Strong cluster properties and geometric expansion of the log-partition function of the Ginibre gas

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L'esprit succombant sous le poids
de la quantité n'a plus d'autre critérium
que l'efficacité.

(S.Weil, Algèbre)

Abstract

We consider the geometric expansion of the log-partition function of a system of interacting Brownian loops in a bounded domain. This expansion generalizes the corresponding one for a classical gas due to one of the authors (see [10]).

It can also be considered as a natural generalization of a famous result of Kac (in "Can one hear the shape of a drum", Amer. Math. Monthly 73 (1966), 1–23), where in the case of one loop only the expansion of the partition function is calculated as the length of the time interval, on which the loop is defined, tends to 0. For the proof of the geometric expansion we need **a new type of decay property of the Ursell function** which can be described in the following way: If one integrates the Ursell function over all configurations $\varepsilon_{x^0} + \mu$ of loops which contain a fixed loop x^0 starting at 0 and at least one loop of μ which touches the outside of the centered ball with radius R, then the integral with respect to x^0 has power decay in R.

I Strong cluster properties

Introduction

The fundamental problems of statistical mechanics require a detailed analysis of the logarithm of the partition function of the system. Since this log-partition function can be represented as an integral of the so called Ursell function with respect to the ideal gas (i.e. the analogue of the Lebesque measure on the configuration space of the system) we have to deal with the problem of estimating Ursell functions and analyzing its decay properties. These questions being of independent interest have already some tradition: In the case of classical systems the main ideas go back to Ruelle [11], Duneau, Jagolnitzer and Souillard (see [1 – 4]) and Minlos, Poghosian [8]. In the case of quantum systems the situation is not so well understood. Basic references are those of Ginibre (see [5, 6]).

Our aim here is to develop them further, thereby combining the ideas of Duneau et al., Ginibre and Minlos, Poghosyan (see [8] as well as Poghosyan [10]). Our main result is theorem 4.3. All other results are in principle well known and explained here for the convenience of the reader. The theorem can be described as follows: Let F be a function of the configurations of the system, possessing the so called **strong cluster property** with respect to some positive parameter a and estimator function h. This property formalizes the idea, that the decay of F as a function of a configuration μ should take into account the seperation of all particles $x, y, z, \ldots$ of μ with respect to each other. Given such a function F it is not hard to see that for each loop x^0 in $\mathbb{R}^\nu$ starting and ending in 0

$$\int d\mu \; z^{|\mu|} \cdot |F(\mu + \varepsilon_{x^0})| < +\infty.$$

(See lemma 4.4) Here z is a positive parameter and the integration $d\mu$ is taken over all configurations of loops with respect to the ideal Ginibre gas. By this we understand the non normalized Poisson point random field in the space of loops in $\mathbb{R}^\nu$ which are realized by means of a Brownian bridge measure. Our theorem 4.4 then gives the rate of convergence of the tail

$$(*) \qquad \int' d\mu \; z^{|\mu|} \cdot |F(\mu + \varepsilon_{x^0})| \text{ as } R \to \infty,$$

when integrated with respect to the Brownian bridge measure dx^0. Here $\int'$ means that the integral is taken over those configurations μ of loops where at least one loop does not stay all the time in $\Lambda_0(R)$, the centered ball in $\mathbb{R}^\nu$ of radius R. We remark that in the classical case (see [10]) it was possible to estimate the rate of convergence of the integral $(*)$ directly, without taking the mean with respect to dx^0. This seems to indicate quantum phenomena.

This result will be used as the main lemma in the second part of this paper where we study the geometric properties of the log-partition function.

1 Conditions on the interactions

We consider a system of n identical particles in a ν-dimensional Euclidean space $\mathbb{R}^\nu$, interacting through a two-body potential ϕ with the following properties:

(1) $\phi : \mathbb{R}^\nu \to \mathbb{R} \cup \{+\infty\}$ is an even function.

(2) There exists a real constant $B \geq 0$ such that for any collection of points $x_i \in \mathbb{R}^\nu$, $i = 1, \ldots, n$,

$$\sum_{i<j} \phi(x_i - x_j) \geq -nB.$$

Let δ be an Euclidean invariant metric on $\mathbb{R}^\nu$. We assume then also that ϕ has the following decay property

$$\|\phi\|_\delta := \int_{\mathbb{R}^\nu} |\phi(u)| \cdot \exp(\delta(0,u))du < +\infty. \tag{3}$$

Denote by $\mathcal{P}_\delta$ the class of translation invariant potentials ϕ satisfying (1), (2) and (3) for some Euclidean invariant metric δ. Important examples of such metrics are

(4) $\delta_e(0,u) = \gamma \cdot |u|, \gamma > 0;$

(5) $\delta_l(0,u) = l \cdot \ln(1 + |u|), l > 0.$

Roughly speaking δ_e corresponds to exponential and δ_l to power decay of the potential ϕ.

2 The Ginibre gas

Let $\beta > 0$ be fixed. The underlying one particle space is the space $\mathcal{C}([0,\beta],\mathbb{R}^\nu)$ of continuous functions provided with the topology of uniform convergence. As such it is a Polish space, i.e. a complete, separable metric space X.

To make the mathematical structure of the Ginibre model more transparent we introduce the following notions in the framework of Polish spaces. (See Matthes et al. [7] e.g.):

Let X be a Polish space. $\mathcal{B}(X)$ denotes the Borel σ-field and $\mathcal{B}_0(X)$ the collection of bounded elements in $\mathcal{B}(X)$. Here $B \subset X$ is called bounded if it is contained in some sphere of the space X. $\mathcal{M}(X)$ is the set of all measures on $(X,\mathcal{B}(X))$ which are finite on $\mathcal{B}_0(X)$ and $\mathcal{M}^{\cdot\cdot}(X)$ denotes the subset of all such measures μ which are integer-valued on $\mathcal{B}_0(X)$. In the following we often write $\mathcal{B},\mathcal{B}_0,\mathcal{M},\mathcal{M}^{\cdot\cdot}$, when it is clear that X is the underlying space.

One can show that for all $\mu \in \mathcal{M}^{\cdot\cdot}$ the set $D_\mu = \{x \in X : \mu\{x\} > 0\}$ is at most countable, and $\mu = \sum\limits_{x\in D_\mu} \mu\{x\}\cdot\varepsilon_x$. Such μ are called **point measures** in X. We denote by $\mathcal{F}$ the smallest σ-field in $\mathcal{M}$, with respect to which the function $\zeta_B : \mathcal{M} \to \mathbb{R}_+, \zeta_B(\mu) = \mu(B), B \in \mathcal{B}_0$, are measurable. For all $\mu \in \mathcal{M}$ and $A \in \mathcal{B}$, let μ_A denote the restriction of μ to $(X, A\cap\mathcal{B})$. Then $\mathfrak{p}_A : \mu \to \mu_A$ is a measurable mapping of $(\mathcal{M},\mathcal{F})$ into itself for all $A \in \mathcal{B}$. It is well-known that $\mathcal{M}^{\cdot\cdot} \in \mathcal{F}$. In $\mathcal{M}^{\cdot\cdot}$ we introduce the σ-field $\mathcal{F}^{\cdot\cdot} = \mathcal{M}^{\cdot\cdot}\cap\mathcal{F}$, i.e. the trace of $\mathcal{F}$ in $\mathcal{M}^{\cdot\cdot}$. This is the σ-field in $\mathcal{M}^{\cdot\cdot}$ generated by the restrictions of ζ_B, $B \in \mathcal{B}_0$, on $\mathcal{M}^{\cdot\cdot}$. One can show that the mapping $\mathcal{X}_1 : X \to \mathcal{M}^{\cdot\cdot}$, $\mathcal{X}_1(x) = \varepsilon_x$, is bimeasurable, i.e. measurable in both directions. Given $A \in \mathcal{B}$ we denote by $\mathcal{F}^{\cdot\cdot}_A$ the σ-field generated by ζ_B, $B \in A\cap\mathcal{B}_0$. $\mathfrak{p}_A$ is then a measurable mapping of $(\mathcal{M}^{\cdot\cdot},\mathcal{F}^{\cdot\cdot})$ into itself satisfying $\mathfrak{p}_A^{-1}(\mathcal{F}^{\cdot\cdot}) = \mathcal{F}^{\cdot\cdot}_A$. Therefore $\mathfrak{p}_A$ transforms a given measure L on $(\mathcal{M}^{\cdot\cdot},\mathcal{F}^{\cdot\cdot})$ into a new measure on $(\mathcal{M}^{\cdot\cdot},\mathcal{F}^{\cdot\cdot})$, which we denote by L_A. To each measure L on $(\mathcal{M}^{\cdot\cdot},\mathcal{F}^{\cdot\cdot})$ we define the **intensity measure** ν_L on $(X,\mathcal{B})$ by

$$\nu_L(A) := \int\limits_{\mathcal{M}^{\cdot\cdot}} \mu(A)L(d\mu), A \in \mathcal{B}. \tag{1}$$

We say that L **is of finite intensity** if $\nu_L \in \mathcal{M}$. Now define a measure on $(\mathcal{M}^{\cdot\cdot},\mathcal{F}^{\cdot\cdot})$ which will be the underlying reference measure: Let $\rho \in \mathcal{M}$ be given and $\mathcal{H}_+ := \mathcal{H}_+(\mathcal{M}^{\cdot\cdot})$ the set of all $\mathcal{F}^{\cdot\cdot}$-measurable, non-negative functions $\varphi : \mathcal{M}^{\cdot\cdot} \to \mathbb{R}_+$.

A measure W_ρ is then defined on $(\mathcal{M}^{\cdot\cdot}, \mathcal{F}^{\cdot\cdot})$ as follows:

$$(2) \qquad W_\rho(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_X \overset{n}{\cdots} \int_X \varphi(\varepsilon_{x_1} + \ldots + \varepsilon_{x_n}) \rho(dx_1) \ldots \rho(dx_n),$$

$\varphi \in \mathcal{H}_+$. Here the first term in the sum corresponding to $n = 0$ is $\varphi(0)$, 0 denoting the vacuum, i.e. the measure 0 in X. Note that W_ρ is supported by $\mathcal{M}^{\cdot\cdot}_f = \{\mu \in \mathcal{M}^{\cdot\cdot} \mid D_\mu \text{ finite}\}$. Moreover, if ρ is **diffuse**, i.e. $\rho\{x\} = 0$ for each $x \in X$, then W_ρ is supported by $\mathcal{M}^{\cdot} = \{\mu \in \mathcal{M}^{\cdot\cdot} \mid \mu\{x\} = 1 \text{ for each } x \in D_\mu\}$. $\mathcal{M}^{\cdot}$ is an element of $\mathcal{F}^{\cdot\cdot}$. A point measure μ in $\mathcal{M}^{\cdot}$ is called **simple**. In $\mathcal{M}^{\cdot}$ we take as σ-field the trace of $\mathcal{F}^{\cdot\cdot}$ in $\mathcal{M}^{\cdot} : \mathcal{F}^{\cdot} := \mathcal{M}^{\cdot} \cap \mathcal{F}^{\cdot\cdot}$.

