Contents

1. Introduction

In this paper we prove the following theorem.

1.1. Theorem. *Let G be a Lie group and M a proper locally linear G-manifold. Then M has the G-homotopy type of a G-CW complex.*

In [E] the above theorem was proved under the additional assumptions that G is a linear Lie group and M has only a finite number of isotropy types. These assumptions were made because of an embedding result that was used in the proof, namely embedding the G-manifold into a euclidean space on which G acts via some representation. Here we construct an embedding into a Banach G-space which allows us to drop the above mentioned two assumptions.

The starting point to construct the embedding was Theorem 6.2 in [Mu] where an equivariant version of the Wojdysławski embedding theorem is stated. In [Mu] the group G is a compact Lie group. However, there is a problem with the construction, see the remark at the end of Chapter 3.

In Chapter 2 we discuss G-invariant metrics for a certain class of proper G-spaces which are then used in Chapter 3 in constructing the embedding. In Chapter 4 we state some properties of G-CW complexes which are needed later in the paper. The definitions of tubular numerable G-coverings and G-nerves given in Chapter 5 follow the ideas presented in [Mu] in the case of a compact Lie group. In Chapter 6 we prove a theorem linking proper locally linear G-manifolds to G-nerves of suitable coverings and a theorem linking G-nerves to G-CW complexes. These results together give us then the above theorem.

We recall that an action of a locally compact Hausdorff topological group G on a space X is *proper*, if for every two points x, y in X there are neighbourhoods U of x and V of y such that the set $G(U|V) = \{g \in G \mid gU \cap V \neq \emptyset\}$ is a relatively compact subset of G. A subset A of a G-space X is called *small* if every point $x \in X$ has a neighbourhood U_x such that $G(U_x|A)$ is relatively compact. All G-spaces are assumed to be Hausdorff spaces.

Note that in [E] we have used the definition of Palais, [Pa1], to define a proper action, which differs slightly from the definition given above. However, the definitions agree if the spaces are for example manifolds or G-CW complexes, see Propositions 1.7 and 4.10 in [E].

In this paper manifolds are assumed to be Hausdorff and second countable.

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2. A G-invariant metric

In the construction of a G-invariant metric the concept of a fundamental set is used. We recall the definition. A subset F of a G-space X is called a *fundamental set* for G in X if (1) $GF = X$ and (2) F is small.

The following lemmas can be found in [Ab–St].

2.1. Lemma. *If F is a fundamental set for G in X then so is its closure $\overline{F}$.*

Proof. This is clear since if $U \subset X$ is open we have $G(U|F) = G(U|\overline{F})$. □

2.2. Lemma. *Let $\mathcal{U}$ be a family of small subsets of a G-space X. Denote by $\pi\colon X \to X/G$ the natural map. If the family $\{\pi(U) \mid U \in \mathcal{U}\}$ of subsets of X/G is locally finite then $\bigcup U$, $U \in \mathcal{U}$, is small.*

Proof. For $x \in X$ let V be a neighbourhood of x in X such that $\pi(V)$ intersects only finitely many $\pi(U)$, $U \in \mathcal{U}$. So $G(V|U)$ is empty for all but a finite number of U's in $\mathcal{U}$. So there is a neighbourhood W of x, $W \subset V$, such that $G(W|U)$ is relatively compact for all $U \in \mathcal{U}$ and empty for all but a finite number of U's, hence $G(W|\bigcup U)$ is relatively compact. □

2.3. Lemma. *Suppose X is a proper G-space and the orbit space X/G is paracompact. Let A be a small subset of X. Then there exists an open fundamental set for G in X which contains A.*

Proof. Since X/G is paracompact the action of G on X is in fact Palais proper, see Proposition 1.7 in [E]. Hence there is a cover $\mathcal{U}$ of X such that U is open and small for every $U \in \mathcal{U}$ and such that $G(U|A)$ is relatively compact for every $U \in \mathcal{U}$. The open cover $\{\pi(U) \mid U \in \mathcal{U}\}$ has a locally finite refinement $\mathcal{V}$ since X/G is paracompact. So there is a map $\mathcal{V} \to \mathcal{U}$, $V \mapsto U_V$, such that $V \subset \pi(U_V)$. For every $V \in \mathcal{V}$ let $W_V = U_V \cap \pi^{-1}(V)$ and let $Y_V = W_V$ if $V \cap \pi(A) = \emptyset$ and $Y_V = G(W_V|A)W_V$ if $V \cap \pi(A) \neq \emptyset$. The sets W_V and Y_V are small, $\pi(W_V) = \pi(Y_V) = V$, so

$$F = \bigcup_{V \in \mathcal{V}} Y_V$$

is small by Lemma 2.2 and $\pi(F) \subset \bigcup \pi(Y_V) = X/G$, hence F is an open fundamental set and F contains A, by definition of the sets Y_V. □

In the next two lemmas we give two results concerning the hypothesis "X/G is paracompact". For other sufficient conditions for paracompactness of the orbit space see for example Proposition 3.2.5 in [Ab–St] and the remark on p. 9 in [Ko].

2.4. Lemma. *Suppose G is a locally compact group and X is a second countable topological manifold on which G acts properly. Then the orbit space X/G is paracompact.*

Proof. By Theorem XI.7.2 in [Du] X is σ-compact. Thus X/G is locally compact and σ-compact. By Theorem XI.7.3 in [Du] X/G is paracompact. □

2.5. Lemma. *Suppose G is a locally compact group and X is a proper G-space on which there exists a G-invariant metric. Then the orbit space X/G is paracompact.*

Proof. If d is a G-invariant metric on X then the formula $D(Gx, Gy) = \inf\{d(gx, hy) \mid g \in G, h \in G\}$ defines a metric on X/G which induces its topology. So X/G is metrizable and hence paracompact. □

The following proposition is the main result of this chapter. The G-invariant metric constructed here is used in Chapter 3 to define an equivariant embedding into a Banach G-space.

2.6. Proposition. *Let X be a proper G-space with X/G paracompact and suppose we are given a metric d on X (which induces the topology of X) with the property*

$$(*) \qquad \forall r > 0 \;\; \forall x \in X : \overline{B}_r^d(x) \text{ is compact},$$

where $\overline{B}_r^d(x) = \{y \in X \mid d(x, y) \le r\}$. Then there exists a G-invariant metric D for X, equivalent with d, with the property that $\overline{B}_1^D(x)$ is compact for every $x \in X$.

