Canonical Matrices for Linear Matrix Problems

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Abstract

We consider a large class of matrix problems, which includes the problem of classifying arbitrary systems of linear mappings. For every matrix problem, we construct Belitskiĭ's algorithm for reducing a matrix to a canonical form, which is the generalization of the Jordan normal form, and study the set C_{mn} of indecomposable canonical $m \times n$ matrices. Considering C_{mn} as a subset in the affine space of m-by-n matrices, we prove that either C_{mn} consists of a finite number of points and straight lines for every $m \times n$, or C_{mn} contains a 2-dimensional plane for a certain $m \times n$.

All matrices are considered over an algebraically closed field k; $k^{m\times n}$ denotes the set of m-by-n matrices over k. The article consists of three sections.

In Section 1 we present Belitskiĭ's algorithm [2] in a form, which is convenient to use in linear algebra. In particular, the algorithm permits to reduce pairs of n-by-n matrices to a canonical form by transformations of simultaneous similarity: $(A, B) \mapsto (S^{-1}AS, S^{-1}BS)$; another solution of this classical problem was given by Friedland [14]. This section uses rudimentary linear algebra (except for the proof of Theorem 1.1; the reader may omit it).

In Section 2 we determine a broad class of matrix problems, which includes the problems of classifying representations of quivers, partially ordered sets and finite dimensional algebras. In Section 3 we get the following description of the set of canonical matrices in the spirit of [16]: if a matrix problem does not 'contain' the canonical form problem for pairs of matrices under simultaneous similarity, then its set of indecomposable canonical $m \times n$ matrices in the affine space $k^{m\times n}$ consists of a finite number of points and straight lines (contrary to [16], these lines are unpunched).

A detailed introduction is given at the beginning of every section. Each introduction may be read independently.

1 Belitskiĭ's algorithm

1.1 Introduction

Every matrix problem is given by a set of admissible transformations that determines an equivalence relation on a certain set of matrices (or sequences of matrices). The question is to find a *canonical form* – i.e., determine a 'nice' set of canonical matrices

*Partially supported by Grant No. UM1-314 of the U.S. Civilian Research and Development Foundation for the Independent States of the Former Soviet Union. E-mail: sergeich@ukrpack.net

such that each equivalence class contains exactly one canonical matrix. Two matrices are then equivalent if and only if they have the same canonical form.

Many matrix problems can be formulated in terms of quivers and their representations, introduced by Gabriel [15] (see also [17]). A *quiver* is a directed graph, its *representation* A is given by assigning to each vertex i a finite dimensional vector space A_i over k and to each arrow $\alpha : i \to j$ a linear mapping $A_\alpha : A_i \to A_j$. For example, the diagonalization theorem, the Jordan normal form, and the matrix pencil theorem give the solution of the canonical form problem for representations of the quivers, respectively,

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[loop right] \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \end{tikzcd}$$

(Analogously, one may study systems of forms and linear mappings as representations of a partially directed graph G, assigning a bilinear form to an undirected edge. As was proved in [27, 29], the problem of classifying representations of G is reduced to the problem of classifying representations of a certain quiver $\bar{G}$. The class of studied matrix problems may be extended by considering quivers with relations [17, 25] and partially directed graphs with relations [29].)

The canonical form problem was solved only for the quivers of so called tame type by Donovan and Freishlich [8] and Nazarova [22], this problem is considered as hopeless for the other quivers (see Section 2). Nevertheless, the matrices of each individual representation of a quiver may be reduced to a canonical form by Belitskiĭ's algorithm [2]. This algorithm and a more known Littlewood algorithm [20] (see also [31, 33]) for reducing matrices to canonical form under unitary similarity have the same conceptual sketch: The matrix is partitioned and successive admissible transformations are applied to reduce the submatrices to some nice form. At each stage, one refines the partition and restricts the set of permissible transformations to those that preserve the already reduced blocks. The process ends in a finite number of steps, producing the canonical form.

We will apply Belitskiĭ's algorithm to the canonical form problem for matrices under Λ-similarity, which is defined as follows. Let Λ be an algebra of $n \times n$ matrices (i.e., a subspace of $k^{n\times n}$ that is closed with respect to multiplication and contains I) and let Λ^* be the set of its nonsingular matrices. We say that two $n \times n$ matrices M and N are Λ-*similar* and write $M \sim_\Lambda N$ if there exists $S \in \Lambda^*$ such that $S^{-1}MS = N$ ($\sim_\Lambda$ is an equivalence relation; see the end of Section 1.2). The problem of classifying representations of each quiver can be formulated in terms of Λ-similarity, where Λ is an algebra of block-diagonal matrices, in which we require some of the diagonal blocks to be equal.

Example 1.1. The problem of classifying representations of the quiver

$$\begin{tikzcd} & 2 \arrow[dr, "\varepsilon"] & \\ 1 \arrow[loop left, "\alpha"] \arrow[ur, "\beta"] \arrow[rr, shift left, "\gamma"] \arrow[rr, shift right, "\delta"'] & & 3 \arrow[loop right, "\zeta"] \end{tikzcd} \tag{1}$$

is the canonical form problem for matrices of the form

$$\begin{bmatrix} A_\alpha & 0 & 0 & 0 \\ A_\beta & 0 & 0 & 0 \\ A_\gamma & 0 & 0 & 0 \\ A_\delta & A_\varepsilon & A_\zeta & 0 \end{bmatrix}$$

under Λ-similarity, where Λ consists of block-diagonal matrices of the form $S_1 \oplus S_2 \oplus S_3 \oplus S_3$.

Example 1.2. By the definition of Gabriel and Roiter [17], a linear matrix problem of size $m \times n$ is given by a pair $(D^*, \mathcal{M})$, where D is a subalgebra of $k^{m\times m} \times k^{n\times n}$ and $\mathcal{M}$ is a subset of $k^{m\times n}$ such that $SAR^{-1} \in \mathcal{M}$ whenever $A \in \mathcal{M}$ and $(S, R) \in D^*$. The question is to classify the orbits of $\mathcal{M}$ under the action $(S, R) : A \mapsto SAR^{-1}$. Clearly, two $m \times n$ matrices A and B belong to the same orbit if and only if $\left[\begin{smallmatrix} 0 & A \\ 0 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right]$ are Λ-similar, where $\Lambda := \{S \oplus R \,|\, (S, R) \in D\}$ is an algebra of $(m + n) \times (m + n)$ matrices.

In Section 1.2 we prove that for every algebra $\Lambda \subset k^{n\times n}$ there exists a nonsingular matrix P such that the algebra $P^{-1}\Lambda P := \{P^{-1}AP \,|\, A \in \Lambda\}$ consists of upper block-triangular matrices, in which some of the diagonal blocks must be equal and off-diagonal blocks satisfy a system of linear equations. The algebra $P^{-1}\Lambda P$ will be called a *reduced matrix algebra.* The Λ-similarity transformations with a matrix M correspond to the $P^{-1}\Lambda P$-similarity transformations with the matrix $P^{-1}MP$, so it suffices to study Λ-similarity transformations given by a reduced matrix algebra Λ.

In Section 1.3 we for every Jordan matrix J construct a matrix $J^\# = P^{-1}JP$ (P is a permutation matrix) such that all commuting with it matrices form a reduced algebra. We call $J^\#$ a *Weir matrix* since its form is determined by the set of its Weir characteristics [21] (Belitskiĭ [2] calls $J^\#$ a modified Jordan matrix; it plays a central role in his algorithm).

In Section 1.4 we construct an algorithm (which is a modification of Belitskiĭ's algorithm [2]) for reducing matrices to canonical form under Λ-similarity, where Λ is a reduced matrix algebra. In Section 1.5 we get a certain description of the set of canonical matrices.

1.2 Reduced matrix algebras

In this section we prove that for every matrix algebra $\Lambda \subset k^{n\times n}$ there exists a nonsingular matrix P such that the algebra $P^{-1}\Lambda P$ is a reduced matrix algebra in the sense of the following definition.

A block matrix $M = [M_{ij}]$, $M_{ij} \in k^{m_i \times n_j}$, will be called an $\underline{m} \times \underline{n}$ *matrix*, where $\underline{m} = (m_1, m_2, \dots)$, $\underline{n} = (n_1, n_2, \dots)$ and $m_i, n_j \in \{0, 1, 2, \dots\}$ (we take into consideration blocks without rows or columns).

Definition 1.1. An algebra Λ of $\underline{n} \times \underline{n}$ matrices, $\underline{n} = (n_1, \dots, n_t)$, will be called a *reduced* $\underline{n} \times \underline{n}$ *algebra* if there exist

(a) an equivalence relation

$$\sim \quad \text{in} \quad T = \{1, \dots, t\}, \tag{2}$$

(b) a family of systems of linear equations

$$\Big\{ \sum_{\mathcal{I}\ni i<j\in\mathcal{J}} c_{ij}^{(l)} x_{ij} = 0, \quad 1 \leqslant l \leqslant q_{\mathcal{I}\mathcal{J}} \Big\}_{\mathcal{I},\mathcal{J}\in T/\sim}, \tag{3}$$

indexed by pairs of equivalence classes, where $c_{ij}^{(l)} \in k$ and $q_{\mathcal{I}\mathcal{J}} \geq 0$, such that Λ consists of all upper block-triangular $\underline{n} \times \underline{n}$ matrices

$$S = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1t} \\ & S_{22} & \ddots & \vdots \\ & & \ddots & S_{t-1,t} \\ 0 & & & S_{tt} \end{bmatrix}, \quad S_{ij} \in k^{n_i \times n_j}, \tag{4}$$

in which diagonal blocks satisfy the condition

$$S_{ii} = S_{jj} \quad \text{whenever} \quad i \sim j\,, \tag{5}$$

and off-diagonal blocks satisfy the equalities

$$\sum_{\mathcal{I} \ni i < j \in \mathcal{J}} c_{ij}^{(l)} S_{ij} = 0, \quad 1 \leqslant l \leqslant q_{\mathcal{I}\mathcal{J}}\,, \tag{6}$$

for each pair $\mathcal{I}, \mathcal{J} \in T/\!\sim$.

Clearly, the sequence $\underline{n} = (n_1, \ldots, n_t)$ and the equivalence relation $\sim$ are uniquely determined by Λ; moreover, $n_i = n_j$ if $i \sim j$.

Example 1.3. Let us consider the classical canonical form problem for pairs of matrices (A, B) under simultaneous similarity (i.e., for representations of the quiver ). Reducing (A, B) to the form (J, C), where J is a Jordan matrix, and restricting the set of permissible transformations to those that preserve J, we obtain the canonical form problem for C under Λ-similarity, where Λ consists of all matrices commuting with J. In the next section, we modify J such that Λ becomes a reduced matrix algebra.

Theorem 1.1. *For every matrix algebra $\Lambda \subset k^{n \times n}$, there exists a nonsingular matrix P such that $P^{-1}\Lambda P$ is a reduced matrix algebra.*

Proof. Let V be a vector space over k and $\Lambda \subset \mathrm{End}_k(V)$ be an algebra of linear operators. We prove briefly that their matrices in a certain basis of V form a reduced algebra (this fact is used only in Section 2.5; the reader may omit the proof if he is not familiar with the theory of algebras).

Let R be the radical of Λ. By the Wedderburn-Malcev theorem [12], there exists a subalgebra $\bar{\Lambda} \subset \Lambda$ such that $\bar{\Lambda} \simeq \Lambda/R$ and $\bar{\Lambda} \cap R = 0$. By the Wedderburn-Artin theorem [12], $\bar{\Lambda} \simeq k^{m_1 \times m_1} \times \cdots \times k^{m_q \times m_q}$. We denote by $e_{ij}^{(\alpha)} \in \bar{\Lambda}$ $(i, j \in \{1, \ldots, m_\alpha\},\ 1 \leqslant \alpha \leqslant q)$ the elements of Λ that correspond to the matrix units of $k^{m_\alpha \times m_\alpha}$. Put $e_\alpha = e_{11}^{\alpha}$, $e = e_1 + \cdots + e_q$, and $V_0 = eV$.

We consider $\Lambda_0 := e\Lambda e$ as a subalgebra of $\mathrm{End}_k(V_0)$, its radical is $R_0 := R \cap \Lambda_0$, and $\Lambda_0/R_0 \simeq k \times \cdots \times k$. Let $R_0^{m-1} \neq 0 = R_0^m$. We choose a basis of $R_0^{m-1}V_0$ formed by vectors $v_1, \ldots, v_{t_1} \in \bigcup_\alpha e_\alpha V_0$, complete it to a basis of $R_0^{m-2}V_0$ by vectors $v_{t_1+1}, \ldots, v_{t_2} \in \bigcup_\alpha e_\alpha V_0$, and so on, until obtain a basis $v_1, \ldots, v_{t_m}$ of V_0. All its vectors have the form $v_i = e_{\alpha_i} v_i$; put $\mathcal{I}_\alpha = \{i \mid \alpha_i = \alpha\}$ for $1 \leqslant \alpha \leqslant q$.

Since $e_\alpha e_\beta = 0$ if $\alpha \neq \beta$, $e_\alpha^2 = e_\alpha$ and e is the unit of Λ_0, the vector space of Λ_0 is the direct sum of all $e_\alpha \Lambda_0 e_\beta$. Moreover, $e_\alpha \Lambda_0 e_\beta = e_\alpha R_0 e_\beta$ for $\alpha \neq \beta$ and $e_\alpha \Lambda_0 e_\alpha = k e_\alpha \oplus e_\alpha R_0 e_\alpha$, hence $\Lambda_0 = (\bigoplus_\alpha k e_\alpha) \oplus (\bigoplus_{\alpha,\beta} e_\alpha R_0 e_\beta)$. The matrix of every linear operator from $e_\alpha R_0 e_\beta$ in the basis $v_1, \ldots, v_{t_m}$ has the form $[a_{ij}]_{i,j=1}^{t_m}$, where $a_{ij} \neq 0$ implies $i < j$ and $(i, j) \in \mathcal{I}_\alpha \times \mathcal{I}_\beta$. Hence the set of matrices $[a_{ij}]$ of linear operators from Λ_0 in the basis $v_1, \dots, v_{t_m}$ may be given by a system of linear equations of the form

$$a_{ij} = 0 \ (i > j), \quad a_{ii} = a_{jj} \ (\{i,j\} \subset \mathcal{I}_\alpha), \qquad \sum_{\mathcal{I}_\alpha \ni i < j \in \mathcal{I}_\beta} c_{ij}^{(l)} a_{ij} = 0 \ (1 \leqslant l \leqslant q_{\alpha\beta}).$$

The matrices of linear operators from Λ in the basis $e_{11}^{(\alpha_1)} v_1, \dots, e_{m_{\alpha_1}1}^{(\alpha_1)} v_1, e_{11}^{(\alpha_2)} v_2, \dots, e_{m_{\alpha_2}1}^{(\alpha_2)} v_2, \dots$ of V have the form (4) and are given by the system of relations (5)–(6). So they form a reduced algebra. $\square$

For every matrix algebra $\Lambda \subset k^{n \times n}$, the set Λ^* of its nonsingular matrices is a group and hence the Λ-similarity is an equivalence relation. Indeed, we may assume that Λ is a reduced matrix algebra. Then every $S \in \Lambda^*$ can be written in the form $D(I - C)$, where D, $C \in \Lambda$ such that D is a block-diagonal and all diagonal blocks of C are zero. Since C is nilpotent, $S^{-1} = (I + C + C^2 + \cdots)D^{-1} \in \Lambda^*$.

Notice also that every finite dimensional algebra is isomorphic to a matrix algebra and hence, by Theorem 1.1, it is isomorphic to a reduced matrix algebra.

1.3 Weyr matrices

Following Belitskiĭ [2], we for every Jordan matrix J define a matrix $J^{\#} = P^{-1}JP$ (P is a permutation matrix) such that all commuting with it matrices form a reduced algebra. We will fix a linear order $\prec$ in k (if k is the field of complex numbers, we may use the lexicographic ordering: $a + bi \prec c + di$ if either $a = c$ and $b < d$, or $a < c$).

Definition 1.2. A *Weyr matrix* is a matrix of the form

$$W = W_{\{\lambda_1\}} \oplus \cdots \oplus W_{\{\lambda_r\}}, \quad \lambda_1 \prec \cdots \prec \lambda_r, \tag{7}$$

where

$$W_{\{\lambda_i\}} = \begin{bmatrix} \lambda_i I_{m_{i1}} & W_{i1} & & 0 \\ & \lambda_i I_{m_{i2}} & \ddots & \\ & & \ddots & W_{i,k_i-1} \\ 0 & & & \lambda_i I_{m_{ik_i}} \end{bmatrix}, \quad W_{ij} = \begin{bmatrix} I \\ 0 \end{bmatrix},$$

$m_{i1} \geqslant \ldots \geqslant m_{ik_i}$. The *standard partition* of W is the $\underline{n} \times \underline{n}$ partition, where $\underline{n} = (\underline{n}_1, \dots, \underline{n}_r)$ and $\underline{n}_i$ is the sequence $m_{i1} - m_{i2}, m_{i2} - m_{i3}, \dots, m_{i,k_i-1} - m_{ik_i}, m_{ik_i}; m_{i2} - m_{i3}, \dots, m_{i,k_i-1} - m_{ik_i}, m_{ik_i}; \dots; m_{ik_i}$ from which all zero components are removed.

The standard partition of W is the most coarse partition for which all diagonal blocks have the form $\lambda_i I$ and all off-diagonal blocks have the form 0 or I.

The matrix W is named a 'Weyr matrix' since $(m_{i1}, m_{i2}, \dots, m_{ik_i})$ is the Weyr characteristic of W (and of every matrix that is similar to W) for λ_i. Recall (see [21] or [33]) that the *Weyr characteristic* of a square matrix A for an eigenvalue λ is the decreasing list $(m_1, m_2, \dots)$, where $m_i := \operatorname{rank}(A - \lambda I)^{i-1} - \operatorname{rank}(A - \lambda I)^i$. Clearly, m_i is the number of Jordan cells $J_l(\lambda)$, $l \geqslant i$, in the Jordan form of A (i.e., $m_i - m_{i+1}$ is the number of $J_i(\lambda)$), so the Jordan form is uniquely, up to permutation of Jordan cells, determined by the set of eigenvalues of A and their Weir characteristics. Taking into account the inequality at the right-hand side of (7), we get the first statement of the following theorem:

Theorem 1.2. *Every square matrix A is similar to exactly one Weyr matrix $A^{\#}$. The matrix $A^{\#}$ is obtained from the Jordan form of A by simultaneous permutations of its rows and columns. All matrices commuting with $A^{\#}$ form a reduced matrix algebra $\Lambda(A^{\#})$ of $\underline{n} \times \underline{n}$ matrices (4) with equalities (6) of the form $S_{ij} = S_{i'j'}$ and $S_{ij} = 0$, where $\underline{n} \times \underline{n}$ is the standard partition of $A^{\#}$.*

To make the proof of the second and the third statements more clear, we begin with an example.

Example 1.4. Let us construct the Weir form $J^{\#}_{\{\lambda\}}$ of the Jordan matrix

$$J_{\{\lambda\}} := \underbrace{J_4(\lambda) \oplus \cdots \oplus J_4(\lambda)}_{p \text{ times}} \oplus \underbrace{J_2(\lambda) \oplus \cdots \oplus J_2(\lambda)}_{q \text{ times}}$$

with a single eigenvalue λ. Gathering Jordan cells of the same size, we first reduce $J_{\{\lambda\}}$ to $J^{+}_{\{\lambda\}} = J_4(\lambda I_p) \oplus J_2(\lambda I_q)$. The matrix $J^{+}_{\{\lambda\}}$ and all commuting with it matrices have the form, respectively,

$$\begin{array}{|cccc|cc|l}
\multicolumn{1}{c}{(11)}&(21)&(31)&\multicolumn{1}{c}{(41)}&(12)&\multicolumn{1}{c}{(22)}&\\
\hline
\lambda I_p & I_p &&&&& (11)\\
& \lambda I_p & I_p &&&& (21)\\
&& \lambda I_p & I_p & 0 && (31)\\
&&& \lambda I_p &&& (41)\\
\hline
& 0 &&& \lambda I_q & I_q & (12)\\
&&&&& \lambda I_q & (22)\\
\hline
\end{array}
\qquad
\begin{array}{|cccc|cc|l}
\multicolumn{1}{c}{(11)}&(21)&(31)&\multicolumn{1}{c}{(41)}&(12)&\multicolumn{1}{c}{(22)}&\\
\hline
A_1 & A_2 & A_3 & A_4 & B_1 & B_2 & (11)\\
& A_1 & A_2 & A_3 && B_1 & (21)\\
&& A_1 & A_2 &&& (31)\\
&&& A_1 &&& (41)\\
\hline
&& C_1 & C_2 & D_1 & D_2 & (12)\\
&&& C_1 && D_1 & (22)\\
\hline
\end{array}$$

Simultaneously permuting strips in these matrices, we get the Weyr matrix $J^{\#}_{\{\lambda\}}$ and all commuting with it matrices (they form a reduced $\underline{n} \times \underline{n}$ algebra $\Lambda(J^{\#}_{\{\lambda\}})$ with equalities (6) of the form $S_{ij} = S_{i'j'}$, $S_{ij} = 0$, and with $\underline{n} = (p, q, p, q, p, p)$):

$$\begin{array}{|cc|cc|c|c|l}
\multicolumn{1}{c}{(11)}&\multicolumn{1}{c}{(12)}&(21)&\multicolumn{1}{c}{(22)}&\multicolumn{1}{c}{(31)}&\multicolumn{1}{c}{(41)}&\\
\hline
\lambda I_p && I_p &&&& (11)\\
& \lambda I_q && I_q &&& (12)\\
\hline
&& \lambda I_p && I_p && (21)\\
&&& \lambda I_q &&& (22)\\
\hline
&&&& \lambda I_p & I_p & (31)\\
\hline
&&&&& \lambda I_p & (41)\\
\hline
\end{array}
\qquad
\begin{array}{|cc|cc|c|c|l}
\multicolumn{1}{c}{(11)}&\multicolumn{1}{c}{(12)}&(21)&\multicolumn{1}{c}{(22)}&\multicolumn{1}{c}{(31)}&\multicolumn{1}{c}{(41)}&\\
\hline
A_1 & B_1 & A_2 & B_2 & A_3 & A_4 & (11)\\
& D_1 && D_2 & C_1 & C_2 & (12)\\
\hline
&& A_1 & B_1 & A_2 & A_3 & (21)\\
&&& D_1 && C_1 & (22)\\
\hline
&&&& A_1 & A_2 & (31)\\
\hline
&&&&& A_1 & (41)\\
\hline
\end{array}$$

Proof of Theorem 1.2. We may suppose that A is a Jordan matrix

$$J = J_{\{\lambda_1\}} \oplus \cdots \oplus J_{\{\lambda_r\}}, \quad \lambda_1 \prec \cdots \prec \lambda_r,$$

where $J_{\{\lambda\}}$ denotes a Jordan matrix with a single eigenvalue λ. Then

$$J^{\#} = J^{\#}_{\{\lambda_1\}} \oplus \cdots \oplus J^{\#}_{\{\lambda_r\}}, \quad \Lambda(J^{\#}) = \Lambda(J^{\#}_{\{\lambda_1\}}) \times \cdots \times \Lambda(J^{\#}_{\{\lambda_r\}});$$

the second since $SJ^{\#} = J^{\#}S$ if and only if $S = S_1 \oplus \cdots \oplus S_r$ and $S_i J^{\#}_{\{\lambda_i\}} = J^{\#}_{\{\lambda_i\}} S_i$.