We observe also that for each $B \in \mathcal{B}_0$ W_{ρ_B} is a finite measure on $(\mathcal{M}^{\cdot\cdot}, \mathcal{F}^{\cdot\cdot})$ with total mass $e^{\rho(B)}$, which is supported by $\mathcal{M}^{\cdot\cdot}_B = \{\mu \in \mathcal{M}^{\cdot\cdot} \mid D_\mu \subset B\}$. The associated probability $P_{\rho_B} = e^{-\rho(B)} W_{\rho_B}$ in $(\mathcal{M}^{\cdot\cdot}, \mathcal{F}^{\cdot\cdot})$ is called the **Poisson process in B with intensity ρ_B**. W_{ρ_B} and P_{ρ_B} are concentrated on $\mathcal{M}^{\cdot}_B = \mathcal{M}^{\cdot\cdot}_B \cap \mathcal{M}^{\cdot}$ if ρ is diffuse. In this case W_{ρ_B} and P_{ρ_B} are called **simple**.

In our basic example $X = \mathcal{C}([0, \beta], \mathbb{R}^\nu)$ we choose $\rho \in \mathcal{M}$ in the following way:

$$(3) \qquad \rho(f) = \int_{\mathbb{R}^\nu} P_{uu}(f) du.$$

Here P_{uu} denotes the law of the Brownian bridge on $X^u_{per} = \{x \in X : x(0) = x(\beta) = u\}$. (For a construction of P_{uu} see [5] e.g.). We remark here that ρ is **diffuse**, i.e. $\rho\{x\} = 0 \; \forall \; x \in X$ and concentrated on $X_{per} = \{x \in X \mid x(0) = x(\beta)\}$. Moreover, $P_{uu} = P_{00} \circ t_u^{-1}$, $t_u : x \mapsto x + u$.

An example of a bounded set in X is the situation which is considered in [5]: Let Λ be a bounded open region in $\mathbb{R}^\nu$ and B_Λ the set of trajectories in X which stay in Λ, i.e.

$$(4) \qquad B_\Lambda = \{x \in X \mid x(t) \in \Lambda \text{ for each } 0 \leq t \leq \beta\}.$$

For a given $B \in \mathcal{B}_0$ and $z > 0$ we call the associated Poisson process with intensity $z \cdot \rho_B$ the **ideal Ginibre gas in B with intensity z**. It is evident from the construction that $P_{z \cdot \rho_B}$ is supported by the set of finite, simple point measures of loops in B.

¿From now on we consider the ideal Ginibre gas $P_{z\rho_B}$ in some given bounded region B of $X = \mathcal{C}([0, \beta], \mathbb{R}^\nu)$ as the underlying reference gas. By means of the potential ϕ we introduce the following energy functional: Let $\mu \in \mathcal{M}^{\cdot}_f(X)$ and $x \in \mu$, i.e. $x \in X$ such that $x \in D_\mu$. Then

(5)
$$E(x,\mu) = \sum_{y \in \mu - \varepsilon_x} \int_0^\beta \phi(x(s) - y(s))ds$$

is called the **energy of** x **in** μ, and the **energy of** $\mu \in \mathcal{M}^{\cdot}_f$ is defined by $\mathcal{U} : \mathcal{M}^{\cdot}_f \to \mathbb{R} \cup \{+\infty\}$, $\mathcal{U}(0) = 0$ and

(6)
$$\mathcal{U}(\mu) = \frac{1}{2} \sum_{x \in \mu} E(x,\mu), \mu \in \mathcal{M}^{\cdot}_f.$$

We remark also that condition (1.2) implies the existence of a measurable function $\hat{x} : \mathcal{M}^{\cdot}_f \to X$, satisfying $\hat{x}(\mu) \in \mu$ and $E(\hat{x}(\mu), \mu) \geq -2\beta B, \mu \in \mathcal{M}^{\cdot}_f$.

Let $f : \mathcal{M}^{\cdot}_f \to \mathbb{R}_+$ denote the **Boltzmann factor** defined by

(7)
$$f(\mu) = exp - \mathcal{U}(\mu), \mu \in \mathcal{M}^{\cdot}_f.$$

The condition (2) in §2 of the potential ϕ implies that $f(\mu) \leq exp\, \mu(X) \cdot \beta \cdot B$, $\mu \in \mathcal{M}^{\cdot}_f$, and therefore the finiteness of the **grand partition function**

(8)
$$Z = Z(\Lambda, z) = W_{z \cdot \rho_\Lambda}(f).$$

Here from now on we write $\rho_\Lambda = \rho_{B_\Lambda}$. Thus we can define

(9)
$$Q_{\Lambda,z} = \frac{1}{Z(\Lambda,z)} \cdot f \cdot W_{z \cdot \rho_\Lambda},$$

which we call the **Ginibre gas in** Λ **with intensity** z **and interaction** ϕ.

3 The non-integrated Kirkwood-Salzburg equation

Let $X, \mathcal{B}$ be given and $\mu \in \mathcal{M}^{\cdot\cdot}_f$. Define another measure on $\mathcal{M}^{\cdot\cdot}_f$ by means of

(1)
$$\grave{W}_\mu(\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\dot{X}^n} \varphi(\varepsilon_{x_1} + \ldots + \varepsilon_{x_n}) \mu(dx_1) \ldots \mu(dx_n), \varphi \in \mathcal{H}_+.$$

Here $\grave{X}^n = \{(x_1, \dots, x_n) \in X^n \mid i \neq j \Rightarrow x_i \neq x_j\}$.

Note that

$$\grave{W}_\mu(\varphi) = \sum_{\nu \subseteq \mu} \varphi(\nu).$$

Here $\nu \subseteq \mu$ means that ν is a subconfiguration of μ, i.e. $\exists\, \eta \in \mathcal{M}_f^{\cdot\cdot} : \nu + \eta = \mu$. Denote by $\mathcal{M}_n^{\cdot\cdot} = \{\mu \in \mathcal{M}_f^{\cdot\cdot} \mid \mu(X) = n\}$, $n \geq 0$, and

(2)

$$\begin{cases} E = \{h : \mathcal{M}_f^{\cdot\cdot} \to \mathbb{C} \text{ measurable such that } h_n := Res_{\mathcal{M}_n^{\cdot\cdot}} h \text{ is bounded for each } n\} \\ E_+ = \{h \in E : h(0) = 0\} \end{cases}$$

Define in E the following multiplication $*$:

$$h_1 * h_2(\mu) = \int_{\mathcal{M}_f^{\cdot\cdot}} h_1(\nu) h_2(\mu - \nu) \grave{W}_\mu(d\nu), h_1, h_2 \in E. \tag{3}$$

$(E, *)$ is a commutative algebra with unit $\mathbf{1} = 1_{\{0\}}$, and E_+ is an ideal in E. We define a mapping $\Gamma : E_+ \to E$ as the $*$**-exponential**

$$\Gamma h = 1 + h + \frac{1}{2!} h * h + \dots. \tag{4}$$

Γ is extended from E_+ to E by $\Gamma(h_0 1 + h_+) = e^{h_0} \cdot \Gamma(h_+), h_+ \in E_+$. This extension satisfies $\Gamma(h_1 + h_2) = \Gamma h_1 * \Gamma h_2, h_1, h_2 \in E$.

We define derivations in E by

$$\begin{cases} D_x h(\mu) & = h(\mu + \varepsilon_x), h \in E. \\ D_{\varepsilon_{x_1} + \dots + \varepsilon_{x_n}} & = D_{x_n} \circ \dots \circ D_{x_1}. \end{cases} \tag{5}$$

D_x is a linear map from E to E and satisfies

$$\begin{cases} D_x(h_1 * h_2) & = D_x h_1 * h_2 + h_1 * D_x h_2 \\ D_x(\Gamma h) & = D_x h * \Gamma h \end{cases} \tag{6}$$

Fix now some $\rho \in \mathcal{M}$ and $B \in \mathcal{B}_0$. One can show (see [6]) that $W_\rho(h_1 * h_2) = W_\rho(h_1) \cdot W_\rho(h_2), h_1, h_2 \in E$. This implies immediately $Z = W_{\rho_B}(f) = exp\, W_{\rho_B}(\Gamma^{-1} f)$

$g = \Gamma^{-1} f$ is called **Ursell-function**. Observe that $g \in E_+$. Consider finally the function $\tilde{\varphi}$, defined by

$$\widetilde{\varphi}_\xi(\mu) = f^{*-1} * D_\xi f(\mu), \mu, \xi \in \mathcal{M}^{\cdot\cdot}_f. \tag{7}$$

One can show that $\tilde{\varphi}$ is a solution of the following so called **non-integrated Kirkwood-Salzburg equation**: For each $\xi, \mu \in \mathcal{M}^{\cdot\cdot}_f \setminus \{0\}$ and $x \in \xi$

(8)

$$\begin{cases} \tilde{\varphi}_\xi(\mu) = \exp\left(-E(x,\mu)\right) \cdot \int_{\mathcal{M}^{\cdot\cdot}_f} K(x,\nu) \cdot \tilde{\varphi}_{\xi-\varepsilon_x+\nu}(\mu-\nu) \grave{\Lambda}_\mu(d\nu). \\ \tilde{\varphi}_0(\mu) = \mathbf{1}(\mu); \tilde{\varphi}_\xi(0) = 1, \text{ if } \xi(X) = 1. \end{cases}$$

Here K is defined by

$$K(x,\nu) = \prod_{y\in\nu} q(x,y) \tag{9}$$

and

$$q(x,y) = \exp\left(-\int_0^\beta \phi(x(t)-y(t))dt\right) - 1. \tag{10}$$

4 Strong cluster estimates

Consider the vector space B of functions $f : X_{per} \times X_{per} \to \mathbb{R}$ which are invariant with respect to the translation group $t_u : (x,y) \mapsto (x+u, y+u)$, $u \in \mathbb{R}^\nu$. For each $f \in B$ and Euclidean invariant metric δ on $\mathbb{R}^\nu$ define

$$\|f\|_\delta = \max\left\{\|f\|_\infty, p(f), p_\delta(f)\right\}, \tag{1}$$

where

$$\|f\|_\infty = \sup_{x,y\in X_{per}} |f(x,y)|, p(f) = \sup_{x\in X_{per}} \int_{X_{per}} |f(x,y)|\rho(dy)$$

and

(2)

$$p_\delta(f) = \inf\left\{C > 0 : \forall\ d > 0 \sup_{R>0} \sup_{x\in B_{\Lambda_0(R)}} \int_{\complement B_{\Lambda_0(R+d)}} |f(x,y)|\rho(dy) \le C \cdot e^{-\delta(0,d)}\right\}.$$

The spaces

$$B_\delta = \{f \in B \mid \|f\|_\delta < +\infty\}, \tag{3}$$

are Banach spaces with norm $\|\cdot\|_\delta$. The elements of the space $B_\delta^+ = \{f \in B_\delta \mid f \geq 0, f \text{ symmetric}\}$ will be called **estimator functions**.

