K-theory and generalized free products of S-algebras: Localization methods

Roland Schwänzl
Fachbereich Mathematik/Informatik, Universität Osnabrück
46069 Osnabrück, Federal Republic of Germany

Ross Staffeldt
Department of Mathematical Sciences, New Mexico State University
Las Cruces NM 88003, USA

Friedhelm Waldhausen
Fakultät für Mathematik, Universität Bielefeld
33615 Bielefeld, Federal Republic of Germany

July 22, 1999

Abstract

A generalized free product diagram of S-algebras is a generalization and stabilization of the diagram of group rings arising from a Seifert-van Kampen situation. Our eventual goal is to obtain a description of the algebraic K-theory of the "large" algebra in a generalized free product diagram in terms of the K-theories of the three smaller algebras. We first provide foundational material on generalized free product diagrams of S-algebras and associated categories of Mayer-Vietoris presentations. We show that the categories of Mayer-Vietoris presentations are categories with cofibrations, weak equivalences, and mapping cylinders. In particular, the hypotheses of the "generic fibration theorem" of Waldhausen (*Algebraic K-theory of spaces,* Lecture Notes in Math. 1126(1985), 318–419) are satisfied for two fundamental notions of weak equivalence, and there is, therefore, a three-term fibration sequence up to homotopy in which the K-theory of Mayer-Vietoris presentations with respect to the fine notion of equivalence is compared with the K-theory with respect to the coarse notion. The rest of the paper is concerned with interpreting these K-theories in terms of the K-theories of the algebras in the generalized free product diagram.

The first interpretative result of the paper uses the additivity theorem to identify the K-theory of Mayer-Vietoris presentations with respect to fine equivalences with the product of the K-theories of the three smaller algebras in a given generalized free product diagram. The second interpretative result uses the approximation theorem to identify the K-theory of Mayer-Vietoris presentations with respect to coarse equivalences with the K-theory of the generalized free product algebra. In order to confirm the hypotheses of the approximation theorem, we develop a localization tool for Mayer-Vietoris presentations which resembles Bousfield's theory of localization of spectra with respect to generalized homology theories. Investigation of the third term in the fibration sequence will be the subject of another work.

1 Introduction

As mentioned in the abstract, a generalized free product of S-algebras is a vast generalization and stablization of the sort of diagram of group rings that arises in a Seifert-van Kampen situation where a space Z is given as the pushout along a map $W \longrightarrow Y$ of a cofibration $W \rightarrowtail X$ when W is path-connected. If we now assume the spaces are pointed, we can define an S-algebra $A = S \wedge |G(W)|_+$, where $|G(W)|$ is the realization of the Kan loop group of the singular complex of W, and S-algebras B, C, and D similarly. In another note [13] we will see that D defined in this manner will be weakly equivalent to the generalized free product of B and C over A. This example is of interest for geometric reasons. For the work [15] on stable K-theory of an S-algebra R, a generalized free product diagram

$$\begin{array}{ccc} S & \longrightarrow & S \vee R \\ \downarrow & & \downarrow \\ S \wedge |G(S^{n+1})|_+ & \longrightarrow & D_n \end{array}$$

appeared. Moreover, if one promotes two unital ring homomorphisms $A \longrightarrow B$ and $A \longrightarrow C$ to S-algebra homomorphisms by passing to associated Eilenberg-MacLane spectra and making everything q-cofibrant, as explained in section 2, then one may form the S-algebra generalized free product $HB *_{HA} HC$ and try to identify its weak homotopy type. To our knowledge, there are hardly any results in this direction valid without very strict hypotheses on the original ring homomorphisms.

Given a generalized free product diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array} \tag{1.1}$$

our goal is to describe the K-theory of the S-algebra D in terms of the K-theories of A, B, and C. Under certain hypotheses on A, B, C, and D we obtain a fibration-up-to-homotopy

$$K(\mathcal{MV}_f^w, v) \longrightarrow K(A) \times K(B) \times K(C) \xrightarrow{\tilde{e}_*} K(D)$$

where $\mathcal{MV}_f^w$ denotes the category of split Mayer-Vietoris presentations of finite cell modules. The map $\tilde{e}$ will be described later in the introduction. The study of $K(\mathcal{MV}_f^w, v)$, commonly called the unreduced K-theory Nil-term, will be continued in another paper [14].

This paper provides the foundational material on generalized free products of S-algebras and develops the K-theory fibration sequence described briefly above. The results of this paper imply those that were announced in section 2 of [15]. For a long time we thought splitting conditions on the arrows in diagram 1.1 starting at A of the type assumed in [15] were necessary, but eventually we realized that they are not needed at all for the results of this paper. However, referring to the earlier paper may help the reader develop a feel for

the combinatorics of the constructions. Proofs that were expected to appear in bibilographic item [12] of [15] appear in the present paper. In [15] we already described the fibration sequence given above and how it is used to make a certain calculation of stable K-theory. In turn, the result of the calculation implies that stable K-theory and topological Hochschild homology coincide. This particular result has been proved for functors with smash product by Bjørn Ian Dundas [4] using a completely different series of reductions. On the other hand, our methods promise to eventually yield new information about the Nil-term in general [14]. The rest of the introduction outlines in more detail the contents of this paper.

In section 2 we study generalized free products of A_∞-algebras using the language of S-algebras [6]. A generalized free product diagram is a commuting diagram 1.1 of S-algebra maps satisfying the universal property of a pushout. In general, we use the notation $D = B *_A C$. For our work on K-theory the interesting and important feature of a generalized free product diagram is that the associated diagram of S-modules

$$\begin{array}{ccc} D \wedge_A D & \longrightarrow & D \wedge_B D \\ \downarrow & & \downarrow \\ D \wedge_C D & \longrightarrow & D \wedge_D D \end{array}$$

is weakly homotopy cocartesian. The smash products appearing in this diagram are those of [6, Definition III.3.1, page 58]. We recall what it means for a diagram to be weakly homotopy cocartesian in section 2; for the purposes of the introduction it suffices to note that the weakly homotopy cocartesian property implies the existence of a long exact sequence of stable homotopy groups

$$\cdots \longrightarrow \pi_q(D \wedge_A D) \longrightarrow \pi_q(D \wedge_B D) \oplus \pi_q(D \wedge_C D) \longrightarrow \pi_q(D \wedge_D D) \longrightarrow \cdots$$

of Mayer-Vietoris type.

The main problems addressed in the section are to give a construction of the algebra D given A, B, C, and the two morphisms from A, and to prove suitable invariance properties of the construction. We develop our own construction of the generalized free product to prove the homotopy cocartesian property, but we use a construction and technical tools developed in [6] to prove homotopy invariance properties.

In section 3 we define a category $\mathcal{MV}$ of Mayer-Vietoris presentations associated to a generalized free product diagram 1.1 of S-algebras. Basically, a Mayer-Vietoris presentation M_- consists of four modules M_A, M_B, M_C, M_D, A-module maps from M_A to M_B and M_C, a B-module map from M_B to M_D, and a C-module map from M_C to M_D such that the diagram

$$M_- = \begin{array}{ccc} M_A & \longrightarrow & M_B \\ \downarrow & & \downarrow \\ M_C & \longrightarrow & M_D \end{array} \quad \text{commutes and} \quad \begin{array}{ccc} M_A \wedge_A D & \longrightarrow & M_B \wedge_B D \\ \downarrow & & \downarrow \\ M_C \wedge_C D & \longrightarrow & M_D \wedge_D D \end{array}$$

is weakly homotopy cocartesian. In this case the weakly homotopy cocartesian property implies the existence of a long exact sequence of stable homotopy groups

$$\cdots \longrightarrow \pi_q(M_A \wedge_A D) \longrightarrow \pi_q(M_B \wedge_B D) \oplus \pi_q(M_C \wedge_C D) \longrightarrow \pi_q(M_D \wedge_D D) \longrightarrow \cdots$$

resembling more closely the standard Mayer-Vietoris sequence in homology. Thus, a generalized free product diagram of S-algebras is trivially a Mayer-Vietoris presentation, and, by the homotopy property of a generalized free product, the diagram in which all four modules are D and all morphisms are identities is also a Mayer-Vietoris presentation.

In section 3 we take up the study of the K-theory of Mayer-Vietoris presentations. For the purposes of K-theory, we have to impose various finiteness conditions on the constituent modules. The precise definitions of the categories of finite and homotopy-finite Mayer-Vietoris presentations $\mathcal{MV}_f$ and $\mathcal{MV}_{hf}$ used for K-theory in this paper are more flexible variants of [15, Definition 2.3, page 169]. Also, we define two notions of weak equivalence of Mayer-Vietoris presentations, one notion which we call a v-equivalence and a coarser notion called a w-equivalence. This just means that every v-equivalence is also a w-equivalence. For a fixed category of Mayer-Vietoris presentations, these two notions of weak equivalence give us two types of K-theory. To compare the two, we check that the hypotheses of the Fibration Theorem [18, Theorem 1.6.4, page 350] are fulfilled. Then there are commuting homotopy cartesian squares

$$\begin{array}{ccc} vS_\bullet\mathcal{MV}_f^w & \longrightarrow & wS_\bullet\mathcal{MV}_f^w \\ \downarrow & & \downarrow \\ vS_\bullet\mathcal{MV}_f & \longrightarrow & wS_\bullet\mathcal{MV}_f \end{array} \qquad \text{and} \qquad \begin{array}{ccc} vS_\bullet\mathcal{MV}_{hf}^w & \longrightarrow & wS_\bullet\mathcal{MV}_{hf}^w \\ \downarrow & & \downarrow \\ vS_\bullet\mathcal{MV}_{hf} & \longrightarrow & wS_\bullet\mathcal{MV}_{hf} \end{array} \tag{1.2}$$

in which the upper righthand corners are contractible. In other words, the sequences of spaces and maps starting from the upper lefthand corners of the squares are fibrations up to homotopy. The categories $\mathcal{MV}_f^w$ and $\mathcal{MV}_{hf}^w$ are the full subcategories of $\mathcal{MV}_f$ and $\mathcal{MV}_{hf}$, respectively, whose objects are the Mayer-Vietoris presentations in which the D-module is weakly contractible. These categories are called the categories of split Mayer-Vietoris presentations.

Also in section 3 we provide several natural variants of the definition of Mayer-Vietoris presentations, including the variant used in [15]. Assembly of results of this paper proves that they all give the same K-theory with respect to a fixed notion of weak equivalence.

In section 4 we analyse the K-theory of the category of Mayer-Vietoris presentations with respect to the fine notion of weak equivalence, obtaining the following result. It implies Theorem 2.4 we announced in [15].

Theorem 4.1. *The three forgetful functors*

$$u_A, u_B, u_C : \mathcal{MV} \longrightarrow \mathcal{M}(A),\, \mathcal{M}(B),\, \mathcal{M}(C)$$

induce homotopy equivalences

$$u_* : vS_\bullet\mathcal{MV}_f \longrightarrow hS_\bullet\mathcal{M}(A)_f \times hS_\bullet\mathcal{M}(B)_f \times hS_\bullet\mathcal{M}(C)_f$$

and

$$u_* : vS_\bullet\mathcal{MV}_{hf} \longrightarrow wS_\bullet\mathcal{M}(A)_{hf} \times wS_\bullet\mathcal{M}(B)_{hf} \times wS_\bullet\mathcal{M}(C)_{hf}$$

Less formally, the K-theory of the category of Mayer-Vietoris presentations with respect to the finer notion of weak equivalence may be identified in a natural way with the product of the usual K-theories associated to the three "smaller" rings in the generalized free product diagram. The main tool used in the proof of this result is the additivity theorem.

In section 5 we prepare to study the K-theory of Mayer-Vietoris presentations with respect to the coarser notion of weak equivalence. The technical tool we develop is analogous to Bousfield localization of spectra. Consequently, this section is both the most technical and the most interesting one of the paper. The idea of using Bousfield localization techniques in K-theory goes back to a paper of the third-named author [19]; the idea of using localization methods to study the K-theory of generalized free products goes back to a preprint of Pierre Vogel [16] that circulated in a draft version.

Our goal is to construct a localization functor

$$L : \mathcal{MV} \longrightarrow \mathcal{MV}$$

that produces from a given Mayer-Vietoris presentation another Mayer-Vietoris presentation in which the weak homotopy type of the original D-module is all that remains of the weak homotopy types of the original four modules. The proof that the functor L behaves precisely in this way is complicated and depends on the direct calculation that the localization of the generalized free product diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array} \qquad \text{is the diagram} \qquad \begin{array}{ccc} D & \xrightarrow{\text{id}} & D \\ {\scriptstyle\text{id}}\downarrow & & \downarrow{\scriptstyle\text{id}} \\ D & \xrightarrow[\text{id}]{} & D \end{array}$$

up to weak homotopy equivalence. (We emphasized the second diagram has the weakly homotopy cocartesian property of a Mayer-Vietoris presentation in the description given above of results in in section 2).

Then we prove in Theorem 5.13 that, if A is connective, then the functor L has a finite approximation property akin to the convergence property formulated in [19] for a Bousfield localization functor of finite type [11], [2]. For a more recent discussion of the convergence property, see [9]. The finite approximation property of L says, for example, that the localization $M_- \longrightarrow L(M_-)$ of a Mayer-Vietoris presentation of finite modules is obtained up to equivalence by taking a filtered colimit of coarse equivalences of Mayer-Vietoris presentations $M_- \longrightarrow M'_-$ under M_-, where $M'_D = M_D$ and the other components of $M_- \longrightarrow M'_-$ are inclusions of finite cell modules, and where each map $M'_- \longrightarrow M''_-$ in the system is a cofibration with finite cofiber.

Finally, in section 6 we use the approximation theorem to prove the following identification theorem. This implies the result we announced as Theorem 2.5 of [15].

Theorem 6.1. *Given a generalized free product situation in which the S-algebras S-algebras A, B, and C are q-cofibrant, the maps to B and C are q-cofibrations, and A, B, C, and D are connective, the forgetful functors*

$$u_D : \mathcal{MV}_f \longrightarrow \mathcal{M}(D)_f \quad \textit{and} \quad u_D : \mathcal{MV}_{hf} \longrightarrow \mathcal{M}(D)_{hf}$$

induce homotopy equivalences

$$wS_\bullet\mathcal{MV}_f \longrightarrow hS_\bullet\mathcal{M}(D)_f \quad \textit{and} \quad wS_\bullet\mathcal{MV}_{hf} \longrightarrow hS_\bullet\mathcal{M}(D)_{hf}.$$

In other words, the K-theory of Mayer-Vietoris presentations with respect to the coarse equivalences is essentially the K-theory of the "large" ring the generalized free product diagram. Our results on the localization functor, especially theorem 5.13, are the keys to confirming the hypotheses of the approximation theorem.

Once we have the identification theorems, we can use the fact that replacing spaces in a homotopy cartesian square by homotopy equivalent ones yields another homotopy cartesian square to reformulate the squares 1.2 as follows. The result is the following theorem, which summarizes the results of the paper.

Theorem 1.1. *Given a generalized free product situation*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in which the S-algebras A, B, and C are q-cofibrant, the maps to B and C are q-cofibrations, and A, B, C, and D are connective, then there are homotopy cartesian squares

$$\begin{array}{ccc} vS_\bullet\mathcal{MV}_f^w & \longrightarrow & wS_\bullet\mathcal{MV}_f^w \\ \downarrow & & \downarrow \\ hS_\bullet\mathcal{M}(A)_f \times hS_\bullet\mathcal{M}(B)_f \times hS_\bullet\mathcal{M}(C)_f & \xrightarrow{\tilde{e}_*} & hS_\bullet\mathcal{M}(D)_f \end{array}$$

and

$$\begin{array}{ccc} vS_\bullet\mathcal{MV}_{hf}^w & \longrightarrow & wS_\bullet\mathcal{MV}_{hf}^w \\ \downarrow & & \downarrow \\ wS_\bullet\mathcal{M}(A)_{hf} \times wS_\bullet\mathcal{M}(B)_{hf} \times wS_\bullet\mathcal{M}(C)_{hf} & \xrightarrow{\tilde{e}_*} & wS_\bullet\mathcal{M}(D)_{hf} \end{array}$$

in which the upper righthand corner terms are contractible. The lefthand vertical arrows are induced by the three forgetful functors from $\mathcal{MV}$ to the categories of A-, B-, and C-modules, and the righthand vertical arrows are induced by the forgetful functor from $\mathcal{MV}$ to D-modules.

The upper horizontal arrows are induced by an inclusion, and the lower horizontal arrows are induced by the functor

$$\tilde{e} : \mathcal{M}(A) \times \mathcal{M}(B) \times \mathcal{M}(C) \longrightarrow \mathcal{M}(D)$$

defined on objects by $\tilde{e}(M_A, M_B, M_C) = \Sigma M_A \wedge_A D \vee M_B \wedge_B D \vee M_C \wedge_C D$, *where* $\Sigma M_A = M_A \wedge (I/\partial I)$ *is the suspension of the* A*-module* M_A.

The proof of this theorem is given at the end of section 6. Of course, it makes essentially no difference whether we work with finite or homotopy finite modules; for some more technical arguments the larger category of homotopy finite modules is occasionally useful.

If we define $K(\mathcal{MV}_f^w, v) = \Omega|vS_\bullet \mathcal{MV}_f^w|$, the loops on the geometric realization of the classifying space of the simplicial category $vS_\bullet \mathcal{MV}_f^w$ and recall that $K(R) = \Omega|hS_\bullet \mathcal{M}(R)_f|$ is a definition of the K-theory of the S-algebra R, then the homotopy cartesian square may also be written as a fibration-up-to-homotopy

$$K(\mathcal{MV}_f^w, v) \longrightarrow K(A) \times K(B) \times K(C) \xrightarrow{\tilde{e}_*} K(D)$$

which, in turn, induces a long exact sequence of homotopy groups

$$\cdots \longrightarrow K_q(\mathcal{MV}_f^w, v) \longrightarrow K_q(A) \times K_q(B) \times K_q(C) \xrightarrow{\tilde{e}_*} K_q(D) \longrightarrow \cdots$$

The study of $K(\mathcal{MV}_f^w, v)$, the unreduced Nil-term, will be continued in the sequel to this paper [14].

2 Generalized free products of S-algebras

Let B and C be S-algebras equipped with S-algebra morphisms $\beta : A \longrightarrow B$ and $\gamma : A \longrightarrow C$, respectively. In standard categorical terminology, B and C are algebras under A. We deal with four main subjects in this section: construction of a generalized free product diagram, the proof of homotopy cocartesian property of a diagram associated with a generalized free product diagram, comparison of constructions, and homotopy invariance properties of the generalized free product.

To prove the homotopy cocartesian property, we need to give a construction of the generalized free product viewing it as the coproduct in the category of algebras under the S-algebra A, so we start with this topic. We plan to derive homotopy invariance properties of our constructions from results of [6], so we need to compare our construction with their construction of the colimit of the diagram

$$C \xleftarrow{\gamma} A \xrightarrow{\beta} B$$

in the category of S-algebras. Of course, the result is that the two constructions are isomorphic. Then we can prove that under some natural cofibrancy conditions, if we vary the

weak homotopy type of the input of our construction, the weak homotopy type of the output is preserved. These homotopy invariance properties are precisely formulated in proposition 2.10 below.

Our construction of the generalized free product of B and C over A uses the monad $\mathbb{T}_A$ on the category of A-bimodules whose algebras are the algebras under A. We will denote the result of this construction by $B *_A C$ and call the completed diagram

$$\begin{array}{ccc} A & \xrightarrow{\beta} & B \\ {\scriptstyle\gamma}\downarrow & & \downarrow \\ C & \longrightarrow & B *_A C \end{array}$$

a generalized free product diagram. Abbreviating $B *_A C = D$, we proceed directly to the proof of the homotopy cocartesian property of

$$\begin{array}{ccc} D \wedge_A D & \longrightarrow & D \wedge_B D \\ \downarrow & & \downarrow \\ D \wedge_C D & \longrightarrow & D \wedge_D D \end{array}$$

under various cofibration hypotheses on the original diagram

$$C \xleftarrow{\gamma} A \xrightarrow{\beta} B.$$

We start by stating some slight extensions of results found in [6]. To this end, let A be an S-algebra, and let $\mathcal{M}_{A^e}$ denote the category of A-bimodules.