So we may restrict ourselves to a Jordan matrix $J_{\{\lambda\}}$ with a single eigenvalue λ; it reduces to the form

$$J^{+}_{\{\lambda\}} = J_{p_1}(\lambda I_{n_1}) \oplus \cdots \oplus J_{p_l}(\lambda I_{n_l}), \quad p_1 > \cdots > p_l. \tag{8}$$

The matrix (8) consists of l horizontal and l vertical strips, the ith strip is divided into p_i substrips. We will index the αth substrip of the ith strip by the pair (α, i). Permuting vertical and horizontal substrips such that they become lexicographically ordered with respect to these pairs,

$$(11), (12), \dots, (1l), (21), (22), \dots, \tag{9}$$

we obtain the Weyr form $J^{\#}_{\{\lambda\}}$ of $J_{\{\lambda\}}$ (see Example 1.4). The partition into substrips is its standard $\underline{n} \times \underline{n}$ partition (i.e. $\underline{n} = (n_1, n_2, \dots, n_l, n_1, n_2, \dots)$).

It is well known (and is proved by direct calculations, see [18, Sect. VIII, §2]) that all matrices commuting with the matrix (8) have the form $C = [C_{ij}]_{i,j=1}^{l}$, where each C_{ij} is of the form

$$\begin{array}{c} \begin{array}{cccccc} (1j) & & \cdots & & (p_j j) & \\ \end{array} \\ \left[\begin{array}{ccccc} & X_1 & X_2 & \cdots & X_{p_i} \\ & & X_1 & \ddots & \vdots \\ & & & \ddots & X_2 \\ 0 & & & & X_1 \end{array}\right] \begin{array}{c} (1i) \\ \vdots \\ \\ (p_i i) \end{array} \\ \text{if } p_i \leqslant p_j \end{array} \qquad \begin{array}{c} \begin{array}{ccccc} (1j) & & \cdots & & (p_j j) \end{array} \\ \left[\begin{array}{cccc} X_1 & X_2 & \cdots & X_{p_j} \\ & X_1 & \ddots & \vdots \\ & & \ddots & X_2 \\ & & & X_1 \\ 0 & & & \end{array}\right] \begin{array}{c} (1i) \\ \\ \vdots \\ \\ (p_i i) \end{array} \\ \text{if } p_i \geqslant p_j \end{array}$$

Hence, if a *nonzero* subblock is located at the intersection of the (α, i) horizontal substrip and the (β, j) vertical substrip, then either $\alpha = \beta$ and $i \leqslant j$, or $\alpha < \beta$. Rating the substrips of C in the lexicographic order (9), we obtain an upper block-triangular $\underline{n} \times \underline{n}$ matrix S that commutes with $J^{\#}_{\{\lambda\}}$. The matrices S form the algebra $\Lambda(J^{\#}_{\{\lambda\}})$, which is a reduced algebra with equations (6) of the form $S_{ij} = S_{i'j'}$ and $S_{ij} = 0$. $\square$

Notice that $J^{\#}_{\{\lambda\}}$ is obtained from

$$J_{\{\lambda\}} = J_{k_1}(\lambda) \oplus \cdots \oplus J_{k_t}(\lambda), \quad k_1 \geqslant \ldots \geqslant k_t, \tag{10}$$

as follows: We collect the first columns of $J_{k_1}(\lambda), \dots, J_{k_t}(\lambda)$ on the first t columns of $J_{\{\lambda\}}$, then permute the rows as well. Next collect the second columns and permute the rows as well, continue the process until $J^{\#}_{\{\lambda\}}$ is achieved.

Remark 1.1. The block-triangular form of $\Lambda(J^{\#})$ is easily explained with the help of Jordan chains. The matrix (10) determines a linear operator $\mathcal{A}$ in the lexicographically ordered basis $\{e_{ij}\}_{i=1}^{t}{}_{j=1}^{k_i}$ such that

$$\mathcal{A} - \lambda \mathbf{1} : e_{ik_i} \mapsto \cdots \mapsto e_{i2} \mapsto e_{i1} \mapsto 0. \tag{11}$$

The matrix $J^{\#}_{\{\lambda\}}$ determines $\mathcal{A}$ in the basis $\{e_{ij}\}$, lexicographically ordered with respect to the pair (j, i):

$$e_{11},\ e_{21}, \dots, e_{t1},\ e_{12},\ e_{22}, \dots \tag{12}$$

Clearly, $S^{-1}J^{\#}_{\{\lambda\}}S = J^{\#}_{\{\lambda\}}$ for a nonsingular matrix S if and only if S is the transition matrix from the basis (12) to another Jordan basis ordered like (12). This transition can be realized by a sequence of operations of the following form: the ith Jordan chain (11) is replaced with $\alpha e_{ik_i} + \beta e_{i,k_{i'}-p} \mapsto \alpha e_{i,k_i-1} + \beta e_{i',k_{i'}-p-1} \mapsto \cdots$, where $\alpha, \beta \in k$, $\alpha \neq 0$, and $p \geqslant \max\{0, k_{i'} - k_i\}$. Since a long chain cannot be added to a more short chain, the matrix S is block-triangular.

1.4 Algorithm

In this section, we give an algorithm for reducing a matrix M to a canonical form under Λ-similarity with a reduced $\underline{n} \times \underline{n}$ algebra Λ.

We apply to M the partition $\underline{n} \times \underline{n}$:

$$M = \begin{bmatrix} M_{11} & \cdots & M_{1t} \\ \cdots & \cdots & \cdots \\ M_{t1} & \cdots & M_{tt} \end{bmatrix}, \quad M_{ij} \in k^{n_i \times n_j}.$$

A block M_{ij} will be called *stable* if it remains invariant under Λ-similarity transformations with M. Then $M_{ij} = a_{ij}I$ whenever $i \sim j$ and $M_{ij} = 0$ (we put $a_{ij} = 0$) whenever $i \not\sim j$ since the equalities $S_{ii}^{-1}M_{ij}S_{jj} = M_{ij}$ must hold for all nonsingular block-diagonal matrices $S = S_{11} \oplus S_{22} \oplus \cdots \oplus S_{tt}$ satisfying (5).

If all the blocks of M are stable, then M is invariant under Λ-similarity, hence M is canonical ($M^{\infty} = M$).

Let there exist a nonstable block. We put the blocks of M in order

$$M_{t1} < M_{t2} < \cdots < M_{tt} < M_{t-1,1} < M_{t-1,2} < \cdots < M_{t-1,t} < \cdots \tag{13}$$

and reduce the first (with respect to this ordering) nonstable block M_{lr}. Let $M' = S^{-1}MS$, where $S \in \Lambda^*$ has the form (4). Then the (l,r) block of the matrix $MS = SM'$ is

$$M_{l1}S_{1r} + M_{l2}S_{2r} + \cdots + M_{lr}S_{rr} = S_{ll}M'_{lr} + S_{l,l+1}M'_{l+1,r} + \cdots + S_{lt}M'_{tr}$$

or, since all $M_{ij} < M_{lr}$ are stable,

$$a_{l1}S_{1r} + \cdots + a_{l,r-1}S_{r-1,r} + M_{lr}S_{rr} = S_{ll}M'_{lr} + S_{l,l+1}a_{l+1,r} + \cdots + S_{lt}a_{tr} \tag{14}$$

(we have removed in (14) all summands with $a_{ij} = 0$; their sizes may differ from the size of M_{lr}).

Let $\mathcal{I}, \mathcal{J} \in T/\!\sim$ be the equivalence classes such that $l \in \mathcal{I}$ and $r \in \mathcal{J}$.

Case I: the $q_{\mathcal{I}\mathcal{J}}$ equalities (6) do not imply

$$a_{l1}S_{1r} + a_{l2}S_{2r} + \cdots + a_{l,r-1}S_{r-1,r} = S_{l,l+1}a_{l+1,r} + \cdots + S_{lt}a_{tr} \tag{15}$$

(i.e., there exists a nonzero admissible addition to M_{lr} from other blocks). Then we make $M'_{lr} = 0$ using $S \in \Lambda^*$ of the form (4) that has the diagonal $S_{ii} = I$ ($1 \leqslant i \leqslant t$) and fits both (6) and (14) with $M'_{lr} = 0$.

Case II: the $q_{\mathcal{I}\mathcal{J}}$ equalities (6) imply (15); $i \not\sim j$. Then (14) simplifies to

$$M_{lr}S_{rr} = S_{ll}M'_{lr}, \tag{16}$$

where S_{rr} and S_{ll} are arbitrary nonsingular matrices. Selecting S, we take

$$M'_{lr} = S_{ll}^{-1}M_{lr}S_{rr} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}.$$

Case III: the $q_{\mathcal{I}\mathcal{J}}$ equalities (6) imply (15); $i \sim j$. Then (14) simplifies to the form (16) with an arbitrary nonsingular matrix $S_{rr} = S_{ll}$; $M'_{lr} = S_{ll}^{-1} M_{lr} S_{rr}$ is taken a Weir matrix.

We restrict ourselves to those admissible transformations with M' that preserve M'_{lr}. Let us prove that they are the Λ'-similarity transformations with

$$\Lambda' := \{S \in \Lambda \,|\, SM' \equiv M'S\}, \tag{17}$$

where $A \equiv B$ means that $\underline{n} \times \underline{n}$ matrices A and B have the same (l, r) block. The transformation $M' \mapsto S^{-1}M'S$, $S \in (\Lambda')^*$, preserves M'_{lr} (i.e. $M' \equiv S^{-1}M'S$) if and only if $SM' \equiv M'S$ since S is upper block-triangular and M' coincides with $S^{-1}M'S$ on the places of all (stable) blocks $M_{ij} < M_{lr}$. The set Λ' is an algebra: let $S, R \in \Lambda'$, then $M'S$ and SM' coincide on the places of all $M_{ij} < M_{lr}$ and R is upper block-triangular, hence $M'SR \equiv SM'R$; analogously, $SM'R \equiv SRM'$ and $SR \in \Lambda'$. The matrix algebra Λ' is a reduced algebra since Λ' consists of all $S \in \Lambda$ satisfying the condition (14) with M'_{lr} instead of M_{lr}.

In Case I, Λ' consists of all $S \in \Lambda$ satisfying (15) (we join it to the system (6)). In Case II, Λ' consists of all $S \in \Lambda$ for which $S_{ll}\left[\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right] S_{rr}$, that is

$$S_{ll} = \begin{bmatrix} P_1 & P_2 \\ 0 & P_3 \end{bmatrix}, \quad S_{rr} = \begin{bmatrix} Q_1 & Q_2 \\ 0 & Q_3 \end{bmatrix}, \quad P_1 = Q_3.$$

In Case III, Λ' consists of all $S \in \Lambda$ for which the blocks S_{ll} and S_{rr} are equal and commute with the Weir matrix M'_{lr}. (It gives an additional partition of $S \in \Lambda$ in Cases II and III; we rewrite (5)–(6) for more small blocks and join the equalities that need for $S_{ll}M'_{lr} = M'_{lr}S_{rr}$.)

In this manner, for every pair (M, Λ) we construct a new pair (M', Λ') with $\Lambda' \subset \Lambda$. If M' is not invariant under Λ'-similarity, then we repeat this construction (with an additional partition of M' in accordance with the structure of Λ') and obtain (M'', Λ''), and so on. Since at every step we reduce a new block, this process ends with a certain pair $(M^{(p)}, \Lambda^{(p)})$ in which all the blocks of $M^{(p)}$ are stable (i.e. $M^{(p)}$ is $\Lambda^{(p)}$-similar only to itself). Putting $(M^\infty, \Lambda^\infty) := (M^{(p)}, \Lambda^{(p)})$, we get the sequence

$$(M^0, \Lambda^0) = (M, \Lambda),\ (M', \Lambda'), \ldots,\ (M^{(p)}, \Lambda^{(p)}) = (M^\infty, \Lambda^\infty), \tag{18}$$

where

$$\Lambda^\infty = \{S \in \Lambda \,|\, M^\infty S = SM^\infty\}. \tag{19}$$

Definition 1.3. The matrix M^∞ will be called the Λ-*canonical form of* M.

Theorem 1.3. *Let $\Lambda \subset k^{n \times n}$ be a reduced matrix algebra. Then $M \sim_\Lambda M^\infty$ for every $M \in k^{n\times n}$ and $M \sim_\Lambda N$ if and only if $M^\infty = N^\infty$.*

Proof. Let Λ be a reduced $\underline{n} \times \underline{n}$ algebra, $M \sim_\Lambda N$, and let M_{lr} be the first nonstable block of M. Then M_{ij} and N_{ij} are stable blocks (moreover, $M_{ij} = N_{ij}$) for all $M_{ij} < M_{lr}$. By reasons of symmetry, N_{lr} is the first nonstable block of N; moreover, M_{lr} and N_{lr} are reduced to the same form: $M'_{lr} = N'_{lr}$. We obtain pairs (M', Λ') and (N', Λ') with the same Λ' and $M' \sim_{\Lambda'} N'$. Hence $M^{(i)} \sim_{\Lambda^{(i)}} N^{(i)}$ for all i, so $M^\infty = N^\infty$. $\square$

Example 1.5. In Example 1.3 we considered the canonical form problem for a pair of matrices under simultaneous similarity. Suppose the first matrix is reduced to the Weir matrix $W = \begin{bmatrix} \lambda I_2 & I_2 \\ 0 & \lambda I_2 \end{bmatrix}$. Preserving W, we may reduce the second matrix by transformations of Λ-similarity, where Λ consists of all 4×4 matrices of the form $\begin{bmatrix} S_1 & S_2 \\ 0 & S_1 \end{bmatrix}$, $S_i \in k^{2\times 2}$. For instance, one of the Λ-canonical matrices is

$$C = \left[\begin{array}{c|c|c} \multirow{2}{*}{C_3} & C_6 & C_7 \\ \cline{2-3} & C_4 & C_5 \\ \hline C_1 & \multicolumn{2}{c}{C_2} \end{array}\right] = \left[\begin{array}{cc|c|c} -1 & 1 & 2 & \emptyset \\ 0 & -1 & 0 & 1 \\ \hline \multicolumn{2}{c|}{3I_2} & \multicolumn{2}{c}{\emptyset} \end{array}\right], \tag{20}$$

where $C_1, \dots, C_7$ are reduced blocks and $C_q = \emptyset$ means that C_q was made zero by additions from other blocks (Case I of the algorithm). Hence, (W, C) may be considered as a canonical pair of matrices under similarity. Notice that $\left[\begin{smallmatrix} W & C \\ 0 & 0 \end{smallmatrix}\right]$ is a canonical matrix with respect to D-similarity, where $D = \{S \oplus S \mid S \in k^{2\times 2}\}$.

Definition 1.4. By the canonical form of a pair of $n \times n$ matrices (A, B) under simultaneous similarity is meant a pair (W, C), where $\left[\begin{smallmatrix} W & C \\ 0 & 0 \end{smallmatrix}\right]$ is the canonical form of the matrix $\left[\begin{smallmatrix} A & B \\ 0 & 0 \end{smallmatrix}\right]$ with respect to D-similarity with $D = \{S \oplus S \mid S \in k^{n\times n}\}$.

Clearly, each pair of matrices is similar to a canonical pair and two pairs of matrices are similar if and only if they reduce to the same canonical pair. The full list of canonical pairs of complex 4×4 matrices under simultaneous similarity was presented in [32].

Remark 1.2. Instead of (13), we may use another linear ordering in the set of blocks, for example, $M_{t1} < M_{t-1,1} < \cdots < M_{11} < M_{t2} < M_{t-1,2} < \cdots$ or $M_{t1} < M_{t-1,1} < M_{t2} < M_{t-2,1} < M_{t-1,2} < M_{t3} < \cdots$. It is necessary only that $(i, j) \lll (i', j')$ implies $M_{ij} < M_{i'j'}$, where $(i, j) \lll (i', j')$ indicates the existence of a nonzero addition from M_{ij} to $M_{i'j'}$ and is defined as follows:

Definition 1.5. Let Λ be a reduced $\underline{n} \times \underline{n}$ algebra. For unequal pairs $(i, j), (i', j') \in T \times T$ (see (2)), we put $(i, j) \lll (i', j')$ if either $i = i'$ and there exists $S \in \Lambda^*$ with $S_{jj'} \neq 0$, or $j = j'$ and there exists $S \in \Lambda^*$ with $S_{i'i} \neq 0$.

1.5 Structured Λ-canonical matrices

The structure of a Λ-canonical matrix M will be more clear if we partition it into boxes $M_1, M_2, \dots$, as it was made in (20).

Definition 1.6. Let $M = M^{(r)}$ for a certain $r \in \{0, 1, \dots, p\}$ (see (18)). We partition its reduced part into *boxes* $M_1, M_2, \dots, M_{q_{r+1}-1}$ as follows: Let $\Lambda^{(l)}$ $(1 \leqslant l \leqslant r)$ be a reduced $\underline{n}^{(l)} \times \underline{n}^{(l)}$ algebra from the sequence (18), we denote by $M_{ij}^{(l)}$ the blocks of M under the $\underline{n}^{(l)} \times \underline{n}^{(l)}$ partition. Then $M_{q_{l+1}}$ for $l \neq p$ denotes the first nonstable block among $M_{ij}^{(l)}$ with respect to $\Lambda^{(l)}$-similarity (it is reduced when $M^{(l)}$ is transformed to $M^{(l+1)}$); $M_{q_l+1} < \cdots < M_{q_{l+1}-1}$ $(q_0 := 0)$ are all the blocks $M_{ij}^{(l)}$ such that

(i) if $l < p$, then $M_{ij}^{(l)} < M_{q_{l+1}}$;

(ii) if $l > 0$, then $M_{ij}^{(l)}$ is not contained in the boxes $M_1, \dots, M_{q_l}$.

(Notice that each box M_i is 0, $\left[\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right]$, or a Weir matrix.) Furthermore, put

$$\Lambda_{q_l} = \Lambda_{q_l+1} = \cdots = \Lambda_{q_{l+1}-1} := \Lambda^{(l)}. \tag{21}$$

Generalizing the equalities (17) and (19), we obtain

$$\Lambda_i = \{S \in \Lambda \mid MS \equiv_i SM\}, \tag{22}$$

where $MS \equiv_i SM$ means that $MS - SM$ is zero on the places of $M_1, \ldots, M_i$.

Definition 1.7. By a *structured Λ-canonical matrix* we mean a Λ-canonical matrix M which is divided into boxes $M_1, M_2, \ldots, M_{q_{p+1}-1}$ and each box M_i that falls into Case I from Section 1.4 (and hence is 0) is marked by $\emptyset$ (see (20)).

Now we give a certain description of the Λ-canonical matrices.

Definition 1.8. By a *part* of a matrix $M = [a_{ij}]_{i,j=1}^n$ is meant an arbitrary set of its entries given with their indices. By a *rectangular part* we mean a part of the form $B = [a_{ij}]$, $p_1 \leqslant i \leqslant p_2$, $q_1 \leqslant j \leqslant q_2$. We consider a partition of M into disjoint rectangular parts (which is not, in general, a partition into substrips, see the matrix (20)) and write, generalizing (13), $B < B'$ if either $p_2 = p_2'$ and $q_1 < q_1'$, or $p_2 > p_2'$.

Definition 1.9. Let $M = [M_{ij}]$ be an $\underline{n} \times \underline{n}$ matrix partitioned into rectangular parts $M_1 < M_2 < \cdots < M_m$ such that this partition refines the partition into the blocks M_{ij}, and let each M_i be equal to 0, $\left[\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right]$, or a Weir matrix. For every $q \in \{0, 1, \ldots, m\}$, we define a subdivision of strips into q-strips as follows: The *0-strips* are the strips of M. Let $q > 0$. We make subdivisions of M into substrips that extend the partitions of $M_1, \ldots, M_q$ into cells 0, I, λI (i.e., the new subdivisions run the length of every boundary of the cells). If a subdivision passes through a cell I or λI from $M_1, \ldots, M_q$, then we construct the perpendicular subdivision such that the cell takes the form

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \lambda I & 0 \\ 0 & \lambda I \end{bmatrix},$$

and repeat this construction for all new divisions until partition $M_1, \ldots, M_q$ into cells 0, I, or λI. The obtained substrips will be called *q-strips of M*; for example, the partition into q-strips of the matrix (20) has the form

$$\left[\begin{array}{cc|cc} -1 & 1 & 2 & 0 \\ 0 & -1 & 0 & 1 \\ \hline 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right] \text{ for } q = 0, 1, 2; \quad \left[\begin{array}{c|c|c|c} -1 & 1 & 2 & 0 \\ \hline 0 & -1 & 0 & 1 \\ \hline 3 & 0 & 0 & 0 \\ \hline 0 & 3 & 0 & 0 \end{array}\right] \text{ for } q = 3, 4, 5, 6, 7.$$

We say that the αth q-strip of an ith (horizontal or vertical) strip *is linked to* the βth q-strip of an jth strip if (i) $\alpha = \beta$ and $i \sim j$ (including $i = j$; see (2)), or if (ii) their intersection is a (new) cell I from $M_1, \ldots, M_q$, or if (iii) they are in the transitive closure of (i) and (ii).

Notice that if M is a Λ-canonical matrix with the boxes $M_1, \ldots, M_{q_{p+1}-1}$ (see Definition 1.6), then $M_1 < \cdots < M_{q_{p+1}-1}$. Moreover, if Λ_q ($1 \leqslant q < q_{p+1}$, see (21)) is a reduced $\underline{n}_q \times \underline{n}_q$ algebra with the equivalence relation $\sim$ (see (2)), then the partition into q-strips is the $\underline{n}_q \times \underline{n}_q$ partition; the ith q-strip is linked with the jth q-strip if and only if $i \sim j$.

Theorem 1.4. *Let Λ be a reduced $\underline{n}\times\underline{n}$ algebra and let M be an arbitrary $\underline{n}\times\underline{n}$ matrix partitioned into rectangular parts $M_1 < M_2 < \dots < M_m$, where each M_i is equal to $\emptyset$ (a marked zero block), $\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$, or a Weir matrix. Then M is a structured Λ-canonical matrix with boxes $M_1, \dots, M_m$ if and only if each M_q $(1 \leqslant q \leqslant m)$ satisfies the following conditions:*

(a) M_q is the intersection of two $(q-1)$-strips.