For a positive constant a and estimator function $h \in B_\delta^+$ we consider the following equation: For $\xi, \mu \in \mathcal{M}_f^{\cdot}(X_{per}) \setminus \{0\}$ and $x_0 \in \xi$

(4)

$$\begin{cases} \psi_{a,h,x_0}(\xi,\mu) & = a \cdot \int\limits_{\mathcal{M}_f^{\cdot}(X_{per})} \prod\limits_{y\in\nu} h(x_0,y)\cdot\psi_{a,h,x_0}(\xi-\varepsilon_{x_0}+\nu,\mu-\nu)\grave{\Lambda}_\mu(d\nu) \\ \psi_{a,h,x_0}(0,\mu) & = \mathbf{1}(\mu); \psi_{a,h,x_0}(\xi,0) = 1 \text{ if } \xi(X)=1. \end{cases}$$

We'll now give an explicit form of the unique solution of (4) in terms of graphs. We consider only non-oriented graphs. A connected graph without closed loops is called a **tree**. A graph, the connected components of which are trees we will call a **forest**.

Let w be a forest constructed on the set $\xi+\mu$, $\xi,\mu \in \mathcal{M}_f^{\cdot}(X_{per}) \setminus \{0\}$. We say that the roots of the forest w are in ξ (or w **is rooted in** ξ) if each tree γ of w has one and only one vertex in ξ. The tree γ may consist only of its root. By $\mathcal{F}(\xi,\mu)$ we'll denote the set of all forests constructed on $\xi+\mu$ and rooted in ξ.

The following lemma describes the solution of (4) and shows in particular that this solution does not depend on the root x_0 chosen in ξ.

Lemma 4.1 *For $a>0$, $h \in B_\delta^+$, and $\xi,\mu \in \mathcal{M}_f^{\cdot}(X_{per})$ such that $\xi \cap \mu = \phi$*

$$\psi_{a,h}(\xi,\mu) = a^{|\xi|+|\mu|} \cdot \sum_{w\in\mathcal{F}(\xi,\mu)} \prod_{[x,y]\in w} h(x,y). \tag{5}$$

Here the product is taken over all **edges** $[x,y]$ of the forest w, and $|\xi| = \xi(X_{per})$.

Proof. Equation (4) always has a unique solution by virtue of its recursive structure. Therefore it is sufficient to verify that the function $\psi_{a,h}$, defined by (5) and satisfying the initial conditions of (4), satisfies this equation. Indeed, let $\xi \in \mathcal{M}_f^{\cdot}(X_{per}) \setminus \{0\}$ and $\mu \in \mathcal{M}_f^{\cdot}(X_{per})$ be disjoint and $\nu \subseteq \mu$. To any $w \in \mathcal{F}(\xi-\varepsilon_x+\nu,\mu-\nu)$ we associate a forest $\tilde{w} \in \mathcal{F}(\xi,\mu)$ by adding the new vertex $\hat{x}(\xi)$ to the set of vertices of w and new edges $[\hat{x}(\xi),y]$,

$y \in \nu$, to the set of edges of w. We denote the set of all such forests $\tilde{w}$ by $\mathcal{F}_\nu(\xi, \mu)$. Now $\nu_1, \nu_2 \subseteq \mu$, $\nu_1 \neq \nu_2$ implies $\mathcal{F}_{\nu_1}(\xi, \mu) \cap \mathcal{F}_{\nu_2}(\xi, \mu) = \phi$. Moreover $\cup_{\nu \subseteq \mu} \mathcal{F}_\nu(\xi, \mu) = \mathcal{F}(\xi, \mu)$. This completes the proof. $\square$

Lemma 4.2 *If $\phi \in \mathcal{P}_{\delta_l}$ then $q \in B_{\delta_l}$ and*

(6)

$$\widetilde{\varphi}_\xi(\mu) \leq \psi_{a,|q|}(\xi, \mu), \xi, \mu \in \mathcal{M}^{\cdot}_f(X_{per}), \xi \neq 0, \xi \cap \mu = \phi, \text{ where } a = e^{2\beta B}.$$

For a proof we remark that $\tilde{\varphi}$ is a solution of the non-integrated Kirkwood-Salzburg equation. Hence

$$\tilde{\varphi}_\xi(\mu) \leq= a \cdot \int_{\mathcal{M}^{\cdot}_f(X_{per})} \prod_{y \in \nu} |q|(\hat{x}(\xi), y) \cdot |\widetilde{\varphi}_{\xi - \varepsilon_{\hat{x}(\xi)} + \nu}(\mu - \nu)| \grave{W}_\mu(d\nu),$$

which implies the lemma by induction on the numbers $|\xi| + |\mu|$. (See [8] e.g.) The fact that $q \in B_{\delta_l}$ is shown in the appendix.

It is well known that the Ursell function $g = \Gamma^{-1} f$ and $\tilde{\varphi}$ are related in the following way. (See [5])

$$D_x g = \widetilde{\varphi}_x, x \in X_{per}. \tag{7}$$

Therefore $g(\mu) = D_x g(\mu - \varepsilon_x) = \tilde{\varphi}_x(\mu - \varepsilon_x)$ if $\mu \in \mathcal{M}^{\cdot}_f(X_{per}) \setminus \{0\}$ and $x \in \mu$. Lemma 4.1 and 4.2 therefore imply for $\mu \in \mathcal{M}^{\cdot}_f(X_{per}) \setminus \{0\}$ and $x \in \mu$

$$g(\mu) \leq e^{2\beta B|\mu|} \cdot \sum_{w \in \mathcal{F}(\varepsilon_x, \mu - \varepsilon_x)} \prod_{[x_1, x_2] \in w} |q|(x_1, x_2). \tag{8}$$

But $\mathcal{F}(\varepsilon_x, \mu - \varepsilon_x) = \mathcal{T}(\mu)$, where $\mathcal{T}(\mu)$ denotes the set of all trees constructed on μ. Thus we arrive at the following **strong cluster estimate** for the Ursell function g:

$$g(\mu) \leq e^{2\beta B|\mu|} \sum_{\gamma \in \mathcal{T}(\mu)} \prod_{[x_1, x_2] \in \gamma} |q|(x_1, x_2), \mu \in \mathcal{M}^{\cdot}_f(X_{per}). \tag{9}$$

Therefore, using the fact that the number of edges of a tree $\gamma \in \mathcal{T}(\mu)$ is $|\mu| - 1$, $g \in \mathcal{C}_{e^{2\beta B}, e^{2\beta B} \cdot |q|}$ in the sense of the following definition. Given $a > 0$ and $h \in B^+_\delta$ let

(10)

$$\mathcal{C}_{a,h} = \{F \in E \mid \forall \mu \in \mathcal{M}^{\cdot}_f(X_{per}) \ |F(\mu)| \leq a \cdot \sum_{\gamma \in \mathcal{T}(\mu)} \prod_{[x_1, x_2] \in \gamma} h(x_1, x_2)\}.$$

Here E denotes the algebra defined above. If $F \in \mathcal{C}_{a,h}$ for some $a \geq 0$ and $h \in B_\delta^+$ we say that F **has a strong cluster property**.

We are now in the position to state the main result of the first part of this paper, which in part II will be used as the main lemma.

Theorem 4.3 *Let δ be an Euclidean invariant metric on $\mathbb{R}^\nu$. If F has a strong cluster property with respect to $a > 0$ and $h \in B_\delta^+$, then if*

$$0 < z < ((2+e) \cdot \|h\|_\delta)^{-1} \tag{11}$$

the following estimate is true for any R and $u \in \mathbb{R}^\nu$:

$$\begin{aligned} &\int\limits_{X_{per}^0} P_{00}(dx^0) \int\limits_{\mathbb{C}\mathcal{M}^{\cdot}_f(\Lambda_u(R))} W_{z\rho}(d\mu) |F(\mu + \varepsilon_{x^0+u})| \leq \\ &\leq \frac{a}{1 - z \cdot e \cdot \|h\|_\delta} \cdot \alpha_\delta(R), \end{aligned} \tag{12}$$

where

$$\alpha_\delta(R) = C_1(\nu, \beta) \cdot \sigma\left(\frac{R}{12}, \frac{\beta}{2}\right) + C_2(\nu, \beta, z, h, \delta) \cdot \exp\left(-\delta\left(0, \frac{2R}{3}\right)\right).$$

Here ρ denotes the measure in X_{per} defined by (3) in §2 and $\mathbb{C}\mathcal{M}^{\cdot}_f(\Lambda_u(R)) = \{\mu \in \mathcal{M}^{\cdot}_f(X_{per}) \setminus \{0\} \bigm| \exists\, x \in \mu : x \notin B_{\Lambda_u(R)}\}$. $\Lambda_u(R)$ denotes the open ball in $\mathbb{R}^\nu$ with radius R centered at u and $B_{\Lambda_u(R)} = \{x \in X_{per} \bigm| x(t) \in \Lambda_u(R)$ for each $0 \leq t \leq \beta\}$. The constants C_1 and C_2 are defined below.

The proof of the theorem is given in a series of lemmas, where we use the following notations and conventions:

$$\begin{cases} dx^0 &= P_{00}(dx^0), d\mu = W_\rho(d\mu), \\ A_R(F) &= \int\limits_{X_{per}^0} dx^0 \int\limits_{\mathbb{C}\mathcal{M}^{\cdot}_f(\Lambda_u(R))} d\mu |F(\mu + \varepsilon_{x^0+u})| \cdot z^{|\mu|}, \\ \tilde{A}_R(x^0, F) &= \int\limits_{\mathbb{C}\mathcal{M}^{\cdot}_f(\Lambda_u(R))} d\mu |F(\mu + \varepsilon_{x^0+u})| \cdot z^{|\mu|}. \end{cases} \tag{13}$$

We can write

$$A_R(F) = \int\limits_{B_{\Lambda_0(R/3)}} \tilde{A}_R(x^0, F) dx^0 + \int\limits_{\mathbb{C}B_{\Lambda_0(R/3)}} \tilde{A}_R(x^0, F) dx^0. \tag{14}$$

To estimate the second term on the right hand side we need the following lemma.