Lemma 2.1. *The category $\mathcal{M}_{A^e}$ is complete and cocomplete.*

Proof. Apply Theorem III.1.1 of [6, page 51] with $R = A^e = A \wedge_S A^{op}$. □

The functor

$$\wedge_A : \mathcal{M}_{A^e} \times \mathcal{M}_{A^e} \longrightarrow \mathcal{M}_{A^e},$$

which sends a pair of bimodules (M, N) to $M \wedge_A N$ [6, Definition III.3.1, page 58] makes $\mathcal{M}_{A^e}$ into a monoidal category with unit A. Let us further write $M^n = M \wedge_A M \wedge_A \cdots \wedge_A M$, where there are n factors M in the smash product. By convention, $M^0 = A$. Then the endofunctor

$$\mathbb{T}_A : \mathcal{M}_{A^e} \longrightarrow \mathcal{M}_{A^e}$$

given by

$$\begin{aligned} \mathbb{T}_A(M) &= \bigvee_{n \geq 0} M^n \\ &= A \vee M \vee (M \wedge_A M) \vee (M \wedge_A M \wedge_A M) \vee \cdots \end{aligned}$$

with the coproduct taken in the category of A-bimodules is a monad on $\mathcal{M}_{A^e}$ whose algebras are exactly the algebras under A.

As usual, the unit $\eta : \mathrm{Id} \longrightarrow \mathbb{T}_A$ is given by the inclusion of $M = M^1$. The product $\mu : \mathbb{T}_A \circ \mathbb{T}_A \longrightarrow \mathbb{T}_A$ is induced by the maps

$$M^{j_1} \wedge_A M^{j_2} \wedge_A \cdots \wedge_A M^{j_k} \longrightarrow M^{j_1+j_2+\cdots+j_k}$$

that are the obvious identifications if each $j_r \geq 1$ and are identifications derived from the identification $X \wedge_A A \wedge_A Y \cong X \wedge_A Y$ if some $j_r = 0$. We think of $\mathbb{T}_A(M)$ as the tensor algebra over A associated to the bimodule M. Since we make no supposition about the left and right multiplications by A here, a $\mathbb{T}_A$-algebra is not an A-algebra in the sense of [6, page 146].

Our next goal is to describe the construction of colimits in the category of S-algebras under A. First we need the following adjointness lemma.

Lemma 2.2. *Let Q, R, and T be S-algebras. Let M be a $(Q-R)$-bimodule, N an $(R-T)$-bimodule, and P be a $(Q-T)$-bimodule. Then the function spectrum $F_Q(M,P)$ inherits a natural $(R-T)$-bimodule structure, and there is a natural homeomorphism of spaces of bimodule morphisms*

$$\mathrm{Hom}_{Q-T}(M \wedge_R N, P) \cong \mathrm{Hom}_{R-T}(N, F_Q(M,P)).$$

Proof. The adjunction formula is well known in homological algebra [8, VI, 8.8, page 194]. The result carries over to the spectrum setting by the same proof as that given in [6, page 66] for Lemma III.6.5. □

Taking $Q = R = T = A$ in the preceding lemma we see that the monoidal category $\mathcal{M}_{A^e}$ is closed in the sense of Definition II.7.1 of [6, page 47].

We also need the following property of $\mathbb{T}_A$.

Lemma 2.3. *The functor $\mathbb{T}_A$ preserves reflexive coequalizers.*

Proof. The notion of reflexive coequalizer is recalled in [6, page 46] in Definition II.6.5. The proof can be read off from the proof of Proposition II.7.2 of [6] by observing that the assumption that the monoidal category be symmetric is superfluous for the functor $\mathbb{T}$, as opposed to the functor $\mathbb{P}$ which is also discussed in the proposition. This proof requires the closedness of $\mathcal{M}_{A^e}$ remarked immediately above. □

Now we are ready to conclude that the category $\mathcal{M}_{A^e}[\mathbb{T}_A]$ of algebras under A is cocomplete. To arrive at this we apply [6, Proposition II.7.4, page 49], which requires us to combine the cocompleteness of $\mathcal{M}_{A^e}$ with lemma 2.3. This technique is also related to results of [7] and [1, Theorem 2, page 319]. Copying [6, Proposition II.7.4] we have the following general recipe for the calculation of colimits in $\mathcal{M}_{A^e}[\mathbb{T}_A]$.

Theorem 2.4. *Let $\{R_i\}$ be a diagram of algebras under A. For each i, let $\eta_i : R_i \longrightarrow \mathbb{T}_A(R_i)$ be the unit map and $\xi_i : \mathbb{T}_A(R_i) \longrightarrow R_i$ be the structure map of the algebra R_i. The following coequalizer diagram of bimodules*

$$\mathbb{T}_A(\operatorname{colim} \mathbb{T}_A R_i) \underset{\mu\circ\mathbb{T}_A(\alpha)}{\overset{\mathbb{T}_A(\operatorname{colim}\xi_i)}{\rightrightarrows}} \mathbb{T}_A(\operatorname{colim} R_i) \longrightarrow \operatorname{colim}_{\mathbb{T}_A} R_i,$$

in which $\alpha : \operatorname{colim} \mathbb{T}_A R_i \longrightarrow \mathbb{T}_A \operatorname{colim} R_i$ is the natural map and the map μ is the multiplication of the monad $\mathbb{T}_A$, is a reflexive coequalizer diagram, via $\mathbb{T}_A(\operatorname{colim} \eta_i)$. Then $\operatorname{colim}_{\mathbb{T}_A} R_i$ has a unique structure of $\mathbb{T}_A$-algebra making the righthand arrow a map of $\mathbb{T}_A$-algebras and the diagram a colimit diagram in $\mathcal{M}_{A^e}[\mathbb{T}_A]$.

□

In particular, a diagram $C \xleftarrow{\gamma} A \xrightarrow{\beta} B$ of S-algebras under A can be completed to a free product diagram

$$\begin{array}{ccc} A & \xrightarrow{\beta} & B \\ {\scriptstyle\gamma}\downarrow & & \downarrow \\ C & \longrightarrow & B *_A C \end{array}$$

in the category of algebras under A. We will see now that under certain cellularity hypotheses the completed diagram has an interesting homotopy property that will be essential for our work. Along the way we look closely at the construction of $B *_A C$ delivered by the preceding theorem.

First we review the definition of a weakly homotopy cocartesian square of S-modules.

Definition 2.5. The commuting diagram of S-modules

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ Y & \xrightarrow[f']{} & Z \end{array}$$

is weakly homotopy cocartesian if the canonical map

$$M(f,g) \longrightarrow Z$$

is a weak homotopy equivalence.

Here $M(f,g)$ is the double mapping cylinder of f and g, defined as the colimit of the diagram

$$\begin{array}{ccccc} & W & & W & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle i_0} \quad {\scriptstyle i_1}\swarrow & & \searrow{\scriptstyle g} \\ X & & W \wedge I_+ & & Y \end{array}$$

in the category of S-modules. If the maps f and g are apparent from the context, we may instead use the notation $M(X, W, Y)$ for the double mapping cylinder. We will also have occasion to use equivalent formulations of the weakly homotopy cocartesian condition as given in the following well-known lemma.

Lemma 2.6. *The diagram of S-modules*

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ Y & \xrightarrow[f']{} & Z \end{array}$$

is weakly homotopy cocartesian if and only if the map

$$C(f) \longrightarrow C(f')$$

of mapping cones induced by the pair (g, g') is a weak homotopy equivalence.

□

Our first major result is the following theorem.

Theorem 2.7. *Let $\beta : A \longrightarrow B$ and $\gamma : A \longrightarrow C$ be a pair of inclusions of S-algebras that are cell left A-modules relative to A. Then $B *_A C$ is a cell left A-, B-, and C-module. Moreover, $D = B *_A C$ satisfies the homotopy property that the diagram*

$$\begin{array}{ccc} D \wedge_A D & \longrightarrow & D \wedge_B D \\ \downarrow & & \downarrow \\ D \wedge_C D & \longrightarrow & D \wedge_D D \end{array}$$

is a weakly homotopy cocartesian diagram of S-modules.

In the following proof we need the monomial word modules in A-bimodules M and N, which are the smash products

$$M \wedge_A N \wedge_A M \wedge_A \cdots \qquad \text{and} \qquad N \wedge_A M \wedge_A N \wedge_A \cdots$$

Indeed, to show D is a cell left A-module, we will show there is an expanding sequence

$$A = D_0 \subset D_1 \subset \cdots \subset D_n \subset \cdots \subset B *_A C = D$$

such that, for $n \geq 1$, D_n/D_{n-1} is the wedge of the two monomial word modules in B/A and C/A.

Proof. We start by reviewing in our special case the construction of theorem 2.4. To simplify notation, we write $\mathbb{T}$ instead of $\mathbb{T}_A$ during this proof. Thus, consider the coequalizer diagram

$$\mathbb{T}(\mathbb{T}B \vee \mathbb{T}C) \underset{f}{\overset{e}{\rightrightarrows}} \mathbb{T}(B \vee C) \longrightarrow B *_A C$$

in $\mathcal{M}_{A^e}$. The map e is induced from the structure maps $\mathbb{T}B \longrightarrow B$ and $\mathbb{T}C \longrightarrow C$, which encode the fact that B and C are algebras under A. The map f is arises as a composite, as follows. First, the inclusions of B and C in $\mathbb{T}(B \vee C)$ induce

$$\mathbb{T}B \vee \mathbb{T}C \longrightarrow \mathbb{T}(B \vee C).$$

Second, $\mathbb{T}$ is a monad, so we have a structure map $\mu : \mathbb{T}(\mathbb{T}(B \vee C)) \longrightarrow \mathbb{T}(B \vee C)$, and we define $f : \mathbb{T}(\mathbb{T}B \vee \mathbb{T}C) \longrightarrow \mathbb{T}(B \vee C)$ to be the composite

$$\mathbb{T}(\mathbb{T}B \vee \mathbb{T}C) \longrightarrow \mathbb{T}(\mathbb{T}(B \vee C)) \xrightarrow{\mu} \mathbb{T}(B \vee C).$$

The essential property of f is that the composition

$$\mathbb{T}(B) \xrightarrow{\eta_1} \mathbb{T}(\mathbb{T}B \vee \mathbb{T}C) \xrightarrow{f} \mathbb{T}(B \vee C)$$

coincides with the inclusion $\mathbb{T}i_1 : \mathbb{T}B \longrightarrow \mathbb{T}(B \vee C)$, a fact which is a consequence of the naturality of the unit η and the multiplication μ, together with the identity $\mu \circ \eta = \mathrm{id}$.

Although we don't need it for the present proof, we also remark upon the map

$$h : \mathbb{T}(B \vee C) \longrightarrow \mathbb{T}(\mathbb{T}B \vee \mathbb{T}C)$$

making the diagram into a reflexive coequalizer diagram. This map h is obtained by applying $\mathbb{T}$ to the inclusion $B \vee C \longrightarrow \mathbb{T}B \vee \mathbb{T}C$. It is clear that the identity $eh = \mathrm{id}$ holds, and it is not much more difficult to check that $fh = \mathrm{id}$ as well.

Returning to the main thrust of the proof, define $\mathbb{T}_{\leq n}(B \vee C)$ to be the A^e-submodule of $\mathbb{T}(B \vee C)$ consisting of wedge summands with no more than n smash product factors B or C. For example,

$$\begin{gathered}\mathbb{T}_{\leq 0}(B \vee C) = A, \\ \mathbb{T}_{\leq 1}(B \vee C) = A \vee B \vee C, \\ \mathbb{T}_{\leq 2}(B \vee C) = A \vee (B \vee C) \vee (B \wedge_A B \vee B \wedge_A C \vee C \wedge_A B \vee C \wedge_A C),\end{gathered}$$

and so on. Define $\mathbb{T}'_{\leq n}(\mathbb{T}B \vee \mathbb{T}C)$ as follows. Substituting $\mathbb{T}B$ and $\mathbb{T}C$ for B and C, respectively, in the previous definition, we obtain that $\mathbb{T}_{\leq n}(\mathbb{T}B \vee \mathbb{T}C)$ is the A^e-submodule of $\mathbb{T}(\mathbb{T}B \vee \mathbb{T}C)$ consisting of wedge summands with no more than n smash product factors $\mathbb{T}B$ or $\mathbb{T}C$. Each of $\mathbb{T}B$ and $\mathbb{T}C$ is also graded by the number of smash product factors. We define $\mathbb{T}'_{\leq n}(\mathbb{T}B \vee \mathbb{T}C)$ to be the A^e submodule of $\mathbb{T}_{\leq n}(\mathbb{T}B \vee \mathbb{T}C)$ consisting of wedge

summands with no more than n smash product factors B or C. Finally, define D_n as the coequalizer of the maps e and f restricted as in the following diagram.

$$\mathbb{T}'_{\leq n}(\mathbb{T}B \vee \mathbb{T}C) \underset{f}{\overset{e}{\rightrightarrows}} \mathbb{T}_{\leq n}(B \vee C) \longrightarrow D_n.$$

Clearly we have maps $D_n \longrightarrow D$ which are compatible with similar maps $D_n \longrightarrow D_{n+1}$ so that $D = \operatorname{colim} D_n$. Since the action maps $\mathbb{T}B \longrightarrow B$ and $\mathbb{T}C \longrightarrow C$ are involved in e, and the inclusions $\mathbb{T}B \longrightarrow \mathbb{T}(B \vee C)$ and $\mathbb{T}C \longrightarrow \mathbb{T}(B \vee C)$ are involved in f, the wedge of the monomial word modules in B and C, where the words are of length no more than n, maps onto D_n. For example,

$$\begin{gathered} A \xrightarrow{\cong} D_0 \\ B \vee C \longrightarrow D_1, \\ B \vee C \vee (B \wedge_A C \vee C \wedge_A B) \longrightarrow D_2, \end{gathered}$$

and so on, are all onto.

Our first objective is to show that the A^e-modules D_n are cell left A-modules, for all n. To this end, let $M \cup_A N$ be the coproduct in the category of left modules under A. This is defined as the pushout

$$\begin{array}{ccc} A \vee A & \longrightarrow & M \vee N \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \cup_A N \end{array}$$

in the category of left A-modules. Thus, the construction just identifies to A the two copies of A in the normal coproduct $M \vee N$. The maps involved are bimodule maps, so that $M \cup_A N$ is, in fact, still an A^e-module, but this extra structure is not of interest at the moment. Worthy of note is the obvious fact that if M and N are relative cell A-modules, then $M \cup_A N$ is also a relative cell A-module. Let

$$W_n^B = B \wedge_A C \wedge_A B \cdots \qquad \text{and} \qquad W_n^C = C \wedge_A B \wedge_A C \cdots$$

be the two monomial word modules of length n in B and C. Put $U_n = W_n^B \cup_A W_n^C$, and observe that U_n is a relative cell left A-module, equipped with a map $U_n \longrightarrow D_n$. Let V_n be the union of the submodules of U_n obtained by replacing a B or a C by an A. The four isomorphisms $A \wedge_A B \longrightarrow B$, $B \wedge_A A \longrightarrow B$, $A \wedge_A C \longrightarrow C$, and $C \wedge_A A \longrightarrow C$ induce a map $V_n \longrightarrow D_{n-1}$ and

$$\begin{array}{ccc} V_n & \longrightarrow & D_{n-1} \\ \downarrow & & \downarrow \\ U_n & \longrightarrow & D_n \end{array}$$

is a pushout. Then

$$D_n/D_{n-1} \cong U_n/V_n \cong \overline{W}_n^B \vee \overline{W}_n^C,$$

where we define $\bar{B} = B/A$ and $\bar{C} = C/A$, and let $\overline{W}_n^B = \bar{B} \wedge_A \bar{C} \wedge_A \cdots$ and $\overline{W}_n^C = \bar{C} \wedge_A \bar{B} \wedge_A \cdots$ be the word monomials in the reduced modules $\bar{B}$ and $\bar{C}$. Since $\bar{B}$ and $\bar{C}$ are cell A-modules, an inductive argument shows that D_n is a cell A-module as well, for all n.

To handle the homotopy theoretic part of the proof, we need variations on the construction of the A^e-submodules D_n to make an inductive argument. Define $B \cdot \mathbb{T}_{\leq n}(B \vee C)$ to be the $(B-A)$-submodule of $\mathbb{T}(B \vee C)$ consisting of wedge summands with no more than $n+1$ smash factors and with B on the left. For example,

$$B \cdot \mathbb{T}_{\leq 0}(B \vee C) = B = B \wedge_A \mathbb{T}_{\leq 0}(B \vee C)$$
$$B \cdot \mathbb{T}_{\leq 1}(B \vee C) = B \vee (B \wedge_A B \vee B \wedge_A C) = B \wedge_A \mathbb{T}_{\leq 1}(B \vee C)$$

and, in general,

$$B \cdot \mathbb{T}_{\leq n}(B \vee C) = B \wedge_A \mathbb{T}_{\leq n}(B \vee C).$$

Define $B \cdot \mathbb{T}'_{\leq n}(\mathbb{T}B \vee \mathbb{T}C)$ by similarly varying the definition of $\mathbb{T}'_{\leq n}(\mathbb{T}B \vee \mathbb{T}C)$, and note that $B \cdot \mathbb{T}'_{\leq n}(\mathbb{T}B \vee \mathbb{T}C) = B \wedge_A \mathbb{T}'_{\leq n}(\mathbb{T}B \vee \mathbb{T}C)$, as well. Define $B \cdot D_n$ to coequalize the restrictions of e and f in the diagram

$$B \cdot \mathbb{T}'_{\leq n}(\mathbb{T}B \vee \mathbb{T}C) \underset{f}{\overset{e}{\rightrightarrows}} B \cdot \mathbb{T}_{\leq n}(B \vee C) \longrightarrow B \cdot D_n.$$

Reasoning as above, we see that the wedge of the monomial word modules in B and C, where B is the initial symbol and the words are of length no more than $n+1$, maps onto $B \cdot D_n$. For example,

$$B \xrightarrow{\cong} B \cdot D_0,$$
$$B \vee B \wedge_A C \longrightarrow B \cdot D_1,$$
$$B \vee B \wedge_A C \vee B \wedge_A C \wedge_A B \longrightarrow B \cdot D_2,$$

and so on, are all onto. Using the notations for word modules introduced above, we observe that there is a map of left B-modules $B \wedge_A W_n^C \longrightarrow B \cdot D_n$ and that there is a pushout diagram

$$\begin{array}{ccc} B \wedge_A (W_n^C \cap V_{n-1}) & \longrightarrow & B \cdot D_{n-1} \\ \downarrow & & \downarrow \\ B \wedge_A W_n^C & \longrightarrow & B \cdot D_n, \end{array}$$

where $W_n^C \cap V_{n-1}$ is the union of the submodules of W_n^C obtained by substituting an A for an occurence of B or C in W_n^C. Consequently,

$$B \cdot D_n / B \cdot D_{n-1} \cong B \wedge_A (W_n^C / W_n^C \cap V_{n-1}) \cong B \wedge_A \overline{W}_n^C.$$

We define $(C - A)$-submodules $C \cdot D_n$ of D in a similar manner, and obtain the analogous formula

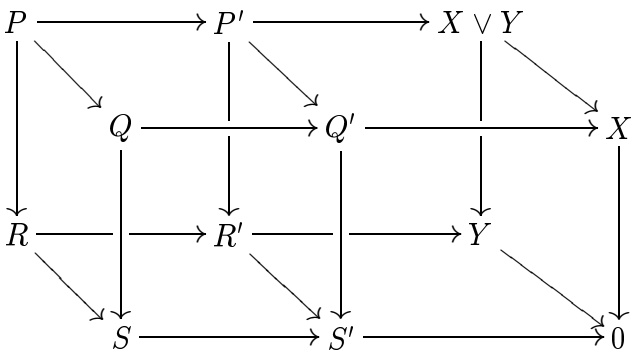
$$C \cdot D_n / C \cdot D_{n-1} \cong C \wedge_A \overline{W}_n^B.$$

Using the preceding formulas and induction, we also find that D is also a relative cell B-module or, respectively, a relative cell C-module.