(b) Let there exist $M' = S^{-1}MS$ (partitioned into rectangular parts conformal to M; $S \in \Lambda^*$) such that $M'_1 = M_1, \dots, M'_{q-1} = M_{q-1}$, but $M'_q \neq M_q$. Then $M_q = \emptyset$.

(c) Let M' from (b) do not exist. Then M_q is a Weir matrix if the horizontal and the vertical $(q-1)$-strips of M_q are linked; $M_q = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$ otherwise.

Proof. This theorem follows immediately from the algorithm of Section 1.4. □

2 Linear matrix problems

2.1 Introduction

In Section 2 we study a large class of matrix problems. In the theory of representations of finite dimensional algebras, similar classes of matrix problems are given by vectorspace categories [25, 34], bocses [26, 5], modules over aggregates [17, 16], or vectroids [3].

Let us define the considered class of matrix problems (in terms of elementary transformations to simplify its use; a more formal definition will be given in Section 2.2). Let $\sim$ be an equivalence relation in $T = \{1, \dots, t\}$. We say that a sequence of nonnegative integers $\underline{n} = (n_1, n_2, \dots, n_t)$ is a *step-sequence* if $i \sim j$ implies $n_i = n_j$. We also say that a $t \times t$ matrix $A = [a_{ij}]$ *links an equivalence class $\mathcal{I} \in T/\!\sim$ to an equivalence class $\mathcal{J} \in T/\!\sim$* if $a_{ij} \neq 0$ implies $(i,j) \in \mathcal{I} \times \mathcal{J}$. Let A link $\mathcal{I}$ to $\mathcal{J}$ and A' link $\mathcal{I}'$ to $\mathcal{J}'$, then $AA' = 0$ if $\mathcal{J} \neq \mathcal{I}'$ and AA' links $\mathcal{I}$ to $\mathcal{J}'$ if $\mathcal{J} = \mathcal{I}'$. For a step-sequence $\underline{n}$ and a linking matrix $A = [a_{ij}] \neq 0$, we denote by $A^{[l,r]}$ the $\underline{n}\times\underline{n}$ matrix that is obtained from A by replacing each entry a_{ij} with the following $n_i \times n_j$ block $A_{ij}^{[l,r]}$: if $a_{ij} \neq 0$ then the (l,r) entry of $A_{ij}^{[l,r]}$ is a_{ij} and the others are zeros (since $\underline{n}$ is a step-sequence, all these block have the same size, say $p \times q$; the numbers l and r must satisfy $1 \leqslant l \leqslant p$ and $1 \leqslant r \leqslant q$), and if $a_{ij} = 0$ then $A_{ij}^{[l,r]} = 0$.

Let a triple

$$(T/\!\sim,\ \{P_i\}_{i=1}^{p},\ \{V_j\}_{j=1}^{q}) \tag{23}$$

consist of the set of equivalence classes of $T = \{1, \dots, t\}$, a finite or empty set of linking nilpotent upper-triangular matrices $P_i \in k^{t\times t}$ and a finite set of linking matrices $V_j \in k^{t\times t}$. Denote by $\mathcal{P}$ the product closure of $\{P_i\}_{i=1}^{p}$ and by $\mathcal{V}$ the closure of $\{V_j\}_{j=1}^{q}$ with respect to multiplication by $\mathcal{P}$ (i.e., $\mathcal{VP} \subset \mathcal{V}$ and $\mathcal{PV} \subset \mathcal{V}$). Since P_i are nilpotent upper-triangular $t \times t$ matrices, $P_{i_1}P_{i_2}\dots P_{i_t} = 0$ for all $i_1, \dots, i_t$. Hence, $\mathcal{P}$ and $\mathcal{V}$ are finite sets consisting of linking nilpotent upper-triangular matrices and, respectively, linking matrices:

$$\mathcal{P} = \{P_{i_1}P_{i_2}\dots P_{i_r} \,|\, r \leqslant t\}, \quad \mathcal{V} = \{PV_jP' \,|\, P, P' \in \{I\} \cup \mathcal{P}\}. \tag{24}$$

For every step-sequence $\underline{n} = (n_1, \dots, n_t)$, we denote by $\mathcal{M}_{\underline{n}\times\underline{n}}$ the vector space generated by all $\underline{n} \times \underline{n}$ matrices of the form $V^{[l,r]}$, $0 \neq V \in \mathcal{V}$.

Definition 2.1. A *linear matrix problem* given by a triple (23) is the canonical form problem for $\underline{n} \times \underline{n}$ matrices $M = [M_{ij}] \in \mathcal{M}_{\underline{n}\times\underline{n}}$ with respect to sequences of the following transformations:

(i) For each equivalence class $\mathcal{I} \in T/\sim$, the same elementary transformations within all the vertical strips $M_{\bullet,i}$, $i \in \mathcal{I}$, then the inverse transformations within the horizontal strips $M_{i,\bullet}$, $i \in \mathcal{I}$.

(ii) For $a \in k$ and a nonzero matrix $P = [p_{ij}] \in \mathcal{P}$ linking $\mathcal{I}$ to $\mathcal{J}$, the transformation $M \mapsto (I + aP^{[l,r]})^{-1}M(I + aP^{[l,r]})$; that is, the addition of ap_{ij} times the lth column of the strip $M_{\bullet,i}$ to the rth column of the strip $M_{\bullet,j}$ simultaneously for all $(i,j) \in \mathcal{I} \times \mathcal{J}$, then the inverse transformations with rows of M.

Example 2.1. As follows from Example 1.1, the problem of classifying representations of the quiver (1) may be given by the triple

$$(\{\{1,\},\{2\},\{3,4\}\},\ \varnothing,\ \{e_{11}, e_{21}, e_{31}, e_{41}, e_{42}, e_{43}\}),$$

where e_{ij} denotes the matrix in which the (i,j) entry is 1 and the others are 0. In the same manner, we may give the problem of classifying representations of each quiver.

Example 2.2. Let $\mathcal{S} = \{p_1, \dots, p_n\}$ be a finite partially ordered set whose elements are indexed such that $p_i < p_j$ implies $i < j$. Its *representation* is a matrix M partitioned into n vertical strips $M_1, \dots, M_n$; we allow arbitrary row-transformations, arbitrary column-transformations within each vertical strip, and additions of linear combinations of columns of M_i to a column of M_j if $p_i < p_j$. (This important for the representation theory notion was given by Nazarova and Roiter [24], see also [17] and [34].) The problem of classifying representations of the poset $\mathcal{S}$ may be given by the triple

$$(\{\{1\},\{2\},\dots,\{n+1\}\}, \{e_{ij}\,|\,p_i < p_j\}, \{e_{n+1,1}, e_{n+1,2}, \dots, e_{n+1,n}\}).$$

Example 2.3. Let us consider Wasow's canonical form problem for an analytic at the point $\varepsilon = 0$ matrix

$$A(\varepsilon) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots, \quad A_i \in \mathbb{C}^{n\times n}, \tag{25}$$

relative to analytic similarity:

$$A(\varepsilon) \mapsto B(\varepsilon) := S(\varepsilon)^{-1}A(\varepsilon)S(\varepsilon), \tag{26}$$

where $S(\varepsilon) = S_0 + \varepsilon S_1 + \cdots$ and $S(\varepsilon)^{-1}$ are analytic matrices at 0. Let us restrict ourselves to the canonical form problem for the first t matrices $A_0, A_1, \dots, A_{t-1}$ in the expansion (25). By (26), $S(\varepsilon)B(\varepsilon) = A(\varepsilon)S(\varepsilon)$, that is $S_0B_0 = A_0S_0, \dots, S_0B_{t-1} +$

$S_1B_{t-2}+\cdots+S_{t-1}B_0 = A_0S_{t-1}+A_1S_{t-2}+\cdots+A_{t-1}S_0$, or in the matrix form

$$\begin{bmatrix} S_0 & S_1 & \cdots & S_{t-1} \\ & S_0 & \ddots & \vdots \\ & & \ddots & S_1 \\ 0 & & & S_0 \end{bmatrix} \begin{bmatrix} B_0 & B_1 & \cdots & B_{t-1} \\ & B_0 & \ddots & \vdots \\ & & \ddots & B_1 \\ 0 & & & B_0 \end{bmatrix} = \begin{bmatrix} A_0 & A_1 & \cdots & A_{t-1} \\ & A_0 & \ddots & \vdots \\ & & \ddots & A_1 \\ 0 & & & A_0 \end{bmatrix} \begin{bmatrix} S_0 & S_1 & \cdots & S_{t-1} \\ & S_0 & \ddots & \vdots \\ & & \ddots & S_1 \\ 0 & & & S_0 \end{bmatrix}.$$

Hence this problem may be given by the following triple of one-element sets:

$$(\{T\},\ \{J_t\},\ \{I_t\}),$$

where $J_t = e_{12}+e_{23}+\cdots+e_{t-1,t}$ is the nilpotent Jordan block. Then all elements of $T=\{1,2,\dots,t\}$ are equivalent, $\mathcal{P}=\{J_t, J_t^2,\dots,J_t^{t-1}\}$ and $\mathcal{V}=\{I_t, J_t,\dots,J_t^{t-1}\}$. This problem is wild even if $t=2$, see [13, 30]. I am thankful to S. Friedland for this example.

In Section 2.2 we give a definition of the linear matrix problems in a form, which is more similar to Gabriel and Roiter's definition (see Example 1.2) and is better suited for Belitskiĭ's algorithm.

In Section 2.3 we prove that every canonical matrix may be decomposed into a direct sum of indecomposable canonical matrices by permutations of its rows and columns. We also investigate the canonical form problem for upper triangular matrices under upper triangular similarity (see [35]).

In Section 2.4 we consider a canonical matrix as a parametric matrix whose parameters are eigenvalues of its Jordan blocks. It enables us to describe a set of canonical matrices having the same structure.

In Section 2.5 we consider linear matrix problems that give matrix problems with independent row and column transformations and prove that the problem of classifying modules over a finite-dimensional algebra may be reduced to such a matrix problem. The reduction is a modification of Drozd's reduction of the problem of classifying modules over an algebra to the problem of classifying representations of bocses [10] (see also Crawley-Boevey [5]). Another reduction of the problem of classifying modules over an algebra to a matrix problem with arbitrary row transformations was given in [16].

2.2 Linear matrix problems and Λ-similarity

In this section we give another definition of the linear matrix problems, which is equivalent to the Definition 2.1 but is often more convenient. The set of admissible transformations will be formulated in terms of Λ-similarity; it simplifies the use of Belitskiĭ's algorithm.

Definition 2.2. An algebra of upper triangular matrices $\Gamma \subset k^{t\times t}$ will be called a *basic matrix algebra* if

$$\begin{bmatrix} a_{11} & \cdots & a_{1t} \\ & \ddots & \vdots \\ 0 & & a_{tt} \end{bmatrix} \in \Gamma \quad \text{implies} \quad \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{tt} \end{bmatrix} \in \Gamma.$$

Lemma 2.1. *(a) Let $\Gamma \subset k^{t\times t}$ be a basic matrix algebra, $\mathcal{D}$ be the set of its diagonal matrices, and $\mathcal{R}$ be the set of its matrices with zero diagonal. Then there exists a basis $E_1, \dots, E_r$ of $\mathcal{D}$ over k consisting of diagonal matrices whose entries are 0 or 1 such that*

$$E_1 + \cdots + E_r = I_t, \quad E_\alpha E_\beta = 0 \ (\alpha \neq \beta), \quad E_\alpha^2 = E_\alpha. \tag{27}$$

As a vector space over k, Γ is a direct sum of subspaces

$$\Gamma = \mathcal{D} \oplus \mathcal{R} = \Big(\bigoplus_{\alpha=1}^{r} kE_\alpha\Big) \oplus \Big(\bigoplus_{\alpha,\beta=1}^{r} E_\alpha \mathcal{R} E_\beta\Big). \tag{28}$$

(b) The set of basic $t \times t$ algebras is the set of reduced $\underline{1} \times \underline{1}$ algebras, where $\underline{1} := (1, 1, \dots, 1)$. A basic $t \times t$ algebra Γ is the reduced $\underline{1} \times \underline{1}$ algebra given by $T/\sim = \{\mathcal{I}_1, \dots, \mathcal{I}_r\}$ (where $\mathcal{I}_\alpha$ is the set of indices defined by $E_\alpha = \sum_{i\in\mathcal{I}_\alpha} e_{ii}$, see (27)) and a family of systems of the form (3) such that for every $\alpha, \beta \in \{1, \dots, r\}$ the solutions of its $(\mathcal{I}_\alpha, \mathcal{I}_\beta)$ system form the space $E_\alpha \mathcal{R} E_\beta$.

Proof. (a) By Definition 2.2, Γ is the direct sum of vector spaces $\mathcal{D}$ and $\mathcal{R}$. Denote by $\mathcal{F}$ the set of diagonal $t \times t$ matrices whose entries are 0 or 1. Let $D \in \mathcal{D}$, then $D = a_1F_1 + \cdots + a_lF_l$, where $a_1, \dots, a_l$ are distinct nonzero elements of k and $F_1, \dots, F_l$ are such matrices from $\mathcal{F}$ that $F_iF_j = 0$ whenever $i \neq j$. The vectors $(a_1, \dots, a_l)$, $(a_1^2, \dots, a_l^2), \dots, (a_1^l, \dots, a_l^l)$ are linearly independent (they form a Vandermonde determinant), hence there exist $b_1, \dots, b_l \in k$ such that $F_1 = b_1D + b_2D^2 + \cdots + b_lD^l \in \mathcal{D}$, analogously $F_2, \dots, F_l \in \mathcal{D}$. It follows that $\mathcal{D} = kE_1 \oplus \cdots \oplus kE_r$, where $E_1, \dots, E_r \in \mathcal{F}$ and satisfy (27). Therefore, $\mathcal{R} = (E_1 + \cdots + E_r)\mathcal{R}(E_1 + \cdots + E_r) = \bigoplus_{\alpha,\beta} E_\alpha \mathcal{R} E_\beta$, we get the decomposition (28). (Notice that (27) is a decomposition of the identity of Γ into a sum of minimal orthogonal idempotents and (28) is the Peirce decomposition of Γ, see [12].) $\square$

Definition 2.3. A *linear matrix problem given by a pair*

$$(\Gamma, \mathcal{M}), \quad \Gamma\mathcal{M} \subset \mathcal{M}, \ \mathcal{M}\Gamma \subset \mathcal{M}, \tag{29}$$

consisting of a basic $t \times t$ algebra Γ and a vector space $\mathcal{M} \subset k^{t\times t}$, is the canonical form problem for matrices $M \in \mathcal{M}_{\underline{n}\times\underline{n}}$ with respect to $\Gamma_{\underline{n}\times\underline{n}}$-similarity transformations

$$M \mapsto S^{-1}MS, \quad S \in \Gamma^*_{\underline{n}\times\underline{n}},$$

where $\Gamma_{\underline{n}\times\underline{n}}$ and $\mathcal{M}_{\underline{n}\times\underline{n}}$ consist of $\underline{n} \times \underline{n}$ matrices whose blocks satisfy the same linear relations as the entries of the $t \times t$ matrices from Γ and $\mathcal{M}$.

More exactly, $\Gamma_{\underline{n}\times\underline{n}}$ is the reduced $\underline{n}\times\underline{n}$ algebra given by the same system (3) and $T/\sim = \{\mathcal{I}_1,\dots,\mathcal{I}_r\}$ as Γ (see Lemma 2.1(b)).[1] Next,

$$\mathcal{M} = \Big(\sum_{\alpha=1}^{r} E_\alpha\Big)\mathcal{M}\Big(\sum_{\beta=1}^{r} E_\beta\Big) = \bigoplus_{\alpha,\beta=1}^{r} E_\alpha \mathcal{M} E_\beta \tag{30}$$

(see (27)), hence there is a system of linear equations

$$\sum_{(i,j)\in\mathcal{I}_\alpha\times\mathcal{I}_\beta} d_{ij}^{(l)} x_{ij} = 0, \quad 1\leqslant l\leqslant p_{\alpha\beta}, \quad \mathcal{I}_\alpha,\mathcal{I}_\beta\in T/\sim, \tag{31}$$

such that $\mathcal{M}$ consists of all matrices $[m_{ij}]_{i,j=1}^{t}$ whose entries satisfy the system (31). Then $\mathcal{M}_{\underline{n}\times\underline{n}}$ ($\underline{n}$ is a step-sequence) denotes the vector space of all $\underline{n}\times\underline{n}$ matrices $[M_{ij}]_{i,j=1}^{t}$ whose blocks satisfy the system (31):

$$\sum_{(i,j)\in\mathcal{I}_\alpha\times\mathcal{I}_\beta} d_{ij}^{(l)} M_{ij} = 0, \quad 1\leqslant l\leqslant p_{\alpha\beta}, \quad \mathcal{I}_\alpha,\mathcal{I}_\beta\in T/\sim .$$

Theorem 2.1. *Definitions 2.1 and 2.3 determine the same class of matrix problems:*

(a) The linear matrix problem given by a triple $(T/\sim,\ \{P_i\}_{i=1}^{p},\ \{V_j\}_{j=1}^{q})$ *may be also given by the pair* $(\Gamma,\mathcal{M})$, *where* Γ *is the basic algebra generated by* $P_1,\dots,P_p$ *and all matrices* $E_{\mathcal{I}} = \sum_{j\in\mathcal{I}} e_{jj}$ $(\mathcal{I}\in T/\sim)$ *and* $\mathcal{M}$ *is the minimal vector space of matrices containing* $V_1,\dots,V_q$ *and closed with respect to multiplication by* $P_1,\dots,P_p$.

(b) The linear matrix problem given by a pair $(\Gamma,\mathcal{M})$ *may be also given by a triple* $(T/\sim,\ \{P_i\}_{i=1}^{p},\ \{V_j\}_{j=1}^{q})$, *where* $T/\sim = \{\mathcal{I}_1,\dots,\mathcal{I}_r\}$ *(see Lemma 2.1(b)),* $\{P_i\}_{i=1}^{p}$ *is the union of bases of the spaces* $E_\alpha\mathcal{R}E_\beta$ *(see (28)), and* $\{V_j\}_{j=1}^{q}$ *is the union of bases of the spaces* $E_\alpha\mathcal{M}E_\beta$ *(see (30)).*

Proof. (a) Let $\underline{n}$ be a step-sequence. We first prove that the set of admissible transformations is the same for both the matrix problems; that is, there exists a sequence of transformations (i)–(ii) from Definition 2.1 transforming M to N (then we write $M\simeq N$) if and only if they are Λ-similar with $\Lambda := \Gamma_{\underline{n}\times\underline{n}}$.

By Definition 2.1, $M\simeq N$ if and only if $S^{-1}MS = N$, where S is a product of matrices of the form

$$I + aE_{\mathcal{I}}^{[l,r]}\ (a\neq -1 \text{ if } l=r), \quad I + bP^{[l,r]}, \tag{32}$$

where $a,b\in k$, $\mathcal{I}\in T/\sim$ and $P\in\mathcal{P}$. Since $S\in\Lambda$, $M\simeq N$ implies $M\sim_\Lambda N$.

Let $M\sim_\Lambda N$, that is $SMS^{-1}=N$ for a nonsingular $S\in\Lambda$. To prove $M\simeq N$, we must expand S^{-1} into factors of the form (32); it suffices to reduce S to I multiplying by matrices (32). The matrix S has the form (4) with $S_{ii}=S_{jj}$ whenever $i\sim j$; we reduce S to the form (4) with $S_{ii}=I_{n_i}$ for all i multiplying by matrices $I + aE_{\mathcal{I}}^{[l,r]}$. Denote by $\mathcal{Q}$ the set of all $\underline{n}\times\underline{n}$ matrices of the form $P^{[l,r]}$, $P\in\mathcal{P}$. Since $\mathcal{Q}\cup\{E_{\mathcal{I}}^{[l,r]}\}_{\mathcal{I}\in T/\sim}$ is product closed, it generates Λ as a vector space. Therefore, $S = I + \sum_{Q\in\mathcal{Q}} a_Q Q$ ($a_Q\in k$). Put $\mathcal{Q}_l = \{Q\in\mathcal{Q}\,|\,Q^l = 0\}$, then $\mathcal{Q}_0=\varnothing$ and $\mathcal{Q}_t=\mathcal{Q}$. Multiplying S by $\prod_{Q\in\mathcal{Q}}(I-a_QQ) = I - \sum_{Q\in\mathcal{Q}} a_QQ + \cdots$, we make $S = I+\cdots$, where the points denote

[1] If $n_1>0,\dots,n_t>0$, then $\Gamma_{\underline{n}\times\underline{n}}$ is Morita equivalent to Γ; moreover, Γ is the basic algebra for $\Gamma_{\underline{n}\times\underline{n}}$ in terms of the theory of algebras, see [12].

a linear combination of products of matrices from $\mathcal{Q}$ and each product consists of at least 2 matrices (so its degree of nilpotency is at most $t-1$). Each product is contained in $\mathcal{Q}_{t-1}$ since $\mathcal{Q}$ is product closed, hence $S = I + \sum_{Q\in\mathcal{Q}_{t-1}} b_Q Q$. In the same way we get $S = I + \sum_{Q\in\mathcal{Q}_{t-2}} c_Q Q$, and so on until obtain $S = I$.

Clearly, the set of reduced $\underline{n} \times \underline{n}$ matrices $\mathcal{M}_{\underline{n}\times\underline{n}}$ is the same for both the matrix problems. □

Further we use Definition 2.3 of linear matrix problems.

2.3 Krull–Schmidt Theorem

In this section we study decompositions of a canonical matrix into a direct sum of indecomposable canonical matrices.

Let a linear matrix problem be given by a pair $(\Gamma, \mathcal{M})$. By the *canonical matrices*, we mean the $\Gamma_{\underline{n}\times\underline{n}}$-canonical matrices $M \in \mathcal{M}_{\underline{n}\times\underline{n}}$ for step-sequences $\underline{n}$. We say that $\underline{n} \times \underline{n}$ matrices M and N are *equivalent* and write $M \simeq N$ if they are $\Gamma_{\underline{n}\times\underline{n}}$-similar. By the *block-direct sum* of an $\underline{m} \times \underline{m}$ matrix $M = [M_{ij}]_{i,j=1}^t$ and an $\underline{n} \times \underline{n}$ matrix $N = [N_{ij}]_{i,j=1}^t$, we mean the $(\underline{m}+\underline{n}) \times (\underline{m}+\underline{n})$ matrix

$$M \uplus N = [M_{ij} \oplus N_{ij}]_{i,j=1}^t.$$

A matrix $M \in \mathcal{M}_{\underline{n}\times\underline{n}}$ is said to be *indecomposable* if $\underline{n} \neq 0$ and $M \simeq M_1 \uplus M_2$ implies that M_1 or M_2 has size 0×0.

