IRREDUCIBLE CHARACTERS FOR ALGEBRAIC GROUPS IN CHARACTERISTIC THREE

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Abstract. In this note, we determine all the irreducible characters for the special orthogonal group $SO(7, K)$ and the symplectic group $Sp(6, K)$ over an algebraically closed field K of characteristic 3.

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The determination of all the irreducible characters is a big theme in the modular representations of the algebraic groups and the related finite groups of Lie type. But so far only a little is known concerning it in the case where the characteristic of the base field less than the Coxeter number. An element $\mathfrak{x}_{(p^n-1)\rho-\lambda} \in \mathfrak{u}_n^-$ for each irreducible module $L(\lambda)$ with $\lambda \in X_n(T)$ was defined in [L2] and [X1]. This element could be used in constructing a certain basis for $L(\lambda)$, computing $\dim L(\lambda)$, and determining $\mathrm{ch}\,(L(\lambda))$. In this way, we determined all the irreducible characters for the special linear groups $SL(5, K)$, $SL(6, K)$ and $SL(7, K)$, the special orthogonal group $SO(7, K)$ and the symplectic group $Sp(6, K)$ over an algebraically closed field K of characteristic 2 in [XY], [YZ1] and [YZ2]. In the present note, we shall work out all the irreducible characters for the special orthogonal group $SO(7, K)$ and the symplectic group $Sp(6, K)$ over an algebraically closed field K of characteristic 3. We shall freely use the notations in [YZ1] without further comments.

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1. Preliminaries

Let G be either the special orthogonal group $SO(7,K)$ or the symplectic group $Sp(6,K)$ over an algebraically closed field K of characteristic 3. Take a Borel subgroup B and a maximal torus T of G with $T\subset B$. Let $X(T)$ be the character group of T, which is also called the weight lattice of G with respect to T. Let $R\subset X(T)$ be the root system associated to (G,T), and choose a positive root system R_+ in such a way that $-R_+$ corresponds to B. We call

$$X(T)_+=\{\lambda\in X(T)\mid\langle\lambda,\alpha^\vee\rangle\geq 0\quad\text{for all }\alpha\in R_+\}$$

the set of dominant weights, where $\alpha^\vee$ is the dual root of α. Let

$$S=\{\alpha_1,\alpha_2,\alpha_3\}$$

be the set of simple roots of G such that

$$\begin{aligned}R_+=\{&\alpha_1,\alpha_2,\alpha_{12}=\alpha_1+\alpha_2,\alpha_{233}=\alpha_2+2\alpha_3,\alpha_{1233}=\alpha_1+\alpha_2+2\alpha_3,\\&\alpha_{12233}=\alpha_1+2\alpha_2+2\alpha_3;\alpha_3,\alpha_{23}=\alpha_2+\alpha_3,\alpha_{123}=\alpha_1+\alpha_2+\alpha_3\}\end{aligned}$$

for $G=SO(7,K)$ and

$$\begin{aligned}R_+=\{&\alpha_1,\alpha_2,\alpha_{12}=\alpha_1+\alpha_2,\alpha_{23}=\alpha_2+\alpha_3,\alpha_{1223}=\alpha_1+2\alpha_2+\alpha_3,\\&\alpha_{123}=\alpha_1+\alpha_2+\alpha_3;\alpha_3,\alpha_{223}=2\alpha_2+\alpha_3,\alpha_{11223}=2\alpha_1+2\alpha_2+\alpha_3\}\end{aligned}$$

for $G=Sp(6,K)$. Let $\omega_i(1\leq i\leq 3)$ be the fundamental weights in $X(T)$ with $\langle\omega_i,\alpha_j^\vee\rangle=\delta_{ij}$, the Kronecker delta, and denote by $\lambda=(\lambda_1,\lambda_2,\lambda_3)$ the weight $\lambda=\lambda_1\omega_1+\lambda_2\omega_2+\lambda_3\omega_3$ with $\lambda_1,\lambda_2,\lambda_3\in\mathbb{Z}$. Then

$$X(T)_+=\{(\lambda_1,\lambda_2,\lambda_3)\in X(T)\mid\lambda_1,\lambda_2.\lambda_3\geq 0\}.$$

Let $W=N_G(T)/T$ be the Weyl group and let W_2 be the affine Weyl group of G. It is well-known that for $\lambda\in X(T)_+$, $H^0(\lambda)$ is the induced G-module from the 1-dimensional B-module K_λ which contains a unique irreducible G-submodule $L(\lambda)$ of the highest weight λ. In this way, $X(T)_+$ parameterizes the finite-dimensional irreducible G-modules. We set $\mathrm{ch}\,(\lambda)=\mathrm{ch}\,(H^0(\lambda))$ and $\mathrm{ch}_3\,(\lambda)=\mathrm{ch}\,(L(\lambda))$ for all $\lambda\in X(T)_+$. Moreover, $\mathrm{ch}\,(\lambda)$ is given by the Weyl character formula, and for $\lambda=(a,b,c)\in X(T)_+$, one has for $G=SO(7,K)$

$$\begin{aligned}\dim H^0(a,b,c)=&\frac{1}{720}(a+1)(b+1)(c+1)(a+b+2)(b+c+2)(2b+c+3)\\&(a+b+c+3)(a+2b+c+4)(2a+2b+c+5),\end{aligned}$$

and for $G = Sp(6, K)$

$$\dim H^0(a,b,c) = \frac{1}{720}(a+1)(b+1)(c+1)(a+b+2)(b+c+2)(b+2c+3)$$
$$(a+b+c+3)(a+b+2c+4)(a+2b+2c+5).$$

Let F^n be the n-th Frobenius morphism of G with $G_n \subset G$ the scheme-theoretic kernel of F^n. Let $V^{[n]}$ be the Frobenius twist for any G-module V. It is well-known that $V^{[n]}$ is trivial regarding as a G_n-module. Moreover, any G-module M has such a form if the action of G_n on M is trivial. Let

$$X_n(T) = \{(\lambda_1, \lambda_2, \lambda_3) \in X(T)_+ \mid \lambda_1, \lambda_2, \lambda_3 < 3^n\}.$$

Then the irreducible G-modules $L(\lambda)$'s with $\lambda \in X_n(T)$ remain irreducible regarded as the G_n-modules. On the other hand, any irreducible G_n-module is isomorphic to exactly one of them.

