Towards an Algorithmic Construction of the Hyperbolic Structure on Surface Bundles*

Heinz Helling Christof Menzel

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1 Introduction

The following theorem is well known:

Theorem (Thurston). Let $S_g^{(r)}$ be an orientable compact surface of Euler–characteristic less than 0. Then the following are equivalent:

(a) $f \in \mathrm{Map}(S_g^{(r)})$ is pseudo–Anosov.

(b) f is neither periodic nor reducible.

(c) The action on Teichmüller–space of $S_g^{(r)}$ induced by f has one (and therefore exactly two) arational fixed points on the boundary of Thurston's compactification of Teichmüller–space.

(d) The fiber bundle $M_f(S_g^{(r)}) = S_g^{(r)} \times [0,1]/(x,0) \sim (f(x),1)$ is hyperbolic.

Take one of (b), (c), (d) as definition for the term "pseudo–Anosov", then the other two are highly nontrivial theorems.

In certain special cases there are versions of (parts of) this theorem that are much easier to prove:

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Theorem. (See e.g. [Casson–Bleiler 1988]) In the case of a closed surface (i.e. $r = 0$) the following implication holds:

If $\chi_{A_f}(t)$, the characteristic polynomial of the Anosov–monodromy A_f of f,

- is irreducible over the integers,
- has no roots of unity as zero, and
- is not a polynomial in a nontrivial power $t^n (n > 1)$ of t,

then f is pseudo–Anosov.

Remarks.

1. The converse might not be true, so by using this theorem you might miss an arbitrarily large number of hyperbolic surface bundles.

2. The theorem uses the fact that the first homology group of the closed surface can be expressed as a direct sum of the first homology groups of parts of the surface (after filling in some punctures). So the behaviour of null–homologous essential closed curves on the surface can be controlled by imposing the premises. This direct–sum–decomposition doesn't work in general (i.e. for $r > 0$), whence in general this easy–to–check criterion is useless.

Of course the characteristic polynomial of the Anosov–monodromy can be used to check whether any essential closed curve that is not null–homologous is mapped to itself by some power of the automorphism of f. So after having ruled out this you might use the following

Theorem. Assume χ_{A_f} has no root of unity as zero. If f is (nonperiodic) reducible, then there is a product of less then g commutators in $\pi_1(S_g^{(r)})$ that is mapped to a conjugate of its own by some power (at most $g!$) of f_*.

This is true since a null–homologous curve C cuts $S_g^{(r)}$ into pieces one of which is a once–punctured surface $S_{g'}^{(1)}$ of a genus g' greater than 0 and less than g. Then C can be expressed as product of commutators of generating elements of $\pi_1(S_{g'}^{(1)})$.

Solving this problem, however, amounts to solving the conjugacy problem.

In case of fiber bundles with fiber the once–punctured torus we have the following

Theorem. Let $r = g = 1$. Then $M_f(S_g^{(r)})$ is hyperbolic (f pseudo–Anosov) if and only if the Anosov–monodromyA_f is a hyperbolic matrix.

This is true since there are no nontrivial null–homologous curves on the once–punctured torus. Therefore an automorphism is nonperiodic irreducible if and only if no element of $H_1(S_1^{(1)}) = \pi_1^{ab}(S_1^{(1)})$ is mapped to itself. Now this can be read of the characteristic polynomial of the Anosov–monodromy, which has only one non–trivial coefficient, $\mathrm{tr}(A_f)$.

Conjecture (Thurston). Every hyperbolic manifold with finite volume is finitely covered by a surface bundle.

Some Algorithms in the present context.

Lee Mosher [Mosher 1986] introduced an algorithm for the classification of pseudo–Anosovs. Given two pseudo–Anosov automorphisms of a surface, he is able to decide (using combinatorial data from a suitable cell–decomposition of the surface) whether powers of the two pseudo–Anosovs are isotopic. He gave no example and no procedure of deciding whether a given automorphism is pseudo–Anosov.

Bestvina and Handel [Bestvina/Handel 1992, 1995] isotope a given automorphism f until it acts nontrivially only on a certain subset $S_g^{(r)^\circ} \subset S_g^{(r)}$, where $S_g^{(r)^\circ}$ possesses a decomposition into “stripes” and “junctions”. These are shrunk to edges and vertices of a graph. The isotopic deformation of $S_g^{(r)^\circ}$ and f and the observation of the induced changes on the graph allows to decide in finitely many steps, whether f is pseudo–Anosov, reducible, or periodic. They provide several examples on $S_2^{(1)}$, $S_0^{(4)}$, and $S_0^{(5)}$. They do not, however, establish the hyperbolic structure explicitly.

The computer algorithm SnapPea by Weeks e.a. [Weeks 1998] constructs hyperbolic manifolds, calculates volumes, fundamental domains, fundamental groups etc. within a remarkably short interval of time. However, there is no hint whether a given manifold is a surface bundle. In fact, the manifolds are sorted by the number of tetrahedra and within such a set by their volume, which indicates the lack of a better criterion.

2 An algebraic algorithm in the present context

The fundamental group of a compact orientable surface of genus g with r punctures is given by

$$\pi_1(S_g^{(r)}) = \langle a_1, \ldots, a_{2g}, \lambda_1, \ldots, \lambda_r \quad | \quad \Pi_{i=1}^g [a_{2i-1}, a_{2i}] \Pi_{j=1}^r \lambda_j \rangle \cong \Phi_{2g+r-1},$$

where Φ_n denotes the free group on n generators.