Using the fact that the smash product with D over A, B, and C preserves cofibrations, we derive from the preceding remarks a list of cofibration sequences

$$D \wedge_A D_{n-1} \rightarrowtail D \wedge_A D_n \twoheadrightarrow D \wedge_A (\overline{W}_n^B \vee \overline{W}_n^C),$$
$$D \wedge_B B \cdot D_{n-1} \rightarrowtail D \wedge_B B \cdot D_n \twoheadrightarrow D \wedge_B (B \wedge_A \overline{W}_n^C),$$
$$D \wedge_C C \cdot D_{n-1} \rightarrowtail D \wedge_C C \cdot D_n \twoheadrightarrow D \wedge_C (C \wedge_A \overline{W}_n^B),$$
$$D \wedge_D D \rightarrowtail D \wedge_D D \twoheadrightarrow 0,$$

which may be assembled for each $n \geq 1$ into a commutative diagram of squares of the following form.



For any $n \geq 1$ the righthand square

$$\begin{array}{ccc} D \wedge_A (\overline{W}_n^B \vee \overline{W}_n^C) & \longrightarrow & D \wedge_B (B \wedge_A \overline{W}_n^C) \\ \downarrow & & \downarrow \\ D \wedge_C (C \wedge_A \overline{W}_n^B) & \longrightarrow & 0 \end{array}$$

is clearly weakly homotopy cocartesian; for $n-1 = 0$ the lefthand square is

$$\begin{array}{ccc} D \wedge_A A & \longrightarrow & D \wedge_B B \\ \downarrow & & \downarrow \\ D \wedge_C C & \longrightarrow & D \wedge_D D, \end{array}$$

which is also clearly weakly homotopy cocartesian. Therefore, inductively assuming the square

$$\begin{array}{ccc} D \wedge_A D_{n-1} & \longrightarrow & D \wedge_B (B \cdot D_{n-1}) \\ \downarrow & & \downarrow \\ D \wedge_C (C \cdot D_{n-1}) & \longrightarrow & D \wedge_D D. \end{array}$$

is weakly homotopy cocartesian, we obtain that the square

$$\begin{array}{ccc} D \wedge_A D_{n} & \longrightarrow & D \wedge_B (B \cdot D_{n}) \\ \downarrow & & \downarrow \\ D \wedge_C (C \cdot D_{n}) & \longrightarrow & D \wedge_D D \end{array}$$

is weakly homotopy cocartesian for all n.

To conclude the argument, we must observe that the square

$$\begin{array}{ccc} D \wedge_A D & \longrightarrow & D \wedge_B D \\ \downarrow & & \downarrow \\ D \wedge_C D & \longrightarrow & D \wedge_D D, \end{array}$$

which is the colimit of the weakly homotopy cocartesian squares above, is also weakly homotopy cocartesian. But this is a simple consequence of the fact that colimits commute. □

Standard methods of homotopical algebra [5] permit us to draw the following corollary from the preceding theorem.

Corollary 2.8. *Suppose that A, B, and C are q-cofibrant S-algebras and that $\beta : A \longrightarrow B$ and $\gamma : A \longrightarrow C$ are q-cofibrations. Construct $D = B *_A C$ using the monad $\mathbb{T}_A$ as above. Then D is a q-cofibrant S-algebra which is q-cofibrant as a left A-, B-, or C module. Also in this case the diagram*

$$\begin{array}{ccc} D \wedge_A D & \longrightarrow & D \wedge_B D \\ \downarrow & & \downarrow \\ D \wedge_C D & \longrightarrow & D \wedge_D D \end{array}$$

is weakly homotopy cocartesian.

In this statement we have used the language of of q-cofibrations and q-cofibrant objects in a closed model category, as adopted in [6, Chapter VII, Section 4]. The reason for this is to distinguish the notion of a cofibration in a closed model category from the notion of cofibration in abstract algebraic K-theory.

Proof. As above, D has filtrations by A-, B-, and C-submodules. The filtration inclusions are appropriate q-cofibrations, so the q-cofibrant properties for D follow.

Next we discuss the homotopy property of the diagram displayed in the statement. Since this time the modules involved in the filtrations are not necessarily cellular, we use the following remarks to reduce to the earlier case. Consider a variant of the factorization construction of [6, Lemma VII.5.2, page 145] used to build a factorization

$$A \longrightarrow B' \longrightarrow B$$

of β into a q-cofibration followed by a trivial q-fibration. Make a construction parallel to that of the lemma using the bar construction $\beta^S(A, \mathbb{T}E, \mathbb{T}CE)$ in place of the pushout $A \amalg_{\mathbb{T}E} \mathbb{T}CE$. (The monad $\mathbb{T}$ and the pushout construction will be recalled below). We take advantage of the weak homotopy equivalences

$$\psi : \beta^S(A, \mathbb{T}E, \mathbb{T}CE) \longrightarrow A \amalg_{\mathbb{T}E} \mathbb{T}CE$$

supplied by Lemma VII.7.3 of [6] to see that at the first stage we are not making any essential changes. Repeating the proof of Lemma VII.7.5 of [6] shows that $\beta^S(A, \mathbb{T}E, \mathbb{T}CE)$ is a left A-cell module. As in [6, Lemma VII.5.2, page 145] we proceed inductively using the bar construction to build a sequence of left cell A-modules parallel to the sequence of pushouts with each term in the bar construction sequence weakly homotopy equivalent to its partner pushout. In the limit we obtain a factorization

$$A \longrightarrow B'' \longrightarrow B$$

as above, where B'' is a left cell A-module by Proposition III.2.6 of [6]. Since B and B'' are q-cofibrant, the map $B'' \longrightarrow B$ has a homotopy inverse. Since the maps $\beta : A \longrightarrow B$ and $A \longrightarrow B''$ are q-cofibrations, B and B'' are actually homotopy equivalent, relative to A. Consequently, there is an induced homotopy equivalence $B''/A \longrightarrow B/A$. We may treat the arrow γ similarly, and find a replacement C'' for C which is a cell left A-module and a homotopy equivalence $C''/A \longrightarrow C/A$. Define $D'' = B'' *_A C''$ and consider the induced map $D'' \longrightarrow D$. This map is filtration preserving, and we can compare the filtrations on the respective auxiliary diagrams as in the statement. Using induction and the induced homotopy equivalences $B''/A \longrightarrow B/A$ and $C''/A \longrightarrow C/A$, the homotopy cocartesian property for $D = B *_A C$ follows from the homotopy cocartesian property for $D'' = B'' *_A C''$. □

Our next goal is to develop some homotopy invariance properties of our construction. For this we choose to compare our construction with the construction of pushout diagrams of S-algebras for which homotopy invariance properties are already known and developed in [6]. Therefore, we proceed to the review of a construction of a pushout diagram of S-algebras, making explicit the recipe called for in [6, page 136]. Given A, B, C, and the maps β and γ from A, a pushout diagram of S-algebras

$$\begin{array}{ccc} A & \xrightarrow{\beta} & B \\ {\scriptstyle\gamma}\downarrow & & \downarrow \\ C & \longrightarrow & B \amalg_A C \end{array}$$

is constructed using [6, Proposition II.7.4, page 48] in the following manner. As an aid to the discussion, for S-modules M, N, and P denote by $N \cup_M P$ the S-module that makes the diagram

$$\begin{array}{ccc} M & \longrightarrow & N \\ \downarrow & & \downarrow \\ P & \longrightarrow & N \cup_M P \end{array}$$

a pushout (coproduct) diagram [6, page 32] in the category of S-modules $\mathcal{M}_S$. Let $\mathbb{T} : \mathcal{M}_S \longrightarrow \mathcal{M}_S$ denote the tensor algebra monad on the category of S-modules [6, page 47]. By the definition of an S-algebra there are S-module maps $\mathbb{T}A \longrightarrow A$, $\mathbb{T}B \longrightarrow B$, and $\mathbb{T}C \longrightarrow C$ codifying the S-algebra structures on A, B, and C, respectively. Let

$$e : \mathbb{T}(\mathbb{T}B \cup_{\mathbb{T}A} \mathbb{T}C) \longrightarrow \mathbb{T}(B \cup_A C)$$

denote the map obtained by applying $\mathbb{T}$ to the obvious map induced by the algebra structures. Denote the canonical maps to the pushout by $i_A : A \longrightarrow B \cup_A C$, $i_B : B \longrightarrow B \cup_A C$, and $i_C : C \longrightarrow B \cup_A C$ and let

$$\alpha : \mathbb{T}B \cup_{\mathbb{T}A} \mathbb{T}C \longrightarrow \mathbb{T}(B \cup_A C)$$

be the unique map such that

$$\mathbb{T}(B) \longrightarrow \mathbb{T}B \cup_{\mathbb{T}A} \mathbb{T}C \xrightarrow{\alpha} \mathbb{T}(B \cup_A C)$$

agrees with $\mathbb{T}(i_B)$ and similarily for A and C. Then define

$$f : \mathbb{T}(\mathbb{T}B \cup_{\mathbb{T}A} \mathbb{T}C) \longrightarrow \mathbb{T}(B \cup_A C)$$

to be the composite $\mathbb{T}(\mathbb{T}B \cup_{\mathbb{T}A} \mathbb{T}C) \xrightarrow{\mathbb{T}(\alpha)} \mathbb{T}(\mathbb{T}(B \cup_A C)) \xrightarrow{\mu} \mathbb{T}(B \cup_A C)$. Finally, define $B \amalg_A C$ by the following coequalizer diagram of S-modules.

$$\mathbb{T}(\mathbb{T}B \cup_{\mathbb{T}A} \mathbb{T}C) \underset{f}{\overset{e}{\rightrightarrows}} \mathbb{T}(B \cup_A C) \longrightarrow B \amalg_A C$$

According to [6, Proposition II.7.4, page 48] the diagram

$$\begin{array}{ccc} A & \xrightarrow{\beta} & B \\ {\scriptstyle\gamma}\downarrow & & \downarrow \\ C & \longrightarrow & B \amalg_A C \end{array}$$

is a coproduct diagram in the category of S-algebras. Note that the S-algebra map $A \longrightarrow B \amalg_A C$ arises from mapping the diagram

$$\mathbb{T}\mathbb{T}A \underset{\mathbb{T}(\mu_A)}{\overset{\mu_{\mathbb{T}A}}{\rightrightarrows}} \mathbb{T}A \xrightarrow{\mu_A} A$$

to the coequalizer diagram above, and similarly for B and C. Consequently the square diagram above commutes.

Let us now confirm that we have just given another formula for $B *_A C$.

Proposition 2.9. *The two diagrams of S-algebras*

$$\begin{array}{ccc} A & \xrightarrow{\beta} & B \\ {\scriptstyle\gamma}\downarrow & & \downarrow \\ C & \longrightarrow & B \amalg_A C \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} A & \xrightarrow{\beta} & B \\ {\scriptstyle\gamma}\downarrow & & \downarrow \\ C & \longrightarrow & B *_A C \end{array}$$

are isomorphic by a map of diagrams respecting the corners A, B and C.

Proof. Basically, all we have to do to prove this result is write out all of the data contained in the descriptions of the two universal properties and realize the results are the same. More formally, our construction of $B *_A C$ is an S-algebra under A equipped with S-algebra maps from B and C, as diagrammed in the statement of the proposition. Consequently, by the universal property of $B \amalg_A C$ in the category of S-algebras, there is a unique map of S-algebras from $B \amalg_A C \longrightarrow B *_A C$ making all triangles commute. Similarly, in the course of describing the construction of $B \amalg_A C$, we observed that $B \amalg_A C$ is also an algebra under A, equipped with maps of algebras under A from B and C, as codified in the other diagram found in the statement of the proposition. Consequently, by the universal property of $B *_A C$ there is a unique map $B *_A C \longrightarrow B \amalg_A C$ of algebras under A compatible with the maps from B and C. Then the usual argument shows that the arrows just constructed are inverse isomorphisms of S-algebras (under A). □

Proposition 2.10. *If the horizontal arrows in the diagram*

$$\begin{array}{ccccc} C & \longleftarrow & A & \longrightarrow & B \\ \downarrow & & {\scriptstyle=}\downarrow & & \downarrow \\ C' & \longleftarrow & A & \longrightarrow & B' \end{array}$$

are q-cofibrations of q-cofibrant S-algebras, and the outer vertical arrows are weak homotopy equivalences, then the induced maps

$$B \amalg_A C \longrightarrow B' \amalg_A C' \quad \textit{and} \quad B *_A C \longrightarrow B' *_A C'$$

are also weak homotopy equivalences

Proof. The result for $B \amalg_A C$ follows by two applications of [6, Proposition VII.7.4, page 153]; the result for $B *_A C$ follows from this and the preceding proposition. □

Corollary 2.11. *Under the hypotheses of the preceding proposition the map on K-theory*

$$K(B *_A C) \longrightarrow K(B' *_A C')$$

is a homotopy equivalence.

Proof. Apply Proposition VI.4.2 of [6, page 114]. □

Now suppose we are given a diagram of S-algebras

$$C \xleftarrow{\gamma} A \xrightarrow{\beta} B$$

that we would like to complete to a generalized free product diagram. Our problem is to ensure that the generalized free product D has the correct weak homotopy type. To achieve this we make the following replacements, along the lines of replacing ordinary topological spaces by CW-complexes or replacing algebras by flat $\mathbf{Z}$-algebras. Factoring the canonical map $S \longrightarrow A$ as $S \longrightarrow A' \longrightarrow A$ where the first arrow is a q-cofibration and the second is a weak equivalence (which is also a q-fibration), we replace the original diagram by

$$C \xleftarrow{\gamma'} A' \xrightarrow{\beta'} B$$

where we have just precomposed by the map from A' to A. Then we factor $A' \longrightarrow B$ as $A' \xrightarrow{\beta''} B' \longrightarrow B$ where β'' is a q-cofibration and $B' \longrightarrow B$ is similarly a weak equivalence (and a q-fibration). We treat the other leg similarly. If we fix the choice of A', then we have just seen that our subsequent constructions are independent of further choices, as far as K-theory is concerned.

This approach to ensuring the good homotopy properties of a generalized free product diagram by deriving them from results already proved in [6] seems to require the replacement of the algebra A in the first step. Possibly this replacement is unnecessary, at the cost of developing results to substitute for those we have chosen to quote.

Remark 2.12. In case $A = S$, the sphere spectrum, then there is an alternative to the model category theoretic approach to finding a well-behaved replacement for it. Namely, according to [6, page 33] the preferred tactic when working homotopically with S is to replace S by $S^0_S = S \wedge_{\mathcal{L}} \mathbb{L}S^0$.

3 Mayer-Vietoris presentations

In this section we make the definitions of the categories of Mayer-Vietoris presentations associated to a generalized free product diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D. \end{array}$$

To define K-theory we need to impose notions of finiteness and to define cofibrations and weak equivalences. An effective comparison of K-theories with respect to two notions of weak equivalence using the Fibration Theorem [18, Theorem 1.6.4, page 350] requires the existence of a suitable cylinder functor T. Therefore, we also construct this gadget.

Underlying the section are some basic notions about categories of modules over an S-algebra R, such as finiteness conditions, cofibration conditions, mapping cylinders, and notions of equivalence. We review briefly these notions before we define the corresponding notions for Mayer-Vietoris presentations. Also, we define a somewhat more rigid notion of Mayer-Vietoris presentation, along the lines of definition 2.3 of [15, page 169]. We obtain that, with respect to a fixed notion of weak equivalence, each variant of of the definition delivers the same K-theory by assembling the results of theorems 3.21, 4.2, and 6.2.

Now let R be an S-algebra; we review module categories over R. Recall the categories $\mathcal{M}(R)_f$ and $\mathcal{M}(R)_{hf}$ of finite and homotopy finite cell R-modules. The prototypes of these categories for simplicial rings were introduced in [18, page 393]; the category $\mathcal{M}(R)_f$ is discussed in [6, page 110], where it is denoted $f\mathcal{C}_R$. An object of $\mathcal{M}(R)_f$ is finite cell R-module [6, page 54], a cofibration in $\mathcal{M}(R)_f$ is a map which is isomorphic to the inclusion of a subcomplex, and a weak equivalence in $\mathcal{M}(R)_f$ is a homotopy equivalence. An object of $\mathcal{M}(R)_{hf}$ is an R-module weakly homotopy equivalent to a finite cell R-module, a cofibration in $\mathcal{M}(R)_{hf}$ is a map satisfying the homotopy extension property, and a weak equivalence in $\mathcal{M}(R)_{hf}$ is a weak homotopy equivalence. With these definitions, the fact that $\mathcal{M}(R)_{hf}$ is a category with cofibrations and weak equivalences depends on [6, Corollary I.3.5, page 16] which implies that weak equivalences in $\mathcal{M}(R)_{hf}$ satisfy the "gluing lemma" (axiom *Weq2*) [18, page 326], which requires pushouts along cofibrations to preserve weak equivalences. The following result is standard; for the proof see [18, page 393] or [6, page 113].

Theorem 3.1. *The inclusion*

$$\mathcal{M}(R)_f \longrightarrow \mathcal{M}(R)_{hf}$$

is an exact functor which induces a homotopy equivalence

$$hS_\bullet\mathcal{M}(R)_f \longrightarrow wS_\bullet\mathcal{M}(R)_{hf}.$$

Thus, the two categories $\mathcal{M}(R)_f$ and $\mathcal{M}(R)_{hf}$ deliver the same K-theory.

□

Now we briefly recall the notion of a cylinder functor in $\mathcal{C}$, a category with cofibrations. This notion is essentially an abstraction of the notion of the reduced mapping cylinder familiar in topology. Before we can give the definition of a cylinder functor on a category with cofibrations we need to recall from [18, page 322] the definitions of two auxiliary categories.

Definition 3.2. Let $\mathcal{C}$ be a category with cofibrations. Then the arrow category of $\mathcal{C}$, $\mathrm{Ar}(C)$ is the category whose objects are morphisms $(f : C \longrightarrow C')$ in $\mathcal{C}$, morphisms

$$(h, h') : (f : C \to C') \longrightarrow (g : D \to D')$$

are pairs of morphisms $h : C \longrightarrow D$ and $h' : C' \longrightarrow D'$ such that the resulting square commutes, and whose cofibrations are morphisms (h, h') such that each component is a cofibration in $\mathcal{C}$.

Definition 3.3. Let $\mathcal{C}$ be a category with cofibrations. The category $F_1\mathcal{C}$ is the full subcategory of $\mathrm{Ar}(\mathcal{C})$ consisting of the objects $(f : C \rightarrowtail C')$ where the morphism is a cofibration in $\mathcal{C}$. A cofibration in $F_1\mathcal{C}$ is cofibration in $\mathrm{Ar}(C)$ satisfying the additional property that

$$D \cup_C C' \rightarrowtail D'$$

is a cofibration.

Here is the definition of a cylinder functor, from [18, page 348].

Definition 3.4. A cylinder functor on $\mathcal{C}$ is a functor defined on $\mathrm{Ar}(\mathcal{C})$, the category of arrows in $\mathcal{C}$, with values in a certain category of diagrams in $\mathcal{C}$. The cylinder functor associates to an arrow $f : C \longrightarrow C'$ a commuting diagram

$$\begin{array}{ccccc} C & \overset{j_1}{\rightarrowtail} & T(f) & \overset{j_2}{\leftarrowtail} & C' \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle =} & \\ & & C' & & \end{array}$$

in which the front inclusion j_1 and the back inclusion j_2 assemble to an exact functor

$$\mathrm{Ar}\mathcal{C} \longrightarrow F_1\mathcal{C}$$
$$(f : C \to C') \mapsto (\, j_1 \vee j_2 : C \vee C' \rightarrowtail T(f)\,)$$

The map p is called the cylinder projection. In case the arrow f is the inclusion of the zero object of $\mathcal{C}$, then we require $T(0 \to C) = C$, and that both the back inclusion and the cylinder projection are the identity map on C.