For $\lambda \in X(T)_+$, one has the unique decomposition

$$\lambda = \lambda^0 + 3^n \lambda^1 \qquad \text{with} \quad \lambda^0 \in X_n(T), \lambda^1 \in X(T)_+.$$

Then the Steinberg tensor product theorem tells us that

$$L(\lambda) \cong L(\lambda^0) \otimes L(\lambda^1)^{[n]}.$$

Therefore we can determine all the characters $\mathrm{ch}_3(\lambda)$ with $\lambda \in X(T)_+$ by using the Steinberg tensor product theorem, provided that all the characters $\mathrm{ch}_3(\lambda)$ with $\lambda \in X_1(T)$ are known.

Recall the strong linkage principle in [1]. We define a strong linkage relation $\mu \uparrow \lambda$ in $X_+(T)$ if $L(\mu)$ occurs as a composition factor in $H^0(\lambda)$. Then $H^0(\lambda)$ is irreducible when λ is a minimal weight in $X(T)_+$ with respect to the partial ordering determined by the strong linkage relations.

Let $\mathfrak{g}$ be the semisimple Lie algebra over $\mathbb{C}$ which has the same type as G, and $\mathfrak{U}$ the universal enveloping algebra of $\mathfrak{g}$. Let $e_\alpha, f_\alpha, h_i (\alpha \in R_+, i = 1,2,3)$ be a Chevalley basis of $\mathfrak{g}$. We also denote $e_{\alpha_I}, f_{\alpha_I}$ by e_I, f_I, respectively, where $I \in \mathcal{B} = \{1, 2, 12, 233, 1233, 12233; 3, 23, 123\}$ for $G = SO(7, K)$ and $I \in \mathcal{C} = \{1, 2, 12, 23, 123, 1223; 3, 223, 11223\}$ for $G = Sp(6, K)$, respectively. The Kostant $\mathbb{Z}$-form $\mathfrak{U}_\mathbb{Z}$ of $\mathfrak{U}$ is the $\mathbb{Z}$-subalgebra of $\mathfrak{U}$ generated by the elements $e_\alpha^{(k)} := e_\alpha^k/k!, f_\alpha^{(k)} := f_\alpha^k/k!$ for $\alpha \in R_+$ and $k \in \mathbb{Z}_+$. Set

$$\binom{h_i + c}{k} := \frac{(h_i + c)(h_i + c - 1)\cdots(h_i + c - k + 1)}{k!}.$$

Then $\binom{h_i+c}{k} \in \mathfrak{U}_\mathbb{Z}$ for $i = 1,2,3$, $c \in \mathbb{Z}$, $k \in \mathbb{Z}_+$. Define $\mathfrak{U}_K := \mathfrak{U}_\mathbb{Z} \otimes K$, called the hyperalgebra over K associated to $\mathfrak{g}$. Let $\mathfrak{U}_K^+, \mathfrak{U}_K^-, \mathfrak{U}_K^0$ be the positive part, negative part and zero part of $\mathfrak{U}_K$, respectively. They are generated by $e_\alpha^{(k)}$, $f_\alpha^{(k)}$ and $\binom{h_i}{k}$, respectively. By abuse of notations, the images in $\mathfrak{U}_K$ of $e_\alpha^{(k)}, f_\alpha^{(k)}, \binom{h_i+c}{k}$, etc. will be denoted by the same notations. The algebra $\mathfrak{U}_K$ is a Hopf algebra which has a triangular decomposition $\mathfrak{U}_K = \mathfrak{U}_K^- \mathfrak{U}_K^0 \mathfrak{U}_K^+$. Given a positive integer n, let $\mathfrak{u}_n$ be the subalgebra of $\mathfrak{U}_K$ generated by the elements $e_\alpha^{(k)}, f_\alpha^{(k)}, \binom{h_i}{k}$ for $\alpha \in R_+, i = 1,2,3$ and $0 \leq k < 3^n$. Denote by $\mathfrak{u}_n^+, \mathfrak{u}_n^-, \mathfrak{u}_n^0$ the positive part, negative part and zero part of $\mathfrak{u}_n$, respectively. Then we also have a triangular decomposition $\mathfrak{u}_n = \mathfrak{u}_n^- \mathfrak{u}_n^0 \mathfrak{u}_n^+$. Given a totally ordering in R_+. It is known that the PBW-type bases for $\mathfrak{U}_K$, resp. $\mathfrak{u}_n$, have the form of

$$\prod_{\alpha \in R_+} f_\alpha^{(a_\alpha)} \prod_{i=1}^{3} \binom{h_i}{b_i} \prod_{\alpha \in R_+} e_\alpha^{(c_\alpha)}$$

with $a_\alpha, b_i, c_\alpha \in \mathbb{Z}_+$ resp. with $0 \leq a_\alpha, b_i, c_\alpha < 3^n$. In particular, $\mathfrak{u} = \mathfrak{u}_1$ is precisely the restricted enveloping algebra of $\mathfrak{g}$.

Let $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in X_n(T)$. We set $\lambda_I = \sum_{i \in I} \lambda_i$ for $I \in \mathcal{B}$ or $\mathcal{C}$. Following [12] and [13], we define an element $\mathfrak{x}_\lambda$ in $\mathfrak{u}_n^-$ by

$$\mathfrak{x}_\lambda = f_3^{(\lambda_3)} f_2^{(\lambda_{23})} f_3^{(\lambda_{223})} f_2^{(\lambda_2)} f_1^{(\lambda_{1223})} f_2^{(\lambda_{123})} f_3^{(\lambda_{11223})} f_2^{(\lambda_{12})} f_1^{(\lambda_1)}$$

for $G = SO(7, K)$, and by

$$\mathfrak{x}_\lambda = f_3^{(\lambda_3)} f_2^{(\lambda_{233})} f_3^{(\lambda_{23})} f_2^{(\lambda_2)} f_1^{(\lambda_{12233})} f_2^{(\lambda_{1233})} f_3^{(\lambda_{123})} f_2^{(\lambda_{12})} f_1^{(\lambda_1)}$$

for $G = Sp(6, K)$. As a special case of [X1, Theorems 6.5 and 6.7], one has

Theorem 1. *Assume that $\mathfrak{g}$ is a simple Lie algebra of the group either $SO(7, K)$ or $Sp(6, K)$ over an algebraically closed field K of characteristic 3. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in X_n(T)$.*