Proof. First we fix a compact symmetric neighbourhood K of e in G, and choose a continuous function $f: G \to [0, 1]$ for which $f(e) = 1$ and $f|G \setminus K \equiv 0$. Denote by $\int_G \cdot \, dg$ a right invariant Haar integral for G. We may assume that $\int_G f \; dg > 1$.

By Lemma 2.3 there exists an open fundamental set F for G in X. Then also KF and $\overline{KF}$ are fundamental sets for G in X and using again Lemma 2.3 we find an open fundamental set B such that $\overline{KF} \subset B$.

For each $x \in \overline{KF}$ choose an open neighbourhood $U_x \subset B$ of x. Then the sets $\{U_x\}_{x \in \overline{KF}}, X \setminus \overline{KF}$ cover X and we choose an open locally finite refinement $\{V_\alpha\}_{\alpha \in \mathcal{A}}$ of this covering. Furthermore by Theorem VII.6.1 in [Du] we find an open covering $\{W_\alpha\}_{\alpha \in \mathcal{A}}$ of X such that $\overline{W_\alpha} \subset V_\alpha$ for every $\alpha \in \mathcal{A}$. We may assume that $W_\alpha \neq \emptyset$ for all $\alpha \in \mathcal{A}$.

For each $\alpha \in \mathcal{A}$ choose a continuous function $\varphi_\alpha: X \to [0, 1]$ with $\varphi_\alpha|\overline{W_\alpha} \equiv 1$ and $\varphi_\alpha|(X \setminus V_\alpha) \equiv 0$. Then define

$$d'(x, y) = d(x, y) + \sum_{\alpha \in \mathcal{A}} |\varphi_\alpha(x) - \varphi_\alpha(y)|.$$

It is easy to see that d' defines a metric on X equivalent to d. This construction is similar to Lemma 2.3 in [Pa2]. Note that the property $(*)$ holds also for the metric d' since $d'(x,y) \geq d(x,y)$ for all $x, y \in X$.

The following observation is the key point in this construction: If now $x \in \overline{KF}$ and $y \in X \setminus B$ then there exists an $\alpha \in \mathcal{A}$ for which $x \in W_\alpha \subset V_\alpha \subset B$. Thus $\varphi_\alpha(x) = 1$ and $\varphi_\alpha(y) = 0$ from which it follows that $d'(x,y) > 1$.

Now we can proceed as in the proof of Theorem 3 in [Ko], p. 9–11. Define

$$r(x) = d'(x, X \setminus B) \quad \text{for} \quad x \in X,$$

$$h(x,y) = \min\{d'(x,y), r(x) + r(y)\} \quad \text{for} \quad x, y \in X$$

and

$$D(x,y) = \int_G h(gx, gy)\, dg.$$

Observe that $r(x) \geq 1$ for every $x \in \overline{KF}$. As in [Ko] we have that D is a G-invariant metric on X and D induces the topology of X.

Finally we prove that $\overline{B}_1^D(x)$ is compact for every $x \in X$. For this it suffices to prove that for every $x \in X$ there exists a compact subset of X containing $\overline{B}_1^D(x)$.

Let $x \in X$. Since the metric D is G-invariant we have $\overline{B}_1^D(gx) = g\overline{B}_1^D(x)$ for every $g \in G$ and hence we may assume that $x \in F$. We denote $C = \cup_{g \in K} \overline{B}_1^{d'}(gx)$. Since K is compact there exists $\max\{d'(x, gx) \mid g \in K\} = \delta$ and we have $C \subset \overline{B}_{\delta+1}^{d'}(x)$ which is compact. If now $y \in X$ is such that $d'(gx, gy) < 1$ for some $g \in K$ then $gy \in C$ and $y \in KC \subset K\overline{B}_{\delta+1}^{d'}(x)$ which is compact.

Thus we have proved that if $y \notin KC$, then $d'(gx, gy) \geq 1$ for all $g \in K$. Moreover since $x \in F$, we have $gx \in \overline{KF}$ and thus $r(gx) \geq 1$ for all $g \in K$. Hence $h(gx, gy) = \min\{d'(gx, gy), r(gx) + r(gy)\} \geq 1$ for all $y \notin KC, g \in K$, and

$$D(x,y) = \int_G h(gx, gy)\, dg \geq \int_G f\, dg > 1$$

for all $y \notin KC$, where the function $f\colon G \to [0,1]$ is defined at the beginning of the proof. This proves that $\overline{B}_1^D(x) \subset KC \subset K\overline{B}_{\delta+1}^{d'}(x)$. □

Remark. Changing the original metric d to d' in the above proof was necessary, since the method in [Ko] doesn't automatically give us the desired property for the metric D.

3. An equivariant embedding into a Banach G-space

Let G and X be as in the previous section and denote by d a G-invariant metric for X such that $\overline{B}_1^d(x)$ is compact for every $x \in X$. In this section we consider the space $D(X)$ of continuous functions $f\colon X \to \mathbb{R}$ vanishing at infinity; that is, for every $\epsilon > 0$ there exists a compact set $K \subset X$ such that $|f(x)| < \epsilon$ for all $x \in X \setminus K$. By defining addition and scalar multiplication in $D(X)$ as usual and equipping $D(X)$ with the norm

$$\|f\| = \max\{|f(x)| \mid x \in X\}$$

$D(X)$ is a Banach space. We define a G-action on $D(X)$ by

$$(gf)(x) = f(g^{-1}x) \tag{$*$}$$

for $g \in G$, $f \in D(X)$ and $x \in X$.

3.1. Proposition. *The action $G \times D(X) \to D(X)$ defined by $(*)$ is continuous.*

Proof. Let $f \in D(X)$. We first prove continuity at the point (e, f). Let $\epsilon > 0$. Fix a compact neighbourhood V of e in G and a compact subset K of X such that $|f(x)| < \epsilon/4$ for all $x \in X \setminus K$.

For $x \in X \setminus VK$ and $g \in V$ we have $g^{-1}x \in X \setminus K$ and thus $|f(g^{-1}x) - f(x)| \leq |f(g^{-1}x)| + |f(x)| < \epsilon/2$. Due to compactness of VK we can find a neighbourhood W of e in G such that $|f(g^{-1}x) - f(x)| < \epsilon/2$ for all $g \in W$ and $x \in VK$. Thus for $g \in V \cap W$ we have $|f(g^{-1}x) - f(x)| < \epsilon/2$ for all $x \in X$.

If now $g \in V \cap W$ and $h \in B_{\epsilon/2}(f)$ we have

$$|h(g^{-1}x) - f(x)| \leq |h(g^{-1}x) - f(g^{-1}x)| + |f(g^{-1}x) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon$$

for all $x \in X$, which proves continuity at the point (e, f).

