1} = \sum_{k=1}^{N+1} h_k$ for all finite sequences $(h_1,\dots,h_{N+1}) \in \mathbb{R}^{N+1}$ satisfying $h_n \in (0,h_0], n = 1,2,\dots,N+1,\ N \in \mathbb{N}$.

Proof: Case $N = 0$ follows from (4.2) with $k = 0$ and case $\alpha = 0$ is a consequence of (4.4) and the uniform boundedness of the $|q_j(hA)|$. We can thus restrict ourselves to the case $\alpha \in (0,1)$ and $N \ge 1$. For $b \in \mathbb{R}$ and $0 \le d < c \le \infty$, define

$$\gamma_b^{c,d} = \{w = b \pm \rho e^{i\theta} \in \mathbb{C} : c \le \rho \le d\}.$$

Note that $\gamma_0 = \gamma_0^{0,\infty}$. In what follows we repeatedly apply Lemma 3.9 as well as inequalities (4.8) (both for r and $q_j; j = 1,2,\dots,m$) and (2.2) without any further notice.

We introduce the abbreviation

$$r_{[k]} = r(h_1a)\cdots r(h_ka)$$

and with $H = \sum_{k=1}^N h_k$ we first prove the slightly weaker estimate

$$|(aI-A)^\alpha r(h_1A)\cdots r(h_NA)q_j(h_{N+1}A)| \le \Omega H^{-\alpha} r_{[N]}. \tag{6.4}$$

There is no loss of generality in assuming that $h_1 \le h_2 \le \dots \le h_N$. We distinguish the following two cases:

Case 1 $h_N \ge H/3$:

Using our $\mathcal{C}$-based operational calculus with w/h_N instead of z the operator $(aI-A)^\alpha r(h_1A)\cdots r(h_NA)q_j(h_{N+1}A)$ can be represented as

$$\frac{1}{2\pi i}\left(\int_{\Gamma_1} + \int_{\Gamma_2}\right)(a - \frac{w}{h_N})^\alpha r(\frac{h_1}{h_N}w)\cdots r(\frac{h_N}{h_N}w)q_j(\frac{h_{N+1}}{h_N}w)(\frac{w}{h_N}I - A)^{-1}\frac{dw}{h_N}$$

where $\Gamma_1 = \gamma^{\delta,\infty}_{ah_N}$ and $\Gamma_2 = \gamma^{0,\delta}_{ah_N}$. Since $r(\infty) = 0$ the integral over Γ_1 can be estimated by

$$
\begin{aligned}
& r(h_1 a)\cdots r(h_{N-1}a)\int_\delta^\infty (\frac{\rho}{h_N})^\alpha \frac{\text{const}}{\rho}\cdot\frac{M}{1+\rho/h_N}\cdot\frac{d\rho}{h_N} \\
\leq \quad & \text{const}\cdot h_N^{-\alpha}\cdot r_{[N]} \leq \text{const}\cdot H^{-\alpha} r_{[N]}.
\end{aligned}
$$

Similarly, the integral over Γ_2 is bounded by

$$
\begin{aligned}
& \text{const}\cdot r(h_1 a)\cdots r(h_N a)\int_0^\delta (\frac{\rho}{h_N})^\alpha e^{-p\rho H/h_N}\frac{d\rho}{\rho} \\
& \quad = \text{const}(pH)^{-\alpha} r_{[N]}\int_0^{\delta pH/h_N} \tau^\alpha e^{-\tau}\frac{d\tau}{\tau} \quad \leq \quad \text{const} H^{-\alpha} r_{[N]}\int_0^\infty \tau^{\alpha-1}e^{-\tau}d\tau.
\end{aligned}
$$

Summing the two contributions, (6.4) follows.

Case 2 $h_N < \frac{H}{3}$:

Then there exists an index $J \in \mathbb{N}$ with the property that

$$\frac{H}{3} < h_1 + \ldots + h_J < \frac{2H}{3}.$$

We use this index and the integration contours $\Gamma_1 = \gamma^{\delta,\infty}_{ah_J}$, $\Gamma_2 = \gamma^{0,\delta}_{ah_J}$ and write the operator $(aI - A)^\alpha r(h_1 A)\cdots r(h_N A) q_j(h_{N+1}A)$ as

$$\frac{1}{2\pi i}\left(\int_{\Gamma_1} + \int_{\Gamma_2}\right)(a - \frac{w}{h_J})^\alpha r(\frac{h_1}{h_J}w)\cdots r(\frac{h_N}{h_J}w) q_j(\frac{h_{N+1}}{h_J}w)(\frac{w}{h_J}I - A)^{-1}\frac{dw}{h_J}$$

The integral over Γ_1 is bounded by

$$
\begin{aligned}
& \int_\delta^\infty (\frac{\rho}{h_J})^\alpha r(h_1 a)\cdots r(h_{J-1}a) r(h_J a)(1-q)\cdots r(h_{N-1}a)(1-q) \\
\cdot \quad & \frac{\text{const}}{\rho h_N/h_J}\cdot\text{const}\cdot\frac{M}{1+\rho/h_J}\frac{d\rho}{h_J} \\
\leq \quad & \text{const}\cdot r_{[N-1]}(1-q)^{N-J}\cdot(\frac{1}{h_J})^\alpha\cdot\frac{h_J}{h_N}\cdot\int_\delta^\infty \rho^{\alpha-2}d\rho \\
= \quad & \text{const}\cdot r_{[N-1]}(1-q)^{N-J}(\frac{h_J}{H})^{1-\alpha}\frac{H}{h_N}H^{-\alpha} \leq \text{const}\cdot r_{[N-1]}H^{-\alpha}\frac{H}{h_N}(1-q)^{N-J}.
\end{aligned}
$$

In view of the simple inequality

$$(N-J)h_N \geq h_{J+1} + \ldots + h_N > \frac{H}{3}$$

the extra factor $h_N^{-1}H(1-q)^{N-J}$ is less than $\sup\{3x(1-q)^x : x \geq 1\} \leq \text{const}(q)$.

Similarly the integral over Γ_2 is bounded by

$$\int_0^\delta (\frac{\rho}{h_J})^\alpha r(h_1 a) e^{-p\rho\frac{h_1}{h_J}}\cdots r(h_J a) e^{-p\rho\frac{h_J}{h_J}} r(h_{J+1}a)\cdots r(h_N a)\cdot\text{const}\cdot\frac{d\rho}{\rho}$$

$$\leq \text{const}\cdot r_{[N]}\int_0^\delta (\frac{\rho}{h_J})^\alpha e^{-p\rho H/(3h_J)}\frac{d\rho}{\rho} = \text{const}\cdot r_{[N]}(pH)^{-\alpha}\int_0^{\delta pH/h_J}\tau^{\alpha-1}e^{-\tau/3}d\tau.$$

Again the sum of the two contributions leads to (6.4) which finishes case 2.

Next we notice that inequality (6.3) is a direct consequence of (6.4) in case $h_{N+1}\leq H$. If $h_{N+1}>H$ we use the curves $\Gamma_1=\gamma^{\delta,\infty}_{ah_{N+1}}$, $\Gamma_2=\gamma^{0,\delta}_{ah_{N+1}}$ and write $(aI-A)^\alpha r(h_1A)\cdots r(h_NA)q_j(h_{N+1}A)$ as

$$\frac{1}{2\pi i}\left(\int_{\Gamma_1}+\int_{\Gamma_2}\right)(a-\frac{w}{h_{N+1}})^\alpha r(\frac{h_1}{h_{N+1}}w)\cdots r(\frac{h_N}{h_{N+1}}w)q_j(w)(\frac{w}{h_{N+1}}I-A)^{-1}\frac{dw}{h_{N+1}}$$

The integral over Γ_1 is bounded by

$$r(h_1a)\cdots r(h_Na)\int_\delta^\infty(\frac{\rho}{h_{N+1}})^\alpha\cdot\frac{\text{const}}{\rho}\cdot\frac{d\rho}{\rho}\leq \text{const}h_{N+1}^{-\alpha}r_{[N]}\leq \text{const}(H+h_{N+1})^{-\alpha}r_{[N]}.$$

Since $h_{N+1}\geq h_N$ the integral over Γ_2 can be estimated by

$$\begin{aligned}
&\text{const}\cdot r_{[N]}\int_0^\delta(\frac{\rho}{h_{N+1}})^\alpha e^{-p\rho H/h_{N+1}}\frac{d\rho}{\rho}\\
\leq\quad & \text{const}\cdot r_{[N]}\int_0^\delta(\frac{\rho}{h_{N+1}})^\alpha e^{-p\rho(H+h_{N+1})/h_{N+1}}e^{p\rho}\frac{d\rho}{\rho}\\
\leq\quad & \text{const}\cdot r_{[N]}(p(H+h_{N+1}))^{-\alpha}e^{\delta p}\int_0^\infty \tau^{\alpha-1}e^{-\tau}d\tau.
\end{aligned}$$