In homotopy theory the cylinder projection p is always a homotopy equivalence. However, in a particular category with cofibrations and weak equivalences, the projection p may not be a weak equivalence. Should the projection p of a cylinder functor on $\mathcal{C}$ always be a weak equivalence, we say that the category of weak equivalences $w\mathcal{C}$ satisfies the cylinder axiom.

Definition 3.5. If $\mathcal{C}$ is a category with cofibrations and weak equivalences $w\mathcal{C}$ equipped with a cylinder functor T, then we say that the weak equivalences $w\mathcal{C}$ satisfy the cylinder axiom if the projection $p : T(f) \longrightarrow C'$ is in $w\mathcal{C}$ for every arrow $f : C \longrightarrow C'$ in $\mathcal{C}$.

One of our goals is to show that the categories of Mayer-Vietoris presentations support a cylinder functor and that both fine and coarse equivalences satisfy the cylinder axiom. As our first item of input we have the fact that our categories of modules support a cylinder functor. In the category $\mathcal{M}(R)$, for example, let $f : M \longrightarrow N$ be an R-module homomorphism, and define $T(f)$ so that the following diagram is a pushout in the category of R-modules.

$$\begin{array}{ccc} M \vee M & = & M \wedge \partial I_+ \rightarrowtail M \wedge I_+ \\ {\scriptstyle \mathrm{id}\vee f}\downarrow & & \downarrow \\ M \vee N & \rightarrowtail & T(f) \end{array}$$

Proposition 3.6. *The construction just described defines a mapping cylinder functor on the categories $\mathcal{M}(R)_f$ and $\mathcal{M}(R)_{hf}$ in the sense of definition 3.4. Moreover, with the notions of weak equivalence we have chosen for these categories, this mapping cylinder functor satisfies the cylinder axiom.*

Proof. The proof is routine. □

Now we are ready to define categories of Mayer-Vietoris presentations. First we give a definition that fixes the homotopy theoretic concept. Then we give two further definitions that impose finiteness conditions, and, finally, two more definitions that impose finiteness conditions and extra cofibration conditions. For the rest of the discussion a generalized free product diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is fixed.

Definition 3.7. A Mayer-Vietoris presentation is a commutative diagram

$$\begin{array}{ccc} M_A & \xrightarrow{\phi_{BA}} & M_B \\ {\scriptstyle \phi_{CA}}\downarrow & & \downarrow{\scriptstyle \phi_{DB}} \\ M_C & \xrightarrow[\phi_{DC}]{} & M_D \end{array}$$

where $M_A \in \mathcal{M}(A)$, $M_B \in \mathcal{M}(B)$, $M_C \in \mathcal{M}(C)$, and $M_D \in \mathcal{M}(D)$. In the diagram ϕ_{BA} and ϕ_{CA} are A-module maps $M_A \longrightarrow M_B$ and $M_A \longrightarrow M_C$, respectively, ϕ_{DB} is a B module map $M_B \longrightarrow M_D$, and ϕ_{DC} is a C-module map $M_C \longrightarrow M_D$

Also, we require that the diagram of D-modules

$$\begin{array}{ccc} M_A \wedge_A D & \longrightarrow & M_B \wedge_B D \\ \downarrow & & \downarrow \\ M_C \wedge_C D & \longrightarrow & M_D \end{array}$$

be weakly homotopy cocartesian. It is usually not necessary to label the structure maps, so, in general, we use the shorthand notations $M_- = (M_A, M_B, M_C, M_D)$ for a Mayer-Vietoris presentation.

A map of Mayer-Vietoris presentations is a quadruple $f_- = (f_A, f_B, f_C, f_D)$ of module maps such that the resulting cubical diagram commutes.

We obtain a category $\mathcal{MV}$.

Recall from section 2 that the diagram

$$\begin{array}{ccc} M_A \wedge_A D & \longrightarrow & M_B \wedge_B D \\ \downarrow & & \downarrow \\ M_C \wedge_C D & \longrightarrow & M_D \end{array}$$

is weakly homotopy cocartesian if the canonical map from the double mapping cylinder

$$M(M_B \wedge_B D, M_A \wedge_A D, M_C \wedge_C D) \longrightarrow M_D$$

is a weak homotopy equivalence. It will save a lot of writing later if we define the telescope of a Mayer-Vietoris presentation by

$$\mathrm{Tel}(M_-) = M(M_B \wedge_B D, M_A \wedge_A D, M_C \wedge_C D).$$

Next we define categories of Mayer-Vietoris presentations

$$\mathcal{MV}_f \longrightarrow \mathcal{MV}_{hf}$$

which we call the category of finite Mayer-Vietoris presentations and the category of homotopy finite Mayer-Vietoris presentations, respectively. For each of these categories we will define cofibrations, mapping cylinders, and two notions of weak equivalence. We will use the term v-equivalence for the finer notion and the term w-equivalences for the coarser. Later in the section we will add extra cofibration conditions to the definition of Mayer-Vietoris presentation, obtaining categories of Mayer-Vietoris presentations

$$\mathcal{MV}_{f,e} \longrightarrow \mathcal{MV}_{hf,e}$$

called the category of finite Mayer-Vietoris presentations with extra cofibration conditions and the category of homotopy-finite Mayer-Vietoris presentations with extra cofibration conditions, respectively. These latter categories are the ones used in [15], but it turns out to be more convenient to work with $\mathcal{MV}_f$ and $\mathcal{MV}_{hf}$. There is, of course, a commutative diagram of exact inclusions

$$\begin{array}{ccc} \mathcal{MV}_{f,e} & \longrightarrow & \mathcal{MV}_{hf,e} \\ \downarrow & & \downarrow \\ \mathcal{MV}_f & \longrightarrow & \mathcal{MV}_{hf} \end{array}$$

for which the induced arrows

$$\begin{array}{ccc} vS_\bullet\mathcal{MV}_{f,e} & \longrightarrow & vS_\bullet\mathcal{MV}_{hf,e} \\ \downarrow & & \downarrow \\ vS_\bullet\mathcal{MV}_f & \longrightarrow & vS_\bullet\mathcal{MV}_{hf} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} wS_\bullet\mathcal{MV}_{f,e} & \longrightarrow & wS_\bullet\mathcal{MV}_{hf,e} \\ \downarrow & & \downarrow \\ wS_\bullet\mathcal{MV}_f & \longrightarrow & wS_\bullet\mathcal{MV}_{hf} \end{array}$$

are all homotopy equivalences. This assertion will be proved by assembling the results of theorems 3.21, 4.2, and 6.2.

We continue with the definitions of the categories of Mayer-Vietoris presentations $\mathcal{MV}_f$ and $\mathcal{MV}_{hf}$.

Definition 3.8. The category of finite Mayer-Vietoris presentations $\mathcal{MV}_f$ is the full subcategory of $\mathcal{MV}$ consisting of Mayer-Vietoris presentations

$$\begin{array}{ccc} M_A & \xrightarrow{\phi_{BA}} & M_B \\ {\scriptstyle \phi_{CA}}\downarrow & & \downarrow{\scriptstyle \phi_{DB}} \\ M_C & \xrightarrow[\phi_{DC}]{} & M_D \end{array}$$

where, additionally, $M_A \in \mathcal{M}(A)_f$, $M_B \in \mathcal{M}(B)_f$, $M_C \in \mathcal{M}(C)_f$, and $M_D \in \mathcal{M}(D)_f$.

Again we use the shorthand notations $M_- = (M_A, M_B, M_C, M_D)$ for a finite Mayer-Vietoris presentation; the context will make the ambient category clear.

If we allow homotopy finite modules in the preceding definition, we obtain the category $\mathcal{MV}_{hf}$. It maybe helpful to note that when we are in $\mathcal{MV}_f$ and the Mayer-Vietoris presentation structure maps are cell maps, the weak homotopy equivalence of D-modules $\mathrm{Tel}(M_-) \longrightarrow M_D$ is actually a homotopy equivalence, because in this case all the D-modules written are cell D-modules.

Now that we have set up the notions of finiteness required to do K-theory, we proceed to define cofibrations.

Definition 3.9. A map of Mayer-Vietoris presentations

$$(f_A, f_B, f_C, f_D) : (M_A, M_B, M_C, M_D) \longrightarrow (M'_A, M'_B, M'_C, M'_D)$$

is a cofibration in $\mathcal{MV}_f$ or $\mathcal{MV}_{hf}$ if each of the four maps f_A, f_B, f_C, and f_D is a cofibration in the appropriate module category $\mathcal{M}(-)_f$ or $\mathcal{M}(-)_{hf}$.

To indicate a cofibration we follow custom and use a feathered arrow $f_- : M_- \rightarrowtail M'_-$.

We must now verify that pushouts exist in the categories of Mayer-Vietoris presentations $\mathcal{MV}_f$ and $\mathcal{MV}_{hf}$. Given a diagram of Mayer-Vietoris presentations

$$\begin{array}{ccc} M_- & \rightarrowtail & M'_- \\ \downarrow & & \\ N_- & & \end{array}$$

we define

$$N'_- = N_- \cup_{M_-} M'_-$$

by using the pushout construction of modules at each corner, obtaining

$$N'_A = N_A \cup_{M_A} M'_A,$$

for example, and we define structure maps using the universal property of the pushout. We must now prove the following proposition.

Proposition 3.10. *The quadruple of modules $N'_- = (N'_A, N'_B, N'_C, N'_D)$ together with these structure maps is a Mayer-Vietoris presentation and the canonical arrow*

$$(N_A, N_B, N_C, N_D) \longrightarrow (N'_A, N'_B, N'_C, N'_D)$$

is a cofibration.

Proof. To see that the arrow $N_- \rightarrowtail N'_-$ in the diagram

$$\begin{array}{ccc} M_- & \rightarrowtail & M'_- \\ \downarrow & & \downarrow \\ N_- & \rightarrowtail & N'_- \end{array}$$

is a cofibration, we need to check that the arrows starting at each corner of N_- are cofibrations. For example, starting at the A-corner, the arrow is the lower arrow in the diagram

$$\begin{array}{ccc} M_A & \rightarrowtail & M'_A \\ \downarrow & & \downarrow \\ N_A & \rightarrowtail & N_A \cup_{M_A} M'_A \end{array}$$

and this is a cofibration of A-modules.

It remains to see that the diagram

$$\begin{array}{ccc} N'_A \wedge_A D & \longrightarrow & N'_B \wedge_B D \\ \downarrow & & \downarrow \\ N'_C \wedge_C D & \longrightarrow & N'_D \end{array}$$

is weakly homotopy cocartesian. That is, we have to show that the canonical arrow

$$\mathrm{Tel}(N'_-) = M(N'_B \wedge_B D, N'_A \wedge_A D, N'_C \wedge_C D) \longrightarrow N'_D$$

is a weak homotopy equivalence.

We start by rewriting the domain of the arrow. First, we observe that $\mathrm{Tel}(N'_-)$ can be written in terms of the mapping cylinder construction defined above before Proposition 3.6. If we let

$$T_{BA}(N'_-) = T(N'_A \wedge_A B \longrightarrow N'_B)$$

and define $T_{CA}(N'_-)$ similarly, then

$$\mathrm{Tel}(N'_-) = M(N'_B \wedge_B D, N'_A \wedge_A, N'_C \wedge_C D) \cong$$
$$\mathrm{colim}(\, T_{BA}(N'_-) \wedge_B D \longleftarrow\!\prec N'_A \wedge_A D \succ\!\longrightarrow T_{CA}(N'_-) \wedge_C D \,)$$

Furthermore, each of the mapping cylinders is a colimit as well. For instance,

$$T_{BA}(N'_-) = \mathrm{colim}(\, N'_A \wedge_A B \wedge I_+ \longleftarrow N'_A \wedge_A B \wedge \partial I_+ \succ\!\longrightarrow N'_A \wedge_A B \vee N'_B \,)$$

and we have a similar description of $T_{CA}(N'_-)$.

Now recall that the modules $N'_A = N_A \cup_{M_A} M'_A$ and $N'_B = N_B \cup_{M_B} M'_B$ are also colimits, and that the operation $\wedge_A$ is defined as a colimit as well. Commuting these colimits among themselves, we obtain that

$$N'_A \wedge_A B \wedge I_+ = \mathrm{colim}(\, N_A \wedge_A B \wedge I_+ \longleftarrow M_A \wedge_A B \wedge I_+ \succ\!\longrightarrow M'_A \wedge_A B \wedge I_+ \,)$$

and similarly for the other terms in the colimit defining $T_{BA}(N'_-)$, Thus, we may represent $T_{BA}(N_-)$ as the iterated colimit of the diagram

$$\begin{array}{ccccc}
M'_A \wedge_A B \wedge I_+ & \longleftarrow\!\prec & M'_A \wedge_A B \wedge \partial I_+ & \longrightarrow & (M'_A \wedge_A B) \vee M'_B \\
\uparrow & & \uparrow & & \uparrow \\
M_A \wedge_A B \wedge I_+ & \longleftarrow\!\prec & M_A \wedge_A B \wedge \partial I_+ & \longrightarrow & (M_A \wedge_A B) \vee M_B \\
\downarrow & & \downarrow & & \downarrow \\
N_A \wedge_A B \wedge I_+ & \longleftarrow\!\prec & N_A \wedge_A B \wedge \partial I_+ & \longrightarrow & (N_A \wedge_A B) \vee N_B
\end{array}$$

where we take the colimits along the columns first. On the other hand, we may take the colimits along the rows first, in which case we obtain another description

$$T_{BA}(N'_-) \cong \mathrm{colim}(\, T_{BA}(N_-) \longleftarrow T_{BA}(M_-) \succ\!\longrightarrow T_{BA}(M'_-) \,)$$

and a similar description of $T_{CA}(N'_-)$. Since $\wedge_A D$ preserves pushouts, there is an isomorphism of D-modules $N'_A \wedge_A D \cong N_A \wedge_A D \cup_{M_A \wedge_A D} M'_A \wedge D$. Consequently, we may interpret

$$\mathrm{Tel}(N'_-) \cong \mathrm{colim}(\, T_{BA}(N'_-) \wedge_B D \longleftarrow\!\prec N'_A \wedge_A D \succ\!\longrightarrow T_{CA}(N'_-) \wedge_C D \,)$$

as the iterated colimit of the following diagram, if we compute the colimit along the columns first.

$$\begin{array}{ccccc}
T_{BA}(N_-) \wedge_B D & \longleftarrow\!\prec & N_A \wedge_A D & \succ\!\longrightarrow & T_{CA}(N_-) \wedge_C D \\
\uparrow & & \uparrow & & \uparrow \\
T_{BA}(M_-) \wedge_B D & \longleftarrow\!\prec & M_A \wedge_A D & \succ\!\longrightarrow & T_{CA}(M_-) \wedge_C D \\
\downarrow & & \downarrow & & \downarrow \\
T_{BA}(M'_-) \wedge_B D & \longleftarrow\!\prec & M'_A \wedge_A D & \succ\!\longrightarrow & T_{CA}(M'_-) \wedge_C D
\end{array}$$

Now we exploit the definition $N'_D = N_D \cup_{M_D} M'_D$ to write N'_D as the iterated colimit of the following diagram, if we compute the colimit along the columns first.

$$\begin{array}{ccccc}
N_D & \longleftarrow & N_D & \longrightarrow & N_D \\
\uparrow & & \uparrow & & \uparrow \\
M_D & \longleftarrow & M_D & \longrightarrow & M_D \\
\downarrow & & \downarrow & & \downarrow \\
M'_D & \longleftarrow & M'_D & \longrightarrow & M'_D
\end{array}$$

The back projection $T_{BA}(N_-) \longrightarrow N_B$ and the structure arrow $\psi^e_{DB} : N_B \wedge_B D \longrightarrow N_D$ combine to provide a map between the upper left corners of the two diagrams, and similar constructions at the other locations assemble to a map of commuting 3-by-3 diagrams. By taking the iterated colimit, computing along the columns first, we obtain the map

$$\mathrm{Tel}(N'_-) \longrightarrow N'_D$$

that we are currently investigating. Of course, we obtain this map, if we also compute the iterated colimit in the other order. Let us examine the steps in this evaluation. First we obtain a diagram

$$\begin{array}{ccccc}
\mathrm{Tel}(N_-) & \longleftarrow & \mathrm{Tel}(M_-) & \longrightarrow & \mathrm{Tel}(M'_-) \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
N_D & \longleftarrow & M_D & \longrightarrow & M'_D
\end{array}$$

in which the vertical arrows are weak homotopy equivalences of D-modules, by hypothesis. By a five lemma argument, or by appeal to the gluing lemma, the induced map on the colimits of the rows is also a weak equivalence. This is what is required to prove that $\mathcal{MV}_f$ and $\mathcal{MV}_{hf}$ are closed under pushouts of cofibrations along arbitrary maps of Mayer-Vietoris presentations. □

Now we turn to the notions of weak equivalence we will be using.

Definition 3.11. Let $(f_A, f_B, f_C, f_D) : (M_A, M_B, M_C, M_D) \longrightarrow (N_A, N_B, N_C, N_D)$ be a map of Mayer-Vietoris presentations. We say that (f_A, f_B, f_C, f_D) is a coarse weak equivalence if the component $f_D : M_D \longrightarrow N_D$ is a weak homotopy equivalence of D-modules. We denote the subcategory of coarse weak equivalences by $w\mathcal{MV}$ and refer to w-maps for brevity.

Restricted to the category $\mathcal{MV}_f$, this means that f_D is a homotopy equivalence of finite cell D-modules; in the category $\mathcal{MV}_{hf}$ this means that f_D is a weak homotopy equivalence of D-modules. We denote these categories of coarse weak equivalences by $w\mathcal{MV}_f$ and $w\mathcal{MV}_{hf}$, respectively.

Definition 3.12. Let $(f_A, f_B, f_C, f_D) : (M_A, M_B, M_C, M_D) \longrightarrow (N_A, N_B, N_C, N_D)$ be a map of Mayer-Vietoris presentations. We say that (f_A, f_B, f_C, f_D) is a fine equivalence if each of the components

$$f_A : M_A \longrightarrow N_A,$$
$$f_B : M_B \longrightarrow N_B,$$
$$f_C : M_C \longrightarrow N_C$$

of the map is a weak homotopy equivalence of modules. We denote the subcategory of fine equivalences by $v\mathcal{MV}$ and refer to v-maps for brevity.

Restricted to the category $\mathcal{MV}_f$ this means that f_A, f_B, and f_C are homotopy equivalences of finite cell modules; in the category $\mathcal{MV}_{hf}$ this means that f_A, f_B, and f_C are all weak homotopy equivalences of modules. We denote these categories of fine weak equivalences by $v\mathcal{MV}_f$ and $v\mathcal{MV}_{hf}$, respectively,

The following proposition is an immediate consequence of the homotopy property of a Mayer-Vietoris presentation, together with the fact that categories of modules over S-algebras are categories with cofibrations and weak equivalences.

Proposition 3.13. *The fine weak equivalences form a subcategory of the category of coarse weak equivalences, $v\mathcal{MV} \subset w\mathcal{MV}$, and selection of either notion makes $\mathcal{MV}_f$ and $\mathcal{MV}_{hf}$ into categories with cofibrations and weak equivalences.*

□

Next we define for Mayer-Vietoris presentations a cylinder functor in the sense of Definition 3.4.