The general case then follows easily. $\square$

3.2. Proposition. *The restricted action $G \times (D(X) \setminus \{0\}) \to D(X) \setminus \{0\}$ is proper (in fact Palais proper).*

Proof. Let $f, h \in D(X) \setminus \{0\}$ and denote $\delta = \|f\|/2 > 0$. Choose a compact subset K of X such that $|h(x)| < \delta/2$ for all $x \in X \setminus K$ and let $x_0 \in X$ for which $|f(x_0)| = \|f\| = 2\delta$.

We prove that $G(B_{\delta/2}(h)|B_\delta(f)) \subset G(K|\{x_0\})$ which is compact. If $f' \in B_\delta(f)$ and $h' \in B_{\delta/2}(h)$ and $f' = gh'$ for some $g \in G$, then $|h'(g^{-1}x_0)| = |f'(x_0)| > \delta$. Thus $|h(g^{-1}x_0)| > \delta/2$, hence $g^{-1}x_0 \in K$ and $g \in G(K|\{x_0\})$. $\square$

Next we define
$$i : X \to D(X)$$
by
$$i(x)(y) = \begin{cases} 1 - d(x, y), & \text{if } 0 \le d(x, y) \le 1 \\ 0, & \text{if } d(x, y) \ge 1. \end{cases}$$
We have $i(x) \in D(X)$ since $\overline{B}_1^d(x)$ is compact for every $x \in X$.

3.3. Proposition. *The function $i: X \to D(X)$ is a G-equivariant embedding and the image $i(X)$ is closed in $D(X)$.*

Proof. It is easy to verify that

$$\|i(x) - i(y)\| \le d(x, y) \quad \text{for all } \ x, y \in X. \tag{1}$$

If $d(x, y) \le 1$, then $\|i(x) - i(y)\| \ge |i(x)(x) - i(y)(x)| = d(x, y)$, that is,

$$\|i(x) - i(y)\| = d(x, y), \text{ if } d(x, y) \le 1. \tag{2}$$

Moreover

$$\|i(x) - i(y)\| \ge |i(x)(x) - i(y)(x)| = 1, \text{ if } d(x, y) > 1. \tag{3}$$

From these observations it follows easily that i is an embedding. That i is a G-map follows from the G-invariance of the metric d.

Finally we prove that the image $i(X)$ is closed in $D(X)$. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in $i(X)$, $f_n \to f \in D(X)$, where $f_n = i(x_n)$, $x_n \in X$. Let $N \in \mathbb{N}$ be such that $\|f_n - f_m\| < 1/2$ for all $n, m > N$. Then by (3) and (2) we have that $d(x_n, x_m) \le 1$ and $d(x_n, x_m) = \|f_n - f_m\|$ for all $n, m > N$. Thus the sequence $(x_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in X. By Theorem XIV.2.3 in [Du] the metric d on X is complete and thus $x_n \to x$ for some $x \in X$. Clearly $i(x) = f$ which completes the proof. $\square$

Remark. In [Kw] and [Mu] the following equivariant version of the Wojdysławski embedding theorem is stated: for a compact Lie group G, every metrizable G-space X can be equivariantly embedded as a closed G-subspace in a convex G-subset of a Banach G-space. In the proof the Banach G-space is the Banach space $B(X)$ of all bounded continuous functions $f: X \to \mathbb{R}$ equipped with the sup-norm. A G-action on $B(X)$ is defined by $(gf)(x) = f(g^{-1}x)$ for $f \in B(X)$, $g \in G$ and $x \in X$. The embedding is then defined by $i: X \to B(X)$, $i(x)(y) = d(x, y)$, where d is a G-invariant bounded metric for X. However, this construction is not correct since the action on $B(X)$ need not be continuous. The embedding theorem itself holds, as follows for example from the results presented in Section 3 of [An–Ma]. The method in [An–Ma] doesn't seem to apply to non-compact groups G. The idea of using $D(X)$ instead of $B(X)$ is due to H. Abels.

We end this chapter with the following theorem, which will be used in proving the main theorem of this article.

3.4. Theorem. *Let G be a Lie group and let X be a proper locally linear G-manifold. Then there exists a Banach G-space B, a convex G-subspace C of B which is a proper G-space, an equivariant embedding i of X into B as a closed subspace such that $i(X) \subset C$ and a G-neighbourhood retraction $r\colon U \to i(X)$, where U is a G-neighbourhood of $i(X)$ in C.*

Proof. Since a second countable topological manifold can be embedded in a euclidean space as a closed subset, we may assume that X has a metric with the property $(*)$ of Proposition 2.6. The orbit space X/G is paracompact by Lemma 2.4.

Let $i\colon X \to B = D(X)$ be the G-embedding constructed as above. The subspace $C = \{f \in D(X) \mid f \geq 0, f \not\equiv 0\}$ is a convex G-invariant subspace of $D(X)$ and $i(X) \subset C$. By Theorem 2.15 in [E] X is a G-ANR. Now C is a paracompact proper G-space. Because the metric on C defined by the max-norm in $D(X)$ is G-invariant, it follows from Lemma 2.5 that C/G is paracompact. The existence of the G-retraction now follows from the definition of a G-ANR, Definition 2.1 in [E]. □

Remark. As was pointed out in the introduction the definition of a proper G-space in [E] is different from the definition used in this paper. However, in the definition of a G-ANR in [E] we only consider G-spaces with paracompact orbit spaces, and for these spaces the definitions agree, see Proposition 1.7(b) in [E].

4. G-CW complexes

For the definition and basic properties of G-CW complexes we refer to [Il1], [Il2] and [E]. In this chapter the group G is a Lie group. The following three propositions are not difficult to verify.

4.1. Proposition. *Let L and A be proper G-CW complexes, $B \subset A$ a G-subcomplex and $\alpha\colon B \to L$ a cellular G-map. Denote $\alpha_n = \alpha|\colon B^n \to L^n$, $n \geq 0$. Then $\overline{L} = A \cup_\alpha L$ is a proper G-CW complex with skeletons*

$$\overline{L}^n = A^n \cup_{\alpha_n} L^n$$

and $L \subset \overline{L}$ is a G-subcomplex. □

4.2. Proposition. *Let $L_0 \subset L_1 \subset \ldots$ be a sequence of proper G-CW complexes such that L_n is a G-subcomplex of L_{n+1} for each $n \geq 0$. If we give*

$L = \cup_{n\geq 0} L_n$ the weak topology with respect to the family $\{L_n\}_{n\geq 0}$ then L is a proper G-CW complex with skeletons

$$L^k = \bigcup_{n\geq 0} L^k_n. \qquad \square$$

The product of two ordinary CW complexes need not be a CW complex in a natural way, see the footnote on p. 401 in [Sp]. With G-CW complexes the situation is even more complicated, see p. 103 in [tD2]. However, we have the following result.