From these two estimates inequality (6.3) follows. ∎

Lemma 6.2 *Consider a Runge–Kutta method which is strictly $A(\theta^c)$-stable, of order $\mathcal{P}$ and satisfies $r(\infty)=0$. Then there exists positive constants Ω and h_0 such that*

$$|r(h_1A)r(h_2A)\cdots r(h_NA)-e^{AH}|\leq\Omega\mathcal{N}h_*^{\mathcal{P}}(|a|+H^{-\mathcal{P}})\qquad(6.5)$$

for all finite sequences $(h_1,h_2,\ldots,h_N)\in\mathbb{R}^N, h_n\in(0,h_0], n=1,2,\ldots,N; N\in\mathbb{N}\setminus\{0\}$ where $H=h_1+\ldots+h_N$, $h_*=\max_{1\leq j\leq N}h_j$ and

$$\mathcal{N}=(\max\{r(h_1a);e^{h_1a}\})\cdot\ldots\cdot(\max\{r(h_Na);e^{h_Na}\}).$$

Proof: We may assume that $h_1\leq h_2\leq\ldots\leq h_N\leq h_0$ where the constant h_0 as well as the constants p,q and δ below are taken from Lemma 3.9. We start with the integral representation

$$\begin{aligned}
r(h_1A)r(h_2A)\cdot\ldots\cdot r(h_NA)-e^{AH}&=\frac{1}{2\pi i}\sum_{k=1}^N[\int_{\gamma_1^k}+\int_{\gamma_2^k}]\\
r(h_1z)\ldots r(h_{k-1}z)\{r(h_kz)-e^{h_kz}\}&e^{h_{k+1}z}\ldots e^{h_Nz}(zI-A)^{-1}dz\\
&+\int_{\gamma_3}[r(h_1z)\cdot\ldots\cdot r(h_Nz)-e^{Hz}](zI-A)^{-1}dz
\end{aligned}$$

where for $k = 1, \dots, N$

$$\gamma_1^k = \{z = a \pm \rho e^{i\theta} \in \mathbb{C} : \rho \le \delta/h_k\}, \gamma_2^k = \{z = a \pm \rho e^{i\theta} \in \mathbb{C} : \delta/h_k \le \rho \le \delta/h_1\}$$
$$\text{and} \quad \gamma_3 = \{z = a \pm \rho e^{i\theta} \in \mathbb{C} : \rho \ge \delta/h_1\}$$

Applying (4.14), the sum of the integrals over γ_1^k $(k = 1, 2, \dots, N)$ is bounded by

$$\begin{aligned}
&\text{const}\sum_{k=1}^{N} \int_0^{\delta/h_k} r(h_1 a)e^{-ph_1\rho} \dots r(h_{k-1}a)e^{-ph_{k-1}\rho}\{|a|h_k^{\mathcal{P}+1} + (h_k\rho)^{\mathcal{P}+1}\} \\
&\cdot \quad \{r(h_k a)\frac{1}{r(h_k a)}e^{-ph_k\rho} \cdot e^{ph_k\rho}\}e^{h_{k+1}a}e^{-ph_{k+1}\rho} \dots e^{h_N a}e^{-ph_N\rho} \cdot \frac{M}{1+\rho}d\rho \\
&\le \quad \text{const} \cdot \mathcal{N}\sum_{k=1}^{N} \int_0^{\delta/h_k} e^{-pH\rho}\{|a|h_k^{\mathcal{P}+1} + (h_k\rho)^{\mathcal{P}+1}\} \cdot \frac{1}{1+\rho}d\rho \\
&\le \quad \text{const} \cdot \mathcal{N}\left(\sum_{k=1}^{N} \int_0^{\infty} e^{-pH\rho}|a|h_k h_N^{\mathcal{P}} d\rho + \int_0^{\infty} e^{-pH\rho} h_k h_N^{\mathcal{P}} \rho^{\mathcal{P}} d\rho\right) \\
&\le \quad \text{const} \cdot \mathcal{N} \cdot h_N^{\mathcal{P}} \int_0^{\infty} e^{-pH\rho} H(|a| + \rho^{\mathcal{P}})d\rho \le \text{const} \cdot \mathcal{N} \cdot h_N^{\mathcal{P}}(|a| + H^{-\mathcal{P}}).
\end{aligned}$$

Integration over γ_2^k is more complicated. We use (4.8), the binomial theorem as well as the simple inequality $\sup\{x^j e^{-p\delta x} : x \ge 0\} \le const(p, \delta, \mathcal{P})$, $j = 0, 1, ..., \mathcal{P}$. The contour γ_2^k is subdivided into smaller pieces. There is no loss of generality in assuming that $e^{-p\delta} \le 1 - q$. After these preparations, the sum of the integrals over γ_2^k $(k = 1, 2, \dots, N)$ can be estimated as follows

$$\begin{aligned}
&\text{const}\sum_{k=2}^{N}\sum_{i=2}^{k} \int_{\delta/h_i}^{\delta/h_{i-1}} r(h_1 a)e^{-ph_1\rho} \dots r(h_{i-1}a)e^{-ph_{i-1}\rho} r(h_i a)(1-q) \cdot \dots \\
&\cdot \quad r(h_{k-1}a)(1-q)\{\frac{1}{h_k\rho} + e^{-ph_k\rho}\}e^{h_{k+1}a}e^{-ph_{k+1}\rho} \dots e^{h_N a}e^{-ph_N\rho} \cdot \frac{M}{1+\rho}d\rho \\
&\le \quad \text{const} \cdot \mathcal{N}\sum_{k=2}^{N}\sum_{i=2}^{k}(1-q)^{k-i} \cdot \int_{\delta/h_i}^{\delta/h_{i-1}} e^{-pH_{i-1}\delta/h_i} \cdot \frac{1}{h_k\rho}e^{-p(H-H_k)\delta/h_i} \cdot \frac{d\rho}{\rho} \\
&\le \quad \text{const} \cdot \mathcal{N}\sum_{k=2}^{N}\sum_{i=2}^{k}(1-q)^{k-i}e^{-p\delta H_{i-1}/h_i} \int_{\delta/h_i}^{\infty} \frac{1}{h_k\rho^2}d\rho(1-q)^{N-k} \\
&\le \quad \text{const} \cdot \mathcal{N}\left(\frac{h_N}{H_n}\right)^{\mathcal{P}} \sum_{k=2}^{N}\sum_{i=2}^{k}(1-q)^{N-i}e^{-p\delta H_{i-1}/h_i} \cdot \frac{h_i}{h_k} \cdot \left(\frac{H}{h_N}\right)^{\mathcal{P}} \\
&\le \quad \text{const} \cdot \mathcal{N}\left(\frac{h_N}{H}\right)^{\mathcal{P}} \sum_{k=2}^{N}\sum_{i=2}^{k}(1-q)^{N-i}e^{-p\delta H_{i-1}/h_i} \\
&\cdot \quad 1 \cdot \sum_{j=0}^{\mathcal{P}} \binom{\mathcal{P}}{j} \left(\frac{H_{i-1}}{h_i}\right)^j \left(\frac{H - H_{i-1}}{h_N}\right)^{\mathcal{P}-j} \\
&\le \quad \text{const} \cdot \mathcal{N}\left(\frac{h_N}{H}\right)^{\mathcal{P}} \sum_{k=2}^{N}\sum_{i=2}^{k}(1-q)^{N-i} \cdot \text{const} \cdot (N - i + 1)^{\mathcal{P}}
\end{aligned}$$

$$
\begin{aligned}
&= \quad \text{const} \cdot \mathcal{N}\left(\frac{h_N}{H}\right)^{\mathcal{P}} \sum\nolimits_{i=2}^{N} \sum\nolimits_{k=i}^{N} (1-q)^{N-i}(N-i+1)^{\mathcal{P}} \\
&= \quad \text{const} \cdot \mathcal{N}\left(\frac{h_N}{H}\right)^{\mathcal{P}} \sum\nolimits_{i=2}^{N} (1-q)^{N-i}(N-i+1)^{\mathcal{P}+1}.
\end{aligned}
$$

Here the last sum is obviously bounded.