Lemma 4.4 *If* $0 < z < (e \cdot \|h\|_\delta)^{-1}$, *then*

$$\sup_{x \in X_{per}} \int\limits_{\mathcal{M}^\cdot_f(X_{per})} d\mu \, |F(\mu + \varepsilon_x)| \cdot z^{|\mu|} \leq \frac{a}{1 - z \cdot e \cdot \|h\|_\delta}. \tag{15}$$

Proof. Since $F \in \mathcal{C}_{a,h}$ on account of translation-invariance of $d\mu$ and h we can write for each $u \in \mathbb{R}^\nu$ and $x^0 \in X^0_{per}$

$$\int\limits_{\mathcal{M}^\cdot_f(X_{per})} d\mu |F(\mu + \varepsilon_{x^0+u})| \leq a \cdot \int\limits_{\mathcal{M}^\cdot_f(X_{per})} z^{|\mu|} \cdot \sum_{\gamma \in \mathcal{T}(\mu + \varepsilon_{x^0})} \prod_{[a,b] \in \gamma} h(a,b) d\mu$$

$$= a \cdot \sum_{n=0}^{\infty} \frac{z^n}{n!} \int\limits_{X^n_{per}} \sum_{\gamma \in \mathcal{T}(\varepsilon_{x^0} + \varepsilon_{x_1} + \ldots + \varepsilon_{x_n})} \prod_{[a,b] \in \gamma} h(a,b) \rho(dx_1) \ldots \rho(dx_n)$$

This can be evaluated by exactly the same method as in [8, Lemma 3], and one obtains

$$\leq a \cdot \sum_{n=0}^{\infty} \frac{z^n}{n!} (n+1)^{n-1} \cdot \|h\|^n_\delta$$

By Stirling's formula $n^n < e^n \cdot n!$, hence $(n+1)^{n-1} < n^n < e^n \cdot n!$.

It follows under the condition $z < (e \cdot \|h\|_\delta)^{-1}$, that

$$\widetilde{A}_R(x^0, F) \leq a \cdot \sum_{n=0}^{\infty} (z \cdot e \cdot \|h\|_\delta)^n = \frac{a}{1 - z \cdot e \cdot \|h\|_\delta}. \tag{16}$$

□

Furthermore we need the following probabilistic estimate which can be found in [5, I, Lemma 6] and [9]:

Lemma 4.5 *If for* $\varepsilon > 0$ *we define*

$$M(\varepsilon, \beta) = \left\{ x \in X : |x(t) - x(s)| > \varepsilon \text{ for some } s,t \in [0, \frac{\beta}{2}] \text{ or } s,t \in \left[\frac{\beta}{2}, \beta\right] \right\}$$

then

$$P_{xy}\left(M\left(\varepsilon, \beta\right)\right) \leq C_1(\nu, \beta) \cdot \sigma\left(\frac{\varepsilon}{4}, \frac{\beta}{2}\right),$$

where $C_1(\nu, \beta) = 2^{\frac{\nu}{2}+2} (\pi \cdot \beta)^{-\frac{\nu}{2}}$ *and*

$$\sigma(\varepsilon, \delta) = \frac{|\Lambda_0(1)|}{\pi^{\nu/2}} \cdot \int\limits_{u > \frac{\varepsilon}{\sqrt{\delta}}} e^{-u^2} \cdot u^{\nu-1} du.$$

Note that $\sigma(\varepsilon, \delta)$ is a function of $\tau = \frac{\varepsilon}{\sqrt{\delta}}$ and hence satisfies the estimate

$$\sigma(\varepsilon, \delta) \leq \overline{C}_\nu \cdot \left(\frac{\varepsilon}{\sqrt{\delta}}\right)^{\nu-1} \cdot exp\left(-\frac{1}{2}\frac{\varepsilon}{\delta}^2\right), \text{ if } \varepsilon \text{ is large enough.}$$

Therefore for large ε $P_{xy}\left(M\left(\varepsilon, \beta\right)\right) \leq \overline{C}_{\nu,\beta}\left(\frac{\varepsilon}{\sqrt{\beta}}\right)^{\nu-1} \cdot \exp\left(-\frac{\varepsilon^2}{16\beta}\right)$, where $\overline{C}_{\nu,\beta}$ is a constant depending only on ν and β.

We can now estimate the second term of (14) by means of lemma 4.4 and lemma 4.5: Since $\mathbb{C}B_{\Lambda_0(R/3)} \subseteq M\left(R/3, \beta\right)$ we obtain

$$\int\limits_{\mathbb{C}B_{\Lambda_0(R/3)}} \widetilde{A}_R(x^0, F)dx^0 \leq \frac{a \cdot P_{00}\left(M\left(\frac{R}{3}, \beta\right)\right)}{1 - z \cdot e \cdot \|h\|_\delta} \leq$$

$$\leq \frac{a \cdot C_1(\nu, \beta)}{1 - z \cdot e \cdot \|h\|_\delta} \cdot \sigma\left(\frac{R}{12}, \frac{\beta}{2}\right). \tag{17}$$

We consider the first term on the right hand side of (14). Let $x^0 \in B_{\Lambda_0(R/3)} \cap X^0_{per}$. Using the strong cluster property of F and translation-invariance we have

$$\int\limits_{\mathbb{C}\mathcal{M}_f^{\cdot}(\Lambda_u(R))} z^{|\mu|} \cdot |F(\mu + \varepsilon_{x^0+u})| d\mu \leq a \cdot \sum_{n=1}^{\infty} \frac{z^n}{n!} \int\limits_{X^n_{per}} 1_{\mathbb{C}\mathcal{M}_f^{\cdot}(\Lambda_0(R))}(\varepsilon_{x_1} + \ldots + \varepsilon_{x_n}) \cdot$$

$$\cdot \sum_{\gamma \in \mathcal{T}(\varepsilon_{x^0} + \varepsilon_{x_1} + \ldots + \varepsilon_{x_n})} \prod_{[a,b]\in\gamma} h(a,b)\rho(dx_1)\ldots\rho(dx_n)$$

$$\leq a \cdot \sum_{n=1}^{\infty} \frac{z^n}{n!} \sum_{j=1}^{n} \int\limits_{A^n_j(R)} \sum_{\gamma \in \mathcal{T}(\varepsilon_{x^0} + \varepsilon_{x_1} + \ldots + \varepsilon_{x_n})} \prod_{[a,b]\in\gamma} h(a,b)\rho(dx_1)\ldots\rho(dx_n). \tag{18}$$

Here

$$A^n_j(R) = \{(x_1, \ldots, x_n) \in X^n_{per} \mid \exists\, t \in [0, \beta] : |x_j(t)| \geq R\}. \tag{19}$$

Let us denote by $\tau_{0,j}(i)$ the chain $(x^0, x_{k_1}, \ldots, x_{k_{i-1}}, x_j)$ where $\tau_{0,j}(1) = (x_0, x_j)$ and let

$$\mathcal{T}(\varepsilon_{x^0} + \varepsilon_{x_1} + \ldots + \varepsilon_{x_n} \mid \tau_{0,j}(i)) = \{\gamma \in \mathcal{T}(\varepsilon_{x^0} + \varepsilon_{x_1} + \ldots + \varepsilon_{x_n}) \mid \tau_{0,j}(i) \text{ is a subgraph of } \gamma\}.$$

It is easy to check that

(20)
$$\mathcal{T}(\varepsilon_{x^0}+\varepsilon_{x_1}+\ldots+\varepsilon_{x_n}) \subseteq \bigcup_{j=1}^{n}\bigcup_{i=1}^{n}{\bigcup}' \mathcal{T}(\varepsilon_{x^0}+\varepsilon_{x_1}+\ldots+\varepsilon_{x_n} \mid \tau_{0,j}(i)),$$

where ${\bigcup}'$ is taken over all ordered sets $(k_1,\ldots,k_{i-1})$, $1 \le k_l \le n$, $k_l \ne j$, $l=1,\ldots,i-1$, $k_l \ne k_s$ if $l \ne s$. Since the graph structure does depend only on say $\{0,1,\ldots,n\}$ and not on $\{x^0,x_1,\ldots,x_n\}$,

$$\int\limits_{\mathbb{C}\mathcal{M}^{\cdot}_{f}(\Lambda_u(R))} z^{|\mu|}\cdot|F(\mu+\varepsilon_{x^0})|d\mu \le a\cdot\sum_{n=1}^{\infty}\frac{z^n}{n!}\sum_{j=1}^{n}\sum_{i=1}^{n}{\sum}'\cdot$$

(21)
$$\cdot\sum_{t\in\mathcal{T}(\{0,1,\ldots,n\}\ \mid\ c_{0,j}(i))}\ \int\limits_{A_j^n(R)}\ \prod_{[a,b]\in\gamma(\varepsilon_{x^0}+\varepsilon_{x_1}+\ldots+\varepsilon_{x_n},t)} h(a,b)\rho(dx_1)\ldots\rho(dx_n).$$

Here the sum Σ' is taken over the same set of $(k_1,\ldots,k_{i-1})$ as in $\cup'$ of formula (20). $c_{0,j}(i)$ denotes the chain $(0,k_1,\ldots,k_{i-1},j)$ and $\mathcal{T}(\{0,1,\ldots,n\}\mid c_{0,j}(i)) = \{t\in\mathcal{T}(\{0,1,\ldots,n\})\mid c_{0,j}(i)$ is a subgraph of $t\}$. Furthermore, $\gamma(\varepsilon_{x^0}+\varepsilon_{x_1}+\ldots+\varepsilon_{x_n},t)$ denotes the tree in $\mathcal{T}(\varepsilon_{x^0}+\varepsilon_{x_1}+\ldots+\varepsilon_{x_n}\mid\tau_{0,j}(i))$ which has the same graph structure as t with $\{0,1,\ldots,n\}$ replaced by $\{x^0,x_1,\ldots,x_n\}$. Since

(22)
$$\int\limits_{A_j^n(R)}\ \prod_{[a,b]\in\gamma(\varepsilon_{x^0}+\varepsilon_{x_1}+\ldots+\varepsilon_{x_n},t)} h(a,b)\rho(dx_1)\ldots\rho(dx_n) = D_i(x^0,R,h)\cdot\|h\|_\delta^{n-i},$$

where

(23)
$$D_i(x^0,R,h) = \int\limits_{\mathbb{C}B_{\Lambda_0(R)}} h^{0i}(x^0,x)\rho(dx)$$

and h^{0i} denotes the convolution of h i times $h^{0i} = h\circ\cdot\overset{i}{\cdot}\cdot\circ h$, where $\circ$ is defined in the appendix. Now it is not difficult to check that for any fixed chain $\tau_{0,j}(i)$

(24)
$$\sum_{t\in\mathcal{T}(\{0,1,\ldots,n\}\ \mid\ c_{0,j}(i))} = \sum_{\substack{(n_1,\ldots,n_{i+1}):\\ n_k\ge 0, n_1+\ldots+n_{i+1}=n-i}} (n-i)!\prod_{k=1}^{i+1}\frac{n_k^{n_k}}{n_k!}.$$