Theorem 2.2. *For every canonical $\underline{n} \times \underline{n}$ matrix M, there exists a permutation matrix $P \in \Gamma_{\underline{n}\times\underline{n}}$ such that*

$$P^{-1}MP = \underbrace{M_1 \uplus \cdots \uplus M_1}_{q_1 \text{ copies}} \uplus \cdots \uplus \underbrace{M_l \uplus \cdots \uplus M_l}_{q_l \text{ copies}}, \tag{33}$$

where M_i are distinct indecomposable canonical matrices. The decomposition (33) is determined by M uniquely up to permutation of summands.

Proof. Let M be a canonical $\underline{n} \times \underline{n}$ matrix. The repeated application of Belitskiĭ's algorithm produces the sequence (18): (M, Λ), (M', Λ'), ..., $(M^{(p)}, \Lambda^{(p)})$, where $\Lambda = \Gamma_{\underline{n}\times\underline{n}}$ and $\Lambda^{(p)} = \{S \in \Lambda \mid MS = SM\}$ (see (19)) are reduced $\underline{n}\times\underline{n}$ and $\underline{m}\times\underline{m}$ algebras; by Definition 1.1(a) Λ and $\Lambda^{(p)}$ determine equivalence relations $\sim$ in $T = \{1, \dots, t\}$ and $\approx$ in $T^{(p)} = \{1, \dots, r\}$. Since M is canonical, $M^{(i)}$ differs from $M^{(i+1)}$ only by additional subdivisions. The strips with respect to the $\underline{m} \times \underline{m}$ partition will be called the *substrips*.

Denote by $\Lambda_0^{(p)}$ the subalgebra of $\Lambda^{(p)}$ consisting of its block-diagonal $\underline{m}\times\underline{m}$ matrices, and let $S \in \Lambda_0^{(p)}$. Then it has the form $S = C_1 \oplus \cdots \oplus C_r$, where $C_\alpha = C_\beta$ if $\alpha \approx \beta$. It may be also considered as a block-diagonal $\underline{n} \times \underline{n}$ matrix $S = S_1 \oplus \cdots \oplus S_t$ from Λ (since $\Lambda^{(p)} \subset \Lambda$); each block S_i is a direct sum of subblocks C_α.

Let $\mathcal{I}$ be an equivalence class from $T^{(p)}/\approx$. In each S_i, we permute its subblocks C_α with $\alpha \in \mathcal{I}$ into the first subblocks:

$$\bar{S}_i = C_{\alpha_1} \oplus \cdots \oplus C_{\alpha_p} \oplus C_{\beta_1} \oplus \cdots \oplus C_{\beta_q}, \quad \alpha_1 < \cdots < \alpha_p, \ \ \beta_1 < \cdots < \beta_q,$$

where $\alpha_1, \dots, \alpha_p \in \mathcal{I}$ and $\beta_1, \dots, \beta_q \notin \mathcal{I}$ (notice that $C_{\alpha_1} = \cdots = C_{\alpha_p}$); it gives the matrix $\bar{S} = Q^{-1}SQ$, where $Q = Q_1 \oplus \cdots \oplus Q_t$ and Q_i are permutation matrices. Let

$i \sim j$, then $S_i = S_j$ (for all $S \in \Lambda$), hence the permutations within S_i and S_j are the same. We have $Q_i = Q_j$ if $i \sim j$, therefore $Q \in \Lambda$.

Making the same permutations of substrips within each strip of M, we get $\bar{M} = Q^{-1}MQ$. Let $M = [M_{ij}]_{i,j=1}^t$ relatively to the $\underline{n} \times \underline{n}$ partition, and let $M = [N_{\alpha\beta}]_{\alpha,\beta=1}^r$ relatively to the $\underline{m} \times \underline{m}$ partition. Since M is canonical, all $N_{\alpha\beta}$ are reduced, hence $N_{\alpha\beta} = 0$ if $\alpha \not\approx \beta$ and $N_{\alpha\beta}$ is a scalar square matrix if $\alpha \approx \beta$. The $\bar{M}$ is obtained from M by gathering all subblocks $N_{\alpha\beta}$, $(\alpha, \beta) \in \mathcal{I} \times \mathcal{I}$, in the left upper cover of every block M_{ij}, hence $\bar{M}_{ij} = A_{ij} \oplus B_{ij}$, where A_{ij} consists of subblocks $N_{\alpha\beta}$, $\alpha, \beta \in \mathcal{I}$, and B_{ij} consists of subblocks $N_{\alpha\beta}$, $\alpha, \beta \notin \mathcal{I}$. We have $\bar{M} = A_1 \uplus B$, where $A_1 = [A_{ij}]$ and $B = [B_{ij}]$. Next apply the same procedure to B; continue the process until get

$$P^{-1}MP = A_1 \uplus \cdots \uplus A_l,$$

where $P \in \Lambda$ is a permutation matrix and the summands A_i correspond to the equivalence classes of $T^{(p)}/\approx$.

The matrix A_1 is canonical. Indeed, M is a canonical matrix, by Definition 1.6, each box X of M has the form $\emptyset$, $\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$, or a Weir matrix. It may be proved that the part of X at the intersection of substrips with indices in $\mathcal{I}$ has the same form and this part is a box of A_1. Furthermore, the matrix A_1 consists of subblocks $N_{\alpha\beta}$, $(\alpha, \beta) \in \mathcal{I} \times \mathcal{I}$, that are scalar matrices of the same size $t_1 \times t_1$. Hence, $A_1 = M_1 \uplus \cdots \uplus M_1$ (t_1 times), where M_1 is canonical. Analogously, $A_i = M_i \uplus \cdots \uplus M_i$ for all i and the matrices M_i are canonical. □

Corollary (Krull–Schmidt theorem). *For every matrix $M \in \mathcal{M}_{\underline{n}\times\underline{n}}$, there exists its decomposition*

$$M \simeq M_1 \uplus \cdots \uplus M_r$$

into a block-direct sum of indecomposable matrices $M_i \in \mathcal{M}_{\underline{n_i}\times\underline{n_i}}$. Moreover, if

$$M \simeq N_1 \uplus \cdots \uplus N_s$$

is another decomposition into a block-direct sum of indecomposable matrices, then $r = s$ and, after a suitable reindexing, $M_1 \simeq N_1, \dots, M_r \simeq N_r$.

Proof. This statement follows from Theorems 1.3 and 2.2. Notice that this statement is a partial case of the Krull–Schmidt theorem [1] for additive categories; namely, for the category of matrices $\cup \mathcal{M}_{\underline{n}\times\underline{n}}$ (the union over all step-sequences $\underline{n}$) whose morphisms from $M \in \mathcal{M}_{\underline{m}\times\underline{m}}$ to $N \in \mathcal{M}_{\underline{n}\times\underline{n}}$ are the matrices $S \in \mathcal{M}_{\underline{m}\times\underline{n}}$ such that $MS = SN$. (The set $\mathcal{M}_{\underline{m}\times\underline{n}}$ of $\underline{m} \times \underline{n}$ matrices is defined like $\mathcal{M}_{\underline{n}\times\underline{n}}$.) □

Example 2.4. Let us consider the *canonical form problem for upper triangular matrices under upper triangular similarity* (see [35] and the references given there). The set Γ^t of all upper triangular $t \times t$ matrices is a reduced $\underline{1} \times \underline{1}$ algebra, so every $A \in \Gamma^t$ is reduced to the Γ^t-canonical form A^∞ by Belitskiĭ's algorithm; moreover, in this case the algorithm is very simplified: All diagonal entries of $A = [a_{ij}]$ are not changed by transformations; the over-diagonal entries are reduced starting with the last but one row:

$$a_{t-1,t};\ a_{t-2,t-1},\ a_{t-2,t};\ a_{t-3,t-2},\ a_{t-3,t-1},\ a_{t-3,t};\ \dots .$$

Let a_{pq} be the first that changes by admissible transformations. If there is a nonzero admissible addition, we make $a_{pq} = 0$; otherwise a_{pq} is reduced by transformations of equivalence or similarity, in the first case me make $a_{pq} \in \{0,1\}$, in the second case a_{pq} is not changed; then we restrict the set of admissible transformations to those that preserve the reduced a_{pq}, and so on. Notice that this reduction is possible for an arbitrary field k, which does not need to be algebraically closed.

Furthermore, Γ^t is a basic $t \times t$ algebra, so we may consider A^∞ as a canonical matrix for the linear matrix problem given by the pair (Γ^t, Γ^t). By Theorem 2.2 and since a permutation $t \times t$ matrix P belongs to Γ^t only if $P = I$, there exists a unique decomposition

$$A^\infty = A_1 \uplus \cdots \uplus A_r,$$

where each A_i is an indecomposable canonical $\underline{n}_i \times \underline{n}_i$ matrix, $\underline{n}_i \in \{0,1\}^t$. Let $t_i \times t_i$ be the size of A_i, then $\Gamma^t_{\underline{n}_i \times \underline{n}_i}$ may be identified with Γ^{t_i}, so M_i is a Γ^{t_i}-canonical matrix.

Let $A^\infty = [a_{ij}]_{i,j=1}^t$, define the graph G_A with vertices $1, \dots, t$ having the edge i—j $(i < j)$ if and only if both $a_{ij} = 1$ and a_{ij} was reduced by equivalence transformations. Then G_A is a union of trees; moreover, G_A is a tree if and only if A^∞ is indecomposable (compare with [29]).

The Krull–Schmidt theorem for this case and a description of nonequivalent indecomposable $t \times t$ matrices for $t \leqslant 6$ was given by Thijsse [35].

2.4 Parametric canonical matrices

Let a linear matrix problem be given by a pair $(\Gamma, \mathcal{M})$. The set $\mathcal{M}$ may be given as the matrix space of all solutions $[m_{ij}]_{i,j=1}^t$ of the system (31) in which the unknowns x_{ij} are disposed like the blocks (13): $x_{t1} \prec x_{t2} \prec \cdots$. The Gauss-Jordan elimination procedure to the system (31) starting with the last unknown reduces the system to the form

$$x_{lr} = \sum_{(i,j) \in \mathcal{N}_f} c_{ij}^{(l,r)} x_{ij}, \quad (l,r) \in \mathcal{N}_d, \tag{34}$$

where $\mathcal{N}_d$ and $\mathcal{N}_f$ are such that $\mathcal{N}_d \cup \mathcal{N}_f = \{1, \dots, t\} \times \{1, \dots, t\}$ and $\mathcal{N}_d \cap \mathcal{N}_f = \varnothing$; the inequality $c_{ij}^{(l,r)} \neq 0$ implies $i \sim l$, $j \sim r$ and $x_{ij} \prec x_{lr}$ (i.e., every unknown x_{lr} with $(l,r) \in \mathcal{N}_d \cap (\mathcal{I} \times \mathcal{J})$ is a linear combination of the preceding unknowns with indices in $\mathcal{N}_f \cap (\mathcal{I} \times \mathcal{J})$).

A block M_{ij} of $M \in \mathcal{M}_{\underline{n} \times \underline{n}}$ will be called *free* if $(i,j) \in \mathcal{N}_f$, *dependent* if $(i,j) \in \mathcal{N}_d$. A box M_i will be called *free* (*dependent*) if it is a part of a free (dependent) block.

Lemma 2.2. *The vector space $\mathcal{M}_{\underline{n} \times \underline{n}}$ consists of all $\underline{n} \times \underline{n}$ matrices $[M_{ij}]_{i,j=1}^t$ whose free blocks are arbitrary and the dependent blocks are their linear combinations given by (34):*

$$M_{lr} = \sum_{(i,j) \in \mathcal{N}_f} c_{ij}^{(l,r)} M_{ij}, \quad (l,r) \in \mathcal{N}_d. \tag{35}$$

On each step of Belitskiĭ's algorithm, the reduced subblock of $M \in \mathcal{M}_{\underline{n} \times \underline{n}}$ belongs to a free block (i.e., all boxes $M_{q_1}, M_{q_2}, \dots$ from Definition 1.6 are subblocks of free blocks).

Proof. Let us prove the second statement. On the lth step of Belitskiĭ's algorithm, we reduce the first nonstable block $M_{\alpha\beta}^{(l)}$ of the matrix $M^{(l)} = [M_{ij}^{(l)}]$ with respect to $\Lambda^{(l)}$-similarity. If $M_{\alpha\beta}^{(l)}$ is a subblock of a dependent block M_{ij}, then $M_{\alpha\beta}^{(l)}$ is a linear combination of already reduced subblocks of blocks preceding to M_{ij}, hence $M_{\alpha\beta}^{(l)}$ is stable, a contradiction. $\square$

We now describe a set of canonical matrices having 'the same form'.

Definition 2.4. Let M be a structured (see Definition 1.7) canonical $\underline{n} \times \underline{n}$ matrix, let $M_{r_1} < \cdots < M_{r_s}$ be those of its free boxes that are Weir matrices (Case III of Belitskiĭ's algorithm), and let $\lambda_{t_{i-1}+1} \prec \cdots \prec \lambda_{t_i}$ be the distinct eigenvalues of M_{r_i}. Considering some of λ_i (resp. all λ_i) as parameters, we obtain a parametric matrix $M(\vec{\lambda})$, $\vec{\lambda} := (\lambda_{i_1}, \dots, \lambda_{i_p})$ (resp. $\vec{\lambda} := (\lambda_1, \dots, \lambda_p)$, $p := t_s$), which will be called a *semi-parametric* (resp. *parametric*) *canonical matrix*. Its *domain of parameters* is the set of all $\vec{a} \in k^p$ such that $M(\vec{a})$ is a structured canonical $\underline{n} \times \underline{n}$ matrix with the same disposition of the boxes $\emptyset$ as in M.

Theorem 2.3. *The domain of parameters $\mathcal{D}$ of a parametric canonical $\underline{n} \times \underline{n}$ matrix $M(\vec{\lambda})$ is given by a system of equations and inequalities of the following three types:*

(i) $f(\vec{\lambda}) = 0$,

(ii) $(d_1(\vec{\lambda}), \dots, d_n(\vec{\lambda})) \neq (0, \dots, 0)$,

(iii) $\lambda_i \prec \lambda_{i+1}$,

where $f, d_j \in k[x_1, \dots, x_p]$.

Proof. Let $M_1 < \cdots < M_m$ be all the boxes of $M(\vec{\lambda})$. Put $\mathcal{A}_0 := k^p$ and denote by $\mathcal{A}_q$ $(1 \leqslant q \leqslant m)$ the set of all $\vec{a} \in k^p$ such that $M(\vec{a})$ coincides with $M(\vec{a})^\infty$ on $M_1, \dots, M_q$. Denote by $\Lambda_q(\vec{a})$ $(1 \leqslant q \leqslant m,\ \vec{a} \in \mathcal{A}_q)$ the subalgebra of $\Lambda := \Gamma_{\underline{n} \times \underline{n}}$ consisting of all $S \in \Lambda$ such that $SM(\vec{a})$ coincides with $M(\vec{a})S$ on the places of $M_1, \dots, M_q$.

We prove that there is a system $\mathcal{S}_q(\vec{\lambda})$ of equations of the form (5) and (6) (in which every $c_{ij}^{(l)}$ is an element of k or a parameter from $M_1, \dots, M_q$; moreover, the equations of each $(\mathcal{I},\ J)$ subsystem of (6) are linearly independent for every $\vec{\lambda} \in \mathcal{A}_q$) such that $\Lambda_q(\vec{a})$, $\vec{a} \in \mathcal{A}_q$, is a reduced $\underline{n}_q \times \underline{n}_q$ algebra given by $\mathcal{S}_q(\vec{a})$. This is obvious for $\Lambda_0(\vec{a}) := \Lambda(\vec{a})$. Let it hold for $q-1$, we prove it for q.

We may assume that M_q is a free box since otherwise $\mathcal{A}_{q-1} = \mathcal{A}_q$ and $\Lambda_q(\vec{a}) = \Lambda_{q-1}(\vec{a})$ for all $\vec{a} \in \mathcal{A}_{q-1}$. Let (l, r) be the indices of M_q as a block of the $\underline{n}_{q-1} \times \underline{n}_{q-1}$ matrix M (i.e. $M_q = M_{lr}$). In accordance with the algorithm of Section 1.4, we consider two cases:

Case 1: $M_q = \emptyset$. Then the equality (15) is not implied by the system $\mathcal{S}_{q-1}(\vec{a})$ (more exactly, by its $(\mathcal{I}, \mathcal{J})$ subsystem with $\mathcal{I} \times \mathcal{J} \ni (l, r)$, see (6)) for all $\vec{a} \in \mathcal{A}_q$. It means that there is a nonzero determinant formed by columns of coefficients of the system (6) $\cup$ (15). Hence, $\mathcal{A}_q$ consists of all $\vec{a} \in \mathcal{A}_{q-1}$ that satisfy the condition (ii), where $d_1(\vec{\lambda}), \dots, d_n(\vec{\lambda})$ are all such determinants; we have $\mathcal{S}_q(\vec{\lambda}) = \mathcal{S}_{q-1}(\vec{\lambda}) \cup (15)$.

Case 2: $M_q \neq \emptyset$. Then (15) is implied by the system $\mathcal{S}_{q-1}(\vec{a})$ for all $\vec{a} \in \mathcal{A}_q$. Hence, $\mathcal{A}_q$ consists of all $\vec{a} \in \mathcal{A}_{q-1}$ that satisfy the conditions $d_1(\vec{a}) = 0, \dots, d_n(\vec{a}) = 0$ of the form (i) and (if M_q is a Weir matrix with the parameters $\lambda_{t_{q-1}+1}, \dots, \lambda_{t_q}$) the conditions $\lambda_{t_{q-1}+1} \prec \cdots \prec \lambda_{t_q}$ of the form (iii). The system $\mathcal{S}_q(\vec{\lambda})$ is obtained from $\mathcal{S}_{q-1}(\vec{\lambda})$ as follows: we rewrite (5)–(6) for more small blocks of Λ_q (every system (6) with $\mathcal{I} \ni l$ or $\mathcal{J} \ni r$ gives several systems with the same coefficients, each of them connects equally disposed subblocks of the blocks S_{ij} with $(i, j) \in \mathcal{I} \times \mathcal{J}$) and join the equations that need for $S_{ll}M_{lr} = M_{lr}S_{rr}$.

Since $\mathcal{A}_0 = k^p$, $\mathcal{A}_q$ $(1 \leqslant q \leqslant m)$ consists of all $\vec{a} \in \mathcal{A}_{q-1}$ that satisfy a certain system of conditions (i)–(iii) and $\mathcal{D} := \mathcal{A}_m$ is the domain of parameters of $M(\vec{\lambda})$. $\square$

Example 2.5. The canonical pair of matrices from Example 1.5 has the parametric form

$$\left(\left[\begin{array}{cc|cc} \lambda_1 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & \lambda_1 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & \lambda_2 & 1 \\ & & 0 & \lambda_2 \end{array}\right],\ \left[\begin{array}{cc|c|c} \mu_2 & 1 & \mu_5 & \emptyset \\ \cline{3-4} 0 & \mu_2 & \mu_3 & \mu_4 \\ \hline \mu_1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\emptyset$}} \\ 0 & \mu_1 & & \end{array}\right]\right).$$

Its domain of parameters is given by the conditions $\lambda_1 \prec \lambda_2$, $\mu_1 \neq 0$, $\mu_3 = 0$, and $\mu_4 \neq \mu_5$.

Remark 2.1. The number of parametric canonical $\underline{n} \times \underline{n}$ matrices is finite for every $\underline{n}$ since there exists a finite number of partitions into boxes, and each box is $\emptyset$, $\left[\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right]$, or a Weir matrix (consisting of 0, 1, and parameters). Therefore, a linear matrix problem for matrices of size $\underline{n} \times \underline{n}$ is reduced to the problem of finding a finite set of parametric canonical matrices and their domains of parameters. Each domain of parameters is given by a system of polynomial equations and inequalities (of the types (i)–(iii)), so it is a semi-algebraic set; moreover, it is locally closed up to the conditions (iii).

2.5 Modules over finite-dimensional algebras

In this section, we consider matrix problems with independent row and column transformations (such problems are called *separated* in [17]) and reduce to them the problem of classifying modules over algebras.

Lemma 2.3. *Let $\Gamma \subset k^{m\times m}$ and $\Delta \subset k^{n\times n}$ be two basic matrix algebras and let $\mathcal{N} \subset k^{m\times n}$ be a vector space such that $\Gamma\mathcal{N} \subset \mathcal{N}$ and $\mathcal{N}\Delta \subset \mathcal{N}$. Denote by $0\diagdown\mathcal{N}$ the vector space of $(m+n)\times(m+n)$ matrices of the form $\left[\begin{smallmatrix} 0 & N \\ 0 & 0 \end{smallmatrix}\right]$, $N \in \mathcal{N}$. Then the pair*

$$(\Gamma \oplus \Delta,\ 0\diagdown\mathcal{N})$$

determines the canonical form problem for matrices $N \in \mathcal{N}_{\underline{m}\times\underline{n}}$ in which the row transformations are given by Γ and the column transformations are given by Δ:

$$N \mapsto CNS, \quad C \in \Gamma^*_{\underline{m}\times\underline{m}},\ S \in \Delta^*_{\underline{n}\times\underline{n}}.$$

Proof. Put $M = \left[\begin{smallmatrix} 0 & N \\ 0 & 0 \end{smallmatrix}\right]$ and apply Definition 2.3. □

In particular, if $\Gamma = k$, then the row transformations are arbitrary; this classification problem is studied intensively in the representation theory as is given by a vectorspace category [25, 34], by a module over an aggregate [17, 16], or by a vectroid [3].

The next theorem shows that the problem of classifying modules over a finite dimensional algebra Γ may be reduced to a linear matrix problem. If the reader is not familiar with the theory of modules (the used results can be found in [12]), he may omit this theorem since it is not used in the next sections. The algebra Γ is isomorphic to a matrix algebra, so by Theorem 1.1 we may assume that Γ is a reduced matrix algebra. Moreover, by the Morita theorem [12], the category of modules over Γ is equivalent to the category of modules over its basic algebra, hence we may assume that Γ is a basic matrix algebra. All modules are taken to be right finite-dimensional.

Theorem 2.4. *Let Γ be a basic $t\times t$ algebra. There is a natural bijection between:*

(i) the set of isoclasses of indecomposable modules over Γ and

(ii) the set of indecomposable $(\Gamma\oplus\Gamma,\ 0\diagdown\mathcal{R})$ canonical matrices without zero $\underline{n}\times\underline{n}$ matrices with $\underline{n}=(0,\dots,0,n_{t+1},\dots,n_{2t})$, where $\mathcal{R}=\operatorname{rad}\Gamma$ (it consists of the matrices from Γ with zero diagonal).