Finally, the integral over γ_3 is bounded by

$$
\begin{aligned}
\text{const} \quad &\cdot \quad \int_{\delta/h_1}^{\infty} [r(h_1 a)(1-q)\ldots r(h_{N-1}a)(1-q)\frac{1}{h_N \rho} \\
&+ \quad e^{h_1 a} e^{-p h_1 \rho} \ldots e^{h_N a} e^{-p h_N \rho}] \cdot \frac{M}{1+\rho} d\rho \\
&\leq \quad \text{const} \cdot \mathcal{N} \int_{\delta/h_1}^{\infty} [(1-q)^{N-1} \frac{1}{h_N \rho^2} + \frac{e^{-pH\rho}}{\rho}] d\rho \\
&\leq \quad \text{const} \cdot \mathcal{N}(1-q)^{N-1} [\frac{h_1}{h_N} + \int_{\delta}^{\infty} \frac{e^{-p\tau}}{\tau} d\tau] \\
&\leq \quad \text{const} \cdot \mathcal{N}(1-q)^{N-1} \leq \text{const} \cdot \mathcal{N}(\frac{h_N}{H})^{\mathcal{P}} N^{\mathcal{P}} (1-q)^{N-1} \leq \text{const}(\frac{h_N}{H})^{\mathcal{P}}.
\end{aligned}
$$

Summing all contributions, (6.5) follows. ■

As a final preparation we derive some α-norm estimates on a finite interval $(0,T]$ similar to Lemma 6.2.

Lemma 6.3 *Under the assumptions of Lemma 6.2 consider a finite interval $(0,T]$ and for stepsizes $h_1,\ldots,h_N$ satisfying $H = \sum_{j=1}^{N} h_j \leq T$ define the following functions in the class $\mathcal{C}_{a,\theta}$*

$$
Q_{1,k}(z) = \frac{e^{zH_k} - 1}{z} e^{z(H-H_k)}, \tag{6.6}
$$

$$
Q_{2,k}(z) = \frac{r(h_1 z)\cdots r(h_k z) - 1}{z} r(h_{k+1}z)\cdots r(h_N z), \tag{6.7}
$$

$$
Q_3(z) = \frac{r(h_1 z)\cdots r(h_N z) - e^{zH}}{z}. \tag{6.8}
$$

Then the following inequalities hold for $h_* = \max_{1\leq j\leq N} h_j$ sufficiently small

$$
|Q_{1,k}(A)|_{0\to\alpha} + |Q_{2,k}(A)|_{0\to\alpha} \leq C \frac{H_k}{(H-H_k)^{\alpha}}, \tag{6.9}
$$

$$
|Q_3(A)|_{0\to\alpha} \leq C \left(\frac{h_*}{H}\right)^{p} H^{1-\alpha}. \tag{6.10}
$$

Proof: Our starting point is the integral representation

$$
(aI - A)^{\alpha} Q_{1,k}(A) = \frac{1}{2\pi i} \int_{\Gamma} (a-z)^{\alpha} \frac{e^{zH_k} - 1}{z} e^{z(H-H_k)} (zI - A)^{-1} dz.
$$

By slightly shifting the integration contour Γ, we can integrate on $\gamma_1 \cup \gamma_2$ where

$$\gamma_1 = \{z = a \pm \rho e^{i\theta} \in \mathbb{C} : \rho \le \delta/H_k\}$$

and

$$\gamma_2 = \{z = a \pm \rho e^{i\theta} \in \mathbb{C} : \rho \ge \delta/H_k\}.$$

The integral over γ_1 can be estimated by

$$\begin{aligned} &\text{const} \int_0^{\delta/H_k} \rho^\alpha H_k e^{-p\rho(H-H_k)} \frac{M}{1+\rho} d\rho \\ \le\ & \text{const} \cdot H_k \int_0^\infty \rho^\alpha e^{-p\rho(H-H_k)} d\rho \ \le\ \text{const} \frac{H_k}{(H-H_k)^\alpha}. \end{aligned}$$

By using the simple inequalities $|a \pm \rho e^{i\theta}| \ge \text{const}(\theta)\rho$, $\rho \ge 0$, and $e^{-x} \le \text{const}(\alpha) x^{-\alpha}$, $x > 0$ the integral over γ_2 is bounded by

$$\begin{aligned} &\text{const} \int_{\delta/H_k}^\infty \rho^\alpha \frac{1}{\rho} e^{-p\rho(H-H_k)} \frac{M}{1+\rho} d\rho \\ \le\ & \text{const} \cdot e^{-p\delta \frac{H-H_k}{H_k}} \int_{\delta/H_k}^\infty \rho^{\alpha-2} d\rho \ \le\ \text{const} \left(\frac{H_k}{H-H_k}\right)^\alpha \cdot H_k^{1-\alpha}. \end{aligned}$$

Consider now the integral expression for $(aI - A)^\alpha Q_{2,k}(A)$

$$\frac{1}{2\pi i} \int_\Gamma (a-z)^\alpha \frac{r(h_1 z) \ldots r(h_k z) - 1}{z} r(h_{k+1} z) \ldots r(h_N z) (zI - A)^{-1} dz.$$

For $z \in \gamma_1$ the expression $z^{-1}(r(h_1 z) \ldots r(h_k z) - 1)$ can be estimated by using telescope summation

$$\frac{r(h_1 z) \ldots r(h_k z) - 1}{z} = \sum_{i=1}^k \frac{r(h_i z) - 1}{z} r(h_{i+1} z) \ldots r(h_k z)$$

and is less then $\text{const} \cdot (h_1 + \ldots + h_k)$. Otherwise, if $z \in \gamma_2$, then the expression $z^{-1}(r(h_1 z) \ldots r(h_k z) - 1)$ is also less then $\text{const} \cdot H_k$ since $|z| \ge \frac{\delta}{H_k} - |a|$. Hence the second integral over $\gamma_1 \cup \gamma_2$ is bounded by

$$\text{const} \cdot H_k \int_0^\infty \rho^\alpha |r(h_{k+1} z) \ldots r(h_N z)| \frac{M}{1+\rho} d\rho$$

with $z = a \pm \rho e^{i\theta}$. The latter integral can be estimated as in the proof of Lemma 6.1 by $\text{const}(H - H_k)^\alpha$. Notice that there is no change in the proof if the stepsizes start with h_{k+1} instead of h_1 and if the q_j-factor is not present.

Finally, for (6.10) we consider the integral

$$\frac{1}{2\pi i} \int_\Gamma (a-z)^\alpha \frac{r(h_1 z) \ldots r(h_N z) - e^{z(h_1 + \ldots + h_N)}}{z} (zI - A)^{-1} dz$$

and show that its norm in X is bounded by $const \cdot h_*^{\mathcal{P}}/H^{\alpha+\mathcal{P}-1}$. This task requires the very same tricks we applied in the proof of Lemma 6.2. We may assume that $h_1 \le h_2 \le \ldots \le h_N = h_*$.

The integral can be rewritten as

$$\sum_{k=1}^{N} h_k \int_{\gamma_1^k \cup \gamma_2^k} (a-z)^\alpha r(h_1 z) \ldots r(h_{k-1} z) \frac{r(h_k z) - e^{h_k z}}{h_k z} e^{h_{k+1} z} \ldots e^{h_N z} (zI-A)^{-1} dz$$

$$+ \int_{\gamma_3} (a-z)^\alpha \frac{r(h_1 z) \ldots r(h_N z) - e^{zH}}{z} (zI-A)^{-1} dz$$

where as in Lemma 6.2

$$\gamma_1^k = \{z = a \pm \rho e^{i\theta} \in \mathbb{C} : \rho \le \delta/h_k\}, \gamma_2^k = \{z = a \pm \rho e^{i\theta} \in \mathbb{C} : \delta/h_k \le \rho \le \delta/h_1\}$$

$$\text{for all} \quad k = 1, 2, \ldots, N \quad \text{and} \quad \gamma_3 = \{z = a \pm \rho e^{i\theta} \in \mathbb{C} : \rho \ge \delta/h_1\}.$$

Arguing as in the proof of Lemma 6.2, the sum of the integrals over γ_1^k $(k = 1, 2, \ldots, N)$ is bounded by

$$\begin{aligned}
& \text{const} \cdot \sum\nolimits_{k=1}^{N} h_k \int_0^{\delta/h_k} \rho^\alpha e^{-pH\rho} \{|a| h_k^{\mathcal{P}} + (h_k \rho)^{\mathcal{P}}\} \frac{M}{1+\rho} d\rho \\
\le \quad & \text{const} \cdot h_*^{\mathcal{P}} \int_0^\infty \rho^\alpha e^{-pH\rho} H(|a| + \rho^{\mathcal{P}-1}) d\rho \\
\le \quad & \text{const} \cdot h_*^{\mathcal{P}}/H^{\alpha+\mathcal{P}-1}.
\end{aligned}$$