4.3. Proposition. *Let X be a locally compact proper G-CW complex, Y a locally compact CW complex and $B \subset Y$ a subcomplex. Then the product $X \times Y$ is a G-CW complex with skeletons*

$$(X \times Y)^n = \bigcup_{k+p=n} X^k \times Y^p$$

and $X \times B$ is a G-subcomplex of $X \times Y$. $\square$

The following is proved in [E].

4.4. Proposition. *Let G be a Lie group and X a G-CW complex. Then X and X/G are paracompact spaces.* $\square$

Next we consider the following commutative diagram of G-spaces and G-maps:

$$\begin{tikzcd}
X_0 \arrow[r, hook] \arrow[d, "f_0"] & X_1 \arrow[r, hook] \arrow[d, "f_1"] & X_2 \arrow[r, hook] \arrow[d, "f_2"] & \cdots \\
Y_0 \arrow[r, hook] & Y_1 \arrow[r, hook] & Y_2 \arrow[r, hook] & \cdots
\end{tikzcd}$$

where the inclusions $i_1, i_2, \ldots$ and $j_1, j_2, \ldots$ are G-cofibrations and the maps $f_0, f_1, \ldots$ are G-homotopy equivalences. Let $X = \cup X_k$ and $Y = \cup Y_k$ with weak topologies with respect to the families $\{X_k\}_{k\geq 0}$ and $\{Y_k\}_{k\geq 0}$, respectively. The spaces X and Y are Hausdorff spaces by Proposition 1.12 in [E]. The G-actions on the spaces X_k and Y_k define actions on the spaces X and Y. Continuity of the actions follows from the fact that the spaces $G \times X$ and $G \times Y$ have the weak topologies with respect to the families $\{G \times X_k\}_{k\geq 0}$ and $\{G \times Y_k\}_{k\geq 0}$, respectively, since G is a locally compact Hausdorff space. The maps f_k define a G-map $f: X \to Y$.

The proof of the following theorem is analogous to the proof of the corresponding non-equivariant theorem, proved in [tD1].

4.5. Theorem. *The map $f: X \to Y$ is a G-homotopy equivalence.* □

4.6. Theorem. *Let (A_n, B_n), $n = 1, 2, \ldots$, be a sequence of proper G-CW pairs. Let K_0 be a proper G-CW complex and suppose that for $n = 1, 2, \ldots$ the G-space K_n is obtained from K_{n-1} as an adjunction space*

$$K_n = A_n \cup_{\varphi_n} K_{n-1},$$

where $\varphi_n: B_n \to K_{n-1}$ is a G-map. If we denote $K = \cup_{n=0}^{\infty} K_n$ and give K the weak topology with respect to the family $\{K_n\}$ then the G-space K has the G-homotopy type of a G-CW complex.

Proof. We shall construct a sequence $L_0, L_1, \ldots$ of proper G-CW complexes and G-homotopy equivalences $f_n: K_n \to L_n$, $n = 0, 1, \ldots$ First define $L_0 = K_0$ and $f_0 = \mathrm{id}$. Suppose $n \geq 1$ and we have constructed $L_0, \ldots, L_{n-1}$ and $f_0, \ldots, f_{n-1}$. Let μ_n be a G-cellular approximation to the G-map $f_{n-1} \circ \varphi_n: B_n \to K_{n-1} \to L_{n-1}$ and define

$$L_n = A_n \cup_{\mu_n} L_{n-1}.$$

By Proposition 4.1 L_n is a proper G-CW complex.

Let

$$\alpha: A_n \cup_{\varphi_n} K_{n-1} \to A_n \cup_{f_{n-1}\varphi_n} L_{n-1}$$

be the canonical extension of the G-map f_{n-1}, that is the G-map induced by f_{n-1} and the identity map on A_n. Since f_{n-1} is a G-homotopy equivalence also α is a G-homotopy equivalence by Proposition 3.3 in [Il3].

Since $f_{n-1} \circ \varphi_n \simeq_G \mu_n$ there is a G-homotopy equivalence

$$\beta: A_n \cup_{f_{n-1}\varphi_n} L_{n-1} \to A_n \cup_{\mu_n} L_{n-1}$$

with $\beta | L_{n-1} = \mathrm{id}_{L_{n-1}}$, by Proposition 3.1 in [Il3].

Then $f_n = \beta \circ \alpha: K_n \to L_n$ is a G-homotopy equivalence extending f_{n-1}. Thus we obtain a commutative diagram

$$\begin{tikzcd}
K_0 \arrow[r, hook] \arrow[d, "f_0"] & K_1 \arrow[r, hook] \arrow[d, "f_1"] & K_2 \arrow[r, hook] \arrow[d, "f_2"] & \cdots \\
L_0 \arrow[r, hook] & L_1 \arrow[r, hook] & L_2 \arrow[r, hook] & \cdots
\end{tikzcd}$$

in which the inclusions are G-cofibrations and the maps $f_0, f_1, \ldots$ are G-homotopy equivalences.

Define $L = \cup_{n \geq 0} L_n$ and give L the weak topology with respect to the family $\{L_n\}_{n \geq 0}$. By Proposition 4.2 L is a proper G-CW complex. By Theorem 4.5 the map $f: K \to L$ induced by the maps f_n is a G-homotopy equivalence. □

5. TN G-coverings and G-nerves

In this chapter we introduce the concepts of a TN (tubular numerable) G-covering and the G-nerve of a TN G-covering, following the ideas in [Mu].

If $\mathcal{U}$ is a covering of a space X then for $A \subset X$ we denote

$$\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} \mid U \cap A \neq \emptyset\},$$

the *star* of A. A covering $\mathcal{U}$ of X is a *star-refinement* of another covering $\mathcal{V}$ of X, if the covering $\{\mathrm{St}(U,\mathcal{U})\}_{U\in\mathcal{U}}$ is a refinement of $\mathcal{V}$. A covering $\mathcal{U}$ of a G-space X is *G-invariant* if each of the sets $U \in \mathcal{U}$ is G-invariant. A covering $\mathcal{U}$ of a G-space X is a *G-covering* if $gU \in \mathcal{U}$ for every $U \in \mathcal{U}$, $g \in G$. Note that the index set for a G-covering can be regarded as a G-set. If $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ is a G-covering of X then define for each $\alpha \in \Lambda/G$

$$\tilde{U}_\alpha = \bigcup_{\lambda\in\alpha} U_\lambda.$$

In other words, if $\lambda \in \alpha$ then we have $\tilde{U}_\alpha = GU_\lambda$. The G-invariant covering $\tilde{\mathcal{U}} = \{\tilde{U}_\alpha\}_{\alpha\in\Lambda/G}$ is called the *saturation* of $\mathcal{U}$.