(i) *The element $\mathfrak{x}_\lambda$ lies in $\mathfrak{u}_n^-$.*

(ii) *Let $\mathfrak{J}_\lambda$ be the left ideal of $\mathfrak{U}_K$ generated by the elements $e_i^{(k)}, \binom{h_i}{k} - \binom{\langle \lambda, \alpha_i^\vee \rangle}{k}, f_i^{(k_i)}$ ($i = 1, 2, 3$, $k \geq 1$, $k_i \geq 3^n$) and the elements $f \in \mathfrak{u}_n^-$ with $f \mathfrak{x}_{(3^n-1)\rho - \lambda} = 0$. Then $\mathfrak{U}_K / \mathfrak{J}_\lambda \cong L(\lambda)$ (Note that $L(\lambda)$ has a $\mathfrak{U}_K$-module structure, which is irreducible).*

(iii) *As a $\mathfrak{u}_n^-$-module, $L(\lambda)$ is isomorphic to $\mathfrak{u}_n^- \mathfrak{x}_{(3^n-1)\rho - \lambda}$.*

By abuse of notations, the images in $\mathfrak{U}_K / \mathfrak{J}_\lambda \cong L(\lambda)$ of $f_i^{(k_i)}$ and $f_I^{(k_I)}$ will be denoted by the same notations. We shall use this theorem to construct certain bases for $L(\lambda)$, to compute $\dim L(\lambda)$, and to determine $\mathrm{ch}\,(L(\lambda))$ ($\lambda \in X_1(T)$) in this note, when G is either $SO(7, K)$ or $Sp(6, K)$.

2. Main Results

From now on, we shall assume $n = 1$. We can easily get the following lemmas:

Lemma 2. *There are* 8 *strong linkage classes in* $X(T)_+$ *for* $G = SO(7, K)$*, whose representatives can be chosen to be* $(0,0,0), (0,0,2), (0,1,0), (2,0,0), (0,0,1), (1,0,1), (2,1,1), (2,2,2)$. *Moreover, one has* $(0,0,0) \uparrow (1,0,0) \uparrow (1,1,0) \uparrow (0,2,0) \uparrow (0,0,4) \uparrow (1,1,2) \uparrow (3,0,2) \uparrow (2,1,2)$, $(0,0,2) \uparrow (1,1,0) \uparrow (2,0,2) \uparrow (1,1,2)$, $(0,1,0) \uparrow (2,1,0) \uparrow (1,2,0) \uparrow (2,2,0)$, $(2,0,0) \uparrow (1,0,2) \uparrow (0,1,2) \uparrow (0,2,2) \uparrow (1,2,2)$, $(0,0,1) \uparrow (0,1,1) \uparrow (0,0,3) \uparrow (1,1,1) \uparrow (1,2,1)$ *and* $(1,0,1) \uparrow (2,0,1) \uparrow (0,2,1) \uparrow (2,2,1)$.

Lemma 3. *There are* 8 *strong linkage classes in* $X(T)_+$ *for* $G = Sp(6, K)$*, whose representatives can be chosen to be* $(0,0,0), (2,0,0), (2,1,0), (0,0,1), (1,0,0), (0,1,1), (2,0,1), (2,2,2)$. *Moreover, one has* $(0,0,0) \uparrow (0,1,0) \uparrow (1,0,1) \uparrow (0,0,2) \uparrow (1,1,1) \uparrow (2,0,2) \uparrow (2,1,2)$, $(1,1,1) \uparrow (0,3,0) \uparrow (2,1,2)$, $(2,0,0) \uparrow (0,2,0) \uparrow (2,2,0) \uparrow (1,2,1)$, $(2,1,0) \uparrow (0,1,2) \uparrow (0,2,2)$, $(0,0,1) \uparrow (1,1,0) \uparrow (1,2,0) \uparrow (0,2,1) \uparrow (0,0,3) \uparrow (1,1,2)$, $(1,0,0) \uparrow (5,0,0) \uparrow (3,2,0) \uparrow (2,2,1)$, $(0,1,1) \uparrow (2,1,1) \uparrow (1,3,0) \uparrow (3,2,0)$ *and* $(2,0,1) \uparrow (1,0,2) \uparrow (1,2,2)$.

Furthermore, the elements $f_i, f_I (1 \le i \le 3$ and $I \in \mathcal{B})$ for $G = SO(7, K)$ satisfy the following commutator relations:

$$\begin{aligned}
f_2 f_3 &= f_3 f_2 + f_{23}, & f_1 f_{23} &= f_{23} f_1 + f_{123},\\
f_{12} f_3 &= f_3 f_{12} + f_{123}, & f_1 f_2 &= f_2 f_1 + f_{12},\\
f_1 f_{233} &= f_{233} f_1 + f_{1233}, & f_{233} f_{12} &= f_{12} f_{233} + f_{12233},\\
f_2 f_{1233} &= f_{1233} f_2 + f_{12233}, & f_3 f_{23} &= f_{23} f_3 + f_{233},\\
f_3 f_{123} &= f_{123} f_3 + f_{1233}, & f_{23} f_{123} &= f_{123} f_{23} + f_{12233},\\
f_I f_{I'} &= f_{I'} f_I \qquad \text{for all the other } I, I' \in \mathcal{B},
\end{aligned}$$

and the elements $f_i, f_I (1 \le i \le 3$ and $I \in \mathcal{C})$ for $G = Sp(6, K)$ satisfy the following commutator relations:

$$\begin{aligned}
f_2 f_3 &= f_3 f_2 + f_{23}, & f_1 f_{223} &= f_{223} f_1 + f_{1223},\\
f_{12} f_3 &= f_3 f_{12} + f_{123}, & f_1 f_2 &= f_2 f_1 + f_{12},\\
f_1 f_{23} &= f_{23} f_1 + f_{123}, & f_{12} f_{23} &= f_{23} f_{12} + f_{1223},\\
f_2 f_{123} &= f_{123} f_2 + f_{1223}, & f_{123} f_{12} &= f_{12} f_{123} + f_{11223},\\
f_{23} f_2 &= f_2 f_{23} + f_{223}, & f_{1223} f_1 &= f_1 f_{1223} + f_{11223},\\
f_I f_{I'} &= f_{I'} f_I \qquad \text{for all the other } I, I' \in \mathcal{C}.
\end{aligned}$$