Proof. We will successively reduce

(a) the problem of classifying, up to isomorphism, modules over a basic matrix algebra $\Gamma\subset k^{t\times t}$

to a linear matrix problem.

Drozd [10] (see also Crawley-Boevey [5]) proposed a method for reducing the problem (a) (with an arbitrary finite-dimensional algebra Γ) to a matrix problem. His method was founded on the following well-known property of projective modules [12, p. 156]:

For every module M over Γ, there exists an exact sequence

$$P\xrightarrow{\varphi}Q\xrightarrow{\psi}M\longrightarrow 0, \tag{36}$$

$$\operatorname{Ker}\varphi\subset\operatorname{rad}P,\quad \operatorname{Im}\varphi\subset\operatorname{rad}Q, \tag{37}$$

where P and Q are projective modules. Moreover, if

$$P'\xrightarrow{\varphi'}Q'\xrightarrow{\psi'}M'\longrightarrow 0$$

is another exact sequence with these properties, then M is isomorphic to M' if and only if there exist isomorphisms $f:P\to P'$ and $g:Q\to Q'$ such that $g\varphi=\varphi' f$.

Hence, the problem (a) reduces to

(b) the problem of classifying triples (P,Q,φ) formed by projective modules P and Q over a basis matrix algebra Γ and a homomorphism $\varphi:P\to Q$ satisfying (37), up to isomorphisms $(f,g):(P,Q,\varphi)\to(P',Q',\varphi')$ given by pairs of isomorphisms $f:P\to P'$ and $g:Q\to Q'$ such that $g\varphi=\varphi' f$.

By Lemma 2.1, Γ is a reduced algebra, it defines an equivalence relation $\sim$ in $T=\{1,\dots,t\}$ (see (2)). Moreover, if $T/\sim\,=\{\mathcal{I}_1,\dots,\mathcal{I}_r\}$, then the matrices $E_\alpha=\sum_{i\in\mathcal{I}_\alpha}e_{ii}$ $(\alpha=1,\dots,r)$ form a decomposition (27) of the identity of Γ into a sum of minimal orthogonal idempotents, and $P_1=E_1\Gamma,\dots,P_r=E_r\Gamma$ are all nonisomorphic indecomposable projective modules over Γ.

Let $\varphi\in\operatorname{Hom}_\Gamma(P_\beta,P_\alpha)$, then φ is given by $F:=\varphi(E_\beta)$. Since $F\in P_\alpha$, $F=E_\alpha F$. Since φ is a homomorphism, $\varphi(E_\beta G)=0$ implies $FG=0$ for every $G\in\Gamma$. Taking $G=I-E_\beta$, we have $F(I-E_\beta)=0$, so $F=FE_\beta=E_\alpha FE_\beta$. Hence we may identify $\operatorname{Hom}_\Gamma(P_\beta,P_\alpha)$ and $E_\alpha\Gamma E_\beta$:

$$\operatorname{Hom}_\Gamma(P_\beta,P_\alpha)=\Gamma_{\alpha\beta}:=E_\alpha\Gamma E_\beta. \tag{38}$$

The set $\mathcal{R}$ of all matrices from Γ with zero diagonal is the radical of Γ; $\operatorname{rad}P_\alpha=P_\alpha\mathcal{R}=E_\alpha\mathcal{R}$. Hence $\varphi\in\operatorname{Hom}_\Gamma(P_\beta,P_\alpha)$ satisfies $\operatorname{Im}\varphi\subset\operatorname{rad}P_\alpha$ if and only if $\varphi(E_\beta)\in\mathcal{R}_{\alpha\beta}:=E_\alpha\mathcal{R}E_\beta$.

Let

$$P=P_1^{(p_1)}\oplus\cdots\oplus P_r^{(p_r)},\quad Q=Q_1^{(q_1)}\oplus\cdots\oplus Q_r^{(q_r)}$$

be two projective modules, where $X^{(i)} := X \oplus \cdots \oplus X$ (i times); we may identify $\mathrm{Hom}_\Gamma(P,Q)$ with the set of block matrices $\Phi = [\Phi_{\alpha\beta}]_{\alpha,\beta=1}^r$, where $\Phi_{\alpha\beta} \in \Gamma_{\alpha\beta}^{q_\alpha \times p_\beta}$ is a $q_\alpha \times p_\beta$ block with entries in $\Gamma_{\alpha\beta}$. Moreover, $\mathrm{Im}\,\Phi \subset \mathrm{rad}\, Q$ if and only if $\Phi_{\alpha\beta} \in \mathcal{R}_{\alpha\beta}^{q_\alpha \times p_\beta}$ for all α, β. The condition $\mathrm{Ker}\,\varphi \subset \mathrm{rad}\, P$ means that exists no decomposition $P = P' \oplus P''$ such that $P'' \neq 0$ and $\varphi(P'') = 0$.

Hence, the problem (b) reduces to

(c) the problem of classifying $\underline{q} \times \underline{p}$ matrices $\Phi = [\Phi_{\alpha\beta}]_{\alpha,\beta=1}^r$, $\Phi_{\alpha\beta} \in \mathcal{R}_{\alpha\beta}^{q_\alpha \times p_\beta}$, up to transformations

$$\Phi \longmapsto C\Phi S, \tag{39}$$

where $C = [C_{\alpha\beta}]_{\alpha,\beta=1}^r$ and $S = [S_{\alpha\beta}]_{\alpha,\beta=1}^r$ are invertible $\underline{q} \times \underline{q}$ and $\underline{p} \times \underline{p}$ matrices, $C_{\alpha\beta} \in \Gamma_{\alpha\beta}^{q_\alpha \times q_\beta}$, and $S_{\alpha\beta} \in \Gamma_{\alpha\beta}^{p_\alpha \times p_\beta}$. The matrices Φ must satisfy the condition: there exists no transformation (39) making a zero column in Φ.

Every element of $\Gamma_{\alpha\beta}$ is an upper triangular matrix $a = [a_{ij}]_{i,j=1}^t$; define its submatrix $\bar{a} = [a_{ij}]_{(i,j) \in \mathcal{I}_\alpha \times \mathcal{I}_\beta}$ (by (38), $a_{ij} = 0$ if $(i,j) \notin \mathcal{I}_\alpha \times \mathcal{I}_\beta$). Let $\Phi = [\Phi_{\alpha\beta}]_{\alpha,\beta=1}^r$ with $\Phi_{\alpha\beta} \in \mathcal{R}_{\alpha\beta}^{q_\alpha \times p_\beta}$; replacing every entry a of $\Phi_{\alpha\beta}$ by the matrix $\bar{a}$ and permuting rows and columns to order them in accordance with their position in Γ, we obtain a matrix $\bar{\Phi}$ from $\Gamma_{\underline{m} \times \underline{n}}$, where $m_i := q_\alpha$ if $i \in \mathcal{I}_\alpha$ and $n_j := p_\beta$ if $j \in \mathcal{I}_\beta$. It reduces the problem (c) to

(d) the problem of classifying $\underline{m} \times \underline{n}$ matrices $N \in \mathcal{R}_{\underline{m} \times \underline{n}}$ ($\underline{m}$ and $\underline{n}$ are step-sequences) up to transformations

$$N \mapsto CNS, \quad C \in \Gamma_{\underline{m} \times \underline{m}}^*,\ S \in \Gamma_{\underline{n} \times \underline{n}}^*. \tag{40}$$

The matrices N must satisfy the condition: do not exist an equivalence class $\mathcal{I} \in T/\sim$ and a transformation (40) making zero the first column in all the ith vertical strips with $i \in \mathcal{I}$.

By Lemma 2.3, the problem (d) is the linear matrix problem given by the pair $(\Gamma \oplus \Gamma,\ 0 \diagdown \mathcal{R})$ with an additional condition on the transformed matrices: they do not reduce to a block-direct sum with a zero summand whose size has the form $\underline{n} \times \underline{n}$, $\underline{n} = (0, \ldots, 0, n_{t+1}, \ldots, n_{2t})$. $\square$

Remark 2.2. The number of nonisomorphic indecomposable modules over an algebra Γ is finite if and only if the set of nonequivalent $n \times n$ matrices over Γ is finite for every integer n (and even if and only if the set of nonequivalent elements is finite in every algebra Λ that is Morita equivalent [12] to Γ; two elements $a, b \in \Lambda$ are said to be equivalent if $a = xby$ for invertible $x, y \in \Lambda$). It follows from the proof of Theorem 2.4 and the second Brauer–Thrall conjecture [17]: the number of nonisomorphic indecomposable modules over an algebra Λ is infinite if and only if there exist infinitely many nonisomorphic indecomposable Λ-modules of the same dimension. The condition (37) does not imply on the finiteness since every exact sequence (36) is the direct sum of an exact sequence $P_1 \to Q_1 \to M \to 0$ that satisfies it and exact sequences of the form $e_i\Gamma \to e_i\Gamma \to 0 \to 0$ and $e_i\Gamma \to 0 \to 0 \to 0$, where $1 = e_1 + \cdots + e_r$ is a decomposition of $1 \in \Gamma$ into a sum of minimal orthogonal idempotents.

3 Tame and wild matrix problems

3.1 Introduction

In this section, we prove the Tame–Wild Theorem in a form approaching to the Third main theorem from [16].

Generalizing the notions of a quiver and its representations, Roiter [26] introduced the notions of a bocs (=bimodule over category with coalgebra structure) and its representations. For each free triangular bocs, Drozd [10] (see also [9, 11]) proved that the problem of classifying its representations satisfies one and only one of the following two conditions (respectively, is of *tame* or *wild* type): (a) all but a finite number of nonisomorphic indecomposable representations of the same dimension belong to a finite number of one-parametric families, (b) this problem 'contains' the problem of classifying pairs of matrices up to simultaneous similarity. It confirmed a conjecture due to Donovan and Freislich [7] states that every finite dimensional algebra is either tame or wild. Drozd's proof was interpreted by Crawley-Boevey [5, 6]. The authors of [16] got a new proof of the Tame–Wild Theorem for matrix problems given by modules over aggregates and studied a geometric structure of the set of nonisomorphic matrices.

The problem of classifying pairs of matrices up to simultaneous similarity (i.e. representations of the quiver $\circlearrowleft\!\cdot\!\circlearrowright$) is used as a measure of complexity since it 'contains' a lot of matrix problems, in particular, the problem of classifying representations of every quiver. For instance, the classes of isomorphic representations of the quiver (1) correspond, in a one-to-one manner, to the classes of similar pairs of the form

$$\left(\begin{bmatrix} I & 0 & 0 & 0\\ 0 & 2I & 0 & 0\\ 0 & 0 & 3I & 0\\ 0 & 0 & 0 & 4I \end{bmatrix}, \begin{bmatrix} A_\alpha & 0 & 0 & 0\\ A_\beta & 0 & 0 & 0\\ A_\gamma & 0 & 0 & 0\\ A_\delta & A_\varepsilon & I & A_\zeta \end{bmatrix}\right). \tag{41}$$

Indeed, if (J, A) and (J, A') are two similar pairs of the form (41), then $S^{-1}JS = J$, $S^{-1}AS = A'$, the first equality implies $S = S_1 \oplus S_2 \oplus S_3 \oplus S_4$ and equating the (4,3) blocks in the second equality gives $S_3 = S_4$ (compare with Example 1.1).

Let $A_1, \dots, A_p \in k^{m\times m}$. For a parametric matrix $M(\lambda_1, \dots, \lambda_p) = [a_{ij} + b_{ij}\lambda_1 + \dots + d_{ij}\lambda_p]$ $(a_{ij}, b_{ij}, \dots, d_{ij} \in k)$, the matrix that is obtained by replacement of its entries with $a_{ij}I_m + b_{ij}A_1 + \dots + d_{ij}A_p$ will be denoted by $M(A_1, \dots, A_p)$.

In this section, we get the following strengthened form of the Tame–Wild Theorem, which is based on an explicit description of the set of canonical matrices.

Theorem 3.1. *Every linear matrix problem satisfies one and only one of the following two conditions (respectively, is of tame or wild type):*

(I) For every step-sequence $\underline{n}$, the set of indecomposable canonical matrices in the affine space of $\underline{n} \times \underline{n}$ matrices consists of a finite number of points and straight lines[2] of the form $\{L(J_m(\lambda)) \mid \lambda \in k\}$, where $L(x) = [a_{ij} + xb_{ij}]$ is a one-parametric $\underline{l} \times \underline{l}$ matrix $(a_{ij}, b_{ij} \in k,\ \underline{l} = \underline{n}/m)$ and $J_m(\lambda)$ is the Jordan cell. Changing m gives a new line of indecomposable canonical matrices $L(J_{m'}(\lambda))$; the number of points of intersections[3] of the line $L(J_m(\lambda))$ with other lines is the same number

[2] Contrary to [16], these lines are unpunched. As was proved by Thomas Brüstle and the author, the number of points and lines is bounded by 4^d, where $d = \dim(\mathcal{V}_{\underline{n}\times\underline{n}})$. This estimate is bases on an explicit form of canonical matrices given in the proof of Theorem 3.1 and is an essential improvement of the estimate from [4], which was based on the article [16].

[3] Hypothesis: this number is equal to 0.

p for all $m>1$ and it is p or $p+1$ for $m=1$.

(II) *There exists a two-parametric $\underline{n}\times\underline{n}$ matrix $P(x,y)=[a_{ij}+xb_{ij}+yc_{ij}]$ ($a_{ij},b_{ij},c_{ij}\in k$) such that the plane $\{P(a,b)\,|\,a,b\in k\}$ consists only of indecomposable canonical matrices. Moreover, a pair (A,B) of $m\times m$ matrices has the canonical form relatively to simultaneous similarity if and only if $P(A,B)$ is a canonical $m\underline{n}\times m\underline{n}$ matrix.*

We will prove Theorem 3.1 analogously to the proof of the Tame–Wild Theorem in [10]: We reduce an indecomposable canonical matrix M to canonical form (making additional partitions into blocks) and meet a free (in the sense of Section 2.4) block P that is reduced by similarity transformations. If there exist infinitely many values of eigenvalues of P for which we cannot simultaneously make zero all free blocks after P, then the matrix problem satisfies the condition (II). If there is no matrix M with such a block P, then the matrix problem satisfies the condition (I). We will consider the first case in Section 3.3 and the second case in Section 3.4. Two technical lemmas are proved in Section 3.2.

3.2 Two technical lemmas

In this section we give two lemmas, which will be used in the proof of Theorem 3.1.

Lemma 3.1. *Let $L=\lambda I_m+F$ and $R=\mu I_n+G$, where F and G are nilpotent upper triangular matrices. For $A\in k^{m\times n}$ and $f(x,y)=\sum_{i,j\geqslant 0}a_{ij}x^iy^j\in k[x,y]$, define*

$$A^f=\sum_{ij}a_{ij}L^iAR^j. \tag{42}$$

Then

(i) *$(A^f)^g=A^{fg}=(A^g)^f$;*

(ii) *$A^f=\sum b_{ij}F^iAG^j$, where $b_{00}=f(\lambda,\mu)$, $b_{01}=\frac{\partial f}{\partial y}(\lambda,\mu),\dots$;*

(iii) *if $f(\lambda,\mu)=0$, then the left lower entry of A^f is 0;*

(iv) *if $f(\lambda,\mu)\neq 0$, then for every $m\times n$ matrix B there exists a unique A such that $A^f=B$ (in particular, $B=0$ implies $A=0$).*

Proof. (ii) $A^f=\sum a_{ij}(\lambda I+F)^iA(\mu I+G)^j=\sum a_{ij}\lambda^i\mu^jA+\sum a_{ij}\lambda^ij\mu^{j-1}AG+\cdots$.

(iii) It follows from (ii).

(iv) Let $f(\lambda,\mu)\neq 0$ and $A\in k^{m\times n}$. By (ii), $B:=A^f=\sum b_{ij}F^iAG^j$, where $b_{00}=f(\lambda,\mu)$. Then $A=b_{00}^{-1}[B-\sum_{i+j\geqslant 1}b_{ij}F^iAG^j]$. Substituting this equality in its right-hand side gives

$$A=b_{00}^{-1}B-b_{00}^{-2}[\sum_{i+j\geqslant 1}b_{ij}F^iBG^j-\sum_{i+j\geqslant 2}c_{ij}F^iAG^j].$$

Repeating this substitution $m+n$ times, we eliminate A on the right since $F^m=G^n=0$ (recall that F and G are nilpotent). □

Lemma 3.2. *Given $\{f_{ij}\}_{i=1,}^{p}{}_{j=1}^{t} \subset k[x,y]$ and an infinite set $\mathcal{D} \subset k \times k$. For $l \in \{1,\dots,p\}$, $(\lambda,\mu) \in \mathcal{D}$ and a family $\mathcal{F} = \{m,n,L,R,N_1,\dots,N_p\}$, where $m,n \in \mathbb{N}$, L and R are nilpotent upper triangular $m \times m$ and $n \times n$ matrices, $N_1,\dots,N_p$ are $m \times n$ matrices, we define a system of matrix equations (see (42))*

$$\mathcal{S}_l = \mathcal{S}_l(\mathcal{F},\lambda,\nu): \quad X_1^{f_{i1}} + \dots + X_t^{f_{it}} = N_i, \quad i = 1,\dots,l. \tag{43}$$

Then there exist an infinite set $\mathcal{D}' \subset \mathcal{D}$, a nonnegative integer $w \leqslant \min(p,t)$, and pairwise distinct $j_1,\dots,j_{t-w} \in \{1,\dots,t\}$ satisfying the conditions:

(i) For each $\mathcal{F}$ and $(\lambda,\mu) \in \mathcal{D}'$, the system $\mathcal{S}_w$ (empty if $w = 0$) is solvable and every $(t-w)$-tuple $S_{j_1}, S_{j_2},\dots,S_{j_{t-w}} \in k^{m\times n}$ is uniquely completed to its solution $(S_1,\dots,S_t)$.

(ii) If $w < p$ then there exist polynomials $d,\varphi \in k[x,y]$ satisfying $d(\lambda,\mu) = 0 \neq \varphi(\lambda,\mu)$ for all $(\lambda,\mu) \in \mathcal{D}'$ and polynomials $h_1,\dots,h_{t-w} \in k[x,y]$ such that for each $\mathcal{F}$ with $N_1 = \dots = N_w = 0$, each $(\lambda,\mu) \in \mathcal{D}'$, and every solution $(S_1,\dots,S_t)$ of $\mathcal{S}_w$, we have

$$(S_{j_1}^{h_1} + \dots + S_{j_{t-w}}^{h_{t-w}})^d = (S_1^{f_{w+1,1}} + \dots + S_t^{f_{w+1,t}})^\varphi \tag{44}$$

(the left-hand side of (44) is zero if $w = t$).

Proof. Step-by-step, we will simplify the system $\mathcal{S}_p$.

The first step: Let there exist a polynomial f_{1j}, say f_{1t}, that is nonzero on an infinite set $\mathcal{D}_1 \subset \mathcal{D}$. By Lemma 3.1(iv), for each $(\lambda,\mu) \in \mathcal{D}_1$ and every $X_1,\dots,X_{t-1}$ there exists a unique X_t such that the first equation of (43) holds. We restrict ourselves to $\mathcal{D}_1$ and such X_t. Subtracting the f_{it}th power of the first equation of (43) from the f_{1t}th power of the ith equation of (43) for all $i > 1$, we obtain the system

$$X_1^{g_{i1}} + \dots + X_{t-1}^{g_{i,t-1}} = N_i^{f_{1t}} - N_1^{f_{it}}, \quad 2 \leqslant i \leqslant w, \tag{45}$$

where $g_{ij} = f_{ij}f_{1t} - f_{1j}f_{it}$. By Lemma 4.1(iv), the system $\mathcal{S}_p$ and the system (45) supplemented by the first equation of $\mathcal{S}_p$ have the same sets of solutions.

The second step: Let there exist a polynomial g_{2j}, say $g_{2,t-1}$, that is nonzero on an infinite set $\mathcal{D}_2 \subset \mathcal{D}_1$. We eliminate X_{t-1} from the equations (45) with $3 \leqslant i \leqslant w$.

The process stops after a certain wth step, $w < p$, if we obtain a system with the first equation of the form

$$X_{j_1}^{r_1} + \dots + X_{j_{t-w}}^{r_{t-w}} = N, \tag{46}$$

where either there is $r_i \neq 0$ and $r_1(\lambda,\mu) = \dots = r_{t-w}(\lambda,\mu) = 0$ for almost all $(\lambda,\mu) \in \mathcal{D}_w$, or $r_1 = \dots = r_{t-w} = 0$, or $w = t$. In the first case, $r_1,\dots,r_{t-w}$ have a common divisor $d(x,y)$ with an infinity of roots in $\mathcal{D}_w$ (we use the following form of the Bezout theorem [19, Sect. 1.3]: two relatively prime polynomials $f_1, f_2 \in k[x,y]$ of degrees d_1 and d_2 have no more than d_1d_2 common roots); we put $h_i = r_i/d$ and $\mathcal{D}' = \{(\lambda,\mu) \in \mathcal{D}_w \,|\, d(\lambda,\mu) = 0\}$. In the second and third cases, the left-hand side of (46) is zero; we put $h_i = 0$ (in the second case), $d = 0$ and $\mathcal{D}' = \mathcal{D}_w$. To prove (44), we apply this process to the system $\mathcal{S}_w$ with $N_1 = \dots = N_w = 0$ supplemented by the equality $X_1^{f_{w+1,1}} + \dots + X_t^{f_{w+1,t}} = X_1^{f_{w+1,1}} + \dots + X_t^{f_{w+1,t}}$. $\square$

3.3 Proof of Theorem 3.1 for wild problems

A subblock of a free (dependent) block will be named a free (dependent) subblock. In this section, we consider a matrix problem given by $(\Gamma, \mathcal{M})$ such that there exists a semi-parametric canonical matrix $M \in \mathcal{M}_{\underline{n}\times\underline{n}}$ having a free box $M_q \neq \emptyset$ with the following property:

$$\text{The horizontal or the vertical } (q-1)\text{-strip of } M_q \text{ is linked (see Definition 1.9) to a } (q-1)\text{-strip containing an } \textit{infinite parameter} \text{ from a free box } M_v,\ v<q, \text{ (i.e., the domain of parameters contains infinitely many vectors with distinct values of this parameter).} \tag{47}$$

We choose such $M \in \mathcal{M}_{\underline{n}\times\underline{n}}$ having the smallest $\sum \underline{n} = n_1 + n_2 + \cdots$ and take its free box $M_q \neq \emptyset$ that is the first with the property (47). Then each $(q-1)$-strip of M is linked to the horizontal or the vertical $(q-1)$-strip containing M_q. Our purpose is to prove that the matrix problem satisfies the condition (II) of Theorem 3.1. Let each of the boxes $M_q,\ M_{q+1}, \dots$ that is free be replaced by 0, and let as many as possible parameters in the boxes $M_1, \dots, M_{q-1}$ be replaced by elements of k (correspondingly we retouch dependent boxes and narrow down the domain of parameters $\mathcal{D}$) such that the property (47) still stands (note that all the parameters of a "new" semi-parametric canonical matrix M are infinite and that $M_q = 0$ but $M_q \neq \emptyset$). The following three cases are possible:

Case 1: The horizontal and the vertical $(q-1)$-strips of M_q are linked to $(q-1)$-strips containing parameters λ_l and λ_r respectively, but not the same parameter.