Let $f \in \mathrm{Map}(S_g^{(r)})$ be an automorphism. We are interested in orientable manifolds only, so we require f to not change the orientation of any boundary component. By passing to a power of f, if necessary, we may assume that f maps every boundary component onto itself. In particular we may isotope f such that $\lambda = \lambda_1$ is fixed pointwise. Let the base point be on λ. Then the mapping class of f is in one–to–one–correspondence with a group isomorphism

$$\begin{aligned} f_* \in \{\varphi \in \mathrm{Aut}(\Phi_{2g+r-1}) \quad | \quad & \varphi(\lambda) = \lambda, \phi(\lambda_i) = \omega_i^{-1} \lambda_i \omega_i \, (i = 2, \ldots, r; \\ & \omega_i \text{ words in } \Phi_{2g+r-1})\} \\ & =: \mathrm{Aut}(\Phi_{2g+r-1}, \lambda). \end{aligned}$$

All information about the surface bundle $M_f(S_g^{(r)})$ is contained in and extractable from a presentation of f_*, in fact, its conjugacy class in $\mathrm{Aut}(\Phi_{2g+r-1}, \lambda)$.

Deciding whether the surface bundle $M_f(S_g^{(r)})$ is hyperbolic now amounts to deciding whether its fundamental group

$$\pi_1(M_f(S_g^{(r)})) = \langle \Phi_{2g+r-1}, \mu \,|\, \mu\omega\mu^{-1} = f_*(\omega) \quad \forall \omega \in \Phi_{2g+r-1} \rangle$$

has a faithful, discrete, admissible (in particular semisimple) (and then by Thurston's hyperbolization theorem automatically cofinite) representation in $(\mathrm{P})\mathrm{SL}_2(\mathbf{C}) \cong \mathrm{Isom}^+(\mathbf{H^3})$.

Now every conjugacy class ρ of semisimple representations of a group G in $\mathrm{SL}_2(\mathbf{C})$ is characterized by its character $s : G \to \mathbf{C}, \gamma \mapsto \mathrm{tr}(\rho(\gamma))$, which in turn is determined by the finite number of values $s(\gamma_i), s(\gamma_i\gamma_j), s(\gamma_i\gamma_j\gamma_k)$, if G is finitely generated by the γ_i (see [Alestalo/Helling 1997], [Helling 1996] and [Menzel 1998] for details).

The condition that the above values determine a point on the character variety of $\pi_1(S_g^{(r)})$ together with the admissibility conditions and the conditions resulting from the relations in $\pi_1(M_f(S_g^{(r)}))$ yield a high degree system of polynomial equations, the so–called trace equations, for those values. By rigidity, there is none or exactly one solution (plus compl. conj.) leading to a faithful, discrete representation.

Faithfulness and discreteness is checked by constructing a Ford domain (or rather trying to construct one).

The steps of the algorithm

(1) Find all solutions of the system of trace equations, i.e. all tuples of complex numbers $s(a_i), s(\lambda_j), s(\text{products of two}), s(\text{products of three})$ solving the system of trace equations, numerically, if necessary.

(2) For every such tuple go through the following steps (3)-(9) until one of (a), (b) or (c) happens:

(a) The corresponding representation of $\pi_1(M_f(S_g^{(r)}))$ is not faithful; seen by finding an element of $\pi_1(S_g^{(r)})$ that is represented by the unit matrix.

(b) The corresponding representation is faithful and non–discrete; seen by finding an element $\gamma \in \pi_1(M_f(S_g^{(r)}))$ the isometric sphere of which is too large (lemma of Shimizu), or a non–suitable element with fixed point ∞.

(c) The corresponding representation is discrete; proof by presenting a Ford domain (Poincaré polyhedron).

(3) Compute the matrices $A_i (i = 1, \ldots, 2g), L_j (j = 2, \ldots, r), M$ representing a_i, λ_j, μ. Let $L = L_1, M$ be a pair of parabolic matrices fixing ∞. Let $\kappa = 1$ (wordlength).

(4) Compute all words $G_\iota^{(\kappa)}$ in A_i, L_j of length $\leq \kappa$.

Has one of them an isometric sphere with radius $> |M_{1,2}|$ or $|L_{1,2}|$? If this is true, the group is not discrete; take the next tuple from step (1) and return to step (3). If this is false, continue.

(5) Do the isometric spheres of $G_\iota^{(\kappa)}$ plus inverses plus translates by M, L cover the boundary of $\mathbf{H^3}$ up to isolated points (which are supposed to be the other cusps of the group, if there are any)? If this is false: increase wordlength κ, return to (4). If this is true, continue.

(6) Compute the polyhedron bounded by these isometric spheres; the projection onto $\mathbf{C}$ yields a tiling of $\mathbf{C}$ with Euclidean polygons. From these polygons, read off the equivalence classes of edges (= relations R_ν) and vertices. One obtains a combinatorial group

$$G^{(\kappa)} := \langle G_\iota^{(\kappa)}, M, L \mid ML = LM, R_\nu \rangle$$

(7) Is the above polyhedron a combinatorial Poincaré polyhedron, i.e. is the polyhedron at least combinatorially a fundamental domain (see the next paragraph for a definition)? If this is false: increase κ, return to (4). If this is true, continue.

(8) Is $G^{(\kappa)} \cong \pi_1(M_f(S_g^{(r)}))$? If this is false: increase κ, return to (4). If this is true, continue.

(9) Is the above polyhedron vertex–generic, i.e. does every vertex on the boundary meet exactly three edges (see next paragraph)? If this is true: $M_f(S_g^{(r)})$ is hyperbolic, the present representation of $\pi_1(M_f(S_g^{(r)}))$ determines the hyperbolic structure, the present polyhedron is a Ford domain. If this is false: increase wordlength κ and the numerical accuracy, return to (3).