Now we can prove our main theorems. Let $e(\nu)=\sum_{w\in W_\nu} w(\nu)$ be the sum of weights of the W-orbit of ν for all $\nu\in X(T)_+$. It is well-known that $\{\mathrm{ch}\,(\nu)|\nu\in X(T)_+\}$, $\{\mathrm{ch}_3\,(\nu)|\nu\in X(T)_+\}$ and $\{e(\nu)|\nu\in X(T)_+\}$ form bases of $\mathbb{Z}[X(T)]^W$, the W-invariant subring of $\mathbb{Z}[X(T)]$, respectively. According to the Weyl character formula, we can get a change of basis matrix $A=(a_{\lambda\nu})_{\lambda,\nu\in X(T)_+}$ from $\{e(\nu)|\nu\in X(T)_+\}$ to $\{\mathrm{ch}\,(\nu)|\nu\in X(T)_+\}$, which is a triangular matrix with 1 on its diagonal, i.e.

$$\mathrm{ch}\,(\lambda)=\sum_{\nu\prec\lambda,\ \nu\in X(T)_+} a_{\lambda\nu}e(\nu)$$

with $a_{\lambda\lambda}=1$. Based on our computation, we can get another change of basis matrix $B=(b_{\lambda\nu})_{\lambda,\nu\in X(T)_+}$ from $\{e(\nu)|\nu\in X(T)_+\}$ to $\{\mathrm{ch}_3\,(\nu)|\nu\in X(T)_+\}$, which is also a triangular matrix with 1 on its diagonal.

Let us mention our computation more detailed. First of all, we compute $\mathfrak{x}_{2\rho-\lambda}$ for any $\lambda\in X_1(T)$. It is well known that for each dominant weight ν of $H^0(\lambda)$, $\beta=\lambda-\nu$ can be expressed in terms of sum of positive roots, and there exist many ways to do so. Each way corresponds to an element $f_\beta\mathfrak{x}_{2\rho-\lambda}$ in $\mathfrak{u}_n$. Then we compute various $f_\beta\mathfrak{x}_{2\rho-\lambda}$. Note that each $f_\beta\mathfrak{x}_{2\rho-\lambda}$ can be write as a linear combination of the basis elements of $\mathfrak{u}_n$ with non-negative integers, and the typical images of all the non-zero $f_\beta\mathfrak{x}_{2\rho-\lambda}$ generate the weight space $L(\lambda)_\nu$ of the irreducible submodule $L(\lambda)$ of $H^0(\lambda)$. Therefore, we can easily determine the dimension of $L(\lambda)_\nu$, provided that we compute the rank of the set of all these non-zero $f_\beta\mathfrak{x}_{2\rho-\lambda}$. It can be reduced to compute the rank of a corresponding matrix. Finally, we obtain the formal character of $L(\lambda)$, which can be written as a linear combination of $e(\nu)'$s with non-negative integers. That is

$$\mathrm{ch}_3\,(\lambda)=\sum_{\nu\prec\lambda,\ \nu\in X(T)_+} b_{\lambda\nu}e(\nu)$$

with $b_{\lambda\lambda}=1$. In this way, we get the second matrix B.

For example, we assume that $G=SO(7,K)$ and $\lambda=(0,1,1)$. Then we have by the Weyl character formula

$$\mathrm{ch}\,(0,1,1)=e(0,1,1)+2e(1,0,1)+5e(0,0,1).$$

It is easy to see that

$$\mathfrak{x}=\mathfrak{x}_{2\rho-\lambda}=\mathfrak{x}_{(211)}=f_3f_2^{(2)}f_3^{(3)}f_2f_1^{(5)}f_2^{(4)}f_3^{(7)}f_2^{(3)}f_1^{(2)}.$$

For $\nu=(0,0,1)$, we have $\lambda-\nu=(0,1,0)=\alpha_1+2\alpha_2+2\alpha_3$. First we compute each of the set $\mathcal{S}=\{f_1f_2f_2f_3f_3\mathfrak{x}, f_1f_2f_3f_{23}\mathfrak{x}, f_1f_{23}f_{23}\mathfrak{x}, f_1f_2f_{233}\mathfrak{x}, f_{12}f_2f_3f_3\mathfrak{x}, f_{12}f_{23}f_3\mathfrak{x}, f_{12}f_{233}\mathfrak{x},\ f_{123}f_2f_3\mathfrak{x}, f_{123}f_{23}\mathfrak{x},\ f_{1233}f_2\mathfrak{x}, f_{12233}\mathfrak{x}\}$. Then we compute the rank of the set $\mathcal{S}$, which is equal to 4. So we have $\dim L(0,1,1)_{(0,0,1)} = 4$. In the same way, we have $\dim L(0,1,1)_{(1,0,1)} = 2$. Finally, we obtain

$$\mathrm{ch}_3\,(0,1,1) = e(0,1,1) + 2e(1,0,1) + 4e(0,0,1).$$

When λ lies in $X(T)_+$ but not in $X_1(T)$, we can also compute the formal character $\mathrm{ch}_3\,(\lambda)$ by using the Steiberg tensor product theorem. For $\lambda \in X(T)_+$, we have the unique decomposition

$$\lambda = \lambda^0 + 3^n\lambda^1 \qquad \text{with} \quad \lambda^0 \in X_1(T), \lambda^1 \in X(T)_+.$$

Then the Steinberg tensor product theorem tells us that

$$\mathrm{ch}_3\,(\lambda) = \mathrm{ch}_3\,(\lambda^0) \cdot \mathrm{ch}_3\,(3^n\lambda^1).$$