Definition 3.14. Let $(f_A, f_B, f_C, f_D) : (M_A, M_B, M_C, M_D) \longrightarrow (N_A, N_B, N_C, N_D)$ be a map of Mayer-Vietoris presentations. Define a quadruple of modules

$$T((f_A, f_B, f_C, f_D)) = (T(f_A), T(f_B), T(f_C), T(f_D))$$

where $T(f_-)$ is the cylinder object of the arrow in the category $\mathcal{M}_-$.

In the following proposition, let $\mathcal{MV}_*$ represent either $\mathcal{MV}_f$ or $\mathcal{MV}_{hf}$.

Proposition 3.15. *The quadruple of modules defined above assembles into a Mayer-Vietoris presentation. Moreover, the front and back inclusions assemble into a cofibration in the category $\mathcal{MV}_*$, the cylinder projections assemble into a map in $\mathcal{MV}_*$, so the resulting diagram satisfies the properties required of a cylinder functor on $\mathcal{MV}_*$. This cylinder functor satisfies the cylinder axiom with respect to v-maps and w-maps.*

Proof. To see that $T(f_-)$ satisfies the homotopy property required of a Mayer-Vietoris presentation recall that by Proposition 3.6 the back projections

$$T(f_-)_A = T(f_A) \longrightarrow N_A$$

and so on, are all weak homotopy equivalences. Then so are the extensions

$$T(f_-)_A \wedge_A D \longrightarrow N_A \wedge_A D$$

using the q-cofibrancy of D as a module over A, B, and C. It follows that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Tel}(T(f_-)) & \longrightarrow & \mathrm{Tel}(N_-) \\ \downarrow & & \downarrow \\ T(f_D) & \longrightarrow & N_D \end{array}$$

in which the horizontal arrows and the righthand vertical arrow are weak homotopy equivalences. Consequently, the remaining arrow is also a weak homotopy equivalence, which says that $T(f_-)$ is a Mayer-Vietoris presentation. Since the construction is carried out corner by corner, we remain in the category $\mathcal{MV}_*$ in which we started.

Since the cofibrations in $\mathcal{MV}_*$ are also defined corner by corner, the front and back inclusions assemble into a cofibration in $\mathcal{MV}_*$, as required.

That the construction satisfies the cylinder axiom for either category of weak equivalences is a trivial consequence of the fact that in any module category the back projection of the cylinder functor on that category is a (weak) homotopy equivalence. □

Finally we come to the definitions of the categories of Mayer-Vietoris presentations with extra cofibration conditions and their properties. Here is the definition of the category $\mathcal{MV}_{f,e}$, one of the categories of Mayer-Vietoris presentations defined in [15].

Definition 3.16. A finite Mayer-Vietoris presentation with extra cofibration conditions is a Mayer-Vietoris presentation

$$\begin{array}{ccc} M_A & \xrightarrow{\phi_{BA}} & M_B \\ {\scriptstyle \phi_{CA}}\downarrow & & \downarrow{\scriptstyle \phi_{DB}} \\ M_C & \xrightarrow[\phi_{DC}]{} & M_D \end{array}$$

where $M_A \in \mathcal{M}(A)_f$, $M_B \in \mathcal{M}(B)_f$, $M_C \in \mathcal{M}(C)_f$, and $M_D \in \mathcal{M}(D)_f$.

Moreover, we require that the cofibration condition in the appropriate category $\mathcal{M}(B)_f$, $\mathcal{M}(C)_f$, or $\mathcal{M}(D)_f$ be satisfied for the extended structure arrows

$$M_A \wedge_A B \rightarrowtail M_B$$
$$M_A \wedge_A C \rightarrowtail M_C$$
$$M_B \wedge_B D \rightarrowtail M_D$$
$$M_C \wedge_C D \rightarrowtail M_D.$$

We will still use the notation $M_- = (M_A, M_B, M_C, M_D)$ for a finite Mayer-Vietoris presentation with extra cofibration conditions; the context will make the ambient category clear.

The full subcategory of $\mathcal{MV}$ consisting of finite Mayer-Vietoris presentations with extra cofibration conditions is denoted $\mathcal{MV}_{f,e}$.

We obtain the definition of the category $\mathcal{MV}_{hf,e}$ by using the categories of homotopy-finite modules $\mathcal{M}(A)_{hf}$, $\mathcal{M}(B)_{hf}$, $\mathcal{M}(C)_{hf}$, and $\mathcal{M}(D)_{hf}$ in place of the categories of finite cell modules used in the definition of $\mathcal{MV}_{f,e}$. Note that this means that the notion of cofibration changes as well.

To sum up, the definitions of the categories $\mathcal{MV}_{f,e}$ and $\mathcal{MV}_{hf,e}$ are obtained from those of the categories $\mathcal{MV}_f$ and $\mathcal{MV}_{hf}$ by adding cofibration requirements on the extended structure arrows. We will see below that this has no effect at the level of K-theory. The extra cofibration conditions on the objects of the categories $\mathcal{MV}_{f,e}$ and $\mathcal{MV}_{hf,e}$, which for quite a while we thought we needed, are suggested by analogy with the most familiar Seifert-van Kampen or Mayer-Vietoris situation. Namely, consider a complex $X = X_0 \cup X_1$, where X_0 and X_1 are subcomplexes with a common subcomplex $X_{01} = X_0 \cap X_1$. Assume that all inclusions induce injections of fundamental groups. Consider also the modules of cellular chains on $\tilde{X}$, $\tilde{X}_0$, $\tilde{X}_1$, and $\tilde{X}_{01}$, the universal covers of these complexes. If A denotes the group ring of the fundamental group of X_{01}, B and C denote the group rings of the fundamental groups of X_0 and X_1, respectively, and D denotes the group ring of the fundamental group of X, then there are algebraic cofibrations

$$C_*(\tilde{X}_{01}) \otimes_A B \rightarrowtail C_*(\tilde{X}_0),$$
$$C_*(\tilde{X}_{01}) \otimes_A C \rightarrowtail C_*(\tilde{X}_1),$$
$$C_*(\tilde{X}_0) \otimes_B D \rightarrowtail C_*(\tilde{X}),$$
$$C_*(\tilde{X}_0) \otimes_B D \rightarrowtail C_*(\tilde{X}),$$

where an algebraic cofibration of complexes is an injection of complexes with the image in each dimension having a complement. However, it is also known that the Seifert-van Kampen theorem works in more general situations. Recall that this is often verified by using mapping cylinders reduce the original diagram of maps to topological cofibrations and by hypothesizing, or arranging, that the new situation is equivalent to the original setup. In our situation there is a similar trade-off. We will now see that the definition of cofibration in the categories $\mathcal{MV}_{f,e}$ and $\mathcal{MV}_{hf,e}$ becomes more complex, but we don't need to work as hard to confirm that the construction of pushouts preserves Mayer-Vietoris presentations.

In the categories $\mathcal{MV}_{f,e}$ and $\mathcal{MV}_{hf,e}$ we define cofibrations of Mayer-Vietoris presentations in terms of the notions of cofibration adopted in the categories $\mathcal{M}(-)_f$ and $\mathcal{M}(-)_{hf}$, as follows. Let $\mathcal{MV}_{*,e}$ denote either $\mathcal{MV}_{f,e}$ or $\mathcal{MV}_{hf,e}$; let $\mathcal{M}(-)_*$ denote either $\mathcal{M}(-)_f$ or $\mathcal{M}(-)_{hf}$, correspondingly.

Definition 3.17. A map $f_- : M_- \longrightarrow M'_-$ is a cofibration in $\mathcal{MV}_{*,e}$ if each component is a cofibration in $\mathcal{M}(-)_*$, and if the canonical arrows

$$M_B \cup_{M_A \wedge_A B} M'_A \wedge_A B \rightarrowtail M'_B,$$
$$M_C \cup_{M_A \wedge_A C} M'_A \wedge_A C \rightarrowtail M'_C$$
$$M_D \cup_{M_B \wedge_B D} M'_B \wedge_B D \rightarrowtail M'_D,$$
$$M_D \cup_{M_C \wedge_C D} M'_C \wedge_C D \rightarrowtail M'_D$$

are cofibrations as well. We use the same feathered arrow $f_- : M_- \rightarrowtail M'_-$ as above to indicate a cofibration.

We emphasize that in the case of finite cell modules a cofibration is a map that is isomorphic to a cellular inclusion, whereas in the case of homotopy finite modules a cofibration is a map that satisfies the homotopy extension property. For the rest of the section we abbreviate the smash product $M_A \wedge_A B$ by $M_A \wedge B$, and so on. Similarly, in the rest of the section the symbol $M_B \wedge D$, for instance, stands for $M_B \wedge_B D$. All the smash products of modules that we need are over an algebra in our generalized free product diagram, and not over S, so the abbreviation shouldn't cause any confusion.

We must now verify that pushouts exist in the categories of Mayer-Vietoris presentations $\mathcal{MV}_{f,e}$ and $\mathcal{MV}_{hf,e}$. We will see that it is relatively easy to check the homotopy condition; on the other hand, checking cofibration conditions is more complicated, as is to be expected. Given a diagram of Mayer-Vietoris presentations

$$\begin{array}{ccc} M_- & \rightarrowtail & M'_- \\ \downarrow & & \\ N_- & & \end{array}$$

we define

$$N'_- = N_- \cup_{M_-} M'_-$$

by using the pushout construction of modules at each corner, obtaining

$$N'_A = N_A \cup_{M_A} M'_A,$$

for example, and we define structure maps using the universal property of the pushout and the exactness of the smash product functor. For example, $N'_A \wedge B \longrightarrow N'_B$ is the composite

$$N_A \cup_{M_A} M'_A \wedge B \cong N_A \wedge B \cup_{M_A \wedge B} M'_A \wedge B \longrightarrow N_B \cup_{M_B} M'_B.$$

For each ambient category $\mathcal{MV}_{f,e}$ and $\mathcal{MV}_{hf,e}$ we must prove the following proposition. It turns out, not unexpectedly, that the same argument proves both cases.

Proposition 3.18. *The quadruple of modules $N'_- = (N'_A, N'_B, N'_C, N'_D)$ together with these structure maps is a Mayer-Vietoris presentation and the canonical arrow*

$$(N_A, N_B, N_C, N_D) \longrightarrow (N'_A, N'_B, N'_C, N'_D)$$

is a cofibration.

Proof. First we verify that the cofibration conditions of a Mayer-Vietoris presentation are satisfied. For example, we check that we have a cofibration

$$N'_A \wedge B \rightarrowtail N'_B$$

Recall that the functor $- \wedge B$ is exact and that we have cofibrations $M_A \wedge B \rightarrowtail M_B$ and $M_B \cup_{M_A \wedge B} M'_A \wedge B \rightarrowtail M'_B$ by hypothesis. Now we factor $N'_A \wedge B \rightarrowtail N'_B$ into a composite of maps which are visibly cofibrations. Recalling that

$$N'_A \wedge B \cong N_A \wedge B \cup_{M_A \wedge B} M'_A \wedge B,$$

by definition of N'_A and exactness of the smash product, the first arrow is the lower arrow in the pushout diagram

$$\begin{array}{ccc} N_A \wedge B & \rightarrowtail & N_B \\ \downarrowtail & & \downarrowtail \\ N_A \wedge B \cup_{M_A \wedge B} M'_A \wedge B & \rightarrowtail & N_B \cup_{M_A \wedge B} M'_A \wedge B. \end{array}$$

Here we have made the identifications

$$\begin{aligned} N_B \cup_{M_A \wedge B} M'_A \wedge B &\cong N_B \cup_{N_A \wedge B} (N_A \wedge B \cup_{M_A \wedge B} M'_A \wedge B) \\ &\cong (N_A \wedge B \cup_{M_A \wedge B} M'_A \wedge B) \cup_{N_A \wedge B} N_B. \end{aligned}$$

We make the alternative identification

$$N_B \cup_{M_A \wedge B} M'_A \wedge B = N_B \cup_{M_B} (M_B \cup_{M_A \wedge B} M'_A \wedge B)$$

and our second arrow is lower one in the diagram

$$\begin{array}{ccc} M_B \cup_{M_A \wedge B} M'_A \wedge B & \rightarrowtail & M'_B \\ \downarrow & & \downarrow \\ N_B \cup_{M_B} (M_B \cup_{M_A \wedge B} M'_A \wedge B) & \rightarrowtail & N_B \cup_{M_B} M'_B = N'_B. \end{array}$$

That the cofibration properties are inherited at the other corners of the diagram goes similarly.

Next we verify that N'_- has the correct homotopy property. That is, we must show that the diagram

$$\begin{array}{ccc} N'_A \wedge D & \rightarrowtail & N'_B \wedge D \\ \downarrow & & \downarrow \\ N'_C \wedge D & \rightarrowtail & N'_D \end{array}$$

is homotopy cocartesian. In this diagram all the arrows are cofibrations, so that

$$T(\, N'_A \wedge D \rightarrowtail N'_C \wedge D \,) \longrightarrow N'_B \wedge D / N'_A \wedge D$$

and

$$T(\, N'_C \wedge D \rightarrowtail N_D \,) \longrightarrow N'_C \wedge D / N'_D$$

are homotopy equivalences. We will show that

$$N'_C \wedge D / N'_A \wedge D \longrightarrow N'_D / N'_B \wedge D,$$

is a weak homotopy equivalence and then we can apply Lemma 2.6 to get the result.

Using the recipe for N'_-, we note that $N'_C \wedge D / N'_A \wedge D$, the cofibre of the lefthand vertical arrow, is the iterated colimit of the diagram

$$\begin{array}{ccccc} N_C \wedge D & \longleftarrow & M_C \wedge D & \rightarrowtail & M'_C \wedge D \\ \uparrow & & \uparrow & & \uparrow \\ N_A \wedge D & \longleftarrow & M_A \wedge D & \rightarrowtail & M_A \wedge D \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longleftarrow & 0 & \rightarrowtail & 0 \end{array}$$

where we take the colimit of the rows first and compute the colimit of the result. Note that we need to know that

$$M_C \wedge D \cup_{M_A \wedge D} M'_A \wedge D \longrightarrow M'_C \wedge D$$

is also a cofibration to make sense of the second stage of the calculation. Similarly, we may compute the colimit by computing along the columns first, obtaining

$$N_C \wedge D / N_A \wedge D \longleftarrow M_C \wedge D / M_A \wedge D \rightarrowtail M'_C \wedge D / M'_A \wedge D,$$

and taking the colimit of this diagram. Turning our attention to the cofibre of the righthand vertical arrow, we obtain a similar diagram and a map of diagrams

$$\begin{array}{ccccc} N_C \wedge D / N_A \wedge D & \longleftarrow & M_C \wedge D / M_A \wedge D & \rightarrowtail & M'_C \wedge D / M'_A \wedge D \\ \downarrow & & \downarrow & & \downarrow \\ N_D / N_B \wedge D & \longleftarrow & M_D / M_B \wedge D & \rightarrowtail & M'_D / M'_B \wedge D \end{array}$$

in which each vertical arrow is a weak equivalence. By the gluing property of weak equivalences, stated as axiom *Weq2* of [18, page 326], the induced map on pushouts, which is the map

$$N'_C \wedge D / N'_A \wedge D \longrightarrow N'_D / N'_B \wedge D,$$

is a weak equivalence also. Therefore, by Lemma 2.6, the diagram we have constructed satisfies the homotopy property required of a Mayer-Vietoris presentation.

Finally, we verify that the arrow $N_- \longrightarrow N'_-$ is a cofibration. That we have, for example, a cofibration $N_A \rightarrowtail N'_A$ at the A-corner follows by construction from the pushout property of cofibrations in the category of A-modules. The crucial matter is that the arrow

$$N_B \cup_{N_A \wedge B} N'_A \wedge B \longrightarrow N'_B$$

should also be a cofibration. By definition and exactness of the smash product with B, we have

$$N'_A \wedge B = N_A \wedge B \cup_{M_A \wedge B} M'_A \wedge B$$

so that

$$\begin{aligned} N_B \cup_{N_A \wedge B} N'_A \wedge B &= N_B \cup_{N_A \wedge B} (N_A \wedge B \cup_{M_A \wedge B} M'_A \wedge B) \\ &\cong N_B \cup_{M_B} (M_B \cup_{M_A \wedge B} M'_A \wedge B). \end{aligned}$$

Also, since $M_- \rightarrowtail M'_-$ is a cofibration, we have a cofibration

$$M_B \cup_{M_A \wedge B} M'_A \wedge B \rightarrowtail M'_B$$

and, therefore, the lower horizontal arrow in the diagram

$$\begin{array}{ccc} M_B \cup_{M_A \wedge B} M'_A \wedge B & \rightarrowtail & M'_B \\ \downarrow & & \downarrow \\ N_B \cup_{M_B} (M_B \cup_{M_A \wedge B} M'_A \wedge B) & \rightarrowtail & N_B \cup_{M_B} M'_B \end{array}$$

is also a cofibration. Combining this fact with the rewriting above proves the claim. The arguments along the other edges are similar. □

The notions of v-map and w-map pass without trouble to notions of weak equivalence on the categories $\mathcal{MV}_{f,e}$ and $\mathcal{MV}_{hf,e}$. Mapping cylinders on these categories may also be defined by using the module category construction at each corner. However, proving that we obtain a cylinder functor in the sense of Definition 3.4 requires a lot of careful work with cofibrations. The details are given below.

Definition 3.19. Let $(f_A, f_B, f_C, f_D) : (M_A, M_B, M_C, M_D) \longrightarrow (N_A, N_B, N_C, N_D)$ be a map of Mayer-Vietoris presentations. Define a quadruple of modules

$$T((f_A, f_B, f_C, f_D)) = (T(f_A), T(f_B), T(f_C), T(f_D))$$

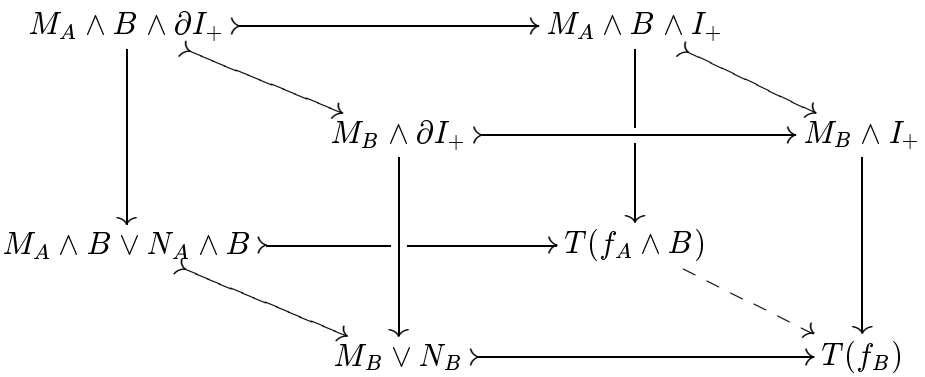
where $T(f_-)$ is the cylinder object of the arrow in the category $\mathcal{M}_-$.