3 Comments on the algorithm

In this paragraph we prove the following theorem (with one gap):

Theorem. Given a surface bundle $M_f(S_g^{(r)})$, the algorithm of paragraph 2 decides in finitely many steps, whether $M_f(S_g^{(r)})$ is hyperbolic or not, except in those cases, where $M_f(S_g^{(r)})$ is hyperbolic and has a Ford domain that is not vertex generic. (In those cases the algorithm finds a numerical diagram of the Ford domain in finitely many steps; however, the final decision whether this numerical diagram is correct is not found due to the lack of numerical precision.)

We comment on the finiteness and effectiveness of every step.

Step (1): It is possible in principle to reduce the system of equations to a system with one equation for every trace variable in this variable alone (e.g. by successively computing resultants). This provides information about the algebraic number field containing all trace values, i.e. at this stage we know at least a finite degree extension of it.

It is possible to compute numerically all simultaneous zeros up to a given accuracy.

Step (2): The fundamental group of $M_f(S_g^{(r)})$ is finitely generated. Now consider a particular representation ρ of this group in $\mathrm{SL}_2(\mathbf{C})$.

If this representation is not faithful, there is a $g \in \pi_1(M_f(S_g^{(r)}))$ with $\rho(g) = \mathbf{1}$. Now g is a word in M and the generators of $\pi_1(S_g^{(r)})$. Using the relations $\mu\omega\mu^{-1} = f(\omega)$ and $\mu^{-1}\omega\mu = f^{-1}(\omega)$ we may commute all powers of μ to the right. Therefore we may w.l.o.g. assume $g = \omega\mu^k$, whence $\rho(\omega) = M^{-k}$. This word ω will be found in finitely many steps.

Now let ρ be faithful and not discrete. Recall that $\langle M, L\rangle \cong \mathbf{Z}^2$ has fixed point ∞. There is a sequence of matrices

$$\left\{ \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \right\}_n \subset \mathrm{Im}\rho$$

converging to $\mathbf{1}$ in $\mathrm{SL}_2(\mathbf{C})$. In particular $c_n \to 0$. Then in finitely many steps we find

- a loxodromic element $\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$, or
- a parabolic element $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \notin \langle M, L\rangle$, or
- a loxodromic element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c \neq 0, 1/|c| > |M_{1,2}|, |L_{1,2}|$.

If, on the contrary, ρ is faithful and discrete, a Ford domain exists. We will now prove a fairly obvious fact about Ford domains of hyperbolic surface bundles, which, however, we could not find in the literature.

Lemma. Let $M_f(S_g^{(r)})$ be a surface bundle, the fiber of which has at least one puncture. Let $M_f(S_g^{(r)})$ be hyperbolic. Let $\rho(\mu)$ fix ∞. Then the Ford domain for $M_f(S_g^{(r)})$ in $\mathbf{H^3}$ is geometrically finite.

Remark. On the first look this statement seems to be clear. For on the one hand it is well known that every complete hyperbolic n-manifold with finite volume is geometrically finite (see e.g. [Ratcliffe, Th. 12.6.3.] and that a hyperbolic surface bundle has finite volume (Thurston hyperbolization theorem). On the other hand it is known that $\sum_{\gamma\in\Gamma\setminus 1} R^4(\gamma) < \infty$ for every (suitably adjusted) Kleinian group of type II (i.e. with nonempty domain of discontinuity in $\partial\mathbf{H}^3$); see e.g. [Maskit 1988], Th. II.B.5). A closer look however shows that these results do not fit together. In particular, if $\Gamma \cong \pi_1(M_f(S_g^{(r)}))$

is supposed to be discrete in $\mathrm{SL}_2(\mathbf{C})$, its domain of discontinuity has to be empty; the geometrical finiteness theorem does not tell you anything about a particular fundamental domain. In fact, we could not find the above lemma in the literature and therefore include a proof.

Proof. A Ford domain $\mathcal{F}$ for $M_f(S_g^{(r)})$ is contained in a Euclidean column above a parallelogram P_0 in $\mathbf{C}$, the sides of which are given by the Euclidean translation by M and L, the generators of the stabilizer of the cusp. All other faces of $\mathcal{F}$ are polygons on isometric spheres of certain group elements.

Now assume that infinitely many of these isometric spheres contribute to $\mathcal{F}$.

Case 1: Let the radii of them be bounded from below (and of course from above). Take some regular compact neighbourhood $K \subset \mathbf{H}^3$ of the floor of the Ford domain, i.e. of all contributing polygonal faces on isometric spheres. Then there are infinitely many $g \in \Gamma$ with $gK \cap K \neq \emptyset$, a contradiction.

Case 2: Now let the radii be not bounded from below. Then $\lim_{i\to\infty} R(g_i) = 0$, where $R(g)$ is the radius of the isometric sphere of g and $i \mapsto g_i$ is a complete enumeration of all elements of $\rho(\pi_1(M_f(S_g^{(r)}))) \setminus \langle M, L\rangle$ with center of isometric sphere inside P_0. Therefore the set of isometric spheres contributing to $\partial\mathcal{F}$ contains a sequence $\{I_j\}$ satisfying $R(I_j) \to 0, C(I_j) \to z_0 \in \mathbf{C}$, where $C(I)$ denotes the center of the isometric sphere I.