Similarly, the sum of the integrals over γ_2^k $(k = 1, 2, \ldots,)$ can be estimated (changing just some ρ-factors) by

$$\begin{aligned}
& \text{const} \sum_{k=2}^{N} \sum_{i=2}^{k} \int_{\delta/h_i}^{\delta/h_{i-1}} \rho^\alpha r(h_1 a) e^{-ph_1\rho} \ldots r(h_{i-1} a) e^{-ph_{i-1}\rho} r(h_i a)(1-q) \ldots \\
\cdot \quad & r(h_{k-1} a)(1-q) \frac{1}{\rho} \{\frac{1}{h_k \rho} + e^{-ph_k \rho}\} e^{h_{k+1} a} e^{-ph_{k+1}\rho} \ldots e^{h_N a} e^{-ph_N \rho} \cdot \frac{M}{1+\rho} d\rho \\
\le \quad & \text{const} \cdot \sum_{k=2}^{N} \sum_{i=2}^{k} (1-q)^{N-i} e^{-p\delta H_{i-1}/h_i} \int_{\delta/h_i}^\infty \frac{\rho^{\alpha-3}}{h_k} d\rho \\
\le \quad & \text{const} \left(\frac{h_N}{H}\right)^{\alpha-1} \sum_{k=2}^{N} \sum_{i=2}^{k} (1-q)^{N-i} e^{-p\delta(H_{i-1})/h_i} \cdot \frac{h_i}{h_k} \cdot h_i^{1-\alpha} \\
\le \quad & \text{const} \left(\frac{h_N}{H}\right)^{\mathcal{P}} \sum_{k=2}^{N} \sum_{i=2}^{k} (1-q)^{N-i} e^{-p\delta H_{i-1}/h_i} \\
\cdot \quad & H^{1-\alpha} \cdot \sum_{j=0}^{\mathcal{P}} \binom{\mathcal{P}}{j} \left(\frac{H_{i-1}}{h_i}\right)^j \left(\frac{H - H_{i-1}}{h_N}\right)^{\mathcal{P}-j}
\end{aligned}$$

$$\leq \quad \text{const} \cdot \frac{h_N^{\mathcal{P}}}{H^{\alpha+\mathcal{P}-1}} \sum_{k=2}^{N} \sum_{i=2}^{k} (1-q)^{N-i} \cdot \text{const} \cdot (N-i+1)^{\mathcal{P}}$$

$$= \quad \text{const} \cdot \frac{h_*^{\mathcal{P}}}{H^{\alpha+\mathcal{P}-1}} \sum_{i=2}^{N} (1-q)^{N-i} (N-i+1)^{\mathcal{P}+1}.$$

Finally, the integral over γ_3 is bounded by

$$\text{const} \int_{\delta/h_1}^{\infty} \rho^{\alpha} \frac{(1-q)^N + (1-q)^N}{\rho} \cdot \frac{1}{\rho} d\rho$$

$$\leq \quad \text{const} \cdot h_1^{1-\alpha} (1-q)^N \leq \text{const} \cdot h_*^{1-\alpha} \left(\frac{h_* N}{H} \right)^{\alpha+\mathcal{P}-1} (1-q)^N$$

$$= \quad \text{const} \cdot \frac{h_*^{\mathcal{P}}}{H^{\alpha+\mathcal{P}-1}} \cdot N^{\alpha+\mathcal{P}-1} (1-q)^N$$

and this ends the proof of the Lemma. ■

The following result is the variable stepsize analogue of Theorem 5.3. Note that (6.11) simplifies to (5.2) if $h_1 = \ldots = h_N = h \in (0, h_0]$ and $h_* = h, H = Nh \leq T$.

Theorem 6.4 *Under the assumptions of Lemma 6.1 consider a fixed time interval* $[0, T)$. *Then there exist constants* $\mathcal{K} > 0$ *and* $h_0 > 0$ *such that for all sequences of positive stepsizes* $(h_1, \ldots, h_N) \in \mathbb{R}^N$ *with*

$$H := h_1 + \ldots + h_N \leq T, \quad h_* := \max_{1 \leq j \leq N} h_j \leq h_0$$

we have the estimate

$$|U_N^v - \bar{u}(h_1 + \cdots + h_N)|_{\alpha} \leq \mathcal{K} \left(\left(\frac{h_*}{H} \right)^{\mathcal{P}} + \frac{h_*}{H^{\alpha}} \log \frac{H}{h_*} \right). \tag{6.11}$$

The constant $\mathcal{K}$ *depends on* $T, \alpha, L, |u_0|_{\alpha}, |f(u_0)|, M, \theta, a$ *as well as on the stability function of the Runge-Kutta method.*

Proof: We use a chain of arguments similar to the case of constant stepsizes and we abbreviate $r_j = r(h_j A)$. As usual we derive a recursive estimate for the errors $|e_k^v|_{\alpha}$ where

$$e_0^v = 0 \quad \text{and} \quad e_k^v = U_k^v - \bar{u}(H_k) \quad \text{for} \quad k = 1, \ldots, N,$$

and then apply discrete Gronwall techniques.

As in the constant stepsize case we use (2.9),(6.1) and decompose

$$e_N^v = I_N^v + II_N^v + III_N^v + IV_N^v + V_N^v$$

where

$$I_N^v = (r_1 \ldots r_N - e^{AH}) u_0,$$

$$II_N^v = \sum_{k=0}^{N-1} h_{k+1}r_{k+2}\dots r_N \sum_{j=1}^{m} q_j(h_{k+1}A)(f(U_{kj}^v) - f(\bar{u}_{kj}^v)),$$

$$III_N^v = \sum_{k=0}^{N-1} h_{k+1}r_{k+2}\dots r_N \sum_{j=1}^{m} q_j(h_{k+1}A)(f(\bar{u}_{kj}^v) - f(\bar{u}(H_{k+1}))),$$

$$IV_N^v = \sum_{k=0}^{N-1} \int_{H_k}^{H_{k+1}} [r_{k+2}\dots r_N \sum_{j=1}^{m} q_j(h_{k+1}A) - \ e^{A(H-s)}] \cdot f(\bar{u}(H_{k+1}))\ ds,$$

$$V_N^v = \sum_{k=0}^{N-1} \int_{H_k}^{H_{k+1}} e^{A(H-s)}(f(\bar{u}(H_{k+1})) - f(\bar{u}(s)))ds.$$

Here the auxiliary stage values $(\bar{u}_{k1}^v, \dots, \bar{u}_{km}^v)^T \in \mathcal{X}^\alpha, k = 0, \dots, N-1$ are uniquely determined by the system

$$\bar{u}_{ki}^v = s_i(h_{k+1}A)\bar{u}(H_k) + h_{k+1}\sum\nolimits_{j=1}^{m} s_{ij}(h_{k+1}A)f(\bar{u}_{kj}^v), \quad i = 1, 2, \dots, m.$$

The terms I_N^v and II_N^v can be estimated as in the constant stepsize case. In fact, a direct application of Lemma 6.2 gives that

$$|I_N^v|_\alpha \le |r_1 \dots r_N - e^{AH}| \cdot |u_0|_\alpha \le \text{const} \cdot h_*^{\mathcal{P}}(|a| + H^{-\mathcal{P}})$$

and therefore, by using $|a|H^{\mathcal{P}} \le |a|T^{\mathcal{P}} \le \text{const}$,

$$|I_N^v|_\alpha \le \text{const}\left(\frac{h_*}{H}\right)^{\mathcal{P}}. \tag{6.12}$$

Similarly, Lemma 6.1 gives that

$$|II_N^v|_\alpha \le \sum_{k=0}^{N-1} h_{k+1} \sum_{j=1}^{m} |(aI - A)^\alpha r_{k+2}\dots r_N q_j(h_{k+1}A)| \cdot |f(U_{kj}^v) - f(\bar{u}_{kj}^v)|$$

and therefore,

$$|II_N^v|_\alpha \le \text{const} \cdot \sum_{k=0}^{N-1} \frac{h_{k+1}}{(H - H_k)^\alpha}|e_k^v|_\alpha. \tag{6.13}$$

The method we used in the constant variable case applies also for estimating III_N^v and V_N^v. In fact, we obtain easily from Lemma 6.1 that

$$|III_N^v|_\alpha \le \text{const} \cdot \sum_{k=0}^{N-1} \frac{h_{k+1}}{(H - H_k)^\alpha} \cdot \max\nolimits_{1\le i\le m}\{|III_{ik}^{1v}|_\alpha + |III_{ik}^{3v}|_\alpha\}$$

where, for each $k = 0, 1, \dots, N-1$ and $i = 1, 2, \dots, m$,

$$|III_{ik}^{1v}|_\alpha + |III_{ik}^{3v}|_\alpha = |s_i(h_{k+1}A)(\bar{u}(H_k) - \bar{u}(H_{k+1}))|_\alpha$$

$$+\ |h_{k+1}\sum\nolimits_{j=1}^{m} s_{ij}(h_{k+1}A)\dot{\bar{u}}(H_{k+1})|_\alpha$$
$$\leq \text{const}\cdot[|\bar{u}(H_k)-\bar{u}(H_{k+1})|_\alpha + h_{k+1}|\dot{\bar{u}}(H_{k+1})|_\alpha].$$