Therefore, we can determine all the characters $\mathrm{ch}_3\,(\lambda)$ with $\lambda \in X(T)_+$, provided that all the characters $\mathrm{ch}_3\,(\lambda)$ with $\lambda \in X_1(T)$ are known. For example, when $G = Sp(6,K)$ and $\lambda = (3,1,0)$, we have

$$\begin{aligned}\mathrm{ch}_3\,(3,1,0) &= \mathrm{ch}_3\,(0,1,0) \cdot \mathrm{ch}_3\,(3,0,0)\\ &= (e(0,1,0) + e(0,0,0)) \cdot e(3,0,0)\\ &= e(3,1,0) + e(2,0,1) + e(1,1,0) + e(3,0,0).\end{aligned}$$

We list the two matrices A and B in the attached tables. When $G = SO(7,K)$, these matrices can be found in Table 1 and 2. When $G = Sp(6,K)$, these matrices can be found in Table 3 and 4. In all these tables, the left column indicates $\mathrm{ch}\,(\lambda)$'s or $\mathrm{ch}_3\,(\lambda)$'s, and the bottom row indicates $e(\nu)$'s. For each weight $\lambda \in X(T)_+$, we can find two rows in tables. The upper row corresponds to all integers $b_{\lambda\nu}$, and the lower row corresponds to all integers $a_{\lambda\nu}$. Therefore, we can easily get the third change of basis matrix $D = AB^{-1}$ from $\{\mathrm{ch}_3\,(\nu) | \nu \in X(T)_+\}$ to $\{\mathrm{ch}\,(\nu) | \nu \in X(T)_+\}$, which is still a triangular matrix with 1 on its diagonal. The matrix D gives the decomposition patterns of various $H^0(\lambda)$ with $\lambda \in X(T)_+$. In this way, we obtain the following theorems.

Theorem 4. *Let $G = SO(7,K)$ and $\Lambda = \{(1,1,0), (0,2,0), (2,0,2), (1,1,2), (2,1,2), (2,1,0), (1,2,0), (2,2,0), (0,1,2), (0,2,2), (1,2,2), (0,1,1), (1,1,1), (1,2,1), (2,0,1), (0,2,1), (2,2,1)\} \subset X_1(T)$. Then $H^0(\lambda)$ is irreducible for all $\lambda \in X_1(T)\backslash\Lambda$. Moreover, one has*

$$\begin{aligned}\mathrm{ch}\,(1,1,0) &= \mathrm{ch}_3\,(1,1,0) + \mathrm{ch}_3\,(0,0,2) + \mathrm{ch}_3\,(1,0,0),\\ \mathrm{ch}\,(0,2,0) &= \mathrm{ch}_3\,(0,2,0) + \mathrm{ch}_3\,(0,0,2) + \mathrm{ch}_3\,(0,0,0),\\ \mathrm{ch}\,(2,0,2) &= \mathrm{ch}_3\,(2,0,2) + \mathrm{ch}_3\,(1,1,0) + \mathrm{ch}_3\,(0,0,2),\end{aligned}$$

$$
\begin{aligned}
\mathrm{ch}\,(1,1,2) &= \mathrm{ch}_3\,(1,1,2) + \mathrm{ch}_3\,(0,0,4) + \mathrm{ch}_3\,(2,0,2) + \mathrm{ch}_3\,(0,2,0)\\
&\quad + \mathrm{ch}_3\,(1,1,0) + \mathrm{ch}_3\,(0,0,2) + \mathrm{ch}_3\,(1,0,0) + \mathrm{ch}_3\,(0,0,0),\\
\mathrm{ch}\,(2,1,2) &= \mathrm{ch}_3\,(2,1,2) + \mathrm{ch}_3\,(3,0,2) + \mathrm{ch}_3\,(1,1,2) + \mathrm{ch}_3\,(0,0,4)\\
&\quad + \mathrm{ch}_3\,(2,0,2) + \mathrm{ch}_3\,(1,1,0) + \mathrm{ch}_3\,(1,0,0) + \mathrm{ch}_3\,(0,0,0),\\
\mathrm{ch}\,(2,1,0) &= \mathrm{ch}_3\,(2,1,0) + \mathrm{ch}_3\,(0,1,0),\\
\mathrm{ch}\,(1,2,0) &= \mathrm{ch}_3\,(1,2,0) + \mathrm{ch}_3\,(0,1,0),\\
\mathrm{ch}\,(2,2,0) &= \mathrm{ch}_3\,(2,2,0) + \mathrm{ch}_3\,(2,1,0),\\
\mathrm{ch}\,(0,1,2) &= \mathrm{ch}_3\,(0,1,2) + \mathrm{ch}_3\,(1,0,2),\\
\mathrm{ch}\,(0,2,2) &= \mathrm{ch}_3\,(0,2,2) + \mathrm{ch}_3\,(2,0,0),\\
\mathrm{ch}\,(1,2,2) &= \mathrm{ch}_3\,(1,2,2) + \mathrm{ch}_3\,(0,2,2),\\
\mathrm{ch}\,(0,1,1) &= \mathrm{ch}_3\,(0,1,1) + \mathrm{ch}_3\,(0,0,1),\\
\mathrm{ch}\,(1,1,1) &= \mathrm{ch}_3\,(1,1,1) + \mathrm{ch}_3\,(0,0,3) + \mathrm{ch}_3\,(0,1,1) + 2\mathrm{ch}_3\,(0,0,1),\\
\mathrm{ch}\,(1,2,1) &= \mathrm{ch}_3\,(1,2,1) + +\mathrm{ch}_3\,(1,1,1) + \mathrm{ch}_3\,(0,0,3) + \mathrm{ch}_3\,(0,0,1),\\
\mathrm{ch}\,(2,0,1) &= \mathrm{ch}_3\,(2,0,1) + \mathrm{ch}_3\,(1,0,1),\\
\mathrm{ch}\,(0,2,1) &= \mathrm{ch}_3\,(0,2,1) + \mathrm{ch}_3\,(1,0,1),\\
\mathrm{ch}\,(2,2,1) &= \mathrm{ch}_3\,(2,2,1) + \mathrm{ch}_3\,(2,0,1);
\end{aligned}
$$