Hence $\partial\mathcal{F} \cap \mathbf{C} \neq \emptyset$. W.l.o.g. $0 \in \partial\mathcal{F} \cap \mathbf{C}$. Since every topological end of $M_f(S_g^{(r)})$ is a cusp, 0 is an isolated point.

Now either $M_f(S_g^{(r)})$ has only one cusp. Then there is a group element g that carries 0 to ∞. Hence the isometric sphere of g covers 0 and therefore almost all I_j, a contradiction.

Or $M_f(S_g^{(r)})$ has more than one cusp. Than 0 is the fixed point of a pair of parabolic matrices. But $\{I_j\}$ is a sequence of spheres not covering 0. Therefore almost all I_j lie below the isometric spheres of the two parabolic elements and their inverses fixing 0, a contradiction.

Hence only a finite set of isometric spheres contribute to $\mathcal{F}$. $\square$

Since the Ford domain of a hyperbolic surface bundle is geometrical finite, all isometric spheres contributing to its boundary are found in finitely many steps. Therefore if we set out to construct a Ford domain for the given representation $\rho(\pi_1(M_f(S_g^{(r)})))$, we either end up with a nontrivial element mapped to **1**, an isometric sphere with too large radius resp. an element with the wrong fixed point, or a geometrically finite polyhedron in finetely many steps. In the latter case, the remaining difficulty is to show that this polyhedron is in fact a fundamental domain. See step (9).

Step (3) – step (6): These steps do not present any problem.

Step (7): In order to check whether the numerically found polyhedron is indeed an approximation of a Poincaré polyhedron for $M_f(S_g^{(r)})$, we have to check the angle relations at every equivalence class of edges and every equivalence class of vertices. (Note that it is not necessary to check, whether every tangency vertex transformation is parabolic (cf. [Maskit 1971]), since in the discrete case this is automatically satisfied, whereas the construction of a Ford domain in the nondiscrete case fails at other steps.) Now the latter are not independent of the former.

From now on every equivalence class and number of elements of equivalence class is understood modulo translations by M and L. We define a polyhedron to be *edge-generic*, if every equivalence class of edges (modulo translations) contains 3 elements. We call a polyhedron *vertex-generic*, if the number of edges meeting in a vertex is always 3. For simplicity, assume now that the polyhedron above is edge-generic. There are similar arguments for other cases, however, up to now we did not need them.

Let v_1 be a vertex with equivalence class $[v_1] = \{v_1, v_2, \ldots, v_n\}$. Let k_i ($k_i \geq 3$) edges meet in v_i. Denote Ω_i the space angle of P at v_i seen from the interior (i.e. from above). Then

$$\Omega_i = \sum_{j=1}^{k_i} \alpha_i^{(j)} + (2 - k_i)\pi,$$

where $\alpha_i^{(j)}$ is the dihedral angle at edge no. j meeting in v_i. Then the following condition has to be satisfied:

$$\begin{aligned} 4\pi &= \sum_{i=1}^{n} \Omega_i \\ &= \sum_{i=1}^{n} \left(\sum_{j=1}^{k_i} \alpha_i^{(j)} + (2 - k_i)\pi \right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_i^{(j)} + 2n\pi - \left(\sum_{i=1}^{n}k_i\right)\pi \\
&= \frac{1}{3}\left(\sum_{i=1}^{n}k_i\right)2\pi + 2n\pi - \left(\sum_{i=1}^{n}k_i\right)\pi \\
\Leftrightarrow 3(4-2n) &= -\left(\sum_{j=1}^{n}k_i\right).
\end{aligned}
$$

Case $n = 4$: Then this equation reads $12 = \sum k_i \Leftrightarrow k_i = 3 \forall i$.

Case $n = 5$: Then there are three possibilities:
$(k_1, k_2, k_3, k_4, k_5) = (3,3,3,3,6), (3,3,3,4,5),$ or $(3,3,4,4,4)$.

It is easy to enumerate all possibilities for higher n, if necessary.

Step (8): The $G_\iota^{(\kappa)}$ are words in A_i, L_j, M, as are R_ν. Use the relations $\mu\omega\mu^{-1} = f_*(\omega)$ and $\mu^{-1}\omega\mu = f_*^{-1}(\omega)$ to commute all M's to the right. Then R_ν is a product of a word in $\rho(\Phi_{2g+r-1})$ and a power of M. In order to have a faithful discrete representation, the latter has to be trivial, since no power of M can be an element of $\rho(\Phi_{2g+r-1})$, or else ρ is not faithful. Therefore the former has to be trivial as well.

Hence it is possible to check algorithmically whether $\langle R_\nu \rangle$ is a subgroup of the group generated by the relations in $\pi_1(M_f(S_g^{(r)}))$. In order to establish the isomorphism, we have to solve the equations $G_\iota^{(\kappa)} =$ word in A_i, L_j, M for A_i, L_j and to derive the relations in $\pi_1(M_f(S_g^{(r)}))$ using the R_ν.

A purely combinatorial proof of this can of course not be done algorithmically (in fact, it is conjectured that even in this rather restricted context the isomorphism problem for combinatorial groups is not solvable, see [Epstein e.a. 1992], Question 2.3.11). Therefore step (8) can be done only by using geometrical arguments. This is an open problem until now.