In view of Lemma 5.1, we have that

$$h_{k+1}|\dot{\bar{u}}(H_{k+1})|_\alpha \leq K\cdot\frac{h_{k+1}}{H_{k+1}} \quad \text{for} \quad k=0,1,\ldots,N-1.$$

The expression $|\bar{u}(H_k)-\bar{u}(H_{k+1})|_\alpha$ requires a little more care. For $k\neq 0$ we can apply Lemma 5.1 again and estimating the integral by a sum from above we obtain that

$$|\bar{u}(H_k)-\bar{u}(H_{k+1})|_\alpha \leq \int_{H_k}^{H_{k+1}}\frac{K}{\tau}d\tau \leq \frac{h_{k+1}}{H_k}, \qquad k\neq 0.$$

On the other hand, using the boundedness of $\{|\bar{u}(\tau)|_\alpha : 0\leq\tau\leq T\}$ guaranteed by (2.10), we obtain that

$$|\bar{u}(H_k)-\bar{u}(H_{k+1})|_\alpha \leq \text{const}, \qquad k=0,1,\ldots,N-1. \tag{6.14}$$

The conclusion is that

$$|III_N^v|_\alpha \leq \text{const}\cdot\sum_{k=0}^{N-1}\frac{h_{k+1}}{(H-H_k)^\alpha}\cdot\min\{\frac{h_{k+1}}{H_k};1\}. \tag{6.15}$$

Applying (2.7) and (2.8), we obtain as in the constant stepsize case that

$$|V_N^v|_\alpha \leq \text{const}\cdot\sum\nolimits_{k=0}^{N-1}\int_{H_k}^{H_{k+1}}\frac{1}{(H-s)^\alpha}\cdot|\bar{u}(H_{k+1})-\bar{u}(s)|_\alpha ds.$$

The argument we used for (6.14) applies to an arbitrary $s\in(H_k,H_{k+1}]$ and yields

$$|\bar{u}(H_{k+1})-\bar{u}(s)|_\alpha \leq \text{const}\cdot\min\{\frac{h_{k+1}}{H_k};1\}.$$

Replacing the remaining integral by two different approximating sums, we conclude that

$$|V_N^v|_\alpha \leq \text{const}\sum_{k=0}^{N-1}\min\{\frac{h_{k+1}}{(H-H_{k+1})^\alpha};(H-H_k)^{1-\alpha}\}\cdot\min\{\frac{h_{k+1}}{H_k};1\}. \tag{6.16}$$

Before passing to the remaining term IV_N^v, we explain briefly the meaning and importance of the $\min\{\ldots;\ldots\}$-type estimates obtained for $|III_N^v|_\alpha$ and $|V_N^v|_\alpha$ above. Depending on the particular choice of the stepsizes, it may happen e.g. that H_k is too small for some k with h_{k+1} relatively large. This shows that the fraction h_{k+1}/H_k *alone* is entirely insufficient as an upper bound. However,

$$\frac{h_{k+1}}{H_k}\leq 2\cdot\frac{h_{k+1}}{H_{k+1}}\leq 2$$

provided that $H_k \geq h_*$. This points to the necessity of considering cases like

$$H_k \leq h_* \text{ and } h_* < H_k < H - h_* \text{ or } H - h_* \leq H_k$$

separately in the sequel.

The remaining term $|IV_N^v|_\alpha$ needs considerably more care. As in the constant stepsize case,we apply Abel's rearrangement trick and rewrite IV_N, $N = 1, 2, \ldots$ as (cf. (5.4))

$$IV_N^v = \sum\nolimits_{k=1}^{N-1} \left(\sum\nolimits_{\ell=N-k}^{N-1} \omega_\ell^v \right) (g_k^v - g_{k+1}^v) + \left(\sum\nolimits_{\ell=0}^{N-1} \omega_\ell^v \right) \cdot g_N^v$$

where, for each $\ell = 0, 1 \ldots, N-1$

$$\omega_\ell^v = h_{N-\ell} r_{N-\ell+1} \ldots r_N \sum\nolimits_{j=1}^{m} q_j(h_{N-\ell} A) - \int_{H_{N-\ell-1}}^{H_{N-\ell}} e^{A(H-s)} ds$$

and for $k = 1, 2, \ldots, N$

$$g_k^v = f(\bar{u}(H_k)).$$

Observe that

$$|IV_N^v|_\alpha \leq \sum_{k=1}^{N-1} \Big| \sum_{\ell=N-k}^{N-1} (aI - A)^\alpha \omega_\ell^v \Big| \cdot |g_k^v - g_{k+1}^v| + \Big| \sum_{\ell=0}^{N-1} (aI - A)^\alpha \omega_\ell^v \Big| \cdot |g_N^v| \quad (6.17)$$

where, combining (2.10) and (5.1),

$$|g_k^v - g_{k+1}^v| \leq \text{const} \cdot \min\{ \frac{h_{k+1}}{H_k}; 1 \} \quad \text{for} \quad k = 1, 2, \ldots, N-1 \quad (6.18)$$

and

$$|g_N^v| \leq \text{const}. \quad (6.19)$$

In order to handle the critical terms

$$|\sum\nolimits_{\ell=N-k}^{N-1} (aI - A)^\alpha \omega_\ell^v|, \quad k = 1, 2, \ldots, N$$

appropriately, we use two different integral representations, namely

$$\sum\nolimits_{\ell=N-k}^{N-1} (aI - A)^\alpha \omega_\ell^v = \frac{1}{2\pi i} \int_\Gamma (a - z)^\alpha \, [\, \frac{r(h_1 z) \ldots r(h_k z) - 1}{z} r(h_{k+1} z) \ldots r(h_N z)$$

$$- \frac{e^{z H_k} - 1}{z} e^{z(H - H_k)} \,] \, (zI - A)^{-1} dz \quad (6.20)$$

for small values of H_k and

$$\sum\nolimits_{\ell=N-k}^{N-1} (aI - A)^\alpha \omega_\ell^v = \frac{1}{2\pi i} \int_\Gamma (a - z)^\alpha \, [\, \frac{r(h_1 z) \ldots r(h_N z) - e^{zH}}{z}$$

$$- \quad \frac{r(h_{k+1}z)\dots r(h_N z) - e^{z(H-H_k)}}{z} \;] \; (zI-A)^{-1} dz \tag{6.21}$$

for the remaining case. Now we apply Lemma 6.3 and obtain the estimates

$$|\sum\nolimits_{\ell=N-k}^{N-1}(aI-A)^\alpha \omega_\ell^v| \le \text{const} \cdot \frac{H_k}{(H-H_k)^\alpha}, \tag{6.22}$$

$$|\sum_{\ell=N-k}^{N-1}(aI-A)^\alpha \omega_\ell^v| \le \text{const}[\frac{h_*^{\mathcal{P}}}{H^{\alpha+\mathcal{P}-1}} + \frac{(\max\{h_j : k+1 \le j \le N\})^{\mathcal{P}}}{(H-H_k)^{\alpha+\mathcal{P}-1}}]. \tag{6.23}$$

Both estimates (6.22), (6.23) are valid for $k = 1, 2, \dots, N-1$. The estimate (6.23) also holds for $k = N$ if the second summand is set to zero in this case.

Next we will show below that (6.15),(6.16) implies the estimate

$$|III_N^v|_\alpha + |V_N^v|_\alpha \le \text{const}\left(\frac{h_*}{H^\alpha} + \frac{h_*}{H^\alpha}\log\frac{H}{h_*}\right) \tag{6.24}$$

and (6.22), (6.23) lead to the same estimate for $|IV_N^v|_\alpha$

$$|IV_N^v|_\alpha \le \text{const}\left(\frac{h_*}{H^\alpha} + \frac{h_*}{H^\alpha}\log\frac{H}{h_*}\right). \tag{6.25}$$

Let us first show how to complete the proof on the basis of these estimates.