Moreover we have

$$\Sigma' 1 = \frac{(n-1)!}{(n-i)!}. \tag{25}$$

To summarize, from (21) – (25) it follows that

$$\begin{aligned}
&\int\limits_{\complement \mathcal{M}^{\cdot}_f(\Lambda_u(R))} z^{|\mu|} \cdot |F(\mu + \varepsilon_{x^0+u})| d\mu \leq a \cdot \sum_{n=1}^{\infty} z^n \cdot \sum_{i=1}^{n} D_i(x^0, R, h) \cdot \|h\|_\delta^{n-i} \cdot \\
&\qquad \cdot \sum_{\substack{(n_1,\ldots,n_{i+1}): \\ n_k \geq 0, n_1+\ldots+n_{i+1}=n-i}} \prod_{k=1}^{i+1} \frac{n_k^{n_k}}{n_k!} \\
&\qquad = a \cdot \sum_{i=1}^{\infty} z^i \cdot D_i(x^0, R, h) \left[\sum_{n=0}^{\infty} (z \cdot \|h\|_\delta)^n \cdot \frac{n^n}{n!} \right]^{i+1} \\
&\qquad \leq \frac{a}{1 - z \cdot e \cdot \|h\|_\delta} \cdot \sum_{i=1}^{\infty} \left(\frac{z}{1 - z \cdot e \cdot \|h\|_\delta} \right)^i \cdot \int\limits_{\complement B_{\Lambda_0(R)}} (h \circ \overset{i}{\ldots} \circ h)(x^0, x) \rho(dx).
\end{aligned} \tag{26}$$

Hence

$$\begin{aligned}
&\int\limits_{\complement \mathcal{M}^{\cdot}_f(\Lambda_u(R))} d\mu \; z^{|\mu|} \cdot |F(\mu + \varepsilon_{x^0+u})| \leq \frac{a}{1 - z \cdot e \cdot \|h\|_\delta} \cdot \\
&\int\limits_{\complement B_{\Lambda_0(R)}} \sum_{i=1}^{\infty} \left(\frac{z}{1 - z \cdot e \cdot \|h\|_\delta} \right)^i \cdot h^{0i}(x^0, x) \rho(dx).
\end{aligned} \tag{30}$$

The interchange of summation and integration is justified by the fact that under the condition (11) the series

$$\mathcal{H}_z = \sum_{i=1}^{\infty} \left(\frac{z}{1 - z \cdot e \cdot \|h\|_\delta} \right)^i \cdot h^{0i}$$

converges uniformly in x^0 and x, if we use here the properties of the convolution operation developed in the appendix.

On the other hand, if z satisfies the condition (11) then $\mathcal{H}_z \in B_\delta^+$, as is shown in the appendix. Therefore the integral in (30) $\int\limits_{\complement B_{\Lambda_0(R)}} \mathcal{H}_z(x^0, x)\rho(dx)$,

$x^0 \in B_{\Lambda_0(R/3)}$, is bounded by $\|\mathcal{H}_z\|_\delta \cdot \exp(-\delta(0, \frac{2R}{3}))$, which does not depend on x^0. Thus for $0 < z < [(2+e)\cdot \|h\|_\delta]^{-1}$

$$\int\limits_{B_{\Lambda_0(R/3)}} dx^0 \int\limits_{\complement \mathcal{M}_f(\Lambda_u(R))} d\mu\; z^{|\mu|} \cdot |F(\mu + \varepsilon_{x^0})| \le \frac{a \cdot C_2}{1 - z \cdot e \cdot \|h\|_\delta} \cdot \exp(-\delta(0, \frac{2R}{3})),$$

where $C_2 = (\pi\beta)^{-\frac{\nu}{2}} \cdot \|\mathcal{H}_z\|_\delta$. Combining this with (17) completes the proof of the theorem. $\square$

5 Appendix

The Banach space

Let B denote the vector space of functions $f : X_{per} \times X_{per} \to \mathbb{R}$ which are invariant with respect to the group $t_u : (x, y) \to (x + u, y + u)$, $u \in \mathbb{R}^\nu$; and define for each $f \in B$ and Euclidean invariant metric δ on R^ν

(A1) $$\|f\|_\delta = \max\{\|f\|_\infty, p(f), p_\delta(f)\},$$

where $\|f\|_\infty = \sup\limits_{x,y\in X_{per}} |f(x,y)|$, $p(f) = \sup\limits_{x \in X_{per}} \int\limits_{X_{per}} |f(x,y)|\rho(dy)$ and

(A2)

$$p_\delta(f) = \inf\left\{ C > 0 \;\Big|\; \forall\; d > 0 \sup_{R>0} \sup_{x\in B_{\Lambda_0(R)}} \int\limits_{\complement B_{\Lambda_0(R+d)}} |f(x,y)|\rho(dy) \le C \cdot \exp(-\delta(0,d)) \right\}.$$

Then $B_\delta = \{f \in B : \|f\|_\delta < +\infty\}$ is a Banach space with the norm $\|\cdot\|_\delta$. For two elements $f, g \in B_\delta$ we define a convolution $f \circ g$ by

(A3) $$f \circ g(x,y) = \int\limits_{X_{per}} f(x,z) \cdot g(z,y)\rho(dz)$$

Let us show first that $\|f \circ g\|_\delta \le 2 \cdot \|f\|_\delta \cdot \|g\|_\delta$, so in particular $\circ$ is a mapping from $\mathcal{M}_l \times \mathcal{M}_l$ into $\mathcal{M}_l$. It is straightforward that $\|f \circ g\|_\infty \le p(f) \cdot \|g\|_\infty$ and that $p(f \circ g) \le p(f) \cdot p(g)$. We show

$$p_\delta(f \circ g) \le 2 \cdot \|f\|_\delta \cdot \|g\|_\delta.$$

Indeed, let $R > 0$, $d > 0$ and $x \in B_{\Lambda(R)}$. Using Fubini's theorem yields

$$\int\limits_{\complement B_{\Lambda_0(R+d)}} |f \circ g|(x,y)\rho(dy) \le \int\limits_{X_{per}} |f(x,z)| \cdot \int\limits_{\complement B_{\Lambda_0(R+d)}} |g(z,y)|\rho(dy)\rho(dz)$$

Decomposing X_{per} into $B_{\Lambda_0\left(R+\frac{d}{2}\right)}$ and $\complement B_{\Lambda_0\left(R+\frac{d}{2}\right)}$ gives the estimate

$$\begin{aligned} &\le p_\delta(g) \cdot \exp(-\delta(0, \frac{d}{2})) \cdot p(f) + p(g) \cdot p_\delta(f) \cdot \exp(-\delta(0, \frac{d}{2})) \\ &\le 2 \cdot \|f\|_\delta \cdot \|g\|_\delta \end{aligned}$$

Thus

$$\|f \circ g\|_\delta \le 2 \cdot \|f\|_\delta \cdot \|g\|_\delta. \tag{A4}$$

Lemma A1 *For any $h \in B_\delta^+$ with $\|h\|_\delta < \frac{1}{2}$ we have $\sum\limits_{i=1}^{\infty} h^{0i} \in B_\delta$.*

Proof. Given $h \in B_\delta^+$ we define a linear operator R_h acting on the space B_δ by $R_h f = f \circ h$, $f \in B_\delta$. Using (A4) it is easy to see that $\|R_h\| \le 2\|h\|_\delta$. Hence if $\|h\|_\delta < \frac{1}{2}$, then $\|R_h\| < 1$ which implies the existence of the inverse operator $(I - R_h)^{-1} : B_\delta \to B_\delta$, where I denotes the identity. Moreover $(I - R_h)^{-1} = \sum\limits_{i=0}^{\infty} R_h^i$. Since $R_h^i(h) = h^{0(i+1)}$ we have

$$\sum_{i=1}^{\infty} h^{0i} = \sum_{i=1}^{\infty} R_h^{i-1}(h) = (I - R_h)^{-1}(h),$$

which completes the proof of the lemma. □

We finally remark that if $h \in B_\delta^+$ and $b > 0$ is a constant satisfying the condition $b < (2\|h\|_\delta)^{-1}$ then $\sum\limits_{i=1}^{\infty} b^i \cdot h^{0i} \in B_\delta^+$.

We remark also that, if $h \in B_\delta^+$ is symmetric then $\mathcal{H}_z$ is symmetric because each h^{0i} is symmetric.

Upper bounds for q

We collect here various bounds for q, which show that if $\phi \in B_{\delta_l}$ then $q \in B_{\delta_l}$ for any $l > 0$. First of all, q is bounded, because ϕ is bounded from below. Therefore $\|q\|_\infty < +\infty$. Moreover,

$$p(q) \le (\pi\beta)^{-\nu/2} \cdot \beta \cdot e^{2\beta B} \cdot \int\limits_{\mathbb{R}^\nu} |\phi(u)| du. \tag{A5}$$

Finally one has

$$p_\delta(q) \le (\pi\beta)^{-\nu/2} \cdot \beta \cdot e^{2\beta B} \cdot c_{l,\nu}, \tag{A6}$$

where $c_{l,\nu}$ is a constant depending only on l and ν. We briefly indicate a proof of (A6). It is implied by the following bound of [5I, (4.7)]:

If $R > 0$, $d > 0$ then

$$\sup_{x\in B_{\Lambda_0(R)}} \int_{\complement B_{\Lambda_0(R+d)}} |q(x,y)|\rho(dy) \le$$

$$\le (\pi\beta)^{-\nu/2} \cdot \beta \cdot e^{2\beta B} \left[\int_{|u|>\frac{d}{2}} |\phi(u)|du + 2^{\frac{\nu}{2}+1} \cdot \sigma\left(\frac{d}{8}, \frac{\beta}{2}\right) \cdot \int_{\mathbb{R}^\nu} |\phi(u)|du \right].$$

Taking into account the decay property (3) in §1 of ϕ allows to write

$$\int_{|u|>\frac{d}{2}} |\phi(u)|du \le \exp\left(-\delta_l\left(0, \frac{d}{2}\right)\right) \cdot \int_{\mathbb{R}^\nu} |\phi(u)| \exp\left(\delta_l(0,u)\right) du.$$

Combining this with the decay property of σ yields (A6).