Step (9):

Lemma. Let a numerical precision be prescribed. Let
$\pi_1 := \langle x_1, \ldots, x_r | \mu x_i \mu^{-1} = f(x_i) \rangle$ be numerically represented via $x_i \mapsto \bar{X}_i$, which is an approximation of a representation $x_i \mapsto X_i$ given by some character. Let $\bar{G}_j = \bar{G}_j(\bar{X}_i)$ be words in $\bar{X}_i$. Let the isometric spheres of $\bar{G}_j$ and there inverses and translates by $M = \bar{M}, \bar{L}$ bound a polyhedron $\bar{P}$, which is combinatorially a Poincaré polyhedron. Let $R_\nu(\bar{G}_j)$ be the relations read

off $\bar{P}$. Assume $\bar{P}$ is vertex–generic and every two vertices of $\bar{P}$ are sufficiently distant (in view of the numerical precision). Assume further that $\langle G_j(X_i)|R_\nu(G_j)\rangle$ is known to be combinatorially isomorphic to π_1. Then the polyhedron P, bounded by the isometric spheres of $G_j(X_i)$ is a fundamental domain for π_1.

Proof. We assume that the precision is sufficient to guarantee that every numerically found vertex between $\bar{G_j}, \bar{G_k}, \bar{G_l}$ is in a small neighbourhood of the corresponding vertex between G_j, G_K, G_l. The same is true for the edges.

Now $G = \langle G_j, M, L|R_\nu, [M, L]\rangle$ is generated by the side pairings of P. Furthermore, all R_ν are strictly correct because of the known isomorphism between the combinatorial groups. Therefore in P all angle relations for edges are satisfied.

Since $\bar{P}$ is sufficiently near P and satisfies the combinatorial condition, so does P. Therefore P satisfies all angle relations in all vertices. At this point we need the fact that $\bar{P}$ is vertex–generic and that two different vertices of $\bar{P}$ are sufficiently distant to guarantee that the corresponding vertices of P are distinct.

Hence P is a Poincaré polyhedron for π_1. $\square$

Up to the open problem in step (8), the theorem of the beginning of this paragraph is now proved.

4 Example

In the following examples we express the trace equations using the coordinates

$$x_i = s(a_i),$$
$$y_{ij} = 2s(a_i a_j) - s(a_i)s(a_j),$$
$$z_{ijk} = s(a_i a_j a_k) - s(a_k a_j a_i).$$

See [Menzel 1998] for details.

Let $g = 2$, $r = 1$. Let $f_4 : (a_1, a_2, a_3, a_4) \mapsto (a_1, a_2, a_3, a_3 a_4)$ and $\alpha_1 : (a_1, a_2, a_3, a_4) \mapsto (a_4^{-1} a_2^{-1} a_4, a_4^{-1} a_2^{-1} a_1 a_4^{-1} a_2 a_4, a_4^{-1} a_2^{-1} a_4^{-1} a_2 a_4, a_4^{-1} a_2^{-1} a_3 a_4)$ be two automorphisms of the once punctured surface of genus 2. The automorphism $\alpha_1^4 f_4^{-1} \alpha_1^3 f_4^2$ is given by

$$(a_1, a_2, a_3, a_4) \mapsto (a_4^{-1} a_2^{-1} a_4, a_4^{-1} a_1 a_4^{-1} a_2 a_4, a_4^{-1} a_2^{-1} a_4^{-1} a_2 a_4, a_4^{-1} a_2^{-1} a_4^{-2} a_3 a_4).$$

From the system of trace equations we obtain

$$x_d = x_c, x_b = x_a = \frac{1}{4}(x_c - 1)(3x_c + 5),$$

$$y_{ab} = \frac{3}{2}(x_c^2 - 7), y_{ac} = y_{bd} = \frac{1}{4}(-9x_c^2 + 30x_c - 29),$$

$$y_{ad} = y_{bc} = \frac{1}{4}(x_c - 1)(7x_c - 11), y_{cd} = \frac{1}{2}(x_c^2 + 2x_c - 9),$$

$$z_{abc} = -z_{abd} = 2(-x_c + 2)(x_c - 3),$$

$$z_{acd} = -z_{bcd} = 2(-2x_c^2 - 3x_c + 15).$$

The trace variable x_c satisfies

$$-11 + 3x_c - x_c^2 + x_c^3 = 0.$$

Substituting $x_c = t + 2$ we get

$$t^3 + 5t^2 + 11t - 1 = 0.$$

This polynomial has one pair of complex roots. We take the one with negative imaginary part.

Let X_i, M, L be the matrices representing a_i, μ, λ. Matrices with large radius of isometric spheres contributing to the boundary of the Ford domain are given by