Case 2: The horizontal or the vertical $(q-1)$-strip of M_q is not linked to all $(q-1)$-strips containing a parameter.

Case 3: The horizontal and the vertical $(q-1)$-strips of M_q are linked to $(q-1)$-strips containing the same parameter λ.

3.3.1 Study Case 1

By Theorem 2.2, the minimality of $\sum \underline{n}$, and since each $(q-1)$-strip of M is linked to a $(q-1)$-strip containing M_q, we have that M is a two-parametric matrix (hence $l, r \in \{1, 2\}$) and, up to permutation of $(q-1)$-strips, it has the form $\hat{H}_l \oplus \hat{H}_r$, where $\hat{H}_l = H_l(J_{s_l}(\lambda_l I))$ and $\hat{H}_r = H_r(J_{s_r}(\lambda_r I))$ lie in the intersection of all $(q-1)$-strips linked to the horizontal and, respectively, the vertical $(q-1)$-strip of M_q, $H_l(a)$ and $H_r(a)$ are indecomposable canonical matrices for all $a \in k$, and

$$J_s(\lambda) := \begin{bmatrix} \lambda I & I & & 0 \\ & \lambda I & \ddots & \\ & & \ddots & I \\ 0 & & & \lambda I \end{bmatrix}.$$

We will assume that the parameters λ_1 and λ_2 are enumerated such that the free boxes M_u and M_v containing λ_1 and, respectively, λ_2 satisfy $u \leqslant v$ (clearly, M_u and M_v are Weir matrices).

Let first $u < v$. Then

$$M_u = A \oplus J_{s_1}(\lambda_1 I) \oplus B, \quad M_v = J_{s_2}(\lambda_2 I), \tag{48}$$

where A and B lie in $\hat{H}_2$ (M_v does not contain summands from $\hat{H}_1$ since each box M_i, $i > u$, that is reduced by similarity transformations belongs to $\hat{H}_1$ or $\hat{H}_2$).

By the $\underline{n}^\star \times \underline{n}^\star$ *partition* of M into blocks $M^\star_{ij}$ (which will be called $\star$-*blocks* and the corresponding strips will be called $\star$-*strips*), we mean the partition obtained from the partition into $(v-1)$-strips by removing the divisions inside of $J_{s_1}(\lambda_1 I)$ and the corresponding divisions inside of the horizontal and vertical $(u-1)$-strips of M_u and all $(u-1)$-strips that are linked with them. Clearly, $J_{s_1}(\lambda_1 I)$ and $J_{s_2}(\lambda_2 I)$ are free $\star$-blocks, the other $\star$-blocks are zero or scalar matrices, and M_q is a part of a $\star$-block. Denote by $\mathcal{I}$ (resp. $\mathcal{J}$) the set of indices of $\star$-strips of $\hat{H}_l$ (resp. $\hat{H}_r$) in $M = [M^\star_{ij}]^e_{i,j=1}$, then $\mathcal{I} \cup \mathcal{J} = \{1, \dots, e\}$ and $\mathcal{I} \cap \mathcal{J} = \varnothing$.

Step 1 (a selection of $M^\star_{\zeta\eta}$). On this step we will select a free $\star$-block $M^\star_{\zeta\eta} > M_v$ with $(\zeta, \eta) \in \mathcal{I} \times \mathcal{J}$ and an infinite set of $(a, b) \in \mathcal{D}$ such that $M^\star_{\zeta\eta}$ cannot be made arbitrary by the transformations of $M(a, b)$ preserving both $M_1, \dots, M_v$ and all $M^\star_{ij} < M^\star_{\zeta\eta}$. Such $M^\star_{\zeta\eta}$ exists since $M_q \neq \emptyset$ is a part of a free $M^\star_{ij}$ with $(i, j) \in \mathcal{I} \times \mathcal{J}$.

Denote by Λ_0 the algebra of all S from $\Lambda := \Gamma_{\underline{n} \times \underline{n}}$ for which MS and SM are coincident on the places of the boxes $M_1, \dots, M_v$ (see (22)). Then the transformation

$$M \longmapsto M' = SMS^{-1}, \quad S \in \Lambda_0^*, \tag{49}$$

preserves $M_1, \dots, M_v$. Note that Λ_0 is an algebra of upper block-triangular $\underline{n}^\star \times \underline{n}^\star$ (and even $\underline{n}_v \times \underline{n}_v$) matrices.

Let $M^\star_{\zeta\eta}$ be selected and let a transformation (49) preserve all $M^\star_{ij} < M^\star_{\zeta\eta}$. Equating the (ζ, η) $\star$-blocks in the equality $M'S = SM$ gives

$$M^\star_{\zeta 1} S^\star_{1\eta} + \dots + M^\star_{\zeta,\eta-1} S^\star_{\eta-1,\eta} + M^{\star\prime}_{\zeta\eta} S^\star_{\eta\eta} = S^\star_{\zeta\zeta} M^\star_{\zeta\eta} + \dots + S^\star_{\zeta e} M^\star_{e\eta}, \tag{50}$$

where $e \times e$ is the number of $\star$-blocks in M. Since

$$M^\star_{ij} \neq 0 \quad \text{implies} \quad (i, j) \in (\mathcal{I} \times \mathcal{I}) \cup (\mathcal{J} \times \mathcal{J}), \tag{51}$$

the equality (50) may contain $S^\star_{ij}$ only if $(i, j) \in \mathcal{I} \times \mathcal{J}$ or $(i, j) = (\eta, \eta)$, hence $M^{\star\prime}_{\zeta\eta}$ is fully determined by M, $S^\star_{\eta\eta}$ and the family of $\star$-blocks

$$S^\star_{\mathcal{I}\mathcal{J}} := \{S^\star_{ij} \mid (i, j) \in \mathcal{I} \times \mathcal{J}\}.$$

We will select $M^\star_{\zeta\eta}$ in the sequence

$$F_1 < F_2 < \dots < F_\delta \tag{52}$$

of all free $M^\star_{ij}$ such that $(i, j) \in \mathcal{I} \times \mathcal{J}$ and $M^\star_{ij} \not\subset M_1 \cup \dots \cup M_v$. For $\alpha \in \{1, \dots, \delta\}$ denote by Λ_α the algebra of all $S \in \Lambda_0$ for which MS and SM coincide on the places of all $M^\star_{ij} \leqslant F_\alpha$. Then the transformation

$$M \longmapsto M' = SMS^{-1}, \quad S \in \Lambda_\alpha^*, \tag{53}$$

preserves $M_1, \dots, M_v$ and all $M^\star_{ij} \leqslant F_\alpha$.

Let us investigate the family $S^\star_{\mathcal{I}\mathcal{J}}$ for each $S \in \Lambda_\alpha^*$.

The algebra $\Lambda = \Gamma_{\underline{n}\times\underline{n}}$ consists of all $\underline{n}\times\underline{n}$ matrices $S=[S_{ij}]$ whose blocks satisfy a system of linear equations of the form (5)–(6) completed by $S_{ij}=0$ $(i>j)$. Let us rewrite this system for more small $\star$-blocks $S^{\star}_{ij}$. The equations that contain blocks from the family $S^{\star}_{\mathcal{I}\mathcal{J}}$ contain no blocks $S^{\star}_{ij}\notin S^{\star}_{\mathcal{I}\mathcal{J}}$. (Indeed, by the definition of Case 1, the $(q-1)$-strips of $\hat{H}_1$ are not linked to the $(q-1)$-strips of $\hat{H}_2$, so the partition into $\mathcal{I}$ and $\mathcal{J}$ is in agreement with the partition $T/\sim$; see (2) and Definition 1.9.) Hence the family $S^{\star}_{\mathcal{I}\mathcal{J}}$ for $S\in\Lambda^{\star}$ is given by a system of equations of the form

$$\sum_{(i,j)\in\mathcal{I}\times\mathcal{J}} \alpha^{(\tau)}_{ij} S^{\star}_{ij} = 0, \quad \tau = 1,\dots,w_1. \tag{54}$$

Denote by $\mathcal{B}_u$ (resp. $\mathcal{B}_{uv}$) the part of M consisting of all entries that are in the intersection of $\cup_{i\leqslant u}M_i$ (resp. $\cup_{u<i\leqslant v}M_i$; by the union of boxes we mean the part of the matrix formed by these boxes) and $\cup_{(i,j)\in\mathcal{I}\times\mathcal{J}}M^{\star}_{ij}$. Let us prove that $\mathcal{B}_u$ and $\mathcal{B}_{uv}$ are unions of $\star$-blocks $M^{\star}_{ij}$, $(i,j)\in\mathcal{I}\times\mathcal{J}$. It is clear for $\mathcal{B}_u$ since the partition into $\star$-strips is a refinement of the partition into $(u-1)$-strips. It is also true for $\mathcal{B}_{uv}$ since $\mathcal{B}_{uv}$ is partitioned into rectangular parts (see Definition 1.8) of the form $[M_{\tau+1}|M_{\tau+2}|\cdots|M_{\tau+s_1}]$ if $r=1$ and $[M^T_{\tau_1}|M^T_{\tau_2}|\cdots|M^T_{\tau_{s_1}}]^T$ if $l=1$ (the indices l and r were defined in the formulation of Case 1); recall that M_i are boxes and M_u has the form (48).

By the definition of the algebra Λ_0, it consists of all $S\in\Lambda$ such that $MS-SM$ is zero on the places of the boxes $M_i\leqslant M_v$. To obtain conditions on the family $S^{\star}_{\mathcal{I}\mathcal{J}}$ of blocks of $S\in\Lambda_0$, by virtue of the statement (51), it suffices to equate zero the blocks of $MS-SM$ on the places of all free $\star$-blocks $M^{\star}_{ij}$ from $\mathcal{B}_u$ and $\mathcal{B}_{uv}$ (notice that some of them may satisfy $M^{\star}_{ij}>M^{\star}_{\zeta\eta}$). Since all free $\star$-blocks of M except for $J_{s_1}(\lambda_1 I)$ and $J_{s_2}(\lambda_2 I)$ are scalar or zero matrices, we obtain a system of equalities of the form

$$\sum_{(i,j)\in\mathcal{I}\times\mathcal{J}} \alpha^{(\tau)}_{ij} S^{\star}_{ij} = 0, \quad \tau = w_1+1,\dots,w_2, \tag{55}$$

for the places from $\mathcal{B}_u$ and

$$\sum_{(i,j)\in\mathcal{I}\times\mathcal{J}} (S^{\star}_{ij})^{g^{(\nu)}_{ij}} = 0, \quad \nu = 1,\dots,w_3, \tag{56}$$

for the places from $\mathcal{B}_{uv}$, where $(S^{\star}_{ij})^{g^{(\nu)}_{ij}}$, $g^{(\nu)}_{ij}\in k[x,y]$ (more precisely, $g^{(\nu)}_{ij}\in k[x]$ if $l=1$ and $g^{(\nu)}_{ij}\in k[y]$ if $r=1$), are given by (42) with $L=J_{s_l}(\lambda_l I)$ and $R=J_{s_r}(\lambda_r I)$.

Applying the Gauss-Jordan elimination algorithm to the system (54)–(55), we choose $S_1,\dots,S_t\in S^{\star}_{\mathcal{I}\mathcal{J}}$ such that they are arbitrary and the other $S^{\star}_{ij}\in S^{\star}_{\mathcal{I}\mathcal{J}}$ are their linear combinations. Rewriting the system (56) for $S_1,\dots,S_t$, we obtain a system of equalities of the form

$$S_1^{f_{i1}}+\cdots+S_t^{f_{it}} = 0, \quad i=1,\dots,w_3. \tag{57}$$

The algebra Λ_α $(1\leqslant\alpha\leqslant\delta)$ consists of all $S\in\Lambda_0$ such that SM and MS have the same blocks on the places of all free $M^{\star}_{xy}\leqslant F_\alpha$:

$$M^{\star}_{x1}S^{\star}_{1y}+\cdots+M^{\star}_{xy}S^{\star}_{yy} = S^{\star}_{xx}M^{\star}_{xy}+\cdots+S^{\star}_{xe}M^{\star}_{ey}. \tag{58}$$

We may omit the equalities (58) for all (x, y) such that $M^\star_{xy}$ is contained in $M_1, \dots, M_v$ (by the definition of Λ_0), or $(x, y) \notin \mathcal{I} \times \mathcal{J}$ (by (51), the equality (58) contains $S^\star_{ij} \in S^\star_{\mathcal{I}\mathcal{J}}$ only if $(x, y) \in \mathcal{I} \times \mathcal{J}$). The remaining equalities (58) correspond to (zero) $M^\star_{xy} \in \{F_1, \dots, F_\alpha\}$ and takes the form

$$S_1^{f_{i1}} + \dots + S_t^{f_{it}} = 0, \quad i = w_3 + 1, \dots, w_3 + \alpha. \tag{59}$$

It follows from the preceding that any sequence of matrices $S_1, \dots, S_t$ is the sequence of the corresponding blocks of a matrix $S \in \Lambda_\alpha$ if and only if the system (57)∪(59) holds for $S_1, \dots, S_t$.

Putting $\alpha = \delta$ (see (52)), $p = w_3 + \delta$ and applying Lemma 3.2 to the system (57)∪(59), we get an infinite $\mathcal{D}' \subset \mathcal{D}$, $w \leqslant \min(p, t)$, and pairwise distinct $j_1, \dots, j_{t-w} \in \{1, \dots, t\}$ satisfying the conditions (i)–(ii) of Lemma 3.2. Since $M_q \neq \emptyset$ is a part of a free $M^\star_{ij}$ with $(i, j) \in \mathcal{I} \times \mathcal{J}$, for every $(\lambda, \nu) \in \mathcal{D}$ we may change its right-hand part to obtain an unsolvable system; therefore $w < p$. We take F_{w+1-w_3} as the desired block $M^\star_{\zeta\eta}$. Since M_q is the first among free boxes $\neq \emptyset$ with the property (47), $M^\star_{\zeta\eta} > M_v$. The equality (50) takes the form

$$S_1^{f_{w+1,1}} + \dots + S_t^{f_{w+1,t}} = S^\star_{\zeta\zeta} M^\star_{\zeta\eta} - M^{\star\prime}_{\zeta\eta} S^\star_{\eta\eta}. \tag{60}$$

Step 2 (a construction of $P(x, y)$). On this step, we construct a two-parametric matrix $P(x, y)$ from the condition (I) of Theorem 3.1.

Recall that M is a two-parametric matrix with the domain of parameters $\mathcal{D} \subset k^2$. Let us fix a pair $(a_1, a_2) \in \mathcal{D}'$ in the following manner. If $d = 0$, then (a_1, a_2) is an arbitrary pair from $\mathcal{D}'$. Let $d \neq 0$, then $d \in k[x, y]$ is a polynomial with an infinity of roots (a_l, a_r) for which $(a_1, a_2) \in \mathcal{D}'$, we may choose an irreducible $d(x, y)$. Furthermore, $\mathcal{D}'$ does not contain infinitely many pairs (a_1, a_2) with the same a_l (otherwise, the lth parameter can be replaced with a_l, but we have already replaced as many as possible parameters by elements of k such that the property (47) still stands); it follows $d \notin k[x]$ and so $d'_y := \partial d / \partial y \neq 0$. Since d is an irreducible polynomial, $(d, d'_y) = 1$; by the Bezout theorem (see the proof of Lemma 3.2), we may chose $(a_1, a_2) \in \mathcal{D}'$ such that

$$d(a_l, a_r) = 0, \quad d'_y(a_l, a_r) \neq 0. \tag{61}$$

We denote by $P(x, y)$ the matrix that is obtained from M by replacement of its $\star$-blocks $J_{s_l}(\lambda_l I)$ and $J_{s_r}(\lambda_r I)$ with Weir matrices

$$L := \mathit{\Pi}(J_1(a_l) \oplus J_3(a_l) \oplus J_5(a_l) \oplus J_7(a_l) \oplus J_9(a_l))\mathit{\Pi}^{-1}, \; R := J_5(a_r I_2) \tag{62}$$

(where $\mathit{\Pi}$ is a permutation matrix, see Theorem 1.2) and the $\star$-block $M^\star_{\zeta\eta}$ with

$$P^\star_{\zeta\eta} = \mathit{\Pi} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{bmatrix}, \quad Q_i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 \\ T_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 1 & y \\ 1 & x \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{63}$$

where Q_i is $(2i-1)$-by-10 (its zero blocks are 1-by-2) and T_i is in the middle row. (Each nonzero free $\star$-block $M^\star_{ij}$ of M, except for $J_{s_l}(\lambda_l I)$ and $J_{s_r}(\lambda_r I)$, is a scalar matrix with

$(i,j) \in (\mathcal{I}\times\mathcal{I}) \cup (\mathcal{J}\times\mathcal{J})$; it is replaced by the scalar matrix $P^{\star}_{ij}$ with the same diagonal having the size $(1+3+5+7+9)\times(1+3+5+7+9)$ if $(i,j)\in\mathcal{I}\times\mathcal{I}$ and 10×10 if $(i,j)\in\mathcal{J}\times\mathcal{J}$.) The dependent blocks are respectively corrected by formulas (35).

Let us enumerate the rows and columns of $J = J_1(a_l)\oplus J_3(a_l)\oplus J_5(a_l)\oplus J_7(a_l)\oplus J_9(a_l)$ and the rows of $[Q_i]_{i=1}^5$ by the pairs of numbers $\langle 1,1\rangle$; $\langle 3,1\rangle$, $\langle 3,2\rangle$, $\langle 3,3\rangle$; $\langle 5,1\rangle$, $\langle 5,2\rangle,\dots,\langle 5,5\rangle;\dots;\langle 9,1\rangle$, $\langle 9,2\rangle,\dots,\langle 9,9\rangle$. Going over to the matrix P, we have permuted them in $L = \varPi J\varPi^{-1}$ and $P^{\star}_{\zeta\eta} = \varPi[Q_i]$ in the following order:

$$\begin{gathered}\langle 9,1\rangle,\langle 7,1\rangle,\langle 5,1\rangle,\langle 3,1\rangle,\langle 1,1\rangle,\langle 9,2\rangle,\langle 7,2\rangle,\langle 5,2\rangle,\langle 3,2\rangle,\\ \langle 9,3\rangle,\langle 7,3\rangle,\langle 5,3\rangle,\langle 3,3\rangle,\langle 9,4\rangle,\langle 7,4\rangle,\langle 5,4\rangle,\\ \langle 9,5\rangle,\langle 7,5\rangle,\langle 5,5\rangle,\langle 9,6\rangle,\langle 7,6\rangle,\langle 9,7\rangle,\langle 7,7\rangle,\langle 9,8\rangle,\langle 9,9\rangle\end{gathered} \tag{64}$$

(see Section 1.3). In the same manner, we will enumerate the rows and columns in every ith $\star$-strip ($i\in\mathcal{I}$) of $P(x,y)$.

We will prove that $P(x,y)$ satisfies the condition (II) of Theorem 3.1. Let (W,B) be a canonical pair of $m\times m$ matrices under simultaneous similarity; put

$$K = P(W,B) \tag{65}$$

and denote by $\bar{Q}_i,\ \bar{T}_i,\ \bar{L},\ \bar{R}$ the blocks of K that correspond to $Q_i,\ T_i,\ L,\ R$ (see (62)) from $P(x,y)$:

$$\bar{L} = \bar{\varPi}\bar{J}\bar{\varPi}^{-1},\quad \bar{R} = J_5(a_r I_{2m}), \tag{66}$$

$$\bar{J} := J_1(a_l I_m)\oplus J_3(a_l I_m)\oplus J_5(a_l I_m)\oplus J_7(a_l I_m)\oplus J_9(a_l I_m), \tag{67}$$

where $\bar{\varPi}$ is a permutation matrix. It suffices to show that K is a canonical matrix (i.e., K is stable relatively to the algorithm of Section 1.4). To prove it, we will construct the partition of K into boxes.

Clearly, the boxes $M_1,\dots,M_u$ of M convert to the boxes $K_1,\dots,K_u$ of K. The box M_v of M is replaced by the box $K_{\bar{v}}$ of K. The numbers v and $\bar{v}$ (i.e., the numbers of boxes in the sequences $M_{u+1},\dots,M_{v-1}$ and $K_{u+1},\dots,K_{\bar{v}-1}$) may be distinct since M_u and K_u may have distinct numbers of cells. The part $K_1\cup\dots\cup K_{\bar{v}}$ of K is in canonical form. The partition of K obtained after reduction of $K_1,\dots,K_{\bar{v}}$ is the partition into $\bar{v}$-strips; the corresponding blocks will be called *$\bar{v}$-blocks*; for instance, $\bar{T}_1,\dots,\bar{T}_5$ are $\bar{v}$-blocks.

The transformations of K that preserve the boxes $K_1,\dots,K_{\bar{v}}$ are

$$K\longmapsto K' = SKS^{-1},\quad S\in\bar{\Lambda}_0^*. \tag{68}$$

For every matrix S from the algebra $\bar{\Lambda}_0$, the family $\mathcal{S}^{\star}_{\mathcal{I}\mathcal{J}}$ of its $\star$-blocks satisfies the system (54)–(56), so $S_1,\dots,S_t\in S^{\star}_{\mathcal{I}\mathcal{J}}$ (which correspond to $S_1,\dots,S_t$ for $S\in\Lambda_0$) are arbitrary satisfying the equations (57) and the other $S^{\star}_{ij}\in S^{\star}_{\mathcal{I}\mathcal{J}}$ are their linear combinations.

Step 3. We prove the following statement:

> Let $p\in\{1,\dots,5\}$ and let the matrix K be reduced by those transformations (68) that preserve all $\bar{v}$-blocks preceding $\bar{T}_p$. Then $\bar{T}_p$ is transformed into $\bar{T}'_p = A_p\bar{T}_pB$, where A_p is an arbitrary nonsingular matrix and B is a nonsingular matrix for which there exist nonsingular matrices $A_{p+1},\dots,A_5$ satisfying $\bar{T}_{p+1} = A_{p+1}\bar{T}_{p+1}B,\dots,\bar{T}_5 = A_5\bar{T}_5B$. (69)

The rows and columns of $P(x,y)$ convert to the *substrips* of $K = P(W,B)$. For every $i \in \mathcal{I}$, we have enumerated the rows and columns in the ith $\star$-strip of $P(x,y)$ by the pairs (64); we will use the same indexing for the substrips in the ith $\star$-strip of K.