Let G be a Lie group and X a proper G-space. If O is an orbit in X and S is a slice at a point $x \in O$ then there exists a G-retraction $r\colon GS \to O$ such that $r^{-1}(x) = S$. The set $T = GS$ is called a *tube about the orbit O*. If V is an open subset of O then the open subset $r^{-1}(V)$ of X is called the *open tube segment of T generated by V*.

By a *tube-segmental G-covering* we mean a G-covering $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ such that for each $\alpha \in \Lambda/G$ the set $\tilde{U}_\alpha = \cup_{\lambda\in\alpha}U_\lambda$ is a tube T_α about an orbit O_α with a G-retraction $r_\alpha\colon T_\alpha \to O_\alpha$ and for each $\lambda \in \alpha$ the set U_λ is a tube segment of T_α. The orbit O_α is called the *central orbit* of T_α. The covering $\tilde{\mathcal{U}} = \{T_\alpha\}_{\alpha\in\Lambda/G}$ is a G-invariant covering of X by tubes.

An open tube-segmental G-covering $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ is a *TN G-covering* (*tubular numerable G-covering*) if there exists a partition of unity $\{p_\alpha\}_{\alpha\in\Lambda/G}$, where the maps $p_\alpha\colon X \to [0,1]$ are G-invariant, such that $\mathrm{supp}(p_\alpha) \subset T_\alpha$ for every $\alpha \in \Lambda/G$.

Next we construct the *G-nerve* of a TN G-covering. Let $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ be a TN G-covering of a proper G-space X, $\tilde{\mathcal{U}} = \{T_\alpha\}_{\alpha\in\Lambda/G}$ the saturation of $\mathcal{U}$ and denote by $r_\alpha\colon T_\alpha \to O_\alpha$ the retractions, $\alpha \in \Lambda/G$. Let $N = N(\tilde{\mathcal{U}})$ denote the nerve of $\tilde{\mathcal{U}}$, that is, a simplicial complex with vertices T_α, $\alpha \in \Lambda/G$ and simplexes by the following condition: if $n \geq 1$ and $\alpha_0, ..., \alpha_n$ are distinct elements of Λ/G and $\cap_{i=0}^n T_{\alpha_i} \neq \emptyset$ then $\{T_{\alpha_0}, ..., T_{\alpha_n}\}$ is an n-simplex of N. Denote by $N_n = N_n(\tilde{\mathcal{U}})$ the set of n-simplexes. We assume that Λ/G is well-ordered.

For each n-simplex $\sigma = \{T_{\alpha_0}, ..., T_{\alpha_n}\}$, where $\alpha_0 < ... < \alpha_n$, of N we define an open subspace K_σ of $O_{\alpha_0} \times ... \times O_{\alpha_n}$ by

$$K_\sigma = \bigcup\{O_{\lambda_0} \times ... \times O_{\lambda_n} \mid \lambda_i \in \alpha_i,\ i = 0, ..., n,\ \cap_{i=0}^n U_{\lambda_i} \neq \emptyset\},$$

where we denote $O_{\lambda_i} = r_{\alpha_i}(U_{\lambda_i}) = U_{\lambda_i} \cap O_{\alpha_i} \subset O_{\alpha_i}$. A G-action on K_σ can be defined by

$$(g, (x_0, ..., x_n)) \mapsto (gx_0, ..., gx_n).$$

We now define a G-space $K_* = K_*(\mathcal{U})$, the *simplicial G-nerve* of $\mathcal{U}$ as follows: The nth space $K_n = K_n(\mathcal{U})$, $n \geq 0$, of K_* is

$$K_n = \sum_{\sigma \in N_n} K_\sigma$$

and the face operators $\partial_i: K_n \to K_{n-1}$, $0 \leq i \leq n$, are defined as follows: Let $\sigma = \{T_{\alpha_0}, ..., T_{\alpha_n}\}$, $\alpha_0 < ... < \alpha_n$, and $x = (x_0, ..., x_n) \in K_\sigma$. Then $\tau_i = \{T_{\alpha_0}, ..., \hat{T}_{\alpha_i}, ..., T_{\alpha_n}\}$, obtained from σ by omitting T_{α_i}, is an $(n-1)$-simplex of N and $(x_0, ..., \hat{x}_i, ..., x_n) \in K_{\tau_i}$. Now define

$$\partial_i(x) = (x_0, ..., \hat{x}_i, ..., x_n) \in K_{\tau_i}.$$

This gives us maps $\partial_i: K_n \to K_{n-1}$, $0 \leq i \leq n$.

The G-actions on the spaces K_σ give us a G-action on every K_n, $n \geq 0$, and the face operators are equivariant maps.

Next we consider the space

$$\sum_{n \geq 0} K_n(\mathcal{U}) \times \Delta^n$$

and the equivalence relation generated by

$$(\partial_i x, t) \sim (x, \delta_i t),$$

where $x \in K_n(\mathcal{U})$, $t \in \Delta^{n-1}$, $\delta_i(t_0, ..., t_{n-1}) = (t_0, ..., t_{i-1}, 0, t_i, ..., t_{n-1}) \in \Delta^n$ and $i \in \{0, ..., n\}$, $n \geq 1$.

We denote the quotient space

$$K(\mathcal{U}) = |K_*(\mathcal{U})| = \left(\sum_{n \geq 0} K_n(\mathcal{U}) \times \Delta^n \right) / \sim,$$

the *geometric G-nerve* of $\mathcal{U}$. Denote by $\pi: \sum_{n \geq 0} K_n(\mathcal{U}) \times \Delta^n \to K(\mathcal{U})$ the projection. The image of $\sum_{i=0}^n K_i(\mathcal{U}) \times \Delta^i$ in $K(\mathcal{U})$ is called the *n-skeleton* of $K(\mathcal{U})$,and denoted by $K^n(\mathcal{U})$.