$$
\begin{array}{lll}
\dim L(0,0,0) = 1, & \dim L(0,1,1) = 104, & \dim L(0,2,2) = 2052,\\
\dim L(1,0,0) = 7, & \dim L(1,1,1) = 384, & \dim L(2,2,0) = 1602,\\
\dim L(0,0,1) = 8, & \dim L(2,0,1) = 120, & \dim L(1,1,2) = 797,\\
\dim L(0,1,0) = 21, & \dim L(1,0,2) = 189, & \dim L(1,2,1) = 2400,\\
\dim L(1,0,1) = 48, & \dim L(2,1,0) = 309, & \dim L(2,1,1) = 1512,\\
\dim L(1,1,0) = 63, & \dim L(1,2,0) = 672, & \dim L(2,2,1) = 7272,\\
\dim L(2,0,0) = 27, & \dim L(0,2,1) = 672, & \dim L(2,1,2) = 2855,\\
\dim L(0,0,2) = 35, & \dim L(0,1,2) = 189, & \dim L(1,2,2) = 5670,\\
\dim L(0,2,0) = 132, & \dim L(2,0,2) = 518, & \dim L(2,2,2) = 19683.
\end{array}
$$

Theorem 5. *Let $G = Sp(6,K)$ and $\Lambda = \{(0,1,0),(1,0,1),(1,1,1),(2,0,2),$ $(2,1,2),(2,2,0),(1,2,1),(1,1,0),(1,2,0),\ (0,2,1),(1,1,2),(2,1,1),(2,2,1)\}$ $\subset X_1(T)$. Then $H^0(\lambda)$ is irreducible for all $\lambda \in X_1(T)\backslash\Lambda$. Moreover, one has*

$$
\begin{aligned}
\mathrm{ch}\,(0,1,0) &= \mathrm{ch}_3\,(0,1,0) + \mathrm{ch}_3\,(0,0,0),\\
\mathrm{ch}\,(1,0,1) &= \mathrm{ch}_3\,(1,0,1) + \mathrm{ch}_3\,(0,1,0),\\
\mathrm{ch}\,(1,1,1) &= \mathrm{ch}_3\,(1,1,1) + \mathrm{ch}_3\,(0,0,2) + \mathrm{ch}_3\,(1,0,1) + \mathrm{ch}_3\,(0,1,0),\\
\mathrm{ch}\,(2,0,2) &= \mathrm{ch}_3\,(2,0,2) + \mathrm{ch}_3\,(1,1,1) + \mathrm{ch}_3\,(0,0,2) + \mathrm{ch}_3\,(0,1,0),
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{ch}\,(2,1,2) &= \mathrm{ch}_3\,(2,1,2) + \mathrm{ch}_3\,(2,0,2) + \mathrm{ch}_3\,(0,3,0) + \mathrm{ch}_3\,(1,1,1)\\
&\qquad + \mathrm{ch}_3\,(1,0,1) + \mathrm{ch}_3\,(0,1,0) + \mathrm{ch}_3\,(0,0,0),\\
\mathrm{ch}\,(2,2,0) &= \mathrm{ch}_3\,(2,2,0) + \mathrm{ch}_3\,(2,0,0),\\
\mathrm{ch}\,(1,2,1) &= \mathrm{ch}_3\,(1,2,1) + \mathrm{ch}_3\,(2,2,0),\\
\mathrm{ch}\,(1,1,0) &= \mathrm{ch}_3\,(1,1,0) + \mathrm{ch}_3\,(0,0,1),\\
\mathrm{ch}\,(1,2,0) &= \mathrm{ch}_3\,(1,2,0) + \mathrm{ch}_3\,(1,1,0) + \mathrm{ch}_3\,(0,0,1),\\
\mathrm{ch}\,(0,2,1) &= \mathrm{ch}_3\,(0,2,1) + \mathrm{ch}_3\,(1,2,0) + \mathrm{ch}_3\,(0,0,1),\\
\mathrm{ch}\,(1,1,2) &= \mathrm{ch}_3\,(1,1,2) + \mathrm{ch}_3\,(0,0,3) + \mathrm{ch}_3\,(0,2,1) + \mathrm{ch}_3\,(1,2,0)\\
&\qquad + \mathrm{ch}_3\,(0,0,1),\\
\mathrm{ch}\,(2,1,1) &= \mathrm{ch}_3\,(2,1,1) + \mathrm{ch}_3\,(0,1,1),\\
\mathrm{ch}\,(2,2,1) &= \mathrm{ch}_3\,(2,2,1) + \mathrm{ch}_3\,(3,2,0) + \mathrm{ch}_3\,(1,3,0) + \mathrm{ch}_3\,(2,1,1)\\
&\qquad + \mathrm{ch}_3\,(0,1,1) + \mathrm{ch}_3\,(1,0,0),
\end{aligned}
$$

$$
\begin{array}{lll}
\dim L(0,0,0) = 1, & \dim L(0,1,1) = 126, & \dim L(2,2,0) = 903,\\
\dim L(1,0,0) = 6, & \dim L(1,1,1) = 358, & \dim L(0,2,2) = 2457,\\
\dim L(0,0,1) = 14, & \dim L(2,0,1) = 216, & \dim L(1,1,2) = 1610,\\
\dim L(0,1,0) = 13, & \dim L(1,0,2) = 378, & \dim L(1,2,1) = 1302,\\
\dim L(1,0,1) = 57, & \dim L(2,1,0) = 189, & \dim L(2,1,1) = 1260,\\
\dim L(1,1,0) = 50, & \dim L(1,2,0) = 286, & \dim L(2,2,1) = 3450,\\
\dim L(2,0,0) = 21, & \dim L(0,2,1) = 316, & \dim L(2,1,2) = 4655,\\
\dim L(0,0,2) = 84, & \dim L(0,1,2) = 594, & \dim L(1,2,2) = 8316,\\
\dim L(0,2,0) = 90, & \dim L(2,0,2) = 623, & \dim L(2,2,2) = 19683.
\end{array}
$$