In view of (6.12) and (6.13) we obtain the inequality

$$|e_N^v|_\alpha \le \text{const}\left(\left(\frac{h_*}{H}\right)^{\mathcal{P}} + \sum_{k=0}^{N-1}\frac{h_{k+1}}{(H-H_k)^\alpha}|e_k^v|_\alpha + \frac{h_*}{H^\alpha} + \frac{h_*}{H^\alpha}\log\frac{H}{h_*}\right). \tag{6.26}$$

Note that the term h_*/H^α can be omitted since it is always dominated by the sum of the first and the last term. Hence we have found a positive constant C satisfying

$$|e_N^v|_\alpha \le C\left(\left(\frac{h_*}{H}\right)^{\mathcal{P}} + \frac{h_*}{H^\alpha}\log\frac{H}{h_*} + \sum_{k=0}^{N-1}\frac{h_{k+1}}{(H-H_k)^\alpha}|e_k^v|_\alpha\right). \tag{6.27}$$

Then an application of the Gronwall Lemma 6.5 below finishes the proof.

In order to prove (6.24) and (6.25), define the indices $I, J, K \in \{1, \dots, N\}$ by

$$H_{I-1} < h_* \le H_I, \quad H_{J-1} < \frac{H}{2} \le H_J, \quad H_{K-1} \le H - h_* < H_K.$$

We claim that the following relations hold

$$I \le J \le K, \quad H - H_{I-1} \ge \frac{H}{2}, \quad H_K \ge \frac{H}{2}. \tag{6.28}$$

Consider first the case $h_* \le \frac{H}{2}$. Then $I \le J$ is obvious and $J \le K$ follows from $H - H_{J-1} \ge H - \frac{H}{2} \ge h_*$. The other two estimates are consequences of

$H - H_{I-1} \geq H - h_* \geq \frac{H}{2}$ and $H - H_K \leq h_* \leq \frac{H}{2}$. In the case $\frac{H}{2} < h_*$ the maximum stepsize occurs exactly once, say $h_k = h_*$. Then $I = J = K = k$ holds and the remainder of (6.28) is trivial.

In order to prove (6.24), we estimate first $|III_N^v|_\alpha$. Starting from (6.15), we obtain by using the defining inequalities for indices I, J as well as (6.28) that

$$\begin{aligned}
|III_N^v|_\alpha &\leq \text{const}\left(\sum_{k=0}^{I-1}\dots + \sum_{k=I}^{J-1}\dots + \sum_{k=J}^{N-1}\dots\right) \leq \text{const}\left[\sum_{k=0}^{I-1}\frac{h_{k+1}}{(H-H_k)^\alpha}\right.\\
&+ \left.\sum_{k=I}^{J-1}\frac{h_{k+1}}{(H-H_k)^\alpha}\cdot\frac{h_{k+1}}{H_k} + \sum_{k=J}^{N-1}\frac{h_{k+1}}{(H-H_k)^\alpha}\cdot\frac{h_{k+1}}{H_k}\right]\\
&\leq \text{const}\left[\sum_{k=0}^{I-1}\frac{h_{k+1}}{(H-H_{I-1})^\alpha} + \sum_{k=I}^{J-1}\frac{h_*}{(H/2)^\alpha}\cdot\frac{2h_{k+1}}{H_{k+1}}\right.\\
&+ \left.\sum_{k=J}^{N-1}\frac{h_{k+1}}{(H-H_k)^\alpha}\cdot\frac{h_*}{H/2}\right]\\
&\leq \text{const}\left[\frac{2h_*}{(H/2)^\alpha} + \frac{h_*}{(H/2)^\alpha}\cdot 2\int_{h_*}^H \frac{1}{\tau}d\tau + \int_{H/2}^H \frac{1}{(H-\tau)^\alpha}d\tau\cdot\frac{h_*}{H/2}\right]\\
&= \text{const}\left[\left(2^{1+\alpha} + \frac{2^\alpha}{1-\alpha}\right)\frac{h_*}{H^\alpha} + 2^{1+\alpha}\cdot\frac{h_*}{H^\alpha}\log\frac{H}{h_*}\right].
\end{aligned}$$

Finding a similar upper bound for $|V_N^v|_\alpha$ is somewhat harder. Set

$$\mathcal{S} = \min\{\frac{h_K}{(H-H_K)^\alpha}; (H-H_{K-1})^{1-\alpha}\}\cdot\min\{\frac{h_K}{H_{K-1}}; 1\}$$

and distinguish two cases according as $3h_* \geq H$ or not. If $3h_* \geq H$, then

$$\mathcal{S} \leq \frac{H - H_{K-1}}{(H-H_{K-1})^\alpha}\cdot 1 \leq \frac{2h_*}{(H/3)^\alpha}.$$

If $3h_* < H$, then

$$\begin{aligned}
\mathcal{S} &\leq \frac{H-H_{K-1}}{(H-H_{K-1})^\alpha}\cdot\frac{h_K}{H_{K-1}}\\
&\leq \frac{2h_*}{(H-H_{K-1})^\alpha}\cdot\frac{h_K}{H/3} \leq \frac{2h_*}{H/3}\cdot\int_{H/3}^H\frac{1}{(H-\tau)^\alpha}d\tau.
\end{aligned}$$

Finally, starting from (6.16), it is readily obtained via (6.28) and the choice of K that

$$\begin{aligned}
|V_N^v|_\alpha &\leq \text{const}\left[\sum_{k=0}^{K-2}\dots + \mathcal{S} + \sum_{k=K}^{N-1}\dots\right]\\
&\leq \text{const}\left[\sum_{k=0}^{K-2}\frac{h_{k+1}}{(H-H_{k+1})^\alpha}\min\{\frac{h_{k+1}}{H_k}; 1\} + \mathcal{S} + \sum_{k=K}^{N-1}\frac{H-H_k}{(H-H_k)^\alpha}\frac{h_{k+1}}{H_k}\right]
\end{aligned}$$

$$\leq \text{ const} \left[\sum_{k=0}^{K-2} \frac{h_{k+1}}{((H-H_k)/2)^\alpha} \min\{\frac{h_{k+1}}{H_k};1\} \; + \; \mathcal{S} + \sum_{k=K}^{N-1} \frac{h_*}{(H-H_k)^\alpha} \frac{h_{k+1}}{H/2} \right]$$

$$\leq \text{ const} \left[\sum_{k=0}^{K-2} \frac{h_{k+1}}{(H-H_k)^\alpha} \min\{\frac{h_{k+1}}{H_k};1\} \; + \; \mathcal{S} + \frac{h_*}{H/2} \int_{H/2}^{H} \frac{1}{(H-\tau)^\alpha} d\tau \right]$$

The first term appears in (6.15) and the other two terms have the desired behavior, so that (6.24) follows.

Our next task is to prove (6.25). The starting point is of course inequality (6.17). As a simple consequence of (6.23), note that

$$|\sum\nolimits_{\ell=N-k}^{N-1} (aI-A)^\alpha \omega_\ell^v| \leq \text{const} \cdot \frac{h_*}{(H-H_k)^\alpha} \quad \text{for each} \quad k = 1, 2, \ldots, N-1.$$

Combining this and (6.22), (6.18), (6.19), (6.28) it follows that

$$\begin{aligned}
|IV_N^v|_\alpha &\leq \text{const} \left[\sum_{k=0}^{I-1} \cdots + \sum_{k=I}^{J-1} \cdots + \sum_{k=J}^{N-1} \cdots + \frac{h_*}{H^\alpha} \right] \\
&\leq \text{ const} \left[\sum_{k=0}^{I-1} \frac{H_k}{(H-H_k)^\alpha} \cdot \frac{h_{k+1}}{H_k} + \sum_{k=I}^{J-1} \frac{h_*}{(H-H_k)^\alpha} \cdot \frac{h_{k+1}}{H_k} \right. \\
&+ \left. \sum_{k=J}^{N-1} \frac{h_*}{(H-H_k)^\alpha} \cdot \frac{h_{k+1}}{H_k} \; + \; \frac{h_*}{H^\alpha} \right] \\
&\leq \text{ const} \left[\sum_{k=0}^{I-1} \frac{h_{k+1}}{(H-H_{I-1})^\alpha} \; + \; \sum_{k=I}^{J-1} \frac{h_*}{(H/2)^\alpha} \cdot \frac{2\,h_{k+1}}{H_{k+1}} \right. \\
&+ \left. \sum_{k=J}^{N-1} \frac{h_*}{(H-H_k)^\alpha} \cdot \frac{h_{k+1}}{H/2} \; + \; \frac{h_*}{H^\alpha} \right] \\
&\leq \text{ const} \left[\frac{2h_*}{(H/2)^\alpha} + \frac{h_*}{(H/2)^\alpha} \cdot 2 \int_{h_*}^{H} \frac{1}{\tau} d\tau + \frac{h_*}{H/2} \cdot \int_{H/2}^{H} \frac{1}{(H-\tau)^\alpha} d\tau + \frac{h_*}{H^\alpha} \right]
\end{aligned}$$

and this ends the proof of (6.25) as well as (6.27). ∎

Lemma 6.5 *For any given constant C and any time interval $(0,T]$ there exist positive constants $h_0, \mathcal{C}, \mathcal{D}$ and E with the following property. Let $(h_1, \ldots, h_N)$ be a sequence of positive stepsizes with $H = \sum_{k=1}^N h_k \leq T$ and $h_* = \max_{1 \leq k \leq N} h_k \leq h_0$ and let $\eta_k, k = 0, \ldots N$ be a sequence of nonnegative numbers satisfying $\eta_0 = 0$ and a recursive estimate*

$$\eta_j \leq C \left(\left(\frac{h_{*j}}{H_j} \right)^{\mathcal{P}} + \frac{h_{*j}}{H_j^\alpha} \log \frac{H_j}{h_{*j}} + \sum_{k=0}^{j-1} \frac{h_{k+1}}{(H_j - H_k)^\alpha} \eta_k \right), \quad j = 1, \ldots, N. \tag{6.29}$$