By analogy with (50), equating in $K'S = SK$ (see (68)) the blocks on the place of $K^\star_{\zeta\eta}$ gives

$$K^{\star\prime}_{\zeta 1}S^\star_{1\eta} + \cdots + K^{\star\prime}_{\zeta\eta}S^\star_{\eta\eta} = S^\star_{\zeta\zeta}K^\star_{\zeta\eta} + \cdots + S^\star_{\zeta e}K^\star_{e\eta} \tag{70}$$

For p from (69) and $i \in \mathcal{I}$, we denote by $C_{\zeta i}$, $\hat{K}'_{\zeta i}$, $\hat{K}_{\zeta i}$ (resp. $D_{\zeta i}$) the matrices that are obtained from $S^\star_{\zeta i}$, $K^{\star\prime}_{\zeta i}$, $K^\star_{\zeta i}$ (resp. $K^\star_{\zeta i}$) by deletion of all horizontal (resp., horizontal and vertical) substrips except for the substrips indexed by $\langle 2p-1, p\rangle$, $\langle 2p-1, p+1\rangle, \dots, \langle 2p-1, 2p-1\rangle$. Then (70) implies

$$\hat{K}'_{\zeta 1}S^\star_{1\eta} + \cdots + \hat{K}'_{\zeta\eta}S^\star_{\eta\eta} = C_{\zeta\zeta}K^\star_{\zeta\eta} + \cdots + C_{\zeta e}K^\star_{e\eta}. \tag{71}$$

The considered in (69) transformations (68) preserve all $\bar{v}$-blocks preceding $\bar{T}_p$. Since $\bar{T}_p$ is a $\bar{v}$-block from the $\langle 2p-1, p\rangle$ substrip of the ζth horizontal $\star$-strip whose substrips are ordered by (64), the block $\hat{K}_{\zeta i}$ $(i < \eta)$ is located in a part of K preserved by these transformations, that is $\hat{K}'_{\zeta i} = \hat{K}_{\zeta i}$. If $\eta > i \in \mathcal{I}$, then $\hat{K}'_{\zeta i}S^\star_{i\eta} = D_{\zeta i}C_{i\eta}$ since $K^\star_{\zeta i}$ is a scalar matrix or $\bar{L}$ (see (66)). If $\eta > i \in \mathcal{J}$, then $\hat{K}'_{\zeta i} = K^\star_{\zeta i} = 0$. So the equality (71) is presented in the form

$$\sum_{i=1}^{\eta-1} D_{\zeta i}C_{i\eta} + \hat{K}'_{\zeta\eta}S^\star_{\eta\eta} = C_{\zeta\zeta}K^\star_{\zeta\eta} + \sum_{i=\zeta+1}^{e} C_{\zeta i}K^\star_{i\eta}. \tag{72}$$

The equality (72) contains C_{ij} only if $(i,j) \in \mathcal{I} \times \mathcal{J}$, so each of them is a part of $S^\star_{ij} \in S^\star_{\mathcal{I}\mathcal{J}}$. We have chosen $S_1, \dots, S_t$ in $S^\star_{\mathcal{I}\mathcal{J}}$ such that they are arbitrary and the others are their linear combinations; let $C_1, \dots, C_t$ be the corresponding parts of $S_1, \dots, S_t$. It is easy to show that $C_1, \dots, C_t$ satisfy the system that is obtained from (57)$\cup$(59) with $w_3 + \alpha = w+1$ by replacing $S_1, \dots, S_t$ with $C_1, \dots, C_t$. Each $D_{\zeta i}$ in (72) is a scalar or zero matrix if $K^\star_{\zeta i}$ is not $\bar{L}$ and $D_{\zeta i}$ is $J_p(a_l I_m)$ otherwise, each $K^\star_{i\eta}$ $(i < \zeta)$ is a scalar or zero matrix or $\bar{R} = J_5(a_r I_{2m})$, so the equality (72) may be rewritten in the form

$$C_1^{f_{w+1,1}} + \cdots + C_t^{f_{w+1,t}} = C_{\zeta\zeta}K^\star_{\zeta\eta} - \hat{K}'_{\zeta\eta}S^\star_{\eta\eta}, \tag{73}$$

where $f_{w+1,j}$ are the same as in (60) and $C_i^{f_{w+1,i}}$ is defined by (42) with $L = J_p(a_l I_m)$ and $R = J_5(a_r I_{2m})$.

By (44),

$$(C_{j_1}^{h_1} + \cdots + C_{j_{t-w}}^{h_{t-w}})^d = (C_1^{f_{w+1,1}} + \cdots + C_t^{f_{w+1,t}})^\varphi, \quad \varphi(a_l, a_r) \neq 0. \tag{74}$$

Put $C := 0$ if $d = 0$ and $C := C_{j_1}^{h_1/d} + \cdots + C_{j_{t-w}}^{h_{t-w}/d}$ if $d \neq 0$. By (73) and (74),

$$C^d = (C_{\zeta\zeta}K^\star_{\zeta\eta} - \hat{K}'_{\zeta\eta}S^\star_{\eta\eta})^\varphi. \tag{75}$$

Let us study the right-hand side of (75). Since $\zeta \in \mathcal{I}$ and $\eta \in \mathcal{J}$, the blocks $S^\star_{\zeta\zeta}$ and $S^\star_{\eta\eta}$ are arbitrary matrices satisfying

$$S^\star_{\zeta\zeta}\bar{L} = \bar{L}S^\star_{\zeta\zeta}, \quad S^\star_{\eta\eta}\bar{R} = \bar{R}S^\star_{\eta\eta}. \tag{76}$$

By (66) and (76), $Z := \bar{\Pi}^{-1} S^{\star}_{\zeta\zeta} \bar{\Pi}$ commutes with $\bar{J}$. Let us partition Z into blocks Z_{ij} $(i,j=1,\dots,5)$ and $X := \bar{\Pi}^{-1}(S^{\star}_{\zeta\zeta} K^{\star}_{\zeta\eta} - K^{\star\prime}_{\zeta\eta} S^{\star}_{\eta\eta}) = Z[\bar{Q}_i] - [\bar{Q}'_i] S^{\star}_{\eta\eta}$ (recall that $K^{\star}_{\zeta\eta} = \bar{\Pi}[\bar{Q}_i]$) into horizontal strips $X_1,\dots,X_5$ in accordance with the partition of $\bar{J}$ into diagonal blocks $J_1(a_l I_m)$, $J_3(a_l I_m),\dots,J_9(a_l I_m)$ (see (67)). Then

$$X_p = Z_{p1}\bar{Q}_1 + \cdots + Z_{p5}\bar{Q}_5 - Q'_p S^{\star}_{\eta\eta}. \tag{77}$$

Since Z commutes with $\bar{J}$, $Z_{pi} J_{2i-1}(a_l I_m) = J_{2p-1}(a_l I_m) Z_{pi}$. Hence Z_{pi} has the form

$$\begin{bmatrix} A_i & & & * \\ & A_i & & \\ & & \ddots & \\ & & & A_i \\ & & & \\ 0 & & & \end{bmatrix} \text{ if } p \geqslant i \text{ and } \begin{bmatrix} & A_i & & & * \\ & & A_i & & \\ & & & \ddots & \\ 0 & & & & A_i \end{bmatrix} \text{ if } p \leqslant i.$$

We look at

$$\bar{Q}_i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 \\ \bar{T}_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ to get } X_p = \begin{bmatrix} * & * & * & * & * \\ \hdashline * & * & * & * & * \\ A_p\bar{T}_p - \bar{T}'_p B & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

where A_p is the diagonal $m \times m$ block of Z_{pp} and B is the diagonal $2m \times 2m$ block of $S^{\star}_{\eta\eta}$ (recall that $S^{\star}_{\eta\eta}$ commutes with $J_5(a_r I_{2m})$). Since X_p is formed by the substrips of $S^{\star}_{\zeta\zeta} K^{\star}_{\zeta\eta} - K^{\star\prime}_{\zeta\eta} S^{\star}_{\eta\eta}$ indexed by the pairs $\langle 2p-1, 1\rangle, \dots, \langle 2p-1, 2p-1\rangle$, the equality (75) implies

$$R := C^d = \begin{bmatrix} A_p\bar{T}_p - \bar{T}'_p B & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\varphi}. \tag{78}$$

Let us prove that

$$A_p \bar{T}_p = \bar{T}'_p B. \tag{79}$$

If $d = 0$, then the equality (79) follows from (78), $\varphi(a_l, a_r) \neq 0$ (see (74)) and Lemma 3.1(iv). Let $d \neq 0$. We partition C and $R = C^d$ into $p \times 5$ blocks C_{ij} and R_{ij} conformal to the block form of the matrix on the right-hand side of (78). By Lemma 3.1(ii) and (78),

$$R = C^d = \sum_{ij} d_{ij} J_p(0I_m)^i C J_5(0I_{2m})^j, \tag{80}$$

$$R_{ij} = 0 \ \text{ if } \ i > 1, \tag{81}$$

where $d_{00} = d(a_l, a_r) = 0$ and $d_{01} = d'_y(a_l, a_r) \neq 0$ (see (61)). Hence $R_{p1} = 0$, it proves (79) for $p = 1$. Let $p \geqslant 2$, then $R_{p2} = d_{01} C_{p1} = 0$ by (80) and $C_{p1} = 0$ by (81). Next, $R_{p3} = d_{01} C_{p2} = 0$ by (80) and $C_{p2} = 0$ by (81), and so on until obtain

$C_{p1} = \cdots = C_{p4} = 0$. By (80), $R_{p-1,1} = 0$, it proves (79) for $p = 2$. Let $p \geqslant 3$, then $R_{p-1,2} = d_{01}C_{p-1,1} = 0$ and $C_{p-1,1} = 0$; further, $R_{p-1,3} = d_{01}C_{p-1,2} = 0$ and $C_{p-1,2} = 0$, and so on until obtain $C_{p-1,1} = \cdots = C_{p-1,3} = 0$. Therefore, $R_{p-2,1} = 0$; we have (79) for $p = 3$ and $C_{p-2,1} = C_{p-2,2} = 0$ otherwise; Analogously, we get (79) for $p = 4$ and $C_{p-3,1} = 0$ otherwise, and, at last, (79) for $p = 5$.

By (69), the considered transformation preserves all $\bar{v}$-blocks preceding $\bar{T}_l$, so we may repeat this reasoning for each $l \in \{p+1, \dots, 5\}$ instead of p and obtain $A_l\bar{T}_l = \bar{T}_l B$. It proves (69).

Step 4 (a construction of $K_{\bar{v}+1}, K_{\bar{v}+2}, \dots$). The boxes $K_1, \dots, K_{\bar{v}}$ were constructed at the end of Step 2. The first nonzero free $\bar{v}$-block of K that is not contained in $K_1 \cup \cdots \cup K_{\bar{v}}$ is $\bar{T}_5 = [0_m\, I_m]$. The preceding it $\bar{v}$-blocks that are not contained in $K_1 \cup \cdots \cup K_{\bar{v}}$ are zero, so they are the boxes $K_{\bar{v}+1}, \dots, K_{v_1-1}$ for a certain $v_1 \in \mathbb{N}$. By the statement (69), the admissible transformations with K that preserve the boxes $K_1, \dots, K_{v_1-1}$ reduce, for $\bar{T}_5$, to the equivalence transformations; therefore, $\bar{T}_5 = [0_m\, I_m]$ is canonical and $K_{v_1} = \bar{T}_5$.

Conformal to the block form of $K_{v_1} = [0_m\, I_m]$, we divide each $\bar{v}$-block of K into two v_1-blocks. The first nonzero free v_1-block that is not contained in $K_1 \cup \cdots \cup K_{v_1}$ is I_m from $\bar{T}_4 = [I_m\, 0_m]$. The preceding it v_1-blocks that are not contained in $K_1 \cup \cdots \cup K_{v_1}$ are the boxes $K_{v_1+1}, K_{v_2}, \dots, K_{v_2-1}$ for a certain $v_2 \in \mathbb{N}$. By the statement (69), the admissible transformations with K that preserve the boxes $K_1, \dots, K_{v_2-1}$ reduce, for $\bar{T}_4$, to the transformations of the form

$$\bar{T}_4 \longmapsto A\bar{T}_4 \begin{bmatrix} B & C \\ 0 & B \end{bmatrix}$$

with nonsingular $m \times m$ matrices A and B. Since the block $\bar{T}_4 = [I_m\, 0_m]$ is canonical under these transformations, we have $\bar{T}_4 = [I_m\, 0_m] = [K_{v_2} | K_{v_2+1}]$; and so on until we get the partition of K into boxes.

It remains to consider the case $u = v$, that is the parameters λ_1 and λ_2 are parameters of a certain free box M_v. Since λ_1 and λ_2 are distinct (by prescribing of Case 1) parameters of the same Weir matrix M_v, $a_1 \neq a_2$ for all (a_1, a_2) from the domain of parameters $\mathcal{D} \subset k^2$. We will assume that the parameters λ_1 and λ_2 are enumerated such that there exists $(a_1, a_2) \in \mathcal{D}$ with $a_1 \prec a_2$, then by Definition 1.2 of Weir matrices $a_1 \prec a_2$ for all $(a_1, a_2) \in \mathcal{D}$. By the minimality of $\sum \underline{n}$, $M_v = J_{s_1}(\lambda_1 I) \oplus J_{s_2}(\lambda_2 I)$, all $(v-1)$-strips are linked, and $M = H(M_v) = \hat{H}_l \oplus \hat{H}_r$, where $H(a)$ is an indecomposable canonical matrix for all $a \in k$, $\hat{H}_l := H(J_{s_l}(\lambda_l I))$ and $\hat{H}_r := H(J_{s_r}(\lambda_r I))$ (i.e. $H = H_l = H_r$, see the beginning of Section 3.3.1). By the $\underline{n}^\star \times \underline{n}^\star$ *partition* of M into blocks $M^\star_{ij}$, we mean the partition into $(v-1)$-strips supplemented by the division of every $(v-1)$-strip into two substrips in accordance with the partition of M_u into subblocks $J_{s_1}(\lambda_1 I)$ and $J_{s_2}(\lambda_2 I)$. Then $J_{s_1}(\lambda_1 I)$ and $J_{s_2}(\lambda_2 I)$ are free $\star$-blocks, the other $\star$-blocks are zero or scalar matrices, and M_q is a part of a $\star$-block. The reasoning in this case is the same as in the case $u < v$ (but with $\mathcal{B}_{uv} = \varnothing$).

3.3.2 Study Case 2

In this case, $M = M(\lambda)$ is a one-parametric matrix with an infinite domain of parameters $\mathcal{D} \subset k$. Up to permutation of $(q-1)$-strips, M has the form $\hat{H}_1 \oplus \hat{H}_2$, where $H_1(a)$ and H_2 are indecomposable canonical matrices for all $a \in k$, $\hat{H}_1 := H_1(J_{s_1}(\lambda I))$, and $\hat{H}_2$ is obtained from H_2 by replacement of its elements h_{ij} with $h_{ij}I_{s_2}$. The matrix $J_{s_1}(\lambda I)$ is a part of M_v (see (47)). Let $l, r \in \{1,2\}$ be such that the horizontal $(q-1)$-strip of M_q crosses $\hat{H}_l$ and its vertical $(q-1)$-strip crosses $\hat{H}_r$. Under the $\star$-partition of M, we mean the partition obtained from the $(q-1)$-partition by removing the divisions inside of $J_{s_1}(\lambda I)$ and the corresponding divisions inside of the horizontal and vertical $(v-1)$-strips of M_v and all $(v-1)$-strips that are linked with them; then M_v is a $\star$-block. Denote by $\mathcal{I}$ (resp. $\mathcal{J}$) the set of indices of $\star$-strips of $\hat{H}_l$ (resp. $\hat{H}_r$) in M.

Let M_z be the last nonzero free box of M (clearly, $z \geqslant v$). Denote by $\mathcal{B}$ the part of M consisting of all entries that are in the intersection of $\cup_{i \leqslant z} M_i$ and $\cup_{(i,j)\in \mathcal{I}\times\mathcal{J}} M^\star_{ij}$. By analogy with Case 1, $\mathcal{B}$ is a union of $\star$-blocks $M^\star_{ij}$, $(i,j) \in \mathcal{I}\times\mathcal{J}$.

Let Λ_0 be the algebra of all $S \in \Lambda$ such that $MS - SM$ is zero on the places of the boxes $M_i \leqslant M_z$. Equating zero the blocks of $MS - SM$ on the places of all free $\star$-blocks $M^\star_{ij}$ from $\mathcal{B}$, we obtain a system of equalities of the form (55)–(56) with $g^{(\tau)}_{ij} \in k[x]$ if $l = 1$ and $g^{(\tau)}_{ij} \in k[y]$ if $l = 2$ ((56) only if $z > v$) for $\star$-blocks of $S = [S^\star_{ij}] \in \Lambda_0$ from the family $S^\star_{\mathcal{IJ}} := \{S^\star_{ij} \,|\, (i,j) \in \mathcal{I}\times\mathcal{J}\}$. Solving the system (54)–(55), we choose $S_1, \dots, S_t \in S^\star_{\mathcal{IJ}}$ such that they are arbitrary and the others are their linear combinations, then we present the system (56) in the form (57).

Let $F_1 < F_2 < \cdots < F_\delta$ be the sequence of all free $M^\star_{ij}$ such that $M^\star_{ij} \not\subset M_1 \cup \cdots \cup M_z$ and $(i,j) \in \mathcal{I}\times\mathcal{J}$, and let Λ_α ($\alpha \in \{1, \dots, \delta\}$) denote the algebra of all $S \in \Lambda_0$ for which MS and SM are coincident on the places of all $M^\star_{ij} \leqslant F_\alpha$; it gives additional conditions (59) on $S^\star_{\mathcal{IJ}}$.

By analogy with Case 1, the transformation (53) preserves all M_i with $i \leqslant z$ and all $M^\star_{ij} \leqslant F_\alpha$; moreover, any sequence of matrices $S_1, \dots, S_t$ is the sequence of the corresponding blocks of a matrix $S \in \Lambda_\alpha$ if and only if the system (57)$\cup$(59) holds.

Putting $\alpha = \delta$, $p = w_3 + \delta$ and applying Lemma 3.2 (more exactly, its analog for $\mathcal{D} \subset k$ and $f_{ij} \in k[x]$ or $f_{ij} \in k[y]$) to (57)$\cup$(59), we get an infinite $\mathcal{D}' \subset \mathcal{D}$, $w \leqslant \min(p,t)$, and $j_1, \dots, j_{t-w} \in \{1, \dots, t\}$ satisfying the conditions (i)–(ii) of Lemma 3.2. Moreover, $w < p$ and the polynomial $d \in k[x] \cup k[y]$ from the condition (ii) is zero since it is zero on the infinite set $\mathcal{D}'$.

Let us fix $a_1, \dots, a_5 \in \mathcal{D}'$, $a_1 \prec a_2 \prec \cdots \prec a_5$ (with respect to the ordering in k, see the beginning of Section 1.3), and denote by $P(x,y)$ the matrix that is obtained from M by replacement of

(i) its $\star$-block $J_{s_1}(\lambda I)$ with $\operatorname{diag}(a_1, a_2, \dots, a_5)$,

(ii) all entries $h_{ij}I_{s_2}$ of $\hat{H}_2$ with $h_{ij}I_2$, and

(iii) $M^\star_{\zeta\eta}$ with T (see (63)) if $l = 1$ and with

$$\begin{bmatrix} 1 & 0 & 1 & x & y \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{if } l = 2,$$

and by the corresponding correction of dependent blocks. Similar to Case 1, we can prove that $P(x,y)$ satisfies the condition (II) of Theorem 3.1.

3.3.3 Study Case 3

The free box M_v is a Weir matrix that is similar to $J_{s_1}(\lambda I) \oplus J_{s_2}(\lambda I)$ ($s_1 \neq s_2$) or $J_s(\lambda I)$, hence it has the form $M_v = \lambda I + F$, where F is a nilpotent upper triangular matrix. Clearly, $M = M(\lambda)$ is a one-parametric matrix with an infinite domain of parameters $\mathcal{D} \subset k$; moreover, $M = H(M_v)$, where $H(a)$ ($a \in k$) is an indecomposable canonical matrix. Under the $\star$-partition we mean the partition into $(v-1)$-strips (then M_v is a $\star$-block).

Step 1 (a construction of $P(x,y)$). Let Λ_{-1} (resp. Λ_0) be the algebra of all $S\in\Lambda$ such that $MS-SM$ is zero on the places of the boxes $M_i<M_v$ (resp. $M_i\leqslant M_v$). Then Λ_{-1} is a reduced $\underline{n}^\star\times\underline{n}^\star$ algebra whose equivalence relation (2) in $T^\star=\{1,\dots,e\}$ is full (i.e. every two elements are equivalent). The blocks of $S\in\Lambda_{-1}$ satisfy a system of equations of the form

$$S^\star_{11}=S^\star_{22}=\cdots=S^\star_{ee}, \tag{82}$$

$$\sum_{i<j}c^{(l)}_{ij}S^\star_{ij}=0,\quad l=1,2,\dots,q_{T^\star T^\star} \tag{83}$$

(see (6)). Solving the system (83), we choose $S_1,\dots,S_t\in\{S^\star_{ij}\,|\,i<j\}$ such that they are arbitrary and the other $S^\star_{ij}$ $(i<j)$ are their linear combinations. The algebra Λ_0 consists of all $S\in\Lambda_{-1}$ for with $S^\star_{11}M_v=M_vS^\star_{11}$.

Let $F_1<F_2<\cdots<F_\delta$ be the sequence of all free $M^\star_{ij}\not\subset M_1\cup\cdots\cup M_v$, and let Λ_α $(\alpha\in\{1,\dots,\delta\})$ denote the algebra of all $S\in\Lambda_0$ for which MS and SM are coincident on the places of all $M^\star_{ij}\leqslant F_\alpha$; it gives conditions on S_i of the form

$$S_1^{f_{i1}}+\cdots+S_t^{f_{it}}=0,\quad i=1,\dots,\alpha, \tag{84}$$

where $f_{ij}\in k[x,y]$ and $S_i^{f_{ij}}$ is defined by (42) with $L=R=M_v$.