II The Geometric expansion of the log-partition function of the Ginibre gas

Introduction and formulation of the main results

We fix a potential ϕ satisfying the conditions of I. 1 with respect to the Euclidean invariant metric $\delta_l, l > \nu$. Thus ϕ has power decay of order faster than $l + \nu$. To understand the problem treated below, the following representation of the log-partition function $ln\, Z(\Lambda, z)$ by means of the so-called Ursell function g is of central importance (see I, §3):

$$ln\, Z(\Lambda, z) = W_{z\rho_\Lambda}(g). \tag{1}$$

The main property of g which allows a detailed analysis of the log-partition function is that g has a **strong cluster property** with respect to the parameter $e^{2\beta B}$ and **estimator function** $|q|$, where $q(x, y) = exp(-\int\limits_0^\beta \phi(x(s) - y(s))ds) - 1$. As we saw above this means that $g \in \mathcal{C}^+_{a,h}$, $a = e^{2\beta B}$ and $h = e^{2\beta B} \cdot |q|$, where $F \in \mathcal{C}^+_{a,h} :\Longleftrightarrow F \in \mathcal{C}_{a,h}$ and $F(0) = 0$.

The aim of part II is to develop a geometric expansion of the log-partition function of the Ginibre gas. In the case of a classical gas this has been obtained by one of the authors. (See [10].)

We consider this problem in a more general setting, so that the underlying mathematical structure is more transparent and in order to be able to consider other examples then Ursell functions. To be capable to do this we have to analyze the decay properties of the estimator function $|q|$. As was shown in I this means: $|q|$ is an element of the Banach space $(B^+_l, \|\cdot\|_l)$, where from now on we write $B_l = B_{\delta_l}$ and $\|u\|_l = \delta_l(0, u)$. The problem of part II can then be formulated in the following way:

Given $z > 0$, $l > \nu$, $a > 0$ and $h \in B^+_l$. If F is in $\mathcal{C}^+_{a,h}$ and **invariant with respect to translations of** $\mathbb{R}^\nu$, we want to develop a geometric expansion of $W_{z\rho_\Lambda}(F)$, in analogy to the classical case treated in [10]. Our result will be as follows:

Let $\mathcal{V}_{(r_1,r_2)}$ be a class of parallel-epipeds $\Lambda = [0, n_1] \times \ldots \times [0, n_\nu]$ in $\mathbb{R}^\nu$ satisfying

$$\min(n_1, \ldots, n_\nu) \geq r_1 \cdot L(\Lambda), diam\, \Lambda \leq r_2 \cdot L(\Lambda), \tag{2}$$

where $L(\Lambda) = |\Lambda|^{1/\nu}$. For any $j = 0, \ldots, \nu$ we denote by $\Lambda^{(j)}$ the family of all j-faces of Λ, and by $|\cdot|_j$ the j-dimensional volume. ($\Lambda^{(\nu)} = \Lambda$, $\Lambda^{(0)}$ is the set of all vertices of Λ.) Instead of $|\cdot|_\nu$ we write $|\cdot|$.

Theorem 1 Let $\Lambda \in \mathcal{V}_{(r_1,r_2)}$, $h \in B_{\delta_l}$ with $l > \nu$, $a > 0$ and $0 < z < [(2+e)\cdot\|h\|_{\delta_l}]^{-1}$. Then for any **translation-invariant** $F \in \mathcal{C}^+_{a,h}$ the following expansion holds:

(3)
$$W_{z\rho_\Lambda}(F) = z \cdot A_0(F) \cdot |\Lambda| + z \cdot \sum_{\lambda \in \Lambda^{(\nu-1)}} A_{1,2}(F) \cdot |\lambda|_{\nu-1} + \ldots + z \cdot \sum_{\lambda \in \Lambda^{(0)}} A_{\nu,\lambda}(F) + R(\Lambda),$$

where $|R(\Lambda)| \leq C \cdot L(\Lambda)^{\nu-l}$ and the coefficients $A_{j,\lambda}(F)$, $\lambda \in \Lambda^{(\nu-j)}$, $j = 0, \ldots, \nu$, do not depend on Λ.

In the case that F is Euclidean-invariant we have the following result as a consequence:

Theorem 2 If in addition to the conditions of theorem 1 F is **Euclidean-invariant**, then

(4)
$$W_{z\rho_\Lambda}(F) = z \cdot A_0(F) \cdot |\Lambda| + z \cdot A_1(F) \cdot \sum_{\lambda \in \Lambda^{(\nu-1)}} |\lambda|_{\nu-1} + \ldots + z \cdot A_\nu(F) + R(\Lambda),$$

where $|R(\Lambda)| \leq C \cdot L(\Lambda)^{\nu-l}$ and the coefficients $A_0, \ldots, A_\nu$ do not depend on Λ.

In both theorems the constant depends on the parameters $a, z, \beta, r_1, r_2, h, \nu$. The coefficients in the expansion are explicitely represented in terms of F and the underlying Ginibre gas in the course of the following proof, which for simplicity is given only for $\nu = 2$. To get this result our main lemma will be theorem 4.3 of I. The geometric expansion of the log-partition function is obtained in the special case when F is the Ursell function.

1 The Proof of Theorem 1

We are given from now on $l > \nu$, $z > 0$, $a > 0$, $h \in B_l^+$ and a function $F \in \mathcal{C}^+_{a,h}$ which is invariant with respect to translations of $\mathbb{R}^\nu$.

The general idea to get the expansion is to decompose $W_{z\rho_\Lambda}(F)$ successively in differences. The leading idea is to isolate 'global' or limiting terms which are independent of the volume Λ. They appear as integrals which are invariant under the full translation group or subgroups of translations in the direction of the edges of Λ.

1. The initial decompositions

To be more precise the main decompositions of $W_{z\rho_\Lambda}(F)$ are as follows:
As an abbreviation we use in the following $\tilde{F}(u, x^0, \mu) = z^{|\mu|} \cdot \frac{F(\mu+\varepsilon_{x^0+u})}{|\mu|+1}$.
If $\Lambda \subseteq \mathbb{R}^\nu$ is a measurable, bounded subset and $z > 0$ we can represent $W_{z\rho_\Lambda}(F)$ as follows using well-known properties of $W_{z\rho_\Lambda}$:

$$W_{z\rho_\Lambda}(F) = z \cdot \int\limits_{\Lambda} du \int\limits_{B_{\Lambda-u}} dx^0 \int\limits_{\mathcal{M}^{\cdot}_f(\Lambda)} d\mu\ \tilde{F}(u, x^0, \mu).$$

Here the integrand depending on $u \in \Lambda$ can be written as a difference

$$\begin{aligned}
(1) \qquad \int\limits_{B_{\Lambda-u}} dx^0 \int\limits_{\mathcal{M}^{\cdot}_f(\Lambda)} d\mu\ \tilde{F}(u, x^0, \mu) &= \\
&= \int\limits_{X^0_{per}} dx^0 \int\limits_{\mathcal{M}^{\cdot}_f(\Lambda)} d\mu\ \tilde{F}(u, x^0, \mu) - \\
&- \int\limits_{\complement B_{\Lambda-u}} dx^0 \int\limits_{\mathcal{M}^{\cdot}_f(\Lambda)} d\mu\ \tilde{F}(u, x^0, \mu) \\
&=: I_1(u) - I_2(u).
\end{aligned}$$

Hence

$$(2) \qquad W_{z\rho_\Lambda}(F) = z \cdot \int\limits_{\Lambda} I_1(u) du - z \cdot \int\limits_{\Lambda} I_2(u) du.$$

In a second step we decompose the terms on the right-hand side once more:

$$(3) \qquad I_1(u) = I_1^A(u) - I_1^B(u),$$

where

$$I_1^A(u) = \int\limits_{X_{per}^0} dx^0 \int\limits_{\mathcal{M}^{\cdot}_f(X_{per})} \tilde{F}(u, x^0, \mu) d\mu$$

$$I_1^B(u) = \int\limits_{X_{per}^0} dx^0 \int\limits_{\complement\mathcal{M}^{\cdot}_f(\Lambda)} \tilde{F}(u, x^0, \mu) d\mu$$

and

(4) $$I_2(u) = I_2^A(u) - I_2^B(u),$$

where

$$I_2^A(u) = \int\limits_{\complement B_{\Lambda-u}} dx^0 \int\limits_{\mathcal{M}^{\cdot}_f(X_{per}} \tilde{F}(u, x^0, \mu) d\mu,$$

$$I_2^B(u) = \int\limits_{\complement B_{\Lambda-u}} dx^0 \int\limits_{\complement\mathcal{M}^{\cdot}_f(\Lambda)} \tilde{F}(u, x^0, \mu) d\mu.$$

In the following we shall show that $I_1^A(u)$ does not depend on u and gives the unique contribution of the volume. All other terms $I_1^B(u)$, $I_2^A(u)$ and $I_2^B(u)$ give contributions of the boundary and angles with some corrections which are estimated from above by $L(\Lambda)^{-(l-\nu)}$. The term $I_1^B(u)$ will be analyzed in all detail. For the others we point out only the main ideas to avoid repetitions.

1.A The contribution of the volume. The pressure

Consider now $I_1^A(u)$. Since both, W_ρ and F, are invariant with respect to translations of u, I_1^A does not depend on u, and thus is equal to

(5) $$A_v^1(F) = \int\limits_{X_{per}^0} dx^0 \int\limits_{\mathcal{M}^{\cdot}_f(X_{per})} d\mu \; z^{|\mu|} \cdot \frac{F(\mu + \varepsilon_{x^0})}{|\mu| + 1}.$$

Therefore

(6) $$z \cdot \int\limits_{\Lambda} I_1(u) du = A_v^1(F) \cdot z \cdot |\Lambda| - z \cdot \int\limits_{\Lambda} I_1^B(u) du.$$

The contribution $A_v^1(F)$ of the volume $|\Lambda|$ is called the **pressure**, if F is the Ursell function g.

1.B Analysis of $z \cdot \int\limits_\Lambda I_1^B(u)du$

Let $\Lambda \in \mathcal{V}_{(r_1,r_2)}$. For simplicity of the argument we consider only the case $\nu = 2$: $\Lambda = [0, n_1] \times [0, n_2]$. In order to decompose $\complement\mathcal{M}^{\cdot}_f(\Lambda)$ we introduce the following half-spaces:

$$\text{(1)} \qquad \begin{cases} R_1 = \{u | u_2 \le 0\}, & R_2 = \{u | u_1 \ge 0\} \\ R_3 = \{u | u_2 \ge 0\}, & R_4 = \{u | u_1 \le 0\}, \end{cases}$$

and

$$\text{(2)} \qquad \begin{cases} \Pi_1 = R_1, & \Pi_2 = R_2 + (n_1, 0) \\ \Pi_3 = R_3 + (0, n_2), & \Pi_4 = R_4. \end{cases}$$

Note that the Π_2 and Π_3 depend on Λ, whereas the R_j are independent of Λ. Finally let $\Pi_{j,j+1} = \Pi_j \cup \Pi_{j+1}$ and $R_{j,j+1} = R_j \cup R_{j+1}$, $j = 1, \ldots, 4$. Here $4 + 1 = 1$.