Proposition 3.20. *The quadruple of modules defined above assembles into a Mayer-Vietoris presentation. Moreover, the front and back inclusions assemble into a cofibration in the category $\mathcal{MV}$, the cylinder projections assemble into a map in $\mathcal{MV}$, so the resulting diagram satisfies the properties required of a cylinder functor on $\mathcal{MV}$. Moreover, this functor satisfies the cylinder axiom with respect to either notion of weak equivalence.*

Proof. Let $(f_A, f_B, f_C, f_D) : (M_A, M_B, M_C, M_D) \longrightarrow (N_A, N_B, N_C, N_D)$ be a map of Mayer-Vietoris presentations, and let $(T(f_A), T(f_B), T(f_C), T(f_D))$ be the quadruple of modules defined above. First we have to show that a number of cofibration conditions are satisfied. For example, we need to show that

$$T(f_A) \wedge B \longrightarrow T(f_B)$$

is a cofibration of B-modules. We observe that smashing with B induces an exact functor from A-modules to B-modules, so that smashing B onto the defining diagram for $T(f_A)$ preserves the pushout property. Consequently, we have $T(f_A) \wedge B \cong T(f_A \wedge B)$. Therefore, it suffices to show that we have a cofibration $T(f_A \wedge B) \longrightarrow T(f_B)$. To see this, consider the cubical diagram



where we display the arrow under discussion as the dashed arrow at the lower right. We show that this arrow is a cofibration by interpreting the diagram as a pushout in the category with cofibrations $F_1\mathcal{M}(B)$ [18, page 322], which is the full subcategory of the arrow category of $\mathcal{M}(B)$ whose objects are the cofibration arrows in $\mathcal{M}(B)$. Since we start with Mayer-Vietoris presentations, all the diagonal arrows in the figure, except possibly for the one at the lower right, are cofibrations of B-modules. Now the upper face of cofibrations has the additional property that the canonical arrow from the pushout of the first three modules to the module at the terminal vertex

$$M_B \wedge \partial I_+ \cup_{M_A \wedge B \wedge \partial I_+} M_A \wedge B \wedge I_+ \rightarrowtail M_B \wedge I_+$$

is a cofibration of B-modules. This observation shows that the upper face represents a cofibration in $F_1\mathcal{M}(B)$, whereas the left face represents some map in $F_1\mathcal{M}(B)$. Now $F_1\mathcal{M}(B)$ is closed under such pushouts, so we obtain the arrow $T(f_A \wedge B) \longrightarrow T(f_B)$ as an object of $F_1\mathcal{M}(B)$. Thus,

$$T(f_A \wedge B) \rightarrowtail T(f_B)$$

as claimed. The arguments for the other cofibration conditions are the same.

We also have to verify that the diagram

$$\begin{array}{ccc} T(f_A) \wedge_A D & \longrightarrow & T(f_B) \wedge_B D \\ \downarrow & & \downarrow \\ T(f_C) \wedge_C D & \longrightarrow & T(f_D) \wedge_D D \end{array}$$

is homotopy cartesian. Using the back projections, map this diagram to the corresponding diagram for (N_A, N_B, N_C, N_D). Since the back projections in the various categories of modules are all homotopy equivalence, the map of diagrams is a homotopy equivalence at each corner. Since the target diagram is homotopy cartesian, so is the source diagram, which proves that the quadruple $(T(f_A), T(f_B), T(f_C), T(f_D))$ is a Mayer-Vietoris presentation.

Now we have to verify that front and back inclusions assemble to a cofibration of Mayer-Vietoris presentations. Let

$$(f_A, f_B, f_C, f_D) : (M_A, M_B, M_C, M_D) \longrightarrow (N_A, N_B, N_C, N_D)$$

be a map of Mayer-Vietoris presentations and construct $(T(f_A), T(f_B), T(f_C), T(f_D))$ as above. To show that the front and back inclusions assemble to a cofibration of Mayer-Vietoris presentations, we must show that

$$(T(f_A) \wedge B) \cup_{((M_A \vee N_A) \wedge B)} (M_B \vee N_B) \rightarrowtail T(f_B)$$

is a cofibration, as well as three other similar arrows. We remark that $T(f_A) \wedge B \cong T(f_A \wedge B)$ to start. In the category of B-modules we have a diagram

$$\begin{array}{ccc} M_A \wedge B & \rightarrowtail & M_B \\ {\scriptstyle f_A \wedge B}\downarrow & & \downarrow{\scriptstyle f_B} \\ N_A \wedge B & \rightarrowtail & N_B \end{array}$$

which we may consider as a cofibration in the category $\mathrm{Ar}\mathcal{M}(B)$. By the first condition on the cylinder functor, the T-construction carries this diagram to a cofibration in the category $F_1\mathcal{M}(B)$, which is a square diagram of cofibrations

$$\begin{array}{ccc} (M_A \vee N_A) \wedge B & \rightarrowtail & M_B \vee N_B \\ \downarrow & & \downarrow \\ T(f_A \wedge B) & \rightarrowtail & T(f_B) \end{array}$$

with the additional property that

$$(T(f_A) \wedge B) \cup_{((M_A \vee N_A) \wedge B)} (M_B \vee N_B) \rightarrowtail T(f_B)$$

is a cofibration. But this is exactly one of the conditions we needed to verify. The completion of the argument that the front and back inclusions of a mapping cylinder in the category of Mayer-Vietoris presentations assemble to a functor from $\mathrm{Ar}\mathcal{MV}$ to $F_1\mathcal{MV}$ proceeds along similar lines. The other parts of the argument are routine. □

Theorem 3.21. *The diagram of exact inclusions of categories*

$$\begin{array}{ccc} \mathcal{MV}_{f,e} & \longrightarrow & \mathcal{MV}_{hf,e} \\ \downarrow & & \downarrow \\ \mathcal{MV}_f & \longrightarrow & \mathcal{MV}_{hf} \end{array}$$

induces the following commuting diagrams in which the vertical arrows are homotopy equivalences.

$$\begin{array}{ccc} vS_\bullet\mathcal{MV}_{f,e} & \longrightarrow & vS_\bullet\mathcal{MV}_{hf,e} \\ \downarrow & & \downarrow \\ vS_\bullet\mathcal{MV}_f & \longrightarrow & vS_\bullet\mathcal{MV}_{hf} \end{array} \qquad \begin{array}{ccc} wS_\bullet\mathcal{MV}_{f,e} & \longrightarrow & wS_\bullet\mathcal{MV}_{hf,e} \\ \downarrow & & \downarrow \\ wS_\bullet\mathcal{MV}_f & \longrightarrow & wS_\bullet\mathcal{MV}_{hf} \end{array}$$

In the next section we will prove that the horizontal arrows in the $vS_\bullet$ diagram are homotopy equivalences in theorem 4.2, drawing the consequence from our identification theorem 4.1 and theorem 3.1. In section 6 we will prove that the horizontal arrows in the $wS_\bullet$ diagram are homotopy equivalences in theorem 6.2 under the hypothesis that A and D in the generalized free product diagram are connective. This theorem uses the corresponding identification theorem 6.1 which also requires these connectivity hypotheses to reduce again to the situation covered by theorem 3.1.

Proof. We prove that the vertical arrows in the v-diagram are homotopy equivalences. The arguments for the vertical arrows in the w-diagram are basically the same, differing only in the last observation. The argument that

$$i_1 : \mathcal{MV}_{f,e} \longrightarrow \mathcal{MV}_f \qquad \text{and} \qquad i_2 : \mathcal{MV}_{hf,e} \longrightarrow \mathcal{MV}_{hf}$$

induce homotopy equivalences

$$vS_\bullet(i_1) : vS_\bullet\mathcal{MV}_{f,e} \longrightarrow vS_\bullet\mathcal{MV}_f \qquad \text{and} \qquad vS_\bullet(i_2) : vS_\bullet\mathcal{MV}_{hf,e} \longrightarrow vS_\bullet\mathcal{MV}_{hf}$$

goes as follows. To start, let (M_A, M_B, M_C, M_D) be a Mayer-Vietoris presentation with extended structure maps $\phi^e_{BA} : M_A \wedge B \longrightarrow M_B$, $\phi^e_{CA} : M_A \wedge C \longrightarrow M_C$, $\phi^e_{DB} : M_B \wedge D \longrightarrow M_D$, and $\phi^e_{DC} : M_C \wedge D \longrightarrow M_D$. Define a new Mayer-Vietoris presentation (M'_A, M'_B, M'_C, M'_D)

with $M'_A = M_A$, $M'_B = T(\phi^e_{BA})$, and $M'_C = T(\phi^e_{CA})$. Extend the back projections $p_B : M'_B \longrightarrow M_B$ and $p_C : M'_C \longrightarrow M_C$ and define maps

$$\psi_{DB} = \phi^e_{DB} \circ (p_B \wedge \mathrm{id}) : M'_B \wedge D \longrightarrow M_D \quad \text{and} \quad \psi_{DC} = \phi^e_{DC} \circ (p_C \wedge \mathrm{id}) : M'_C \wedge D \longrightarrow M_D.$$

Now define $M'_D = T(\psi_{DB} \vee \psi_{DC})$. The structure map $\phi'_{BA} : M'_A \longrightarrow M'_B$ is the canonical inclusion $M_A \longrightarrow M_A \wedge B$ followed by the front inclusion $M_A \wedge B \rightarrowtail M'_B$, and the other structure maps are defined similarly. Clearly the quadruple of modules (M'_A, M'_B, M'_C, M'_D) is a Mayer-Vietoris presentation satisfying the extra cofibration conditions. The construction is also functorial in M_-, and so we obtain functors

$$r_1 : \mathcal{MV}_f \longrightarrow \mathcal{MV}_{f,e} \qquad \text{and} \qquad r_2 : \mathcal{MV}_{hf} \longrightarrow \mathcal{MV}_{hf,e}.$$

Since the mapping cylinder construction for modules satisfies the cylinder axiom, the back projections combine to provide natural transformations

$$\eta_1 : i_1 \circ r_1 \to \mathrm{Id} : \mathcal{MV}_f \longrightarrow \mathcal{MV}_f \qquad \text{and} \qquad \eta_2 : i_2 \circ r_2 \to \mathrm{Id} : \mathcal{MV}_{hf} \longrightarrow \mathcal{MV}_{hf}.$$

Both of these natural transformations are v-maps, so they induce homotopies $vS_\bullet(i_1 \circ r_1) \simeq vS_\bullet(\mathrm{Id})$ and $vS_\bullet(i_2 \circ r_2) \simeq vS_\bullet(\mathrm{Id})$, by Proposition 1.3.1 of [18, page 330]. Therefore, $vS_\bullet(i_1)$ and $vS_\bullet(i_2)$ have right homotopy inverses. Similarly, the compositions $r_1 \circ i_1$ and $r_2 \circ i_2$ induce selfmaps of the categories $\mathcal{MV}_{f,e}$ and $\mathcal{MV}_{hf,e}$ that admit natural transformations to the identity through v-maps, so $vS_\bullet(i_1)$ and $vS_\bullet(i_2)$ also have left homotopy inverses. We conclude that $vS_\bullet(i_1)$ and $vS_\bullet(i_2)$ are homotopy equivalences, as claimed, with homotopy inverses induced by the functors r_1 and r_2 defined above. □

The main reason for setting up the cylinder functors on the categories of Mayer-Vietoris presentations defined above is that the Fibration Theorem [18, Theorem 1.6.4, page 350] applies so that we obtain homotopy cartesian squares

$$\begin{array}{ccc} vS_\bullet\mathcal{MV}^w_f & \longrightarrow & wS_\bullet\mathcal{MV}^w_f \\ \downarrow & & \downarrow \\ vS_\bullet\mathcal{MV}_f & \longrightarrow & wS_\bullet\mathcal{MV}_f \end{array} \qquad \text{and} \qquad \begin{array}{ccc} vS_\bullet\mathcal{MV}^w_{hf} & \longrightarrow & wS_\bullet\mathcal{MV}^w_{hf} \\ \downarrow & & \downarrow \\ vS_\bullet\mathcal{MV}_{hf} & \longrightarrow & wS_\bullet\mathcal{MV}_{hf} \end{array} \tag{3.1}$$

in which the upper righthand corners are contractible. We also obtain variants using the Mayer-Vietoris presentations with extra cofibration conditions. The rest of the paper is devoted to an analysis of the lower terms in each square.

The categories $\mathcal{MV}^w_f$ and $\mathcal{MV}^w_{hf}$ in the upper left corners are the full subcategories of $\mathcal{MV}_f$ and $\mathcal{MV}_{hf}$, respectively, whose objects are the Mayer-Vietoris presentations in which the D-module is weakly contractible. These categories are called the categories of split Mayer-Vietoris presentations; the analysis of the upper lefthand corners is left to a later paper [14].

4 Mayer-Vietoris presentations and fine equivalences

The object of the section is to prove the following identification theorem.

Theorem 4.1. *The three forgetful functors*

$$u_A, u_B, u_C : \mathcal{MV} \longrightarrow \mathcal{M}(A),\, \mathcal{M}(B),\, \mathcal{M}(C)$$

induce homotopy equivalences

$$u_* : vS_\bullet \mathcal{MV}_f \longrightarrow hS_\bullet \mathcal{M}(A)_f \times hS_\bullet \mathcal{M}(B)_f \times hS_\bullet \mathcal{M}(C)_f$$

and

$$u_* : vS_\bullet \mathcal{MV}_{hf} \longrightarrow wS_\bullet \mathcal{M}(A)_{hf} \times wS_\bullet \mathcal{M}(B)_{hf} \times wS_\bullet \mathcal{M}(C)_{hf}$$

The proof uses formulation (4) of the additivity theorem [18, page 331] to take apart the identity functors on $vS_\bullet \mathcal{MV}_f$ and $vS_\bullet \mathcal{MV}_{hf}$. The proof also depends heavily on the properties of the mapping cylinder functor defined in section 3. The proof of theorem 1 in [12] uses the same ingredients and is practically dual to the proof we give below. That paper uses a notion of Mayer-Vietoris presentation associated to a generalized free product of ordinary rings and other particular arguments that do not seem to generalize to S-algebras. Granting for the moment theorem 4.1 we can now prove the following result comparing possible definitions of K-theory with respect to fine equivalences. This result is complementary to theorems 3.21 and 6.2.

Theorem 4.2. *The inclusion functors*

$$j_1 : \mathcal{MV}_{f,e} \longrightarrow \mathcal{MV}_{hf,e} \qquad \textit{and} \qquad j_2 : \mathcal{MV}_f \longrightarrow \mathcal{MV}_{hf}$$

induce homotopy equivalences

$$vS_\bullet(j_1) : vS_\bullet \mathcal{MV}_{f,e} \longrightarrow vS_\bullet \mathcal{MV}_{hf,e} \quad \textit{and} \quad vS_\bullet(j_2) : vS_\bullet \mathcal{MV}_f \longrightarrow vS_\bullet \mathcal{MV}_{hf}.$$

Proof. By the preceding theorem the horizontal arrows in the diagram

$$\begin{array}{ccc} vS_\bullet \mathcal{MV}_f & \longrightarrow & hS_\bullet \mathcal{M}(A)_f \times hS_\bullet \mathcal{M}(B)_f \times hS_\bullet \mathcal{M}(C)_f \\ \downarrow & & \downarrow \\ vS_\bullet \mathcal{MV}_{hf} & \longrightarrow & wS_\bullet \mathcal{M}(A)_{hf} \times wS_\bullet \mathcal{M}(B)_{hf} \times wS_\bullet \mathcal{M}(C)_{hf} \end{array}$$

are homotopy equivalences. By theorem 3.1 the righthand vertical arrow is a homotopy equivalence. The desired result for $vS_\bullet(j_2)$ follows.

We also have a commutative diagram

$$\begin{array}{ccc} vS_\bullet \mathcal{MV}_{f,e} & \longrightarrow & vS_\bullet \mathcal{MV}_{hf,e} \\ \downarrow & & \downarrow \\ vS_\bullet \mathcal{MV}_f & \longrightarrow & vS_\bullet \mathcal{MV}_{hf} \end{array}$$

in which we proved the vertical arrows are homotopy equivalences in Theorem 3.21 and in which we have just proved the lower arrow is a homotopy equivalence. Therefore, the upper arrow $vS_\bullet(j_1)$ is also a homotopy equivalence. □

Proof of Theorem 4.1. For either case the proof is the same. Therefore, we let $\mathcal{MV}_*$ represent one of the categories $\mathcal{MV}_f$ or $\mathcal{MV}_{hf}$; similarly, let $\mathcal{M}(-)_*$ denote one of the corresponding module categories $\mathcal{M}_f$ or $\mathcal{M}_{hf}$. First we observe that there are sections to the forgetful functors defined above, given as follows. We have $s_A : \mathcal{M}(A)_* \longrightarrow \mathcal{MV}_*$ which associates to the A-module M_A the Mayer-Vietoris presentation $(M_A, M_A \wedge B, M_A \wedge C, M_A \wedge D)$. In other words, $s_A(M_A)$ is the smash product of M_A with the canonical Mayer-Vietoris presentation defined by the free product situation. Diagramatically,

$$s_A(M_A) = \begin{array}{ccc} M_A & \longrightarrow & M_A \wedge_A B \\ \downarrow & & \downarrow \\ M_A \wedge_A C & \longrightarrow & M_A \wedge_A D \end{array}$$

For use later in the proof, note now that the structure maps of the original Mayer-Vietoris presentation $M_- = (M_A, M_B, M_C, M_D)$ yield a natural transformation

$$s_A \circ u_A \longrightarrow \mathrm{Id}.$$

The sections $s_B : \mathcal{M}(B)_* \longrightarrow \mathcal{MV}_*$ and $s_C : \mathcal{M}(C)_* \longrightarrow \mathcal{MV}_*$ are more simply defined. The section s_B associates to the B-module M_B the Mayer-Vietoris presentation $(0, M_B, 0, M_B \wedge D)$, and the section s_C associates to the C-module M_C the Mayer-Vietoris presentation $(0, 0, M_C, M_C \wedge D)$. Thus, diagramatically,

$$s_B(M_B) = \begin{array}{ccc} 0 & \longrightarrow & M_B \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & M_B \wedge_B D \end{array}$$

and

$$s_C(M_C) = \begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ M_C & \longrightarrow & M_C \wedge_C D. \end{array}$$

The wedge operation on the module categories induces a functor

$$\vee : v\mathcal{MV}_* \vee v\mathcal{MV}_* \longrightarrow v\mathcal{MV}_*$$

which, in turn, induces on the realization of the nerve of $vS_\bullet\mathcal{MV}_*$ a homotopy commutative H-space structure also denoted $\vee$. We may use this operation on the space $|vS_\bullet\mathcal{MV}_*|$ to add the three sections, obtaining a map of spaces

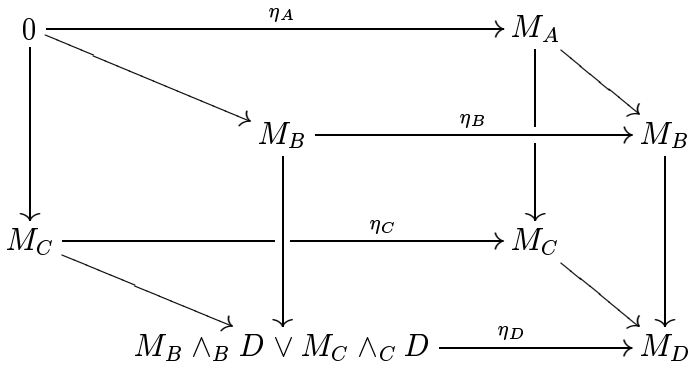
$$s_* : |wS_\bullet\mathcal{M}(A)_*| \times |wS_\bullet\mathcal{M}(B)_*| \times |wS_\bullet\mathcal{M}(C)_*| \longrightarrow |vS_\bullet\mathcal{MV}_*|,$$

where the wS should be read as hS if $* = f$ or as wS if $* = hf$. If we denote by $\beta : \mathcal{M}(A)_* \longrightarrow \mathcal{M}(B)_*$ the extension of scalars functor from A-modules to B-modules that associates to the A-module M_A the B-module $M_A \wedge_A B$, and if we define $\gamma : \mathcal{M}(A)_* \longrightarrow \mathcal{M}(C)_*$ similarly, then it is easy to verify that the composite self map of $|hS_\bullet\mathcal{M}(A)_*| \times |hS_\bullet\mathcal{M}(B)_*| \times |hS_\bullet\mathcal{M}(C)_*|$ given by $u_* \circ s_*$ is represented on homotopy groups by the following matrix.

$$\begin{pmatrix} \mathrm{id} & 0 & 0 \\ \beta_* & \mathrm{id} & 0 \\ \gamma_* & 0 & \mathrm{id} \end{pmatrix}$$

Clearly, this matrix represents an isomorphism. Thus, we see that u_* has a right homotopy inverse.