$$G_1 := X_1^{-1} X_2 X_1, G_2 := X_1^{-1} X_3^{-1} X_4, G_3 := X_1^{-1} X_3^{-1} X_4^2,$$

$$G_4 := X_1^{-1} X_2^{-1}, G_5 := X_4^{-1} X_3 X_4^{-1} X_3 X_4.$$

This implies

$$X_1 = G_4^{-1} G_1^{-1}, X_2 = G_4^{-1} G_1 G_4, X_3 = G_2^{-1} G_3 G_2^{-1} G_1 G_4, X_4 = G_2^{-1} G_3.$$

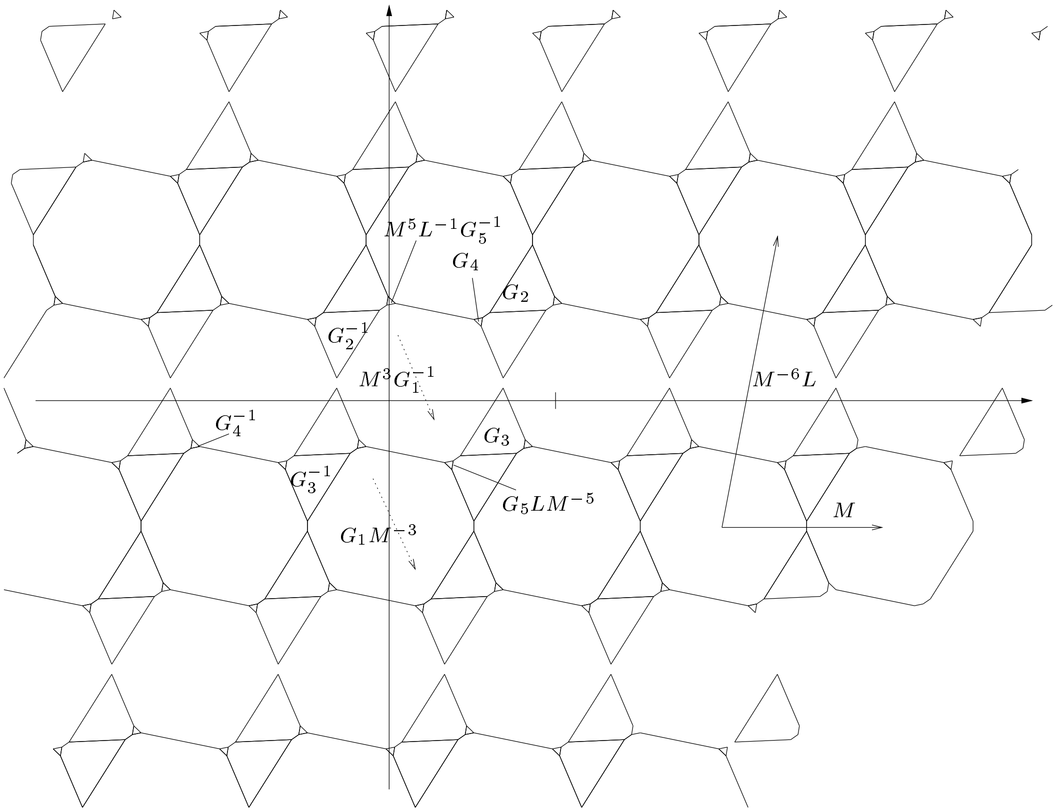

Figure 1: Projected Ford domain for $M_f(S_g^{(r)})$ with $f = \alpha_1^4 f_4^{-1} \alpha_1^3 f_4^2$; group elements belonging to the isometric spheres are inscribed. One edge is missing for technical reasons only. The dashed arrows indicate the invariant direction in the only case, where this is not obvious. The two arrows on the right indicate the action of the parabolic stabilizors of ∞.

From the numerical diagram (see figure 1) we read off the following relations for the G_i:

$$R_1: \quad MG_1M^2L^{-1}G_5^{-1}LM^{-3}G_1^{-1} \tag{1}$$
$$R_2: \quad G_4^{-1}M^{-1}G_1MG_1^{-1} \tag{2}$$
$$R_3: \quad G_5LM^{-4}G_3M^4G_1^{-1} \tag{3}$$
$$R_4: \quad M^{-1}G_2^{-1}M^{-4}LG_1^{-1}M^5L^{-1}G_5^{-1} \tag{4}$$
$$R_5: \quad MG_3M^2G_4M^{-3}LG_1^{-1} \tag{5}$$

$$\begin{aligned} R_6 :&\quad M^{-1}G_2^{-1}M^{-1}G_4^{-1}M^2G_1^{-1} && (6)\\ R_7 :&\quad LM^{-3}G_1^{-1}G_3^{-1}M^3G_1^{-1} && (7)\\ R_8 :&\quad MG_2MG_2LM^{-2}G_1^{-1} && (8)\\ R_9 :&\quad G_2LM^{-3}G_1^{-1}M^3G_1^{-1} && (9)\\ R_{10} :&\quad M^{-1}G_3^{-1}M^{-1}G_3^{-1}M^2G_1^{-1} && (10) \end{aligned}$$

In the appendix we show the remaining part of the isomorphy

$$\langle a_1, \dots, a_4, \mu \mid \mu\omega\mu^{-1} = f(\omega)\, \forall\omega\rangle \cong \langle G_i, M, L \mid [M, L], R_\nu\rangle,$$

which cannot be done algorithmically.

Since the numerical diagram shows a vertex–generic polyhedron, this is in fact a Poincaré polyhedron and a fundamental domain for the corresponding surface bundle.

Its volume has been computed numerically:

$$\mathrm{vol}(M_{\alpha^4 f_4^{-1}\alpha^3 f_4^2}(S_2^{(1)})) = 3.17729...$$

This manifold is homeomorphic to the manifold `m036` shown by `SnapPea`.

Appendix

We have to show that the relations $S_i : \mu\, a_i\, \mu^{-1} = f_*(a_i)$ follow from the R_ν. Start with defining relations in $\pi_1(M_{\alpha_1^4 f_4^{-1}\alpha_1^3 f_4^2}(S_2^{(1)})$:

$$\begin{aligned} S_1 &: \mu\, a_1^{-1}\, \mu^{-1}\, a_4^{-1}\, a_2^{-1}\, a_4,\\ S_2 &: \mu\, a_2^{-1}\, \mu^{-1}\, a_4^{-1}\, a_1\, a_4^{-1}\, a_2\, a_4,\\ S_3 &: \mu\, a_3^{-1}\, \mu^{-1}\, a_4^{-1}\, a_2^{-1}\, a_4^{-1}\, a_2\, a_4,\\ S_4 &: \mu\, a_4^{-1}\, \mu^{-1}\, a_4^{-1}\, a_2^{-1}\, a_4^{-1}\, a_4^{-1}\, a_3\, a_4. \end{aligned}$$