Furthermore we denote $\pi(x, t) = |x; t|$, also if $x = (x_{\alpha_0}, ..., x_{\alpha_n}) \in K_\sigma \subset O_{\alpha_0} \times ... \times O_{\alpha_n}$ and $t = (t_0, ..., t_n) \in \Delta^n$, denote $\pi(x, t) = |x_{\alpha_0}, ..., x_{\alpha_n}; t_0, ..., t_n|$.

The G-actions on the spaces K_n fit together and give an action on $K(\mathcal{U})$ by $g|x; t| = |gx; t|$. Since G is locally compact the action on the quotient space is continuous.

Since there are no identifications between points of $K_0(\mathcal{U})$ we have that $K^0(\mathcal{U}) = K_0(\mathcal{U}) = \sum_{\alpha\in\Lambda/G} O_\alpha$. Let $n \geq 1$. Each $\bar{t} \in \partial\Delta^n$ is of the form $\bar{t} = \delta_i(t)$ for some $t \in \Delta^{n-1}$, $i \in \{0, ..., n\}$. For each $n \geq 1$ we can define a continuous map $\alpha_n: K_n \times \partial\Delta^n \to K^{n-1}$ by

$$\alpha_n(x, \delta_i(t)) = |\partial_i x; t|.$$

We obtain a commutative diagram

$$\begin{tikzcd} K_n \times \partial\Delta^n \arrow[r, "\mathrm{id}\times\iota_n"] \arrow[d, "\alpha_n"'] & K_n \times \Delta^n \arrow[d, "\pi|"] \\ K^{n-1} \arrow[r, "i_{n-1}"'] & K^n \end{tikzcd}$$

which is a pushout, see p. 95–96 in [tD2], and K^n can be regarded as the adjunction space $(K_n \times \Delta^n) \cup_{\alpha_n} K^{n-1}$. The map ι_n is a cofibration and thus $\mathrm{id} \times \iota_n$ is a G-cofibration and $i_{n-1}: K^{n-1} \to K^n$ is a G-cofibration by Proposition II.1.2 in [tD2]. The spaces K^n are Hausdorff spaces by Proposition 1.11 in [E].

The topology of $K(\mathcal{U})$ is the weak topology with respect to the filtration $K^0(\mathcal{U}) \subset K^1(\mathcal{U}) \subset ...$ and thus $K(\mathcal{U})$ is a Hausdorff space, see Proposition 1.12 in [E].

5.1. Proposition. *Let $\mathcal{U} = \{U_\lambda\}_{\lambda\in\Lambda}$ be a TN G-covering of a proper G-space X with a G-invariant partition of unity $\{p_\alpha\}_{\alpha\in\Lambda/G}$ subordinate to $\mathcal{U}$. Define $P: X \to K(\mathcal{U})$ by*

$$P(x) = |r_{\alpha_0}(x), ..., r_{\alpha_n}(x); p_{\alpha_0}(x), ..., p_{\alpha_n}(x)|,$$

where $x \in X$ and $\{\alpha_0, ..., \alpha_n\} = \{\alpha \in \Lambda/G \mid p_\alpha(x) \neq 0\}$, $\alpha_0 < ... < \alpha_n$. Then P is a continuous G-map.

Proof. Denote $\sigma = \{T_{\alpha_0}, ..., T_{\alpha_n}\} \in N(\tilde{\mathcal{U}})$. Since $p_{\alpha_i}(x) > 0$ we see that $x \in T_{\alpha_i}$ and thus there exists a $\lambda_i \in \alpha_i$ for which $x \in U_{\lambda_i}$, $i = 0, ..., n$. Thus $x \in \cap_{i=0}^n U_{\lambda_i}$ and hence $(r_{\alpha_0}(x), ..., r_{\alpha_n}(x)) \in K_\sigma$ by the definition of K_σ. Thus $P(x) \in K(\mathcal{U})$.

To show the continuity of P we first consider the following simplicial space, compare [Se]. The nth space $X_{\tilde{\mathcal{U}}^n}$ is given by

$$\sum_{\sigma\in N_n(\tilde{\mathcal{U}})} T_\sigma, \quad \text{where } T_\sigma = \bigcap_{\alpha\in\sigma} T_\alpha \neq \emptyset,$$

that is,

$$X_{\tilde{\mathcal{U}}^n} = \bigcup_{\sigma\in N_n(\tilde{\mathcal{U}})} (T_\sigma \times \{\sigma\}).$$

The face maps are defined as follows: If $n \geq 1$, $(x, \sigma) \in X_{\tilde{\mathcal{U}}^n}$ and $i \in \{0, ..., n\}$, then $x \in T_\sigma$ and T_σ is of the form $T_\sigma = T_{\alpha_0} \cap ... \cap T_{\alpha_n}$, where $\alpha_0 < ... < \alpha_n \in \Lambda/G$. Now define $\tau_i = \{T_{\alpha_0}, ..., \hat{T}_{\alpha_i}, ..., T_{\alpha_n}\}$, that is, τ_i is obtained from σ by omitting the ith term, and $\partial_i(x, \sigma) = (x, \tau_i) \in X_{\tilde{\mathcal{U}}^{n-1}}$.

As before we define an equivalence relation on the space $\sum_{n \geq 0} X_{\tilde{\mathcal{U}}^n} \times \Delta^n$ generated by the relations $(\partial_i x; t) \sim (x; \delta_i t)$ and consider the quotient space

$$BX_{\tilde{\mathcal{U}}} = \sum_{n \geq 0} X_{\tilde{\mathcal{U}}^n} \times \Delta^n / \sim .$$

Denote by $\tilde{\pi}$ the projection.