Furthermore we consider for $j = 1, \ldots, 4$

(3)
$$\begin{cases} G_j = \{\mu \in \mathcal{M}^{\cdot}_f(X_{per}) \mid \mu \notin \mathcal{M}^{\cdot}_f(\complement\Pi_j) \text{ and } \mu \in \mathcal{M}^{\cdot}_f(\complement\Pi_{j+2})\} \\ \overline{G}_j = \{\mu \in \mathcal{M}^{\cdot}_f(X_{per}) \mid \mu \notin \mathcal{M}^{\cdot}_f(\complement\Pi_j)\}; \\ G_{j,j+1} = G_j \cap G_{j+1}, \\ \overline{G}_{j,j+1} = \overline{G}_j \cap \overline{G}_{j+1}. \end{cases}$$

G_j^0, $\overline{G}_j^0$, $G_{j,j+1}^0$, $\overline{G}_{j,j+1}^0$ are defined analogously with Π_j replaced by R_j. They are independant of Λ whereas the sets of (3) do depend on Λ.

Here we use the convention that $4 + 1 = 1$, $4 + 2 = 2$ and $3 + 2 = 1$.

We decompose $\complement\mathcal{M}^{\cdot}_f(\Lambda)$ as follows:

$$\text{(4)} \qquad \complement\mathcal{M}^{\cdot}_f(\Lambda) = \bigcup_{j=1}^{4} (G_j \setminus G_{j,j+1}) \cup Q(\Lambda),$$

where $Q(\Lambda) = \complement\mathcal{M}^{\cdot}_f(\Lambda) \setminus \bigcup\limits_{j=1}^{4} G_j$.

As a result we obtain another important decomposition:

$$
\begin{aligned}
z\int\limits_{\Lambda} I_1^B(u)du = &\sum_{j=1}^{4} z\cdot\int\limits_{\Lambda} du \int\limits_{X^0_{per}} dx^0 \int\limits_{G_j} d\mu\ \tilde{F}(u,x^0,\mu) - \\
&-\sum_{j=1}^{4} z\cdot\int\limits_{\Lambda} du \int\limits_{X^0_{per}} dx^0 \int\limits_{G_{j,j+1}} d\mu\ \tilde{F}(u,x^0,\mu) + \\
&+z\cdot\int\limits_{\Lambda} du \int\limits_{X^0_{per}} dx^0 \int\limits_{Q(\Lambda)} d\mu\ \tilde{F}(u,x^0,\mu).
\end{aligned}
\tag{5}
$$

In what follows we analyze the different terms on the right-hand side of (5).

1.B.1 The contribution $A_b^{1,B}(F)$ of the boundary term

Our aim here is to show that the four summands of the **first sum** on the right-hand side of (5) contain contributions of the four boundaries of the parallel-epiped Λ. Consider for instance the integrand of the first summand with respect to u. This can be decomposed as follows:

Representing G_1 as $G_1 = \overline{G}_1 \setminus (\overline{G}_1 \setminus G_1)$ gives

(6)

$$
\begin{aligned}
&z\int\limits_{\Lambda} du \int\limits_{X^0_{per}} dx^0 \int\limits_{G_1} d\mu\tilde{F}(u,x^0,\mu) = z\int\limits_0^{n_1} du_1 \int\limits_0^{\infty} du_2 \int\limits_{X^0_{per}} dx^0 \int\limits_{\overline{G}_1} d\mu\tilde{F}(u,x^0,\mu) - \\
&-z\int\limits_0^{n_1} du_1 \int\limits_{n_2}^{\infty} du_2 \int\limits_{X^0_{per}} dx^0 \int\limits_{\overline{G}_1} d\mu\tilde{F}(u,x^0,\mu) - z\int\limits_{\Lambda} du \int\limits_{X^0_{per}} dx^0 \int\limits_{\overline{G}_1\setminus G_1} d\mu\tilde{F}(u,x^0,\mu).
\end{aligned}
$$

The integrand with respect to u_1 of the first term on the right-hand side of (6) does not depend on u_1, due to the u_1-invariance of F and W_ρ as well as $\overline{G}_1$, and hence equals

$$
A_b^{1,B}(F;1) = \int\limits_0^{\infty} du_2 \int\limits_{X^0_{per}} dx^0 \int\limits_{\overline{G}^0_1} d\mu\ z^{|\mu|}\cdot\frac{F(\mu+\varepsilon_{x^0+(0,u_2)})}{|\mu|+1}.
\tag{7}
$$

Therefore the first term on the right-hand side of (6) is $A_b^{1,B}(F;1) \cdot z \cdot n_1$, which is given by the length n_1 of that side of the boundary of Λ which defines the event $\overline{G}_1$. It is obvious that $A_b^{1,B}(F;1)$ is independant of Λ.

Now applying the main lemma to the integrand with respect to u_2 of the second term on the right-hand side of (6) we obtain that

$$z \cdot \int\limits_0^{n_1} du_1 \int\limits_{n_2}^{\infty} du_2 \int\limits_{X^0_{per}} dx^0 \int\limits_{\overline{G}_1} d\mu |\tilde{F}(u, x^0, \mu)| \leq z \cdot C \cdot \int\limits_0^{n_1} du_1 \int\limits_{n_2}^{\infty} du_2 \frac{1}{u_2^l} \leq C_1 L(\Lambda)^{-l+2},$$

where $C_1 = C_1(a, z, \beta, h, r_1, r_2) > 0$. With respect to the third term on the right-hand side of (6) we note that $\overline{G}_1 \setminus G_1 \subset Q(\Lambda)$. Thus it is enough to estimate the term corresponding to $Q(\Lambda)$. This is done below in 1.B.3.

In a similar way one can analyze the contributions of G_2, G_3 and G_4. **To summarize:** The contribution of the boundary is

$$\begin{aligned}
&\sum_{j=1}^{4} z \cdot \int\limits_{\Lambda} du \int\limits_{X^0_{per}} dx^0 \int\limits_{G_j} d\mu \; z^{|\mu|} \cdot \frac{F(\mu + \varepsilon_{x^0+u})}{|\mu| + 1} = \\
&= z \cdot n_1 \cdot \left[A_b^{1,B}(F;1) + A_b^{1,B}(F;3) \right] + z \cdot n_2 \cdot \left[A_b^{1,B}(F;2) + A_b^{1,B}(F;4) \right] + R_b^{1,B}(\Lambda),
\end{aligned}$$

where $|R_b^{1,B}(\Lambda)| \leq C_1' \cdot L(\Lambda)^{-(l-2)}$. Here for $j = 1, \ldots, 4$ the $A_b^{1,B}(F;j)$ are defined in a similar way as $A_b^{1,B}(F;1)$.

1.B.2 The contribution $A_a^{1,B}(F)$ of the angles

The aim now is to show that the four summands of the second sum of the decomposition (5) in II.1.B give contributions of the four angles of Λ. The summand corresponding to $G_{j,j+1}$ will give the contribution of the associated angle $\mathbb{C}\Pi_{j,j+1}$.

We proceed as before: Representing $G_{j,j+1}$ as $\overline{G}_{j,j+1} \setminus [\overline{G}_{j,j+1} \setminus G_{j,j+1}]$ we decompose this summand as a difference of one term built on $\overline{G}_{j,j+1}$ and one built on $\overline{G}_{j,j+1} \setminus G_{j,j+1}$. The term corresponding to $\overline{G}_{j,j+1}$ is decomposed using the representation $\Lambda = \mathbb{C}\Pi_{j,j+1} \setminus [\mathbb{C}\Pi_{j,j+1} \setminus \Lambda]$:

$$
\begin{aligned}
& z \cdot \int\limits_{\Lambda} du \int\limits_{X^0_{per}} dx^0 \int\limits_{G_{j,j+1}} d\mu\ \tilde{F}(u,x^0,\mu) \\
(8) \qquad & = z \cdot A_a^{1,B}(F;j) - z \cdot \int\limits_{(\mathbb{C}\Pi_{j,j+1})\setminus\Lambda} du \int\limits_{X^0_{per}} dx^0 \int\limits_{\overline{G}_{j,j+1}} d\mu\ \tilde{F}(u,x^0,\mu), \\
& - z \cdot \int\limits_{\Lambda} du \int\limits_{X^0_{per}} dx^0 \int\limits_{\overline{G}_{j,j+1}} \setminus G_{j,j+1} d\mu\ \tilde{F}(u,x^0,\mu),
\end{aligned}
$$

where

$$
(9) \qquad A_a^{1,B}(F;j) = \int\limits_{\mathbb{C}R_{j,j+1}} du \int\limits_{X^0_{per}} dx^0 \int\limits_{\overline{G}^0_{j,j+1}} d\mu\ \tilde{F}(u,x^0,\mu).
$$

Here in (2) we used once more translation-invariance. Consider the second term on the right-hand side of (8).

If $u \in (\mathbb{C}\Pi_{j,j+1}) \setminus \Lambda$ then $\overline{G}_{j,j+1} \subseteq \mathbb{C}\mathcal{M}^{\cdot}_f(\Lambda_u(\|u\|_\infty))$, where $\|u\|_\infty = \max(|u_1|,|u_2|)$. Thus the absolute value of the second term on the right-hand of (8) is dominated by

$$
z \cdot \int\limits_{\mathbb{C}\Lambda_u(r_1 \cdot L(\Lambda))} du \int\limits_{X^0_{per}} dx^0 \int\limits_{\mathbb{C}\mathcal{M}^{\cdot}_f(\Lambda_u(\|u\|_\infty))} d\mu |F(\mu + \varepsilon_{x^0+u})|
$$

Therefore the main lemma yields that this is

$$
\leq C \cdot \int\limits_{\mathbb{C}\Lambda_0(r_1 \cdot L(\Lambda))} du \|u\|_\infty^{-l} \leq C \cdot L(\Lambda)^{-l+2}.
$$

Since $\overline{G}_{j,j+1} \setminus G_{j,j+1} \subset Q(\Lambda)$, the estimate of the third term on the right-hand side of (8) can be found below under 1.B.3.

To summarize: The contribution of the angles in 1.B (5)

$$
\begin{aligned}
& \sum_{j=1}^{4} z \cdot \int\limits_{\Lambda} du \int\limits_{G_{j,j+1}} d\mu\ z^{|\mu|} \cdot \frac{F(\mu + \varepsilon_{x^0+u})}{|\mu|+1} = \\
& = z \cdot \sum_{j=1}^{4} A_a^{1,B}(F;j) + R_a^{1,B}(\Lambda),
\end{aligned}
$$

where $|R_a^{1,B}(\Lambda)| \leq C \cdot L(\Lambda)^{-l+2}$. Here C is a constant depending on z, β, a, h, r_1, r_2 and ν.