Replace a_1 with $\mu^{-1}\, a_4^{-1}\, a_2^{-1}\, a_4\, \mu$ according to S_1, replace a_3 with $\mu^{-1}\, a_4^{-1}\, a_2^{-1}\, a_4^{-1}\, a_2\, a_4\, \mu$ according to S_3. Then S_1, S_3 disappear. The remaining relators are:

$$
\begin{aligned}
S_2 &: \mu\, a_2^{-1}\, \mu^{-1}\, a_4^{-1}\, \mu^{-1}\, a_4^{-1}\, a_2^{-1}\, a_4\, \mu\, a_4^{-1}\, a_2\, a_4, \\
S_4 &: \mu\, a_4^{-1}\, \mu^{-1}\, a_4^{-1}\, a_2^{-1}\, a_4^{-2}\, \mu^{-1}\, a_4^{-1}\, a_2^{-1}\, a_4^{-1}\, a_2\, a_4\, \mu\, a_4.
\end{aligned}
$$

Shift S_2 left by 2, shift S_4 left by 1, replace a_2 with $g_4^{-1}\, g_1\, g_4$, a_4 with $g_2^{-1}\, g_3$ according to p. 13:

$$
\begin{aligned}
S_2 &: \mu^{-1}\, g_3^{-1}\, g_2 \mu^{-1}\, g_3^{-1}\, g_2\, g_4^{-1} g_1^{-1}\, g_4\, g_2^{-1}\, g_3\, \cdot \\
&\quad \cdot \mu\, g_3^{-1}\, g_2\, g_4^{-1} g_1\, g_4\, g_2^{-1}\, g_3\, \mu\, g_4^{-1}\, g_1^{-1}\, g_4, \\
S_4 &: g_3^{-1}\, g_2\, \mu^{-1}\, g_3^{-1}\, g_2\, g_4^{-1}\, g_1^{-1} g_4\, g_3^{-1}\, g_2\, g_3^{-1}\, \cdot \\
&\quad \cdot g_2\, \mu^{-1}\, g_3^{-1}\, g_2\, g_4^{-1}\, g_1^{-1}\, g_4\, g_3^{-1}\, g_2\, g_4^{-1}\, g_1\, \cdot \\
&\quad \cdot g_4\, g_2^{-1}\, g_3\, \mu\, g_2^{-1}\, g_3\, \mu.
\end{aligned}
$$

The relators R_ν are:

$$
\begin{aligned}
R_0 &: \mu^{-1}\, \lambda^{-1}\, \mu\, \lambda, \\
R_1 &: \mu\, g_1\, \mu^2 \lambda^{-1}\, g_5^{-1}\, \lambda\, \mu^{-3} g_1^{-1}, \\
R_2 &: g_4^{-1}\, \mu^{-1}\, g_1\, \mu\, g_1^{-1}, \\
R_3 &: g_5\, \lambda\, \mu^{-4}\, g_3\, \mu^4\, g_1^{-1}, \\
R_4 &: \mu^{-1}\, g_2^{-1}\, \mu^{-4}\, \lambda\, g_1^{-1}\, \mu^5\, \lambda^{-1}\, g_5^{-1}, \\
R_5 &: \mu\, g_3\, \mu^2\, g_4\, \mu^{-3} \lambda\, g_1^{-1}, \\
R_6 &: \mu^{-1}\, g_2^{-1}\, \mu^{-1}\, g_4^{-1}\, \mu^2\, g_1^{-1}, \\
R_7 &: \lambda\, \mu^{-3} g_1^{-1}\, g_3^{-1}\, \mu^3\, g_1^{-1}, \\
R_8 &: \mu\, g_2\, \mu\, g_2\, \lambda\, \mu^{-2} g_1^{-1}, \\
R_9 &: g_2\, \lambda\, \mu^{-3} g_1^{-1}\, \mu^3\, g_1^{-1}, \\
R_{10} &: \mu^{-1}\, g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu^2\, g_1^{-1}.
\end{aligned}
$$

Replace g_5 with $\lambda\, \mu^{-3}\, g_1^{-1}\, \mu\, g_1\, \mu^2\, \lambda^{-1}$ according to R_1, replace λ with $g_1\, \mu^{-4} g_3^{-1}\, \mu^2\, g_1^{-1}\, \mu^{-1}\, g_1\, \mu^3$ according to R_3, replace g_4 with $\mu^{-1}\, g_1\, \mu\, g_1^{-1}$ according to R_2. Then R_1, R_2, R_3 disappear. The R_ν now read:

$$
\begin{aligned}
R_0 &: \mu^{-1}\, g_1^{-1}\, \mu\, g_1\, \mu^{-2} g_3\, \mu^4\, g_1^{-1}\, \mu\, g_1\, \mu^{-4} g_3^{-1}\, \mu^2\, g_1^{-1}\, \mu^{-1}\, g_1, \\
R_4 &: \mu^{-1}\, g_2^{-1}\, \mu^{-4} g_1\, \mu^{-4} g_3^{-1}\, \mu^2\, g_1^{-1}\, \mu^{-1}\, g_1\, \mu^3\, g_1^{-1}\, \mu\, g_3\, \mu^4\, g_1^{-1}, \\
R_5 &: \mu\, g_3\, \mu\, g_1\, \mu\, g_1^{-1}\, \mu^{-3} g_1\, \mu^{-4} g_3^{-1}\, \mu^2\, g_1^{-1}\, \mu^{-1}\, g_1\, \mu^3\, g_1^{-1}, \\
R_6 &: \mu^{-1}\, g_2^{-1}\, \mu^{-1}\, g_1\, \mu^{-1}\, g_1^{-1}\, \mu^3\, g_1^{-1}, \\
R_7 &: \mu^{-1}\, g_3^{-1}\, \mu^2\, g_1^{-1}\, \mu^{-1}\, g_3^{-1}, \\
R_8 &: \mu\, g_2\, \mu\, g_2\, g_1\, \mu^{-4} g_3^{-1}\, \mu^2\, g_1^{-1}\, \mu^{-1}\, g_1\, \mu\, g_1^{-1}, \\
R_9 &: g_2\, g_1\, \mu^{-4} g_3^{-1}\, \mu^2\, g_1^{-1}\, \mu^2\, g_1^{-1}, \\
R_{10} &: \mu^{-1}\, g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu^2\, g_1^{-1}.
\end{aligned}
$$