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Table 1

000	1																				
	1																				
100	1	1																			
	1	1																			
010	3	1	1																		
	3	1	1																		
200	3	1	1	1																	
	3	1	1	1																	
002	3	2	1		1																
	3	2	1		1																
110	1	2	1	1	1	1															
	5	5	2	1	2	1															
300	1						1														
	3	3	1	1	1	1	1														
102	9	6	5	2	3	1		1													
	9	6	5	2	3	1		1													
020	4	2	3	2	1	1		1	1												
	8	4	4	2	2	1		1	1												
210	9	6	6	5	3	2	1	2	1	1											
	12	7	7	5	3	2	1	2	1	1											
012	3	4	2	1	3	2		1	1		1										
	12	10	7	3	6	3		2	1		1										
400	1	1		1			1			1		1									
	6	3	3	3	1	1	1	1	1	1		1									
202	12	10	8	5	7	4	2	3	1	1	1		1								
	16	14	10	6	9	5	2	3	1	1	1		1								
004					1			1			1			1							
	8	6	5	3	4	2		2	1		1			1							
120	12	14	8	6	9	6	2	3	2	1	2		1		1						
	15	15	9	6	9	6	2	3	2	1	2		1		1						
310	3	1	1	1		1	3			1		1	1			1					
	15	15	9	7	9	7	5	3	2	2	2	1	2		1	1					
112	13	10	11	7	7	5	1	5	4	2	2		1	1	1		1				
	35	27	24	15	18	11	3	10	6	3	4		2	2	1		1				
030	3						1											1			
	17	11	11	7	7	5	2	5	4	2	2		1	1	1		1	1			
302	3	2	1	2	1	1	3	1		2		2	1			1			1		
	28	22	20	14	15	10	6	9	5	5	3	2	3	1	1	1	1		1		
104			2		3	2		3	2		3		1	3			1			1	
	24	21	17	11	15	10	3	8	5	2	5		2	3	1		1			1	
220	24	17	17	13	12	9	5	9	7	5	4	2	3	3	2	1	2	1	1		1
	33	23	23	18	15	11	6	11	8	6	4	2	3	3	2	1	2	1	1		1
	0	1	0	2	0	1	3	1	0	2	0	4	2	0	1	3	1	0	3	1	2
	0	0	1	0	0	1	0	0	2	1	1	0	0	0	2	1	1	3	0	0	2
	0	0	0	0	2	0	0	2	0	0	2	0	2	4	0	0	2	0	2	4	0

022	30	28	22	14	21	14	5	11	8	4	7		4	3	3		2	1	
	33	29	23	15	21	14	5	11	8	4	7		4	3	3		2	1	
212	31	32	25	18	25	19	9	13	10	6	10	1	7	4	5	2	3	1	1
	62	57	46	33	42	30	15	23	15	11	14	3	10	6	6	3	4	1	2
014		4	4	4	4	4		4	4	2	4		2	4	2		2		
	32	27	24	17	20	14	5	13	9	5	8		4	6	3		3	1	
130	3	3		1			1			1		1			1			1	
	35	35	25	19	25	19	10	13	10	7	10	2	7	4	6	2	3	2	1
204		4	4	4	6	4		5	4	2	4		2	3	2		2		1
	51	43	39	29	33	24	11	22	16	10	13	3	8	9	5	2	5	1	2
122	60	47	47	37	36	28	16	28	21	16	16	5	12	12	9	4	9	5	4
	90	75	69	51	57	42	21	39	29	20	23	5	16	15	12	4	11	6	4
	0	1	0	2	0	1	3	1	0	2	0	4	2	0	1	3	1	0	3
	0	0	1	0	0	1	0	0	2	1	1	0	0	0	2	1	1	3	0
	0	0	0	0	2	0	0	2	0	0	2	0	2	4	0	0	2	0	2

022	1		1					
	1		1					
212	2	1	1	1				
	2	1	1	1				
014	2		1		1			
	2		1		1			
130			1			1		
	2	1	2	1		1		
204	2		1	1	1		1	
	3	1	1	1	1		1	
122	4	3	3	2	2	1	1	1
	5	3	4	2	2	1	1	1
	1	2	0	2	0	1	2	1
	0	2	2	1	1	3	0	2
	4	0	2	2	4	0	4	2

Table 2

001	1																			
	1																			
101	3	1																		
	3	1																		
011	4	2	1																	
	5	2	1																	
201	3	2	1	1																
	6	3	1	1																
003					1															
	4	2	1		1															
111	8	6	3	2	1	1														
	14	8	4	2	2	1														
301	1			1			1													
	10	6	3	3	1	1	1													
103			1		1			1												
	13	8	5	2	3	1		1												
021	12	8	6	3	3	2		1	1											
	15	9	6	3	3	2		1	1											
211	27	18	11	8	6	4	2	2	1	1										
	27	18	11	8	6	4	2	2	1	1										
013		1	1		3	1		1			1									
	18	12	8	4	6	3		2	1		1									
401	3	2		3		1	3			1		1								
	15	10	6	6	3	3	3	1	1	1		1								
203		1	2		3	1		1			1		1							
	27	19	13	8	9	5	2	3	1	1	1		1							
005	1	1	2	1	3	1		2	1		1			1						
	10	7	5	3	4	2		2	1		1			1						
121	31	22	16	11	11	8	3	5	4	2	2		1		1					
	40	28	19	13	13	9	3	5	4	2	2		1		1					
311	4	4	3	4		2	4	1		2		2	1			1				
	44	32	22	18	14	11	8	6	4	4	2	2	2		1	1				
113	1	3	3	1	1	2		2	1	1	1		1	1			1			
	52	38	28	18	20	13	4	10	6	3	4		2	2	1		1			
031	3			1			1		1						1			1		
	35	25	19	13	13	10	4	7	6	3	3		2	1	2		1	1		
303					3														1	
	46	35	26	19	19	13	8	9	5	5	3	2	3	1	1	1	1		1	
221	72	55	41	32	30	23	13	16	12	9	7	3	5	3	4	2	2	1	1	1
	75	57	42	33	30	23	13	16	12	9	7	3	5	3	4	2	2	1	1	1
	0	1	0	2	0	1	3	1	0	2	0	4	2	0	1	3	1	0	3	2
	0	0	1	0	0	1	0	0	2	1	1	0	0	0	2	1	1	3	0	2
	1	1	1	1	3	1	1	3	1	1	3	1	3	5	1	1	3	1	3	1