A G-action on $BX_{\tilde{\mathcal{U}}}$ can be defined by

$$g|(x, \sigma); t| = |(gx, \sigma); t|.$$

Define a function $P': X \to BX_{\tilde{\mathcal{U}}}$ as follows: If $x \in X$ and $\{\alpha_0, ..., \alpha_n\} = \{\alpha \in \Lambda/G \mid p_\alpha(x) \neq 0\}$, $\alpha_0 < ... < \alpha_n$, then denote $\sigma = \{T_{\alpha_0}, ..., T_{\alpha_n}\}$ and

$$P'(x) = |(x, \sigma); p_{\alpha_0}(x), ..., p_{\alpha_n}(x)|.$$

We now prove that P' is continuous. Let V' be a neighbourhood of x such that $p_\alpha | V' \equiv 0$ for all but a finite number of α. Let $\{\beta_0, ..., \beta_m\} = \{\beta \in \Lambda/G \mid p_\beta(y) \neq 0$ for some $y \in V'\}$. If $x \notin T_{\beta_j}$ for some $\beta_j \in \{\beta_0, ..., \beta_m\}$ we can find a neighbourhood of x in which $p_{\beta_j} \equiv 0$ and we exclude β_j. Thus we may assume that $x \in T_{\beta_j}$ for all $j = 0, ..., m$. Assume $\beta_0 < ... < \beta_m$. Now $V = V' \cap \cap_{j=0}^m T_{\beta_j}$ is a neighbourhood of x. Let $\sigma' = \{T_{\beta_0}, ..., T_{\beta_m}\}$. Then we can define

$$\varphi: V \to X_{\tilde{\mathcal{U}}^m} \times \Delta^m$$

by

$$y \mapsto ((y, \sigma'); p_{\beta_0}(y), ..., p_{\beta_m}(y)),$$

which is clearly continuous. Now

$$P' | V = \tilde{\pi} \circ \varphi$$

and hence P' is continuous at x. That P' is a G-map follows from

$$P'(gx) = |(gx, \sigma); p_{\alpha_0}(gx), ..., p_{\alpha_n}(gx)| = |(gx, \sigma); p_{\alpha_0}(x), ..., p_{\alpha_n}(x)| = gP'(x).$$

Next we define a map

$$f_*: X_{\tilde{\mathcal{U}}^*} = \sum_{n \geq 0} \sum_{\sigma \in N_n(\tilde{\mathcal{U}})} (T_\sigma \times \{\sigma\}) \to K_*(\mathcal{U})$$

as follows: if now $(x,\sigma) \in X_{\tilde{\mathcal{U}}^n}$, where $x \in T_\sigma = T_{\alpha_0} \cap ... \cap T_{\alpha_n}$, $\alpha_0 < ... < \alpha_n \in \Lambda/G$, then define $f_n(x,\sigma) = (r_{\alpha_0}(x), ..., r_{\alpha_n}(x)) \in K_\sigma \subset K_n(\mathcal{U})$. Clearly f_* is a continuous G-map.

Now f_* satisfies

$$f_*(\partial_i(x,\sigma)) = \partial_i f_*(x,\sigma),$$

where ∂_i on the left side denotes the face map in $X_{\tilde{\mathcal{U}}^*}$ and on the right side the face map in $K_*(\mathcal{U})$. From this it follows that f_* defines a continuous map

$$|f_*|: BX_{\tilde{\mathcal{U}}} \to K(\mathcal{U})$$

such that the diagram

$$\begin{tikzcd}
\sum_{n\geq 0} X_{\tilde{\mathcal{U}}^n} \times \Delta^n \arrow[r, "\sum f_n \times \mathrm{id}"] \arrow[d, "\tilde{\pi}"'] & \sum_{n\geq 0} K_n(\mathcal{U}) \times \Delta^n \arrow[d, "\pi"] \\
BX_{\tilde{\mathcal{U}}} \arrow[r, "|f_*|"'] & K(\mathcal{U})
\end{tikzcd}$$

commutes. Clearly $|f_*|$ is a G-map.

Now $P = |f_*| \circ P': X \to BX_{\tilde{\mathcal{U}}} \to K(\mathcal{U})$, which proves that P is a continuous G-map. □

Later we shall need also the following result concerning G-coverings.

5.2. Proposition. *Let $\mathcal{U} = \{U_\alpha \mid \alpha \in \mathcal{A}\}$ be an open G-covering of a proper G-space X and suppose we have a G-invariant metric d for X. Then there exists an open G-covering $\mathcal{W}$ of X which is a star-refinement of $\mathcal{U}$.*

Proof. We shall use the concept of a locally starring sequence of coverings, see Definition VIII.3.6 in [Du]. Now it follows from the proof of Theorem IX.5.3 in [Du] that in a metric space there is one sequence $\{\mathcal{B}_n \mid n \in \mathbb{Z}^+\}$ that is locally starring for every open covering, namely $\mathcal{B}_n = \{B(x,1/n) \mid x \in X\}$. Since our metric d is G-invariant, these coverings $\mathcal{B}_n$ are G-coverings.

Next we look at the proof of Theorem VIII.3.7 in [Du]. Now the sequence $\{\mathcal{B}_n\}_{n\in\mathbb{Z}^+}$ is locally starring for the given covering $\mathcal{U}$. Form a new covering $\mathcal{V}$ by

$$(*) \qquad \mathcal{V} = \{V \Subset X \mid \exists n : [V \subset B \in \mathcal{B}_n] \wedge [\mathrm{St}(V,\mathcal{B}_n) \subset \text{some } U_\alpha]\}.$$

As in [Du], $\mathcal{V}$ is an open covering and a barycentric refinement of $\mathcal{U}$. It is also easy to check that $\mathcal{V}$ is a G-covering, since the coverings $\mathcal{U}$ and $\mathcal{B}_n$ are G-coverings. Then define $\mathcal{W}$ from $\{\mathcal{V},\mathcal{B}_n\}$ as $\mathcal{V}$ was defined from $\{\mathcal{U},\mathcal{B}_n\}$ in $(*)$. Then $\mathcal{W}$ is an open G-covering and a barycentric refinement of $\mathcal{V}$. By VIII.3.4 in [Du] $\mathcal{W}$ is a star-refinement of $\mathcal{U}$. □

6. Proof of the main theorem

Let X be a space, $H = (h_t)_{t \in I} : X \times I \to X$ a homotopy and $\mathcal{U}$ a covering of X. We say that (h_t) is a $\mathcal{U}$-homotopy, if for every $x \in X$ there exists a $U \in \mathcal{U}$ such that $h_t(x) \in U$ for all $t \in I$. We say that X is $\mathcal{U}$-dominated by a space Y if there exist maps $f : X \to Y$ and $g : Y \to X$ and a $\mathcal{U}$-homotopy between $g \circ f$ and id_X.

6.1. Proposition. *Let G be a Lie group and X a proper locally linear G-manifold. Then, for any open G-covering $\mathcal{V}$ of X there exists a TN G-covering $\mathcal{U}$ of X such that X is $\mathcal{V}$-G-dominated by the G-nerve $K(\mathcal{U})$ of $\mathcal{U}$.*

Proof. Let $B \supset C \supset U$ and $r : U \to X$ be as in Theorem 3.4. We think of X as a subset of B. Since C is locally convex and U is a G-neighbourhood of X in C, there is a G-covering $\mathcal{W} = \{W_\beta\}$ of X by convex open sets in U which is a refinement of the G-covering $\{r^{-1}(V)\}_{V \in \mathcal{V}}$: For each $x \in X$ choose a convex neighbourhood $U_x \subset U$ for which $U_x \subset r^{-1}(V)$ for some $V \in \mathcal{V}$. Then the covering $\{gU_x\}_{g \in G, x \in X}$ can be chosen as $\mathcal{W}$.