Putting $p=\delta$, taking $\{(a,a)\,|\,a\in\mathcal{D}\}$ instead of $\mathcal{D}$ and applying Lemma 3.2 to (84) with $\alpha:=\delta$, we get an infinite $\mathcal{D}'\subset\mathcal{D}$, $w\leqslant\min(p,t)$, $j_1,\dots,j_{t-w}\in\{1,\dots,t\}$, and $d,\varphi,h_1,\dots,h_{t-w}\in k[x,y]$. Since $d(a,a)=0$ for all $a\in\mathcal{D}'$ and $\mathcal{D}'$ is infinite, $d(x,y)$ is divisible by $x-y$ by the Bezout theorem (see the proof of Lemma 3.2). We may take

$$d(x,y)=x-y. \tag{85}$$

Let us fix an arbitrary $a\in\mathcal{D}'$ and denote by $P(x,y)$ the matrix that is obtained from M by replacement of its $\star$-blocks M_v and $M^\star_{\zeta\eta}$ with

$$P_v=\begin{bmatrix}aI_2&0&I_2&0\\0&aI_1&0&0\\0&0&aI_2&I_2\\0&0&0&aI_2\end{bmatrix}\quad\text{and}\quad P^\star_{\zeta\eta}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\T&0&0&0\\0&Q&0&0\end{bmatrix}, \tag{86}$$

where

$$Q=\begin{bmatrix}1\\0\end{bmatrix},\quad T=\begin{bmatrix}x&y\\1&0\end{bmatrix}, \tag{87}$$

and by the corresponding correction of dependent blocks. (P_v is a Weir matrix that is similar to $J_1(a)\oplus J_3(aI_2)$.) We prove that $P(x,y)$ satisfies the condition (II) of Theorem 3.1. Let (W,B) be a canonical pair of $m\times m$ matrices under simultaneous similarity, put $K=P(W,B)$ and denote by $\bar{Q}$ and $\bar{T}$ the blocks of K that correspond to Q and T. It suffices to show that K is a canonical matrix.

Step 2 (a construction of $K_1,\dots,K_{v_1}$). The boxes $M_1,\dots,M_v$ of M become the boxes $K_1,\dots,K_v$ of K.

Let us consider the algebra $\bar{\Lambda}_{-1}$ for the matrix K. For each $S \in \bar{\Lambda}_{-1}$, its $\star$-blocks satisfy the system (83), so we may choose $S_1, \dots, S_t \in \{S^\star_{ij} \,|\, i < j\}$ (on the same places as for Λ_{-1}) that are arbitrary and the other $S^\star_{ij}$ $(i < j)$ are their linear combinations. A matrix $S \in \bar{\Lambda}_{-1}$ belongs to $\bar{\Lambda}_0$ if and only if $S^\star_{11} = S^\star_{22} = \cdots$ (see (82)) commutes with K_v, that is

$$S^\star_{11} = S^\star_{22} = \cdots = S^\star_{ee} = \begin{bmatrix} A_0 & B_2 & A_1 & A_2 \\ & B_0 & 0 & B_1 \\ & & A_0 & A_1 \\ 0 & & & A_0 \end{bmatrix} \tag{88}$$

by (86) and by analogy with Example 1.4.

The first nonzero free v-block of K that is not contained in $K_1 \cup \cdots \cup K_v$ is $\bar{Q}$ (see (87)). The preceding it v-blocks that are not contained in $K_1 \cup \cdots \cup K_v$ are the boxes $K_{v+1}, \dots, K_{v_1 - 1}$ for a certain $v_1 \in \mathbb{N}$.

The block $\bar{Q}$ is reduced by the transformations

$$K \longmapsto K' = SKS^{-1}, \quad S \in \bar{\Lambda}^*_0 \tag{89}$$

with the matrix K, which preserve its boxes $K_1, \dots, K_v$. Each $\star$-strip of $P(x, y)$ consists of 7 rows or columns (since $P_v \in k^{7\times 7}$, see (86)); they become the *substrips* of the corresponding $\star$-strip of K. Denote by C_{ij}, $\hat{K}'_{ij}$, $\hat{K}_{ij}$ (resp. D_{ij}) the matrices that are obtained from $S^\star_{ij}$, $K^{\star\prime}_{ij}$, $K^\star_{ij}$ (resp. $K^\star_{ij}$) by elimination of the first 5 horizontal (resp., horizontal and vertical) substrips; notice that $\bar{Q}$ is contained in the remaining 6th and 7th substrips of $K^\star_{\zeta\eta}$. The equation (70) implies (71). Since all $K^\star_{ij} < K^\star_{\zeta\eta}$ are upper triangular, the equation (71) implies (72).

The equality (72) takes the form (73), where $C_1, \dots, C_t$ are the corresponding parts of $S_1, \dots, S_t$; $f_{w+1,j}$ are the same as in (84) and $C_j^{f_{w+1,j}}$ is defined by (42) with $L = aI_{2m}$ (a part of K_v) and $R = K_v$.

By (44) and (85),

$$(C_{j_1}^{h_1} + \cdots + C_{j_{t-w}}^{h_{t-w}})^{x-y} = (C_1^{f_{w+1,1}} + \cdots + C_t^{f_{w+1,t}})^\varphi, \quad \varphi(a, a) \neq 0.$$

Setting $C := C_{j_1}^{h_1/(x-y)} + \cdots + C_{j_{t-w}}^{h_{t-w}/(x-y)}$ and using (73), we obtain

$$C^{x-y} = (C_{\zeta\zeta} K^\star_{\zeta\eta} - \hat{K}'_{\zeta\eta} S^\star_{\eta\eta})^\varphi. \tag{90}$$

As follows from the form of the second matrix in (86) and from (88),

$$S^\star_{\zeta\zeta} K^\star_{\zeta\eta} - K^{\star\prime}_{\zeta\eta} S^\star_{\eta\eta} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ A_0\bar{T} - \bar{T}' A_0 & * & * & * \\ 0 & A_0 \bar{Q} - \bar{Q}' B_0 & 0 & * \end{bmatrix}. \tag{91}$$

Looking at the form of the matrix K_v (see (86)), we have

$$(S^\star_{\zeta\zeta} K^\star_{\zeta\eta} - K^{\star\prime}_{\zeta\eta} S^\star_{\eta\eta})^\varphi = \varphi(a, a) \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ A_0\bar{T} - \bar{T}' A_0 & * & * & * \\ 0 & A_0 \bar{Q} - \bar{Q}' B_0 & 0 & * \end{bmatrix}$$

and

$$K_v D - D K_v = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ D_{41} & * & * & * \\ 0 & 0 & -D_{41} & * \end{bmatrix} \tag{92}$$

for an arbitrary block matrix $D = [D_{ij}]$. So the equality (90) can be presented in the form

$$\begin{bmatrix} 0 & 0 & -D_{41} & * \end{bmatrix} = \varphi(a,a) \begin{bmatrix} 0 & A_0\bar{Q} - \bar{Q}'B_0 & 0 & * \end{bmatrix}, \tag{93}$$

where $C = [D_{41}\, D_{42}\, D_{43}\, D_{44}]$. Since $\varphi(a,a) \neq 0$, it follows $A_0\bar{Q} - \bar{Q}'B_0 = 0$ and $\bar{Q}' = A_0\bar{Q}B_0^{-1}$. Therefore, the block $\bar{Q}$ is reduced by elementary transformations. Since $\bar{Q} = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$ is canonical, $K_{v_1} := \bar{Q}$ is a box.

Step 3 (a construction of $K_{v_1+1}, \dots, K_{v_2}$). The partition into v_1-strips coincides with the partition into substrips, so the v_1-blocks are the subblocks of K corresponding to the entries of P. The first nonzero free subblock of K that is not contained in $K_1 \cup \dots \cup K_{v_1}$ is $\bar{T}_{21} = I_m$ from $\bar{T} = [\bar{T}_{ij}]_{i,j=1}^2$. The preceding it subblocks that are not contained in $K_1 \cup \dots \cup K_{v_1}$ are the boxes $K_{v_1+1}, \dots, K_{v_2-1}$ for a certain $v_2 \in \mathbb{N}$.

Let a transformation (89) preserve the boxes $K_1, \dots, K_{v_2-1}$. Denote by C_{ij}, $\hat{K}'_{ij}$, $\hat{K}_{ij}$ (resp. D_{ij}) the matrices that are obtained from $S^\star_{ij}, K^{\star\prime}_{ij}$, $K^\star_{ij}$ (resp. $K^\star_{ij}$) by elimination of the first 4 horizontal (resp., horizontal and vertical) substrips; notice that $\bar{T}_{21} = I_m$ is contained in the 5th horizontal substrip of $K^\star_{\zeta\eta}$. Let $C_1, \dots, C_t$ be the corresponding parts of $S_1, \dots, S_t$. Similar to Step 2, we have the equalities (73) and (90). As follows from (91) and (92), the equality (90) may be presented in the form

$$\begin{bmatrix} (D_{41})_2 & * & * & * \\ 0 & 0 & -D_{41} & * \end{bmatrix} = \varphi(a,a) \begin{bmatrix} (A_0\bar{T} - \bar{T}'A_0)_2 & * & * & * \\ 0 & A_0\bar{Q} - \bar{Q}'B_0 & 0 & * \end{bmatrix} \tag{94}$$

(compare with (93)), where $(D_{41})_2$ and $(A_0\bar{T} - \bar{T}'A_0)_2$ are the lower substrips of D_{41} and $A_0\bar{T} - \bar{T}'A_0$. It follows that $A_0\bar{Q} - \bar{Q}'B_0 = 0$, $D_{41} = 0$, and so $(A_0\bar{T} - \bar{T}'A_0)_2 = 0$. But $\bar{Q} = \bar{Q}' = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$, hence

$$A_0 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \tag{95}$$

and we have $A_{22}\bar{T}_{21} - \bar{T}'_{21}A_{11} = 0$, so $\bar{T}_{21}$ is reduced by equivalence transformations. Therefore, $\bar{T}_{21} = I_m$ is canonical and $K_{v_2} = \bar{T}_{21} = I_m$.

Step 4 (a construction of K_{v_2}, $K_{v_2+1}, \dots$). The partition into v_2-strips coincides with the partition into substrips. The first nonzero free subblock of K that is not contained in $K_1 \cup \dots \cup K_{v_2}$ is $\bar{T}_{11} = W$ from $\bar{T}$. The preceding it subblocks that are not contained in $K_1 \cup \dots \cup K_{v_2}$ are the boxes $K_{v_2+1}, \dots, K_{v_3-1}$ for a certain $v_3 \in \mathbb{N}$.

Let a transformation (89) preserve the boxes $K_1, \dots, K_{v_3-1}$. Denote by C_{ij}, $\hat{K}'_{ij}$, $\hat{K}_{ij}$ (resp. D_{ij}) the matrices that are obtained from $S^\star_{ij}$, $K^{\star\prime}_{ij}$, $K^\star_{ij}$ (resp. $K^\star_{ij}$) by elimination of the first 3 horizontal (resp., horizontal and vertical) substrips. In this case, instead of (94) we get the equality

$$\begin{bmatrix} D_{41} & * & * & * \\ 0 & 0 & -D_{41} & * \end{bmatrix} = \varphi(a,a) \begin{bmatrix} A_0\bar{T} - \bar{T}'A_0 & * & * & * \\ 0 & A_0\bar{Q} - \bar{Q}'B_0 & 0 & * \end{bmatrix},$$

so $A_0\bar{T} - \bar{T}'A_0 = 0$, where A_0 is of the form (95). Since $[\bar{T}_{21}\, \bar{T}_{22}] = [\bar{T}'_{21}\, \bar{T}'_{22}] = [I_m\, 0_m]$, we have $A_{11} = A_{22}$ and $A_{12} = 0$, so $A_{11}\bar{T}_{11} - \bar{T}'_{11}A_{11} = 0$ and $\bar{T}_{11}$ is reduced by similarity transformations. Since $\bar{T}_{11} = W$ is a Weir matrix, it is canonical and $K_{v_3} = W$.

Furthermore, $A_{11}\bar{T}_{12} - \bar{T}'_{12}A_{11} = 0$, where A_{11} commutes with W, hence $\bar{T}_{12} = B$ is canonical too. It proves that K is a canonical matrix.

3.4 Proof of Theorem 3.1 for tame problems

In this section, we consider a matrix problem given by $(\Gamma, \mathcal{M})$, for which there exists no semi-parametric canonical matrix M having a free box $M_q \neq \emptyset$ with the property (47). Our purpose is to prove that the matrix problem satisfies the condition (I) of Theorem 3.1.

Let $\underline{n}$ be a step-sequence. By Remark 2.1, the number of parametric canonical $\underline{n} \times \underline{n}$ matrices is finite. Let M be a parametric canonical $\underline{n} \times \underline{n}$ matrix and one of its parameters is a finite parameter λ; that is, the set of λ-components in the domain of parameters is a finite set $\{a_1, \dots, a_r\}$. Putting $\lambda = a_1, \dots, a_r$ gives r semi-parametric canonical matrices. Repeating this procedure, we obtain a finite number of semi-parametric canonical $\underline{n} \times \underline{n}$ matrices having only infinite parameters.

Let M be an indecomposable semi-parametric canonical $\underline{n} \times \underline{n}$ matrix that has no finite parameters but has an infinite parameter, and let M_v be the first among its boxes with parameters (then M_v is free). By the property (47), if a v-strip is linked with a v-strip containing a parameter λ from M_v, then it does not contain a free box $M_i > M_v$ such that $M_i \neq \emptyset$. Since M is indecomposable, it follows that all its v-strips are linked, all free boxes $M_i > M_v$ are equal to $\emptyset$, and $M_v = J_m(\lambda)$. Hence, all free v-blocks excepting M_v are scalar matrices and $M = L(J_m(\lambda))$, where $L(\lambda) = [a_{ij} + \lambda b_{ij}]$ is a semi-parametric canonical matrix with a free 1×1 box $M_v = [\lambda]$ and all free boxes after if are 1×1 matrices of the form $\emptyset$.

Let $\mathcal{D}_m \subset k$ be the domain of parameters of M. By the property (47), $\mathcal{D}_m$ is a cofinite set (i.e. $k \setminus \mathcal{D}_m$ is finite).

If $a \notin \mathcal{D}_m$, then the matrix $M(a)$ is canonical, but there exists a free box $M_q > M_v$ such that $M_q \neq \emptyset$. This box M_q is the zero 1×1 matrix; since M is indecomposable, all its rows and columns are linked, so M_q is reduced by similarity transformations. Replacing it by the parametric box $[\mu]$, we obtain a straight line of indecomposable canonical matrices that intersects $\{M(\lambda) \,|\, \lambda \in k\}$ in the point $M(a)$. Hence, each $M(a)$, $a \notin \mathcal{D}_m$, is a point of intersection with a straight line of indecomposable canonical matrices.

Let $M(a)$, $a \in \mathcal{D}_m$, be a point of intersection too; that is, there exists a line $\{N(\mu) \,|\, \mu \in k\}$ of indecomposable canonical matrices such that $M(a) = N(b)$ for a certain $b \in k$. Then $M(\lambda)$ has a free box M_u $(u < v)$ that is a Weir matrix, b is its eigenvalue, and $N(\mu)$ is obtained from $M(a)$ by replacement of b with μ. Since $M(\lambda)$ and $N(\mu)$ coincide on $M_1 \cup \dots \cup M_{u-1}$, $M_u = N_u$ for $\mu = b$. By analogy with the structure of $M(\lambda)$, all free boxes $N_i > N_u$ are zero, hence $M_v = 0$ if $\lambda = a$. Since $M_v = J_m(\lambda)$, $M(a)$ with $a \in \mathcal{D}_m$ can be a point of intersection only if $m = 1$ and $\lambda = 0$.

Replacing m by an arbitrary integer n gives a new semi-parametric canonical matrix $L(J_n(\lambda))$ with the domain of parameters $\mathcal{D}_n$. To prove that the condition (I) of Theorem 3.1 holds, it suffices to show that $\mathcal{D}_m = \mathcal{D}_n$. Moreover, it suffices to show that $\mathcal{D}_m = \mathcal{D}_1$.

Let first $a \in \mathcal{D}_1$. By analogy with Section 3.3.3, under the $\star$-partition we mean the partition into $(v-1)$-strips. Then $a \in \mathcal{D}_m$ if and only if all free $\star$-blocks after M_v in $M(a)$ are $\emptyset$. The $\star$-blocks of every $S \in \Lambda^*_{-1}$ (see Section 3.3.3) satisfy the system (82)–(83), where c^l_{ij} do not depend on m and a. Solving the system (83), we choose $S_1, \dots, S_t \in \{S^\star_{ij} \,|\, i < j\}$ such that they are arbitrary and the other $S^\star_{ij}$ $(i<j)$ are their linear combinations.

Let $F_1 < F_2 < \dots < F_\delta$ be the sequence of all free $M^\star_{ij} \not\subset M_1 \cup \dots \cup M_v$ and let K be obtained from M by replacing $F_1, \dots, F_\delta$ with arbitrary $m \times m$ matrices $G_1, \dots, G_\delta$. To prove that $a \in \mathcal{D}_m$, we must show that $F_1 = \dots = F_\delta = \emptyset$ for $M(a)$; that is, there exists $S \in \Lambda^*_0$ such that $G'_1 = \dots = G'_\delta = 0$ in $K' := SKS^{-1}$. It suffices to consider the case $G_1 = \dots = G_{q-1} = 0 \neq G_q$ $(q \in \{1, \dots, \delta\})$ and to show that there exists $S \in \Lambda^*_{-1}$ with $S^\star_{11} = S^\star_{22} = \dots = I_m$ (then $S \in \Lambda^*_0$) such that $G'_1 = \dots = G'_{q-1} = G'_q = 0$. It means that the $\star$-blocks $S_1, \dots, S_t$ of S satisfy a system of equations that is obtained by equating in $K'S = SK$ the blocks on the places of $G_1, \dots, G_q$:

$$S_1^{f_{l1}} + \dots + S_t^{f_{lt}} = 0, \quad l = 1, \dots, q-1, \tag{96}$$

$$S_1^{f_{q1}} + \dots + S_t^{f_{qt}} = G_q^\varphi, \tag{97}$$

where $\varphi(a,a) \neq 0$ and $S_j^{f_{ij}}$ is defined by (42) with $L = R = J_m(a)$. Notice that the polynomials f_{ij} are the same for all $m \in \mathbb{N}$ and a.

Taking 1 instead of m, we obtain the system

$$f_{l1}(a,a)s_1 + \dots + f_{lt}(a,a)s_t = 0, \quad l = 1, \dots, q-1, \tag{98}$$

$$f_{q1}(a,a)s_1 + \dots + f_{qt}(a,a)s_t = g \tag{99}$$

and, since $a \in \mathcal{D}_1$, the system (98)–(99) is solvable with respect to $s_1, \dots, s_t$ for all $g \in k$. It holds for all q, so the rows of $F := [f_{ij}(a,a)]$ are linearly independent.

Let $S_r = [s^{(r)}_{ij}]^m_{i,j=1}$ and $G^\varphi_q = [g_{ij}]^m_{i,j=1}$. Since $L = R = J_m(a)$, the system of q matrix equations (96)–(97) is equivalent to the m^2 systems of q linear equations relatively to the entries of $S_1, \dots, S_t$, each of them is obtained by equating the (i,j) entries for the corresponding $i, j \in \{1, \dots, m\}$ and has the form:

$$f_{l1}(a,a)s^{(1)}_{ij} + \dots + f_{lt}(a,a)s^{(t)}_{ij} = d^{(l)}_{ij}, \quad l = 1, \dots, q, \tag{100}$$

where $d^{(l)}_{ij}$ is a linear combination of $s^{(1)}_{i'j'}, \dots, s^{(t)}_{i'j'}$, $(i',j') \in \{(1,j), \dots, (i-1,j)\} \cup \{(i,j+1), (i,j+2), \dots\}$, and (only if $l = q$) g_{ij}. Since the rows of $F = [f_{ij}(a,a)]$ are linearly independent, the system (100) for $(i,j) = (m,1)$ is solvable. Let $\bar{s}_{m1} = (\bar{s}^{(1)}_{m1}, \dots, \bar{s}^{(t)}_{m1})$ be its solution. Knowing $\bar{s}_{m1}$, we calculate $d^{(l)}_{m-1,1}$ and $d^{(l)}_{m2}$ and solve the system (100) for $(i,j) = (m-1,1)$ and for $(i,j) = (m,2)$. Then we calculate $d^{(l)}_{ij}$, $i - j = m-2$, and solve (100) for $(i,j) = (m-2,1)$, $(m-1,2)$, $(m,3)$, and so on, until obtain a solution $\bar{S}_1, \dots, \bar{S}_t$ of (96), a contradiction. Hence $a \in \mathcal{D}_m$. Clearly also that $a \in \mathcal{D}_m$ implies $a \in \mathcal{D}_1$. It proves Theorem 3.1.

Remark 3.1. Basing on the proof of Theorem 3.1, we can give a more comprehensive than in this theorem description of the set of canonical matrices. For simplicity, we restrict ourselves to the case $\mathcal{M} = k^{t \times t}$.

Namely, a linear matrix problem given by a pair $(\Gamma, k^{t\times t})$ satisfies one and only one of the following two conditions (respectively, is of tame or wild type):

(I) For every step-sequence $\underline{n}$, there exist a finite set of semi-parametric canonical $\underline{n}\times\underline{n}$ matrices $M_{\underline{n},i}(\lambda)$, $1 \leqslant i \leqslant t_{\underline{n}}$, (in which one entry is the parameter λ and the other entries are elements of k, $[\lambda]$ is a box and all boxes after it are 1×1 boxes of the form $\emptyset$) such that their domains of parameters $\mathcal{D}_{\underline{n},i}$ are cofinite subsets in k and

  (a) for every m, the semi-parametric canonical $m\underline{n} \times m\underline{n}$ matrices $M_{\underline{n},i}(J_m(\lambda))$ (in which all boxes $\leqslant J_m(\lambda)$ correspond to the boxes $\leqslant [\lambda]$ of $M_{\underline{n},i}(\lambda)$ and all boxes after $J_m(\lambda)$ are 1×1 boxes of the form $\emptyset$) have the same domain of parameters $\mathcal{D}_{\underline{n},i}$,

  (b) for every $\underline{n}'$, the set of matrices of the form $M_{\underline{n},i}(J_m(a))$, $m\underline{n} = \underline{n}'$, $a \in \mathcal{D}_{\underline{n},i}$, is a cofinite subset in the set of indecomposable structured canonical $\underline{n}' \times \underline{n}'$ matrices.

(II) There exists an $\underline{n} \times \underline{n}$ matrix $P(\alpha, \beta)$ (in which two entries are the parameters α and β and the other entries are elements of k) such that

  (a) two pairs of $m \times m$ matrices (A, B) and (C, D) are similar if and only if $P(A, B) \simeq P(C, D)$,

  (b) if $(W,\, B)$ is a canonical pair of $m\times m$ matrices under similarity (see Definition 1.4), then $P(W,\, B)$ is a canonical $m\underline{n} \times m\underline{n}$ matrix.

The author wishes to thanks P. Gabriel, Yu. A. Drozd, C. M. Ringel and A. V. Roiter for the opportunity to speak at their seminars and stimulating discussions. The author is grateful to G. R. Belitskiĭ, T. Brüstle, S. Friedland and D. I. Merino for helpful suggestions and comments.

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