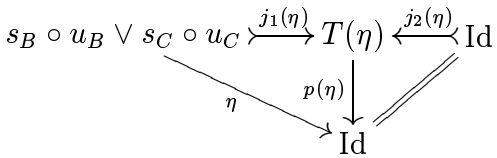
We complete the proof that u_* is a homotopy equivalence by finding a left homotopy inverse for it. Here the additivity theorem and the cylinder functor T come into play. First observe that the structure maps of a Mayer-Vietoris presentation may be used to make a commutative diagram



which represents a canonical natural transformation

$$\eta : s_B \circ u_B \vee s_C \circ u_C \longrightarrow \mathrm{Id}$$

of functors $vS_\bullet\mathcal{MV}_* \longrightarrow vS_\bullet\mathcal{MV}_*$. The component η_D is not a cofibration, in general. Applying the mapping cylinder T to the components of η, we factor η into a cofibration $j_1(\eta)$ followed by a v-map $p(\eta)$. We display all these functors and the natural transformations between them in the following diagram.



Since the natural transformation

$$p(\eta) : T(\eta) \longrightarrow \mathrm{Id}$$

is a v-map, we may apply Proposition 1.3.1 of [18, page 330] to deduce that

$$vS_\bullet T(\eta) \simeq vS_\bullet \mathrm{Id} : vS_\bullet \mathcal{MV}_* \longrightarrow vS_\bullet \mathcal{MV}_*.$$

Now we apply formulation (4) of the additivity theorem [18, page 331] to analyse the map on $vS_\bullet \mathcal{MV}_*$ induced by the functor $T(\eta)$. Let us write

$$F = T(\eta) \quad \text{and} \quad F' = s_B \circ u_B \vee s_C \circ u_C$$

and define an exact functor $F'' = F/F' : \mathcal{MV}_* \longrightarrow \mathcal{MV}_*$ by taking quotients. Thus, we obtain a cofibration sequence of exact functors on $\mathcal{MV}_*$

$$F' \rightarrowtail F \twoheadrightarrow F''$$

and formulation (4) of the additivity theorem says that there is a homotopy

$$vS_\bullet F \simeq vS_\bullet F' \vee vS_\bullet F''$$

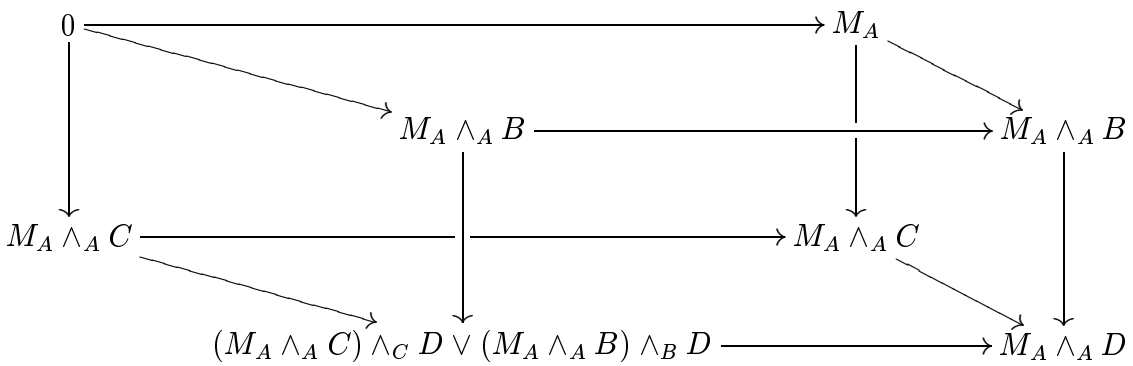
of induced self-maps of $vS_\bullet \mathcal{MV}_*$. By definition, $vS_\bullet F'$ captures part of $s_* \circ u_*$, and $vS_\bullet F = vS_\bullet T(\eta) \simeq vS_\bullet \mathrm{Id}$, by the result at the end of the preceding paragraph, so it remains to identify the map on $vS_\bullet \mathcal{MV}_*$ induced by the functor F''.

For this, consider the diagram

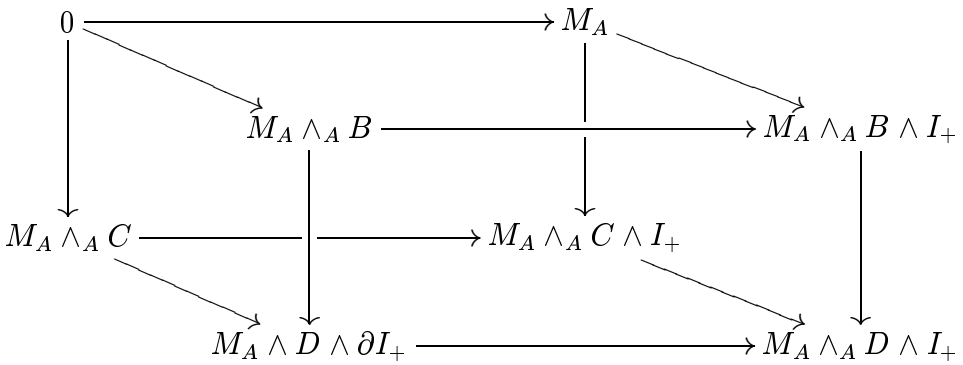
$$\begin{array}{ccccc} F' \circ s_A \circ u_A & \rightarrowtail & F \circ s_A \circ u_A & \twoheadrightarrow & F'' \circ s_A \circ u_A \\ \downarrow & & \downarrow & & \downarrow \\ F' & \rightarrowtail & F & \twoheadrightarrow & F'' \end{array}$$

obtained by applying the three functors F, F', and F'' to the components of the natural transformation $s_A \circ u_A \longrightarrow \mathrm{Id}$ mentioned at the beginning of the proof. We will now look at the diagrams represented by $F' \circ s_A \circ u_A \rightarrowtail F \circ s_A \circ u_A$ and $F' \rightarrowtail F$ so that we can compute the respective quotients and identify the arrow $F'' \circ s_A \circ u_A \longrightarrow F''$ at the right of the two by three diagram above.

We compute $F' \circ s_A \circ u_A \rightarrowtail F \circ s_A \circ u_A$ step by step, as follows. Recalling that this arrow is derived from a transformation $F' \circ s_A \circ u_A \longrightarrow \mathrm{Id} \circ s_A \circ u_A$ by applying the T-functor to the arrow, we first compute that the value of the natural transformation $F' \circ s_A \circ u_A \longrightarrow s_A \circ u_A$ is the following diagram, where the components of the natural transformation appear as the horizontal arrows.

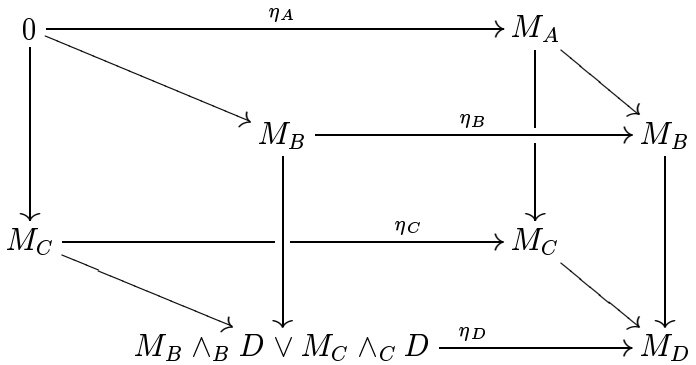


In this situation also, the arrow connecting the D-modules is not a cofibration, so this diagram does not depict a cofibration of Mayer-Vietoris presentations. However, applying F converts the horizontal arrows into a cofibration of Mayer-Vietoris presentations using the mapping cylinder construction. When we apply the recipe for mapping cylinders and make the identification $(M_A \wedge_A C) \wedge_C D \vee (M_A \wedge_A B) \wedge_B D \cong M_A \wedge D \wedge \partial I_+$, we compute that the value of the natural transformation $F' \circ s_A \circ u_A \rightarrowtail F \circ s_A \circ u_A$ is the following diagram.

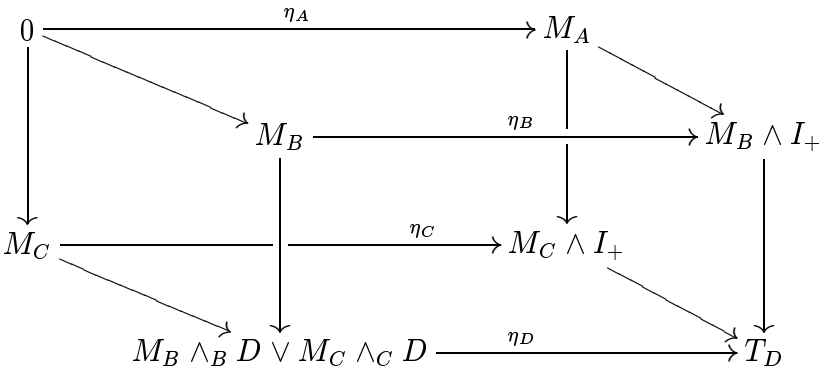


We will shortly compute $F'' \circ s_A \circ u_A$ by taking quotients of the horizontal arrows in this diagram.

We treat $F' \rightarrowtail F$ by means of a similar two step computation. First, we identify $F' = s_B \circ u_B \vee s_C \circ u_C \longrightarrow Id$ with the following diagram.

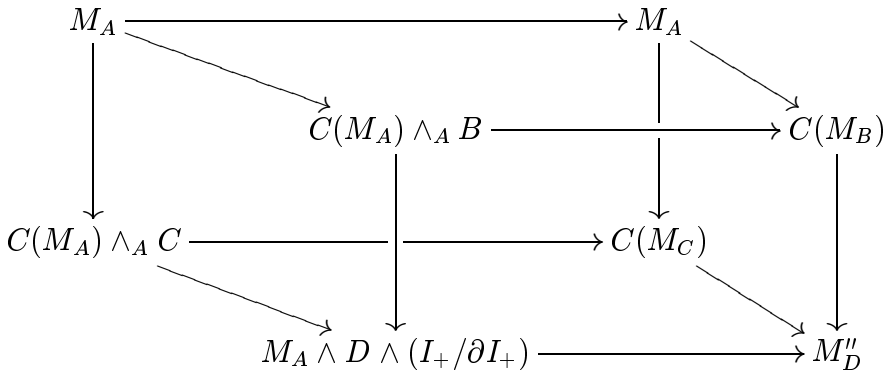


If we apply the recipe for mapping cylinders, we obtain a diagram



where $T_D = T(M_B \wedge_B D \vee M_C \wedge_C D \longrightarrow M_D)$.

Now it is easy to identify the quotients $F'' \circ s_A \circ u_A$ and F'' and the natural transformation $F'' \circ s_A \circ u_A \longrightarrow F''$ between them. The transformation is represented by the horizontal arrows in the following diagram.



In the diagram $C(?) = ? \wedge I_+/? \wedge 0_+$ is the cone construction on a given module, and $M''_D = T_D/(M_B \wedge_B D \vee M_C \wedge_C D)$. Since the cone construction delivers a contractible module, the homotopy cocartesian property of Mayer-Vietoris presentations and the five lemma imply that the natural transformation $F'' \circ s_A \circ u_A \longrightarrow F''$ is a v-equivalence. We apply once more Proposition 1.3.1 of [18, page 330] to deduce that

$$vS_\bullet(F'' \circ s_A \circ u_A) \simeq vS_\bullet F''.$$

Chaining together the homotopies we have produced and plugging in definitions yields

$$vS_\bullet \mathrm{Id} \simeq vS_\bullet F \simeq vS_\bullet F'' \vee vS_\bullet F' \simeq vS_\bullet(F'' \circ s_A \circ u_A) \vee vS_\bullet(s_B \circ u_B) \vee vS_\bullet(s_C \circ u_C)$$

Thus, we see that u_* has a left homotopy inverse as well. Since we have already shown that u_* has a right homotopy inverse, we conclude that u_* is a homotopy equivalence. □

Thinking of u_* as a three-by-one matrix and writing the left homotopy inverse we have just exhibited as a one-by-three matrix, one may write the preceding homotopy as follows.

$$\begin{pmatrix}(F'' \circ s_A)_* & (s_B)_* & (s_C)_*\end{pmatrix} \circ u_* \simeq \mathrm{id}$$

Thus, the homotopy inverse of u_* is obtained from the three sections of the forgetful functors by "twisting" the section associated to the ground S-algebra A and adding on the sections associated to the other rings B and C.

Remark 4.3. For use in the proof of theorem 1.1, note that the D-module in the Mayer-Vietoris presentation $F'' \circ s_A(M_A)$ is $\Sigma M_A \wedge_A D$, the suspension of M_A extended to be a D-module.

5 Localization of Mayer-Vietoris presentations

Underlying the material of this section is a fixed generalized free product diagram

$$\begin{array}{ccc} A & \xrightarrow{\beta} & B \\ {\scriptstyle\gamma}\downarrow & & \downarrow \\ C & \longrightarrow & D. \end{array}$$

of q-cofibrant S-algebras, where β and γ are q-cofibrations, as in the situation of Corollary 2.8. We will have occasion to use the fact that D is q-cofibrant as an A-, B-, and C-module. The object of the section is the production of a functor $L : \mathcal{MV} \longrightarrow \mathcal{MV}$ reminiscent of a Bousfield localization functor on a category of spectra, accompanied by development of some of its properties.

Before we construct L and undertake its study, let us sketch some features of Bousfield's theory of localization of spectra. For a detailed presentation of the theory using the machinery of S-modules, we refer the reader to Chapter VII of [6]; for sample calculations of localizations of spectra we refer the reader to [2] and [3]. In the category of S-modules one starts with a fixed S-module E, the coefficients of a homology theory. An E_*-equivalence is defined to be a map $f : X \longrightarrow Y$ of S-modules such that the induced map

$$f_* : E_*(X) = \pi_*(E \wedge X) \longrightarrow \pi_*(E \wedge Y) = E_*(Y)$$

is an isomorphism. The main theorem of the subject states that one has for each S-module X a universal E_*-equivalence $\lambda_X : X \longrightarrow X_E$, where X_E is a spectrum with the property that every map $Z \longrightarrow X_E$ from a cell S-module Z satisfying $E_*(Z) = 0$ is null homotopic. This property of X_E is that of being E_*-local, and the map $\lambda_X : X \longrightarrow X_E$ is called the localization of X with respect to the homology theory E. Interpreting the homotopy property of X_E as a statement that all features of X invisible to E_*-homology theory have been thrown away, what remains in X_E is, therefore, that part of X that can be studied by E_*-homology methods.

Construction of the localization is usually done rather abstractly, so that the result is functorial in X. In his work Bousfield also addressed the problem of describing the localizations, at least up to homotopy, in some special cases. His most important result in this direction is Corollary 4.10 of [3, page 270] which in essence describes up to homotopy the localization of spectra with respect to periodic complex K-theory. Interpreting his result, one finds that the K_*-localization of an S-module X is a colimit of finite cellular extensions of X. In [19] Waldhausen identified this property of K_*-localization as particularly useful for K-theory and named it the convergence property of a localization. In that paper the convergence property played an essential role in the study of the linearization map $A(*) \longrightarrow K(\mathbf{Z})$.

Returning to our situation, the fundamental objects are Mayer-Vietoris presentations and the homology theory is $E_*(M_-) = \pi_* M_D$. Thus, the Mayer-Vietoris presentations M_- that are invisible to this homology theory are those for which M_D is weakly contractible. Imitating the construction of a Bousfield localization functor, we obtain a functor $L : \mathcal{MV} \longrightarrow \mathcal{MV}$.

Intuitively, if we discard from M_- that which is not seen by our homology theory, what remains must be essentially the the module M_D. We formally state this result as Corollary 5.9 below. We also prove that L satisfies a variant of the convergence property, which we find instructive to think of as a finite approximation property. This result is Theorem 5.13, which is also proved in this section.

In the following definition the symbol I denotes the unit interval with base point 0, and for a space X the inclusion $X \longrightarrow X \wedge I$ is induced by the map $i_1 : X \longrightarrow X \times I$ given by the formula $i_1(x) = (x, 1)$.

Definition 5.1. The Mayer-Vietoris presentation

$$\begin{array}{ccc} S^n \wedge A & \longrightarrow & S^n \wedge I \wedge B \\ \downarrow & & \downarrow \\ S^n \wedge C & \longrightarrow & S^n \wedge I \wedge D \end{array}$$

is called a standard A-n-sphere coned in B, or an (A, B) n-sphere, for short. A standard (A, C) n-sphere is similarly defined.

If $M_- = (M_A, M_B, M_C, M_D)$ is a Mayer-Vietoris presentation, then a map of Mayer-Vietoris presentations

$$\lambda_- = (\lambda_A, \bar{\lambda}_B, \lambda_C, \bar{\lambda}_D) : (S^n \wedge A, S^n \wedge I \wedge B, S^n \wedge C, S^n \wedge I \wedge D) \longrightarrow (M_A, M_B, M_C, M_D)$$

is called an A n-sphere in M_-, coned in M_B, or, briefly, an (A, B) n-sphere in M_-.

Note that an (A, B) n-sphere in a Mayer-Vietoris presentation

$$M_- = \begin{array}{ccc} M_A & \xrightarrow{\phi_{BA}} & M_B \\ \downarrow{\scriptstyle \phi_{CA}} & & \downarrow{\scriptstyle \phi_{DB}} \\ M_C & \xrightarrow[\phi_{DC}]{} & M_D \end{array}$$

is completely specified by giving a commuting diagram

$$\begin{array}{ccc} S^n & \longrightarrow & S^n \wedge I \\ {\scriptstyle \lambda}\downarrow & & \downarrow{\scriptstyle \bar{\lambda}} \\ M_A & \xrightarrow[\phi_{BA}]{} & M_B \end{array}$$

We have chosen our terminology to reflect precisely this fact.

The standard (A, B) $(n+1)$-cell is the target of following cofibration of Mayer-Vietoris presentations.

$$(S^n \wedge A, S^n \wedge I \wedge B, S^n \wedge C, S^n \wedge I \wedge D) \rightarrowtail (S^n \wedge I \wedge A, S^n \wedge I \wedge B, S^n \wedge I \wedge C, S^n \wedge I \wedge D)$$

which we refer to as the inclusion of the standard (A, B) n-sphere into the standard (A, B) $(n+1)$-cell. The inclusion of the standard (A, C) n-sphere into the standard (A, C) $(n+1)$-cell is defined similarly. Note that the prefixes (A, B) and (A, C) as applied to the cell are just formal distinctions, for both cells are just the smash product of the space $S^n \wedge I$ with the standard Mayer-Vietoris presentation that is the generalized free product diagram.

Given a standard (A, B) n-sphere in a Mayer-Vietoris presentation M_-, we may therefore take the pushout of the cofibration along the n-sphere. This defines the operation of a attaching an (A, B) $(n+1)$-cell along an (A, B) n-sphere in a Mayer Vietoris presentation M_-. The diagram that illustrates this operation follows.

$$\begin{array}{ccc} (S^n \wedge A, S^n \wedge I \wedge B, S^n \wedge C, S^n \wedge I \wedge D) & \longrightarrow & (M_A, M_B, M_C, M_D) \\ \downarrow & & \downarrow \\ (S^n \wedge I \wedge A, S^n \wedge I \wedge B, S^n \wedge I \wedge C, S^n \wedge I \wedge D) & \longrightarrow & (M'_A, M'_B, M'_C, M'_D) \end{array}$$

Note that if $(M_A, M_B, M_C, M_D) \longrightarrow (M'_A, M'_B, M'_C, M'_D)$ is obtained from M_- by attaching an (A, B) $(n+1)$-cell along an (A, B) n-sphere, then there are canonical identifications

$$\rho_B : M'_B \longrightarrow M_B \quad \text{and} \quad \rho_D : M'_D \longrightarrow M_D$$

defined by noting that $M'_B = M_B \cup_{S^n \wedge I \wedge B} S^n \wedge I \wedge B$ and similarly for M'_D. Making use of these identifications, we always choose the pushout construction to satisfy $M'_B = M_B$ and $M'_D = M_D$. To sum up, attaching an (A, B) $(n+1)$-cell along an (A, B) n-sphere generally alters the homotopy types of M_A and M_C, while always preserving M_B and M_D in an especially strong sense.