The S_i read:

$$
\begin{aligned}
S_2 &: \mu^{-1}\, g_3^{-1}\, g_2\, \mu^{-1}\, g_3^{-1}\, g_2\, g_1\, \mu^{-1}\, g_1^{-1}\, \mu\, g_1^{-1} \mu^{-1}\, g_1\, \mu\, g_1^{-1}\, g_2^{-1} \cdot \\
&\quad \cdot g_3\, \mu\, g_3^{-1}\, g_2\, g_1\, \mu^{-1}\, g_1^{-1}\, \mu\, g_1\, \mu^{-1}\, g_1\, \mu\, g_1^{-1}\, g_2^{-1}\, g_3\, \mu \cdot \\
&\quad \cdot g_1\, \mu^{-1}\, g_1^{-1}\, \mu\, g_1^{-1}\, \mu^{-1}\, g_1\, \mu\, g_1^{-1}, \\
S_4 &: g_3^{-1}\, g_2\, \mu^{-1}\, g_3^{-1}\, g_2\, g_1\, \mu^{-1}\, g_1^{-1}\, \mu\, g_1^{-1}\, \mu^{-1}\, g_1\, \mu\, g_1^{-1}\, g_3^{-1} \cdot \\
&\quad \cdot g_2\, g_3^{-1}\, g_2\, \mu^{-1}\, g_3^{-1}\, g_2\, g_1\, \mu^{-1}\, g_1^{-1}\, \mu\, g_1^{-1}\, \mu^{-1}\, g_1\, \mu\, g_1^{-1} \cdot \\
&\quad \cdot g_3^{-1}\, g_2\, g_1\, \mu^{-1}\, g_1^{-1}\, \mu\, g_1\, \mu^{-1}\, g_1\, \mu\, g_1^{-1}\, g_2^{-1}\, g_3\, \mu\, g_2^{-1}\, g_3\, \mu.
\end{aligned}
$$

Replace g_1 with $\mu^{-1}\, g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu^2$ according to R_7. Then R_9 is:

$$
R_9 \;:\; g_2\, \mu^{-1}\, g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu^{-1}\, g_3\, \mu\, g_3\, \mu\, g_3\, \mu.
$$

Replace g_2 with $\mu^{-1}\, g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu\, g_3\, \mu\, g_3\, \mu$ according to R_9. We get:

$$
R_8 \;:\; g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu\, g_3\, \mu\, g_3\, \mu\, g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu^{-2} g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu\, g_3\, \mu\, g_3\, \mu,
$$

$$
\begin{aligned}
S_2 &: 1, \\
S_4 &: \mu^{-1}\, g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu\, g_3\, \mu\, g_3\, \mu\, g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu^{-1} \cdot \\
&\quad \cdot g_3^{-1}\, \mu\, g_3\, \mu\, g_3\, \mu\, g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu^{-2} g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu\, g_3\, \mu\, g_3\, \mu\, g_3^{-1} \cdot \\
&\quad \cdot \mu^{-1}\, g_3^{-1}\, \mu\, g_3\, \mu\, g_3\, \mu\, g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu^{-1}.
\end{aligned}
$$

Rotate R_8 left by 11:

$$\begin{aligned} R_8 \quad : \quad & \mu^{-1}\, g_3^{-1}\, \mu^{-2} g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu\, g_3\, \mu\, g_3\, \mu\, g_3^{-1} \cdot \\ & \cdot \mu^{-1}\, g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu\, g_3\, \mu\, g_3\, \mu\, g_3^{-1}. \end{aligned}$$

Shift S_4 left by 22:

$$\begin{aligned} S_4 \quad : \quad & \mu^{-1}\, g_3^{-1}\, \mu^{-2} g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu\, g_3\, \mu\, g_3\, \mu\, g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu\, g_3 \cdot \\ & \cdot \mu\, g_3\, \mu\, g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu\, g_3\, \mu\, g_3\, \mu\, g_3^{-1}\, \mu^{-1} \cdot \\ & \cdot g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu\, g_3\, \mu\, g_3\, \mu\, g_3^{-1}. \end{aligned}$$

Replace S_4 with $R_8^{-1} S_4$, shift left by 6:

$$\begin{aligned} S_4 \quad : \quad & \mu^{-1}\, g_3^{-1}\, \mu^{-2} g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu\, g_3\, \mu\, g_3\, \mu\, g_3^{-1} \cdot \\ & \cdot \mu^{-1}\, g_3^{-1}\, \mu^{-1}\, g_3^{-1}\, \mu\, g_3\, \mu\, g_3\, \mu\, g_3^{-1}. \end{aligned}$$

Now $S_4 = R_8$.

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