Table 3

000	1																					
	1																					
010	1	1																				
	2	1																				
200	3	1	1																			
	3	1	1																			
101	3	2	1	1																		
	4	3	1	1																		
020	6	3	2	1	1																	
	6	3	2	1	1																	
210	9	6	4	2	1	1																
	9	6	4	2	1	1																
002	4	2	2	1	1		1															
	4	2	2	1	1		1															
400			1			1		1														
	6	3	3	1	1	1		1														
111	8	7	5	4	3	2	1		1													
	16	12	8	6	4	2	2		1													
030	1									1												
	11	8	5	4	3	2	1		1	1												
301			1	1		1		1			1											
	13	11	7	6	3	3	1	1	1		1											
220	21	16	13	9	7	5	3	2	2	1	1	1										
	24	17	14	9	7	5	3	2	2	1	1	1										
012	12	10	7	6	4	3	3		2	1			1									
	12	10	7	6	4	3	3		2	1			1									
410		2	3	1	2	3		3	1		1	1		1								
	20	15	12	8	6	6	2	4	2	1	2	1		1								
202	9	6	7	5	5	3	4	2	2	1	1	1	1		1							
	22	16	14	10	9	5	6	2	3	1	1	1	1		1							
121	18	16	11	11	8	6	6	1	5	3	2	1	2		1	1						
	39	32	24	20	15	11	9	3	7	4	3	2	2		1	1						
040	1	1			1				1	1		1				1	1					
	22	16	13	10	9	6	5	3	4	3	2	2	1		1	1	1					
311		4	5	5	4	5	3	5	2		3	2		2	1			1				
	48	40	32	26	20	16	12	8	9	4	6	4	2	2	2	1		1				
103			1			1		1					1						1			
	18	16	12	11	8	7	6	2	5	3	2	1	3		1	1			1			
230	3	1	1		1	1			1	3		1	1			1	1			1		
	48	39	32	25	20	17	12	8	10	7	6	5	3	2	2	2	1	1		1		
022	33	26	23	18	16	11	12	5	8	5	4	4	4		3	2	1		1		1	
	33	26	23	18	16	11	12	5	8	5	4	4	4		3	2	1		1		1	
212	45	42	35	31	26	22	19	10	16	10	9	7	8	3	5	4	1	2	2	1	1	1
	68	58	48	41	34	27	24	12	19	12	10	8	9	3	6	4	1	2	2	1	1	1
	0	0	2	1	0	2	0	4	1	0	3	2	0	4	2	1	0	3	1	2	0	2
	0	1	0	0	2	1	0	0	1	3	0	2	1	1	0	2	4	1	0	3	2	1
	0	0	0	1	0	0	2	0	1	0	1	0	2	0	2	1	0	1	3	0	2	2

Table 4

100	1																			
	1																			
001	1	1																		
	1	1																		
110	3	1	1																	
	4	2	1																	
300				1																
	3	1	1	1																
011	5	3	2		1															
	5	3	2		1															
201	7	6	3	1	1	1														
	7	6	3	1	1	1														
120	7	5	4	2	2	1	1													
	11	7	5	2	2	1	1													
310			1	1		1		1												
	12	8	6	4	2	2	1	1												
102	9	6	5	2	3	1	1		1											
	9	6	5	2	3	1	1		1											
500	1		1	3		1	1	1		1										
	6	3	3	3	1	1	1	1		1										
021	5	5	3	1	2	2	1		1		1									
	13	11	7	3	4	3	2		1		1									
211	22	18	14	8	8	6	4	2	2		1	1								
	27	21	16	8	9	6	4	2	2		1	1								
003				1									1							
	5	5	3	2	2	2	1		1		1		1							
130	1						1				1			1						
	24	18	14	8	8	6	5	2	2		2	1		1						
401	1	3	2	3	2	2	1	2		1		1			1					
	18	16	11	7	6	6	3	3	1	1	1	1			1					
320	2	3	4	6	2	3	2	3	1	2		1			1	1				
	32	24	20	14	11	9	7	5	3	2	2	2		1	1	1				
112	19	15	14	8	10	7	6	3	5		3	2	1	1			1			
	32	26	21	12	14	10	8	3	6		4	2	2	1			1			
031	1	1			1		1				1	1		1				1		
	29	23	19	11	13	9	8	4	5		4	3	1	2			1	1		
302	2	3	3	4	3	2	2	2	1	2		1			1	1			1	
	32	25	22	14	15	10	9	5	6	2	3	3	1	1	1	1	1		1	
221	35	33	25	15	19	16	12	6	9	1	8	5	3	3	2	1	2	1	1	1
	66	57	45	29	30	25	19	11	12	3	10	7	3	4	3	2	2	1	1	1
	1	0	1	3	0	2	1	3	1	5	0	2	0	1	4	3	1	0	3	2
	0	0	1	0	1	0	2	1	0	0	2	1	0	3	0	2	1	3	0	2
	0	1	0	0	1	1	0	0	2	0	1	1	3	0	1	0	2	1	2	1

013			1	1		1		1	1				1				1			
	21	17	15	10	11	8	7	4	6		4	3	3	2			2	1		
140	3	1	2		1	2	3	1	2		3	1		3		1	1	1		1
	46	37	31	21	21	17	15	9	9	3	8	6	2	5	2	2	2	2	1	1
203	1		1	3		1	1	1	1	1	1		3				1			
	36	33	26	18	19	17	13	7	10	2	9	5	6	3	2	1	3	1	1	1
122	78	66	57	39	42	33	29	17	22	5	17	13	9	9	4	4	7	4	3	2
	78	66	57	39	42	33	29	17	22	5	17	13	9	9	4	4	7	4	3	2
	1	0	1	3	0	2	1	3	1	5	0	2	0	1	4	3	1	0	3	2
	0	0	1	0	1	0	2	1	0	0	2	1	0	3	0	2	1	3	0	2
	0	1	0	0	1	1	0	0	2	0	1	1	3	0	1	0	2	1	2	1

013	1			
	1			
140		1		
		1		
203	1		1	
	1		1	
122	2	1	1	1
	2	1	1	1
	0	1	2	1
	1	4	0	2
	3	0	3	2