Next we construct a TN G-covering $\mathcal{U}$ of X which is a star-refinement of the open G-covering $\{W_\beta \cap X\}$ of X: First by Lemma 5.2 we can choose an open G-covering $\{W'_\gamma\}_{\gamma \in \Gamma}$ of X which is a star-refinement of $\{W_\beta \cap X\}$. For each $x \in X$ choose a W'_γ containing x. Then there exists a slice S_x at x and a neighbourhood D_x of e in G such that $D_x S_x \subset W'_\gamma$. Now $D_x S_x$ is an open tube segment in GS_x and $\{gD_x S_x\}_{g \in G, x \in X}$ is an open G-covering of X which refines $\{W'_\gamma\}_{\gamma \in \Gamma}$. Now $\{GS_x\}_{x \in X}$ is an open G-invariant covering of X and $\{\pi(GS_x)\}$ is an open covering of X/G. Since X/G is paracompact there exists a partition of unity subordinate to this covering. This gives us a G-invariant partition of unity on X, subordinate to the covering $\{GS_x\}_{x \in X}$.

Denote the obtained TN G-covering by $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ and the partition of unity by $\{p_\alpha\}_{\alpha \in \Lambda/G}$.

Let $K = K(\mathcal{U})$, the G-nerve of $\mathcal{U}$, and $P : X \to K$ the G-map defined in Proposition 5.1. We can define a continuous G-map $q' : K \to U$ by setting

$$q'(y) = \sum_{i=0}^{n} t_i x_{\alpha_i},$$

where $y = |x_{\alpha_0}, ..., x_{\alpha_n}; t_0, ..., t_n| \in K$.

Clearly $q'(K) \subset C$ since C is convex. We shall prove that $q'(K) \subset U$: By definition there exist $\{U_{\lambda_0}, ..., U_{\lambda_n}\} \subset \mathcal{U}$ such that $\lambda_i \in \alpha_i$, $x_{\alpha_i} \in O_{\lambda_i} \subset U_{\lambda_i}$, $i = 0, ..., n$ and $\cap_{i=0}^n U_{\lambda_i} \neq \emptyset$. Choose $x \in \cap_{i=0}^n U_{\lambda_i}$. Then $x \in \cup_{i=0}^n U_{\lambda_i} \subset \mathrm{St}(x, \mathcal{U})$. Since $\mathcal{U}$ is a star-refinement of $\{W_\beta \cap X\}$ there exists a β such that

$$\{x_{\alpha_0}, ..., x_{\alpha_n}\} \subset \mathrm{St}(x, \mathcal{U}) \subset W_\beta.$$

Since W_β is convex, we have

$$q'(y) = \sum_{i=0}^{n} t_i x_{\alpha_i} \in W_\beta \subset U,$$

which proves the claim.

Then define

$$q = r \circ q' \colon K \to U \to X$$

and a $\mathcal{V}$-G-homotopy $(h_t)_{t \in I} \colon q \circ P \simeq_G \mathrm{id}_X$ by

$$h_t(x) = r(tx + (1-t)(q' \circ P(x))), \;\; x \in X, \;\; t \in I.$$

We have to verify that $tx + (1-t)(q' \circ P(x)) \in U$: Let $x \in X$ and $\{\alpha_0, ..., \alpha_n\} = \{\alpha \in \Lambda/G \mid p_\alpha(x) > 0\}$. Then there exist $\{U_{\lambda_0}, ..., U_{\lambda_n}\} \subset \mathcal{U}$ such that $\lambda_i \in \alpha_i$, $i = 0, ..., n$, and $x \in \cap_{i=0}^{n} U_{\lambda_i}$. Furthermore there exists a β such that

$$\{x, r_{\alpha_0}(x), ..., r_{\alpha_n}(x)\} \subset \cup_{i=0}^{n} U_{\lambda_i} \subset \mathrm{St}(x, \mathcal{U}) \subset W_\beta.$$

Since W_β is convex, we see that

$$tx + (1-t)(q' \circ P(x)) = tx + (1-t) \sum_{i=0}^{n} p_{\alpha_i}(x) r_{\alpha_i}(x) \in W_\beta \subset U.$$

Since there exists a $V \in \mathcal{V}$ such that $W_\beta \subset r^{-1}V$, we have $h_t(x) \in r(W_\beta) \subset V$ for every $t \in I$. □

6.2. Theorem. *The G-nerve $K(\mathcal{U})$ of a TN G-covering of a proper G-space X has the G-homotopy type of a G-CW complex.*

Proof. We first prove that each space $K_n(\mathcal{U})$ is a G-CW complex. Now $K_n = \sum_{\sigma \in N_n} K_\sigma$, where each K_σ is an open G-subspace of a space of the form $O_{\alpha_0} \times ... \times O_{\alpha_n}$, where $O_{\alpha_0}, ..., O_{\alpha_n}$ are orbits in X and hence K_σ is a smooth proper G-manifold. Thus it follows from [Il4] that K_σ is a locally compact proper G-CW complex and hence K_n is a locally compact proper G-CW complex.

By Proposition 4.3 $(K_n \times \Delta^n, K_n \times \partial\Delta^n)$ is a proper G-CW complex pair (A_n, B_n). Now K^0 is a proper G-CW complex and for every $n = 1, 2, ...$ K^n can be regarded as an adjunction space $K^n = A_n \cup_{\alpha_n} K^{n-1}$, where $\alpha_n \colon B_n = K_n \times \partial\Delta^n \to K^{n-1}$ is a G-map, as in Chapter 5. The G-nerve $K(\mathcal{U}) = \cup_{n=0}^{\infty} K^n$ has the weak topology with respect to the family $\{K^n\}$ and thus the claim follows from Theorem 4.6. □

6.3. Theorem. *Let G be a Lie group and M a proper locally linear G-manifold. Then M has the G-homotopy type of a G-CW complex.*

Proof. We have proved that M is G-dominated by the G-nerve $K(\mathcal{U})$ of a TN-G-covering $\mathcal{U}$ of M. By the previous theorem $K(\mathcal{U})$ has the G-homotopy type of a G-CW complex. Thus M is dominated by a G-CW complex and the theorem follows from Theorem 4.18 in [E]. □

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