1.B.3 The contribution of $Q(\Lambda)$

To finish the analysis of the integral $z \cdot \int\limits_\Lambda I_1(u)du$ it remains to estimate the third term on the right-hand side of (5) in II.1.B which is built on $Q(\Lambda)$. Again using the main lemma we find that

$$\begin{aligned}
&z \cdot \int\limits_\Lambda du \int\limits_{X^0_{per}} dx^0 \int\limits_{Q(\Lambda)} d\mu \ z^{|\mu|} \cdot \frac{|F(\mu + \varepsilon_{x^0+u})|}{|\mu|+1} \leq \\
\leq &z \cdot \int\limits_\Lambda du \int\limits_{X^0_{per}} dx^0 \int\limits_{\mathbb{C}\mathcal{M}_f^{\cdot}(\Lambda_0(\frac{r_1}{2} \cdot L(\Lambda)))} d\mu \ z^{|\mu|} \cdot |F(\mu + \varepsilon_{x^0})| \\
\leq &C \cdot L(\Lambda)^{-l+2}.
\end{aligned}$$

To summarize the above analysis of the first integral on the right-hand side of (2) in II §1, our result is the following:

$$\begin{aligned}
z \cdot \int\limits_\Lambda I_1(u)du = z \cdot A_v^1(F) \cdot |\Lambda| - \sum_{j=1}^4 l_j(\partial\Lambda) \cdot z \cdot A_b^{1,B}(F;j) + \\
+ \sum_{j=1}^4 z \cdot A_a^{1,B}(F;j) + R^{(1)}(\Lambda),
\end{aligned}$$

where $|R^{(1)}(\Lambda)| \leq C \cdot L(\Lambda)^{-l+2}$ with a constant C depending on a, z, β, h, r_1, r_2 and ν.

2 The analysis of the second term of the initial decomposition (2) in §1 of $\Lambda_{z\rho_\Lambda}(F)$

We now regard the integral $z \cdot \int\limits_\Lambda I_2(u)du$ of decomposition (2) in §1 of II. We proceed in an analogous way as we did for the first term. Now we are only concerned with simple loops instead of configurations of loops. We use the decomposition (4) from §1, i.e. $I_2(u) = I_2^A(u) - I_2^B(u)$.

2.A The analysis of I_2^A

Let us first consider $I_2^A(u)$ and introduce in analogy to (3) in §1.B the following notations with the same conventions on j.

$$\begin{aligned}
&H_j(u) = \{x^0 \in X^0_{per} \mid x^0 + u \notin B_{\mathbb{C}\Pi_j} \text{ and } x^0 + u \in B_{\mathbb{C}\Pi_{j+2}}\}, u \in \Lambda;\\
&\overline{H}_j(u) = \{x^0 \in X^0_{per} \mid x^0 + u \notin B_{\mathbb{C}\Pi_j}\}, u \in \mathbb{R}^2.\\
&H_{j,j+1}(u) = H_j(u) \cap H_{j+1}(u), u \in \Lambda;\\
&\overline{H}_{j,j+1}(u) = \overline{H}_j(u) \cap \overline{H}_{j+1}(u), u \in \mathbb{R}^2.
\end{aligned}$$

Note that one has the following invariance properties with respect to translations in direction of the axes:

$$\begin{aligned}
H_1(u + (a_1, 0)) = H_1(u), H_3(u + (a_1, 0)) = H_3(u),\\
H_2(u + (0, a_2)) = H_2(u), H_4(u + (0, a_2)) = H_4(u).
\end{aligned}$$

The same is true for the $\overline{H}_j(u)$. We decompose $\mathbb{C}B_{\Lambda-u}$, $u \in \Lambda$, as follows:

$$\mathbb{C}B_{\Lambda-u} = \bigcup_{j=1}^{4}(H_j(u) \setminus H_{j,j+1}(u)) \cup Q_0(u), \tag{1}$$

where $Q_0(u) = \mathbb{C}B_{\Lambda-u} \setminus \bigcup\limits_{j=1}^{4} H_j(u)$. This implies that

$$\begin{aligned}
I_2^A(u) = \sum_{j=1}^{4} \int\limits_{H_j(u)} dx^0 \hat{F}(x^0) - \\
- \sum_{j=1}^{4} \int\limits_{H_{j,j+1}(u)} dx^0 \hat{F}(x^0) + \int\limits_{Q_0(u)} dx^0 \hat{F}(x^0)
\end{aligned} \tag{2}$$

where

$$\hat{F}(x^0) = \int\limits_{\mathcal{M}^{\cdot}_f(X_{per})} d\mu\ z^{|\mu|} \cdot \frac{F(\mu + \varepsilon_{x^0})}{|\mu| + 1}. \tag{3}$$

Here we used that $\hat{F}(x^0 + u) = \hat{F}(x^0)$, $u \in \mathbb{R}^2$. $\hat{F}$ is bounded by lemma 4.4 of I.

2.A.1 Further contributions $A_b^{2,A}$ of the boundary

We consider the first term of the first sum of $I_2^A(u)$ for simplicity. Since $H_1(u) \subseteq \overline{H}_1(u)$, we can decompose it as follows:

$$(1) \qquad z \cdot \int_{\Lambda} du \int_{H_1(u)} dx^0 \hat{F}(x^0) =$$
$$= z \cdot \int_{\Lambda} du \int_{\overline{H}_1(u)} dx^0 \hat{F}(x^0) - z \cdot \int_{\Lambda} du \int_{\overline{H}_1(u) \setminus H_1(u)} dx^0 \hat{F}(x^0).$$

The invariance property of $\overline{H}_1(u)$ in the first term on the right-hand side of (1) enables us to write

$$(2) \qquad z \cdot \int_{\Lambda} du \int_{\overline{H}_1(u)} dx^0 \hat{F}(x^0) = z \cdot \int_0^{n_1} du_1 \int_0^{\infty} du_2 \int_{\overline{H}_1(0,u_2)} dx^0 \hat{F}(x^0) -$$
$$- z \cdot \int_0^{n_1} du_1 \int_{n_2}^{\infty} du_2 \int_{\overline{H}_1(0,u_2)} dx^0 \hat{F}(x^0).$$

It is evident by translation-invariance that the integral

$$(3) \qquad A_b^{2,A}(F;1) = \int_0^{\infty} du_2 \int_{\overline{H}_1^0(0,u_2)} dx^0 \hat{F}(x^0)$$

does not depend on Λ.Here $\overline{H}_j^0(u)$ is defined as $\overline{H}_j(u)$ by replacing Π_j by R_j.

Hence

$$(4) \qquad z \cdot \int_0^{n_1} du_1 \int_0^{\infty} du_2 \int_{\overline{H}_1^0(0,u_2)} dx^0 \hat{F}(x^0) = z \cdot n_1 \cdot A_b^{2,A}(F;1).$$

Similarly to §1 we can analyze the second term on the right-hand side of (2).

To summarize: We find that

$$z \cdot \int_{\Lambda} I_2^A(u) du = z \cdot \sum_{j=1}^{4} l_j(\partial\, \Lambda) \cdot z \cdot A_b^{2,A}(F;j) - \sum_{j=1}^{4} z \cdot A_a^{2,A}(F;j) + R_1^A(\Lambda),$$

where $A_a^{2,A}(F;j) = \int\limits_{\mathbb{C}R_{j,j+1}} du \int\limits_{\overline{H}^0_{j,j+1}(u)} dx^0 \hat{F}(x^0)$ and $|R_1^A(\Lambda)| \leq C \cdot L(\Lambda)^{-l+2}$.

$\overline{H}^0_{j,j+1}$ is defined as $\overline{H}_{j,j+1}$ but now with $R_{j,j+1}$ instead of $\Pi_{j,j+1}$.

2.B The analysis of I_2^B

To finish the geometric expansion of the log-partition function it remains to consider the term

$$(1) \qquad I_2^B(u) = \int\limits_{\mathbb{C}B_{\Lambda-u}} dx^0 \int\limits_{\mathbb{C}\mathcal{M}_f^{\cdot}(\Lambda)} d\mu \ z^{|\mu|} \cdot \frac{F(\mu+\varepsilon_{x^0+u})}{|\mu|+1}, u \in \Lambda.$$

We'll decompose now both, $\mathbb{C}B_{\Lambda-u}$ and $\mathbb{C}\mathcal{M}_f^{\cdot}(\Lambda)$, using its decompositions (1) of §2.A respectively (4) of §1.B. As a result we obtain

$$z\int\limits_{\Lambda} I_2^B(u)du = \sum_{j=1}^{4} l_j(\partial\ \Lambda) z A_b^{2,B}(F,j) -$$
$$-\sum_{j=1}^{4} z A_a^{2,B}(F,j) + R_2^B(\Lambda),$$

where $|R_2^B(\Lambda)| \leq \text{const } L(\Lambda)^{-l+2}$ and

$$A_b^{2,B}(F;1) = \int\limits_0^\infty du_2 \int\limits_{\overline{H}^0_1(0,u_2)} dx^0 \int\limits_{\overline{G}^0_1} d\mu \ z^{|\mu|} \frac{F(\mu+\varepsilon_{x^0+(0,u_2)})}{|\mu|+1}$$

$$A_a^{2,B}(F,j) = \int\limits_{\mathbb{C}R_{j,j+1}} du \{ \int\limits_{\overline{H}^0_j(u)} dx^0 \int\limits_{\overline{G}^0_{j+1}} d\mu \tilde{F}(u,x^0,\mu) - \int\limits_{\overline{H}^0_j(u)} dx^0 \int\limits_{\overline{G}^0_{j,j+1}} d\mu \tilde{F}(u,x^0,\mu)$$
$$+ \int\limits_{\overline{H}^0_{j+1}(u)} dx^0 \int\limits_{\overline{G}^0_j} d\mu \tilde{F}(u,x^0,\mu) - \int\limits_{\overline{H}^0_{j+1}} dx^0 \int\limits_{\overline{G}^0_{j,j+1}} d\mu \tilde{F}(u,x^0,\mu)$$
$$- \int\limits_{\overline{H}^0_{j,j+1}} dx^0 (\int\limits_{\overline{G}^0_j} d\mu \tilde{F}(u,x^0,\mu) + \int\limits_{\overline{G}^0_{j+1}} d\mu \tilde{F}(u,x^0,\mu)) +$$
$$+ \int\limits_{\overline{H}^0_{j,j+1}} dx^0 \int\limits_{\overline{G}^0_{j,j+1}} d\mu \tilde{F}(u,x^0,\mu)\}.$$

Concluding remarks

Theorem 1 and Theorem 2 remain true for the case of Λ from a naturally defined class of convex polyhedrons in $\mathbb{R}^\nu$. Moreover, using arguments of [8] one can prove similar strong decay properties also for the derivatives of the Ursell functions under smoothness conditions on the potential ϕ. This in combination with methods developed in [10] can be used to obtain a geometric expansion for the case of convex bounded Λ with smooth boundary.

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