Then the following estimate holds for $k = 1, \ldots, N$

$$\eta_k \leq \mathcal{C}\left(\left(\frac{h_{*k}}{H_k}\right)^{\mathcal{P}} + \mathcal{D} \cdot \frac{h_{*k}}{H_k^\alpha} \log \frac{H_k}{h_{*k}}\right) \cdot e^{EH_k}. \tag{6.30}$$

Proof: Generalizing (5.10) and (5.11) it is sufficient to prove the following two estimates

$$\sum_{k=1}^{N-1} \frac{h_{k+1}}{(H-H_k)^\alpha}\left(\frac{h_{*k}}{H_k}\right)^{\mathcal{P}} \cdot e^{-E(H-H_k)} \leq B \cdot \frac{h_*}{H^\alpha} \log \frac{H}{h_*} + \varepsilon\left(\frac{h_*}{H}\right)^{\mathcal{P}}, \tag{6.31}$$

$$\sum_{k=1}^{N-1} \frac{h_{k+1}}{(H-H_k)^\alpha}\left(\frac{h_{*k}}{H_k^\alpha} \log \frac{H_k}{h_{*k}}\right) \cdot e^{-E(H-H_k)} \leq \varepsilon \frac{h_*}{H^\alpha} \log \frac{H}{h_*} + \varepsilon\left(\frac{h_*}{H}\right)^{\mathcal{P}}, \tag{6.32}$$

where B is a fixed positive constant and $\varepsilon = \varepsilon(E)$ is a further positive constant that depends only on E.

In contrast to the constant stepsize case, it is not true in general that $\varepsilon(E) \to 0$ as $E \to \infty$. However, what we need is only that (cf. (5.14))

$$C(1 + \mathcal{C}\varepsilon + \mathcal{C}\mathcal{D}\varepsilon) \leq \mathcal{C} \quad \text{and} \quad C(1 + \mathcal{C}B + \mathcal{C}\mathcal{D}\varepsilon) \leq \mathcal{C}\mathcal{D}. \tag{6.33}$$

Define

$$\sigma_0 = \min\{\sigma > 0 \ : \ \sigma \geq 2C + 1 \text{ and } \sigma^2 \geq (CB + 2^{-1})\sigma + C\}.$$

Set $\mathcal{C} = \mathcal{D} = \sigma_0$ and assume for the moment that

$$\varepsilon \leq (2C\sigma_0)^{-1}. \tag{6.34}$$

It is elementary to check that (6.33) is implied by (6.34). Hence it is enough to prove that, for some E and h_0 suitably chosen, both (6.31) and (6.32) hold true with some ε satisfying (6.34).

By using (6.28), the estimate for (5.13) and the defining inequalities for the indices I and J from the proof of Theorem 6.4, we see that the left-hand side of (6.31) is not greater than

$$\begin{aligned}
& \sum\nolimits_{k=1}^{I-1} \frac{h_{k+1}}{(H-H_k)^\alpha} \cdot 1 \cdot e^{-E(H-H_k)} + \sum\nolimits_{k=I}^{J-1} \frac{h_{k+1}}{(H-H_k)^\alpha} \cdot \frac{h_{*k}}{H_k} \cdot 1 \\
+ \quad & \sum\nolimits_{k=J}^{N-1} \frac{h_{k+1}}{(H-H_k)^\alpha} \cdot \left(\frac{h_*}{H}\right)^{\mathcal{P}} \cdot \left(\frac{H}{H_k}\right)^{\mathcal{P}} \cdot e^{-E(H-H_k)} \\
\leq \quad & \sum\nolimits_{k=1}^{I-1} \frac{h_{k+1}}{(H/2)^\alpha} \cdot e^{-EH/2} + \sum\nolimits_{k=I}^{J-1} \frac{h_{k+1}}{(H/2)^\alpha} \cdot \frac{2\ h_*}{H_{k+1}} \\
+ \quad & \sum\nolimits_{k=J}^{N-1} \frac{h_{k+1}}{(H-H_k)^\alpha} \cdot e^{-E(H-H_k)} \cdot 2^{\mathcal{P}} \cdot \left(\frac{h_*}{H}\right)^{\mathcal{P}} \\
\leq \quad & \frac{2h_*}{(H/2)^\alpha} \cdot e^{-EH/2} + \frac{2h_*}{(H/2)^\alpha} \int_{h_*}^{H} \frac{1}{\tau} d\tau + \int_{H/2}^{H} \frac{e^{-E(H-\tau)}}{(H-\tau)^\alpha} d\tau \cdot 2^{\mathcal{P}} \cdot \left(\frac{h_*}{H}\right)^{\mathcal{P}} \\
\leq \quad & 2^{1+\alpha} \frac{h_*}{H^\alpha} \cdot e^{-EH/2} + 2^{1+\alpha} \cdot \frac{h_*}{H^\alpha} \log \frac{H}{h_*} + \text{const} \cdot \frac{1}{E^{1-\alpha}} \cdot \left(\frac{h_*}{H}\right)^{\mathcal{P}}.
\end{aligned}$$

The first term can be dominated by the second and the third, since for each $E > 0$ we have

$$\frac{h_*}{H^\alpha} \cdot e^{-EH/2} \leq \left(\frac{2(1-\alpha)}{eE}\right)^{1-\alpha} 3^{\mathcal{P}} \left(\frac{h_*}{H}\right)^{\mathcal{P}} \quad \text{if} \quad H < 3h_* \tag{6.35}$$

and trivially

$$\frac{h_*}{H^\alpha} \cdot e^{-EH/2} \leq \frac{h_*}{H^\alpha} \log \frac{H}{h_*} \quad \text{if} \quad H \geq 3h_*. \tag{6.36}$$

In order to tackle inequality (6.32), define

$$\rho_k = \frac{h_{*k} \cdot \log(H_k/h_{*k})}{h_* \cdot \log(H/h_*)} \quad \text{for} \quad k = 1, \ldots, N-1.$$

It is important to note that $\rho_k \leq 1$. This is clear if $h_{*k} = h_*$. Otherwise $h_* = \beta h_{*k}$, $H_k = \gamma h_{*k}$ and $H \geq H_k + h_* = (\gamma + \beta) h_{*k}$ with some $\beta > 1$ and $\gamma \geq 1$. By passing to the new variable $\delta = \beta^{-1}\gamma$, we see that

$$\rho_k \leq \sup\{\frac{\log \gamma}{\gamma \cdot \log((1+\delta)^{1/\delta})} \ : \ \delta \in (0, \gamma] \text{ and } \gamma \geq 1\} = \sup_{\gamma \geq 1} \frac{\log \gamma}{\log(\gamma + 1)} = 1.$$

Applying (6.28),(6.35),(6.36) and the estimate for (5.13) we obtain via $\rho_k \leq 1$ that the left-hand side of inequality (6.32) is not greater than

$$\begin{aligned}
&\sum_{k=1}^{I-1} \frac{h_{k+1}}{(H-H_k)^\alpha} \cdot h_*^{1-\alpha} \cdot \sup\{x^{-\alpha} \log x \ : \ x \geq 1\} \cdot e^{-E(H-H_k)} \\
&\quad + \sum_{k=I}^{N-1} \frac{h_{k+1}}{(H-H_k)^\alpha} \cdot \frac{H^\alpha}{H_k^\alpha} \left(\frac{h_*}{H^\alpha} \log \frac{H}{h_*}\right) \cdot e^{-E(H-H_k)} \\
&\leq \sum_{k=1}^{I-1} \frac{h_{k+1}}{(H/2)^\alpha} \cdot h_*^{1-\alpha} \frac{1}{e\alpha} \cdot e^{-EH/2} \\
&\quad + \sum_{k=I}^{N-1} H^\alpha \cdot \frac{h_{k+1}}{(H-H_k)^\alpha} \frac{e^{-E(H-H_k)}}{((H_{k+1})/2)^\alpha} \cdot \frac{h_*}{H^\alpha} \log \frac{H}{h_*} \\
&\leq h_*^{1-\alpha} \cdot \text{const} \cdot \frac{h_*}{H^\alpha} \cdot e^{-EH/2} \ + \ 2^\alpha \int_0^H H^\alpha \cdot \frac{e^{-E(H-\tau)}}{(H-\tau)^\alpha} \cdot \frac{1}{\tau^\alpha} d\tau \cdot \frac{h_*}{H^\alpha} \log \frac{H}{h_*} \\
&\leq \text{const} \left(h_*^{1-\alpha} \frac{h_*}{H^\alpha} \cdot e^{-EH/2} + \frac{1}{E^{1-\alpha}} \cdot \frac{h_*}{H^\alpha} \log \frac{H}{h_*}\right).
\end{aligned}$$

Recalling that the first term has already been investigated before, the desired result follows easily. We first set the constant B in (6.31) and then satisfy the restriction (6.34) for ε by choosing E sufficiently large and h_0 sufficiently small. ■

A Appendix

Following Section VII.9 of [6] (and correcting several misprints there) we explain briefly the appearence of the "strange" term $\tau(\infty)I$ in formula (3.8) and show how the operational calculus for unbounded operators can be traced back to the standard Dunford–Gelfand operational calculus for bounded operators. Then the relation between the Dunford–Gelfand calculus and our function class $\mathcal{C}_{a,\theta}$ is discussed.

By letting $w = \mathcal{H}(z) = (z - z_0)^{-1}$ with some fixed $z_0 \notin \sigma(A)$, $\mathcal{H}(\infty) = 0$ and assuming (without loss of generality) that $z_0 \notin D^\tau$, the integral in (3.8) transforms as

$$\int_{\gamma^\tau} \tau(z)(zI - A)^{-1} dz = \int_{\mathcal{H}(\gamma^\tau)} \tau(\mathcal{H}^{-1}(w))(\mathcal{H}^{-1}(w)I - A)^{-1} \frac{dw}{-w^2}.$$

Set $B = (A - z_0 I)^{-1}$. The key observation is that $\sigma(B) = \mathcal{H}(\sigma(A)) \cup \{0\}$ and that $(\mathcal{H}^{-1}(w)I - A)^{-1} = wI - w^2(wI - B)^{-1}$ for all $w \notin \sigma(B)$. Consequently, our integral equals to

$$-\int_{\mathcal{H}(\gamma^\tau)} \frac{\tau(\mathcal{H}^{-1}(w))}{w} dw \cdot I + \int_{\mathcal{H}(\gamma^\tau)} \tau(\mathcal{H}^{-1}(w))(wI - B)^{-1} dw.$$

Note that $\mathcal{H}(\gamma^\tau)$ contains a unique smooth Jordan curve c_0 encircling the origin ($c_0 = \mathcal{H}(\{$ the boundary of the unbounded component of $D^\tau\})$) and therefore, by repeated use of Cauchy formula for analytic functions,

$$-\int_{\mathcal{H}(\gamma^\tau)} \frac{\tau(\mathcal{H}^{-1}(w))}{w} dw = -\int_{c_0} \frac{\tau(\mathcal{H}^{-1}(w))}{w} dw = \tau(\mathcal{H}^{-1}(0)) = -\tau(\infty).$$

We conclude that (3.8) is equivalent to

$$\tau(A) = \frac{1}{2\pi i} \int_{\mathcal{H}(\gamma^\tau)} \tau(\mathcal{H}^{-1}(w)) \cdot (wI - B)^{-1} dw$$

where $B \in L(X, X)$, the function $\tau(\mathcal{H}^{-1}(\cdot)) : \mathbb{C} \hookrightarrow \mathbb{C}$ is analytic on the open neighborhood $\mathcal{H}(U)$ of $\sigma(B)$, $\mathcal{H}(D^\tau)$ is a closed neighborhood of $\sigma(B)$ in $\mathcal{H}(U)$ and $\mathcal{H}(\gamma^\tau) = \partial\mathcal{H}(D^\tau)$ consists of a finite number of positively oriented (pairwise nonintersecting) smooth Jordan curves. But this is exactly the framework of the Dunford–Gelfand calculus.

On the other hand, (3.10) goes over into

$$\eta(A) = \frac{1}{2\pi i} \int_{\Gamma'} \frac{\eta(\mathcal{H}^{-1}(w))}{w} dw I + \frac{1}{2\pi i} \int_{\Gamma'} \eta(\mathcal{H}^{-1}(w)) \cdot (wI - B)^{-1} dw$$

where $\Gamma' = \mathcal{H}(\Gamma) \cup \{0\}$ is a piecewise smooth Jordan curve with positive orientation. In view of the growth order condition imposed on each member of our function class $\mathcal{C}_{a,\theta}$, we have that

$$\limsup \left\{ |w|^{-k} \cdot |\eta(\mathcal{H}^{-1}(w))| \big| w \in \mathcal{H}(\Gamma \cup \Sigma_{a,\theta}), w \to 0 \right\}$$

is finite. It follows immediately that the first integral is zero. (In fact, for any $r > 0$ sufficiently small

$$c_r = \mathcal{H}(\Gamma \cup \Sigma_{a,\theta}) \cap \{w \in \mathbb{C} \big| |w| = r\}$$

is a circular arc. Instead of integrating over Γ', we may integrate over $c_r \cup \{w \in \Gamma' \big| |w| > r\}$ and over $c_r \cup \{w \in \Gamma' \big| |w| < r\}$ separately. By Cauchy formula, the integral over $c_r \cup \{w \in \Gamma' \big| |w| > r\}$ is zero. The integral over $c_r \cup \{w \in \Gamma' \big| |w| < r\}$ is bounded by $const \cdot r^{\kappa}$. By letting $r \to 0$, the result follows.) Hence

$$\eta(A) = \frac{1}{2\pi i} \int_{\Gamma'} \eta(\mathcal{H}^{-1}(w))(wI - B)^{-1} dw$$

which does not fit into the framework of Dunford–Gelfand calculus: Since $0 \in \Gamma' \cap \sigma(B)$, we are facing an integral with a (weak) singularity.

It seems to be an interesting problem to characterize those function classes for which classical Dunford–Gelfand calculus extends to. This is a problem Zorn lemma applies to but e.g. we do not know if the operational calculus extends to a function ring containing both $\mathcal{C}_{a,\theta}$ for some a, θ and the Taylor class $\mathcal{T}$.

B Appendix

The following results are analogous to those on pp.316-317 of [6]. They show that, parallel to the $\mathcal{T}$-based operational calculus, also the $\mathcal{C}$-based operational calculus extends to polynomials. The parameters $a \in \mathbb{R}$ and $\theta \in (\frac{\pi}{2}, \pi)$ will be fixed in the following. The proofs of the results below can be easily given along the lines of those in Chapter X of [4], or [24] or [6].

Proposition B.1 *Let $f, g \in \mathcal{C}_{a,\theta}$ and assume that $f = pg$ for some polynomial p of degree n. Then $g(A)x \in \mathcal{D}(A^n)$ and $f(A)x = p(A)g(A)x$ for each $x \in X$.*

Proposition B.2 *Let $f \in \mathcal{C}_{a,\theta}$ and let p be a polynomial of degree n. Then $f(A)x \in \mathcal{D}(A^n)$ and $f(A)p(A)x = p(A)f(A)x$ for each $x \in \mathcal{D}(A^n)$.*

Proposition B.3 *Let p be a polynomial of degree n. Then*

$$p(A)x = \frac{1}{2\pi i} \int_{\Gamma} \frac{p(z)}{(z-a)^{n+1}} (A - aI)^{n+1} (zI - A)^{-1} x dz$$

for each $x \in \mathcal{D}(A^{n+1})$.

Proposition B.4 *Let $f \in \mathcal{C}_{a,\theta}$ and assume that $1/((\cdot - a)^{n+1} f(\cdot)) \in \mathcal{C}_{a,\theta}$ for some $n \in \mathbb{N}$. Then $f(A)$ is invertible, $\mathcal{D}(A^{n+1}) \subset \mathcal{D}([f(A)]^{-1})$ and*

$$[f(A)]^{-1} x = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{(z-a)^{n+1} f(z)} (A - aI)^{n+1} (zI - A)^{-1} x dz$$

for each $x \in \mathcal{D}(A^{n+1})$.

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