Proposition 5.2. *Let $M_- = (M_A, M_B, M_C, M_D)$ be a Mayer-Vietoris presentation. Then there is a functor*

$$L_{AB} : \mathcal{MV} \longrightarrow \mathcal{MV}$$

together with a natural transformation $\mathrm{Id} \longrightarrow L_{AB}$ *with the following properties.*

1. *The map $M_- \longrightarrow M'_- = L_{AB}(M_-)$ is obtained by attaching (A, B)-cells to M_- and is therefore a cofibration of Mayer-Vietoris presentations.*
2. *The structure map $\phi'_{BA} : M'_A \longrightarrow M'_B$ is a weak homotopy equivalence of A-modules.*
3. *The map $M_D \longrightarrow M'_D$ is the identity.*
4. *The functor L_{AB} preserves cofibrations.*

Proof. Set $M^0_- = M_-$. We will define $M'_- = L_{AB}(M_-)$ as the colimit of a sequence of cofibrations

$$M_- = M^0_- \rightarrowtail M^1_- \rightarrowtail M^2_- \rightarrowtail \cdots$$

where we construct the terms in the sequence inductively as follows.

Assume M_-^k has been defined for some $k \geq 0$. Consider the family of maps $\lambda_\alpha : S^{n_\alpha} \longrightarrow M_A^k$ for which there is an extension $\bar{\lambda}_\alpha : S^{n_\alpha} \wedge I \longrightarrow M_B$ of the composite $\phi_{BA} \circ \lambda_\alpha$. As remarked following definition 5.1 each such situation gives rise to an (A, B) sphere in M_-^k along which we attach a cell. Writing

$$U^k = \bigvee_{\alpha \in A^k} (S^{n_\alpha} \wedge A, S^{n_\alpha} \wedge I \wedge B, S^{n_\alpha} \wedge C, S^{n_\alpha} \wedge I \wedge D)$$

and V^k for the corresponding wedge of (A, B) cells, we define M_-^{k+1} by means of the following pushout diagram.

$$\begin{array}{ccc} U^k & \rightarrowtail & V^k \\ \downarrow & & \downarrow \\ M_-^k & \rightarrowtail & M_-^{k+1} \end{array}$$

Therefore, mathematical induction gives us a sequence of cofibrations.

Now we define

$$M'_- = L_{AB}(M_-) = \operatorname{colim} M_-^k$$

where we compute the colimit component by component. Recall that the B-module and the D-module as we pass from stage to stage are kept constant. Therefore, we clearly have $M'_B = M_B$ and $M'_D = M_D$.

Next we verify that $\pi_*(M'_B, M'_A) = 0$. Consider a diagram

$$\begin{array}{ccc} S^n & \longrightarrow & S^n \wedge I \\ \downarrow & & \downarrow \\ M'_A & \longrightarrow & M_B \end{array}$$

representing an element of the $(n+1)$-st relative homotopy group. Since $M'_A = \operatorname{colim} M_A^k$ is the colimit of a sequence of cell inclusions relative to M_A, Proposition III.1.7 of [6, page 53] applies, and there is an integer k such that the diagram above factors through a diagram

$$\begin{array}{ccc} S^n & \longrightarrow & S^n \wedge I \\ \downarrow & & \downarrow \\ M_A^k & \longrightarrow & M_B \end{array}$$

But this is an example of a map which is made null homotopic rel M_A^k in the production of M_A^{k+1}. Therefore, we conclude that $\pi_*(M'_B, M'_A) = 0$.

To see that L preserves cofibrations argue by induction, supposing that $M_- \rightarrowtail M'_-$ is a cofibration to begin with and that we have a cofibration $M_-^k \rightarrowtail (M'_-)^k$. At each stage in the construction, we would be considering a diagram of the following sort.

$$\begin{array}{ccccc} N_- & \longleftarrow & U & \rightarrowtail & V \\ \downarrow & & \downarrow & & \downarrow \\ N'_- & \longleftarrow & U' & \rightarrowtail & V' \end{array}$$

This diagram is of a type we have met before, in section 3, having the property that the righthand square of cofibrations satisfies that $V \cup_U U' \rightarrowtail V'$ is also a cofibration. Using this fact, it is easy to show that the map on pushouts is a cofibration. Finally, passage to the limit preserves cofibrations. □

We also have the following companion proposition, whose proof is exactly the same as the proof just given.

Proposition 5.3. *Let $M_- = (M_A, M_B, M_C, M_D)$ be a Mayer-Vietoris presentation. Then there is a functor*

$$L_{AC} : \mathcal{MV} \longrightarrow \mathcal{MV}$$

together with a natural transformation Id $\longrightarrow L_{AC}$ *with the following properties.*

1. *The map $M_- \longrightarrow M'_- = L_{AC}(M_-)$ is obtained by attaching (A, C)-cells to M_- and is therefore a cofibration of Mayer-Vietoris presentations.*
2. *The structure map $\phi'_{CA} : M'_A \longrightarrow M'_C$ is a weak homotopy equivalence of A-modules.*
3. *The map $M_D \longrightarrow M'_D$ is the identity.*
4. *The functor L_{AC} preserves cofibrations.*

□

We can combine the results of propositions 5.2 and 5.3 to obtain the localization functor we want to exploit.

Theorem 5.4. *Let $M_- = (M_A, M_B, M_C, M_D)$ be a Mayer-Vietoris presentation. There is a functor*

$$L : \mathcal{MV} \longrightarrow \mathcal{MV}$$

together with a natural transformation Id $\longrightarrow L$ *with the following properties.*

1. *The arrow $M_- \longrightarrow L(M_-) = M'_-$ is a cofibration of Mayer-Vietoris presentations.*

2. *The structure maps* $\phi'_{BA} : M'_A \longrightarrow M'_B$ *and* $\phi'_{CA} : M'_A \longrightarrow M'_C$ *are both homotopy equivalences.*

3. *The map* $M_D \longrightarrow M'_D$ *is the identity.*

4. *The functor* L *preserves cofibrations.*

Proof. Given a Mayer-Vietoris presentation M_- we consider the sequence of cofibrations

$$M_- \rightarrowtail L_{AB}(M_-) \rightarrowtail L_{AC}L_{AB}(M_-) \rightarrowtail L_{AB}L_{AC}L_{AB}(M_-) \rightarrowtail \cdots$$

and take the colimit to define $L(M_-)$.

In greater detail, define a sequence of Mayer-Vietoris presentations M^n_-, where

$$M^n_- = \begin{cases} (L_{AC}L_{AB})^k(M_-) & \text{if } n = 2k, \\ L_{AB}(L_{AC}L_{AB})^k(M_-) & \text{if } n = 2k+1. \end{cases}$$

Using the natural transformations from the identity functor to the functors L_{AB} and L_{AC} gives us the following sequence of cofibrations.

$$M_- = M^0_- \rightarrowtail M^1_- \rightarrowtail M^2_- \rightarrowtail \cdots$$

We define

$$L(M_-) = \operatorname{colim} M^n_-$$

and let $M_- \longrightarrow L(M_-)$ be the canonical map. Since a colimit of cofibrations is a cofibration, the canonical map is a cofibration. Similarly, L preserves cofibrations. Also, at all stages of the sequence the module at the D-corner is M_D, mapped by the identity, so the third property is also clearly satisfied. In the rest of the proof we will write $M'_- = L(M_-)$.

For the second item listed in the statement of the proposition, consider, for example, the relative homotopy group $\pi_n(M'_B, M'_A)$ and let the diagram

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ \downarrow & & \downarrow \\ M'_A & \longrightarrow & M'_B \end{array}$$

represent some element of this homotopy group. By compactness of the disc and Proposition III.1.7 of [6, page 53] again, there is a stage where the representative map factors as

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ \downarrow & & \downarrow \\ M^N_A & \longrightarrow & M^N_B \end{array}$$

followed by $(M_B^N, M_A^N) \rightarrowtail (M'_B, M'_A)$. If N is odd, then $M_A^N \longrightarrow M_B^N$ induces an isomorphism on homotopy groups, and, therefore, the homotopy element represented by the first diagram is already trivial in $\pi_n(M_A^N, M_B^N)$. On the other hand, if N is even, then the homotopy element represented by the diagram becomes trivial in M_-^{N+1}, by definition of M_-^{N+1} and the basic property of the functor L_{AB}. In any case, we obtain

$$\pi_n(M'_B, M'_A) = 0$$

for any n. The argument that $\pi_n(M'_C, M'_A) = 0$ is similar. □

There are, of course, situations where the applying the localization functor makes no essential change in a Mayer-Vietoris presentation.

Lemma 5.5. *Let $M = (M_A, M_B, M_C, M_D)$ be a Mayer-Vietoris presentation in which the structure map*

$$\phi_{BA} : M_A \longrightarrow M_B$$

is a weak homotopy equivalence. Then all four components of the canonical map

$$M \longrightarrow L_{AB}(M)$$

are weak homotopy equivalences.

Proof. Write, as usual, $M' = L_{AB}(M)$. Recall that the functor L_{BA} is obtained by attaching (A, B) cells to M along (A, B) spheres, which are spheres in M_A which cone off in M_B. That the arrows $M_B \longrightarrow M'_B$ and $M_D \longrightarrow M'_D$ are identities and, therefore, weak homotopy equivalences is an immediate consequence.

To see that $M_A \longrightarrow M'_A$ is a weak homotopy equivalence, consider the following diagram.

$$\begin{array}{ccc} M_A & \longrightarrow & M'_A \\ {\scriptstyle \phi_{BA}}\downarrow & & \downarrow{\scriptstyle \phi'_{BA}} \\ M_B & \longrightarrow & M'_B \end{array}$$

We have just recalled that the lower arrow is an identity. By construction the righthand vertical arrow is a weak homotopy equivalence, and by hypothesis the lefthand vertical arrow is also a weak homotopy equivalence. Therefore, since the diagram commutes, the arrow $M_A \longrightarrow M'_A$ is a weak homotopy equivalence as well. In particular, note that the cofiber M'_A/M_A is weakly contractible. However, more is true. By construction M'_A/M_A is a cell A-module. So, if it is weakly contractible, it is actually contractible.

To see that $M_C \longrightarrow M'_C$ is a homotopy equivalence, we note that the definition of (A, B) n-cell attachment implies that $M'_C/M_C = (M'_A/M_A) \wedge C$. Since we just observed that M'_A/M_A is contractible, M'_C/M_C is also contractible. Therefore, $M_C \longrightarrow M'_C$ is a weak homotopy equivalence as well. □

There is, of course, a parallel statement for the localization functor L_{AC}, just as proposition 5.3 is parallel to proposition 5.2. In this case we leave the statement to the reader.

We may combine the two variants, obtaining the following result about the functor L.

Lemma 5.6. *Let $M = (M_A, M_B, M_C, M_D)$ be a Mayer-Vietoris presentation in which the structure maps*

$$\phi_{BA} : M_A \longrightarrow M_B \quad \text{and} \quad \phi_{CA} : M_A \longrightarrow M_C$$

are weak homotopy equivalences. Then all four components of the canonical map

$$M \longrightarrow L(M)$$

are weak homotopy equivalences.

Proof. This follows easily by combining lemma 5.5 with its analogue □

Here is a concrete example of the localization process. This example figures in the proof of theorem 5.8 below, which is the main result concerning the output of the functor L. Let E_- denote the elementary Mayer-Vietoris presentation

$$E_- = \quad \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

The following result identifies $L(E_-)$ up to homotopy.

Proposition 5.7. *Let E_- be the elementary Mayer-Vietoris presentation, as above, and assume that the structure maps are all q-cofibrations of S-algebras. Then there is a canonical map of Mayer-Vietoris presentations*

$$L(E_-) \longrightarrow (D_A, D_B, D_C, D_D)$$

which is a homotopy equivalence at each corner. Here the notation (D_A, D_B, D_C, D_D) emphasizes that D is to be viewed as an A-module, a B-module, a C-module, and, finally, as a D-module.

Proof. In section 2 we gave three filtrations on D as a left module over the algebras A, B, and C, respectively. Similar filtrations of D as a right module appear in this proof and are absolutely crucial to the explanation. First we have

$$B = D_0 \cdot B \rightarrowtail D_1 \cdot B \rightarrowtail D_2 \cdot B \rightarrowtail \cdots \rightarrowtail D$$

where $D_n \cdot B / D_{n-1} \cdot B \cong \widetilde{W}_n^C \wedge_A B$. Here we write $\bar{B} = B/A$ and $\bar{C} = C/A$ and let $\widetilde{W}_n^C = \cdots \wedge_A \bar{B} \wedge_A \bar{C}$ be the word monomial having length n in the reduced modules $\bar{B}$

and $\bar{C}$ and ending in $\bar{C}$. We think of $D_n \cdot B$ as the right B-submodule of D generated by monomials of length less than or equal to n. Similarly, we have

$$C = D_0 \cdot C \rightarrowtail D_1 \cdot C \rightarrowtail D_2 \cdot C \rightarrowtail \cdots \rightarrowtail D$$

where $D_n \cdot C / D_{n-1} \cdot C \cong \widetilde{W}_n^B \wedge_A C$. Here $\widetilde{W}_n^B = \cdots \wedge_A \bar{C} \wedge_A \bar{B}$ is the word monomial having length n in the reduced modules $\bar{B}$ and $\bar{C}$ and ending in $\bar{B}$. Also, we can define a filtration of D by right A-modules

$$A = D_0 \rightarrowtail D_1 \rightarrowtail D_2 \rightarrowtail \cdots \rightarrowtail D$$

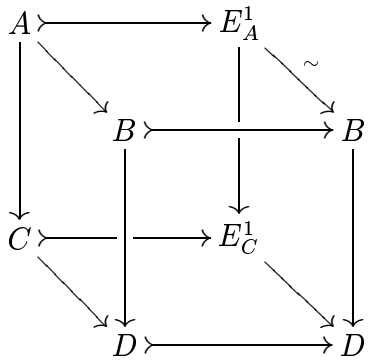
where $D_n / D_{n-1} \cong \widetilde{W}_n^B \vee \widetilde{W}_n^C$.

We are going to produce a map of sequences of cofibrations of Mayer-Vietoris presentations

$$\begin{array}{ccccccccc} E_- = E_-^0 & \rightarrowtail & E_-^1 & \rightarrowtail & E_-^2 & \rightarrowtail & \cdots & \rightarrowtail & E' \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ E_- = F_-^0 & \rightarrowtail & F_-^1 & \rightarrowtail & F_-^2 & \rightarrowtail & \cdots & \rightarrowtail & D_- \end{array}$$

where the upper row comes from the definition of L, and the lower row is constructed using the filtrations on D. At each stage the vertical arrows will be weak homotopy equivalences between corners of the Mayer-Vietoris presentations. Then passage to the limit will finish the proof.

We introduce the technique by studying the cofibration $E_-^0 \rightarrowtail E_-^1$. This step is slightly simpler than successive ones. Diagrammatically we have



so that $E_A^1 / A \xrightarrow{\sim} B/A = \bar{B}$ is a weak homotopy equivalence. Now the map $E_C^1 \longrightarrow D$ can be described by looking at stages in the construction of the domain.

$$\begin{aligned} E_C^{1,i+1} &= E_C^{1,i} \cup_{U^i \wedge_A C} V^i \wedge_A C \\ &\cong (E_A^{1,i} \cup_{U^i} V^i) \wedge_A C \\ &\longrightarrow B \wedge_A C \\ &\longrightarrow D \wedge_A D \longrightarrow D \end{aligned}$$

shows that $E^1_C \longrightarrow D$ factors through $D_1 \cdot C$, the right C-submodule of D generated by monomials of length less than or equal to 1. Since $D_0 \cdot B = B$, we can make a diagram

$$F^1_- = \begin{array}{ccc} D_0 \cdot B_A & \longrightarrow & D_0 \cdot B \\ \downarrow & & \downarrow \\ D_1 \cdot C & \longrightarrow & D \end{array}$$

and a map of diagrams

$$\begin{array}{ccc} E_- & \rightarrowtail & E^1_- \\ = \downarrow & & \downarrow \\ E_- & \rightarrowtail & F^1_- \end{array}$$

where the components of $E_- = F^0_- \longrightarrow F^1_-$ are cofibrations. We need to show that F^1_- is a Mayer-Vietoris presentation and that the components of $E^1_- \longrightarrow F^1_-$ are weak homotopy equivalences. These facts will follow after we show that the square F^1_-/F^0_- has the weakly homotopy cocartesian property of a Mayer-Vietoris presentation and after we show that the map of squares $E^1_-/E^0_- \longrightarrow F^1_-/F^0_-$ is a weak homotopy equivalence at each corner. We obtain what we need from these facts using the five lemma.

As just seen, we need to show that

$$\begin{array}{ccc} (D_0 \cdot B_A/A) \wedge_A D & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ (D_1 \cdot C/C) \wedge_C D & \longrightarrow & 0 \end{array}$$

is weakly homotopy cocartesian. It suffices to show that the lefthand vertical arrow is a weak homotopy equivalence. But, by definition of the filtrations, we have cofibration sequences

$$A \longrightarrow D_0 \cdot B \longrightarrow \bar{B}$$
$$C \longrightarrow D_1 \cdot C \longrightarrow \bar{B} \wedge_A C$$

so that $(D_0 \cdot B_A/A) \wedge_A D \cong \bar{B} \wedge_A D$ and $(D_1 \cdot C/C) \wedge_C D \cong \bar{B} \wedge_A C \wedge_C D \cong \bar{B} \wedge_A D$, so the lefthand vertical arrow is a weak homotopy equivalence, as needed.

To see that $E^1_- \longrightarrow F^1_-$ is a weak homotopy equivalence at each corner, it suffices to check this for $E^1_-/E^0_- \longrightarrow F^1_-/F^0_-$, since $E^0_- = F^0_-$. This reduces further to checking that

$$E^1_A/E^0_A \longrightarrow F^1_A/F^0_A \qquad \text{and} \qquad E^1_C/E^0_C \longrightarrow F^1_C/F^0_C$$

are weak homotopy equivalences. But the first of these arrows is just $E^1_A/A \longrightarrow B/A$ induced from the weak homotopy equivalence $E^1_A \longrightarrow B$. By studying the construction as above and commuting colimits, the second arrow may be identified with

$$(E^1_A/E^0_A) \wedge_A C \longrightarrow (F^1_A/F^0_A) \wedge_A C = (B/A) \wedge_A C.$$

Since C is q-cofibrant as an A-module, smashing with C preserves weak equivalences, [6, page 60], so the second arrow is also a weak homotopy equivalence.

Let $n \geq 2$. For n even, define

$$F^n_- = \begin{array}{ccc} D_{n-1} \cdot C_A & \longrightarrow & D_n \cdot B \\ \downarrow & & \downarrow \\ D_{n-1} \cdot C & \longrightarrow & D \end{array}$$

and for n odd define

$$F^n_- = \begin{array}{ccc} D_{n-1} \cdot B_A & \longrightarrow & D_{n-1} \cdot B \\ \downarrow & & \downarrow \\ D_n \cdot C & \longrightarrow & D \end{array}$$

The notations $D_{n-1} \cdot C_A$ and $D_{n-1} \cdot B_A$ emphasize that these modules are to be viewed as A-modules, by restriction. There are canonical and compatible maps $F^n_- \longrightarrow (D_A, D_B, D_C, D_D)$ which induce an isomorphism of diagrams $\mathrm{colim}_n F^n_- \cong (D_A, D_B, D_C, D_D)$. We claim that F^n_- is a Mayer-Vietoris presentation and that $E^n_- \longrightarrow D_-$ factors through a map $E^n_- \longrightarrow F^n_-$ that is a weak homotopy equivalence at each corner. We will prove all these assertions by induction on n.

Before we launch into the details, let us make some rough notes on the passage from E^n_- to E^{n+1}_- by application of L_{AB} or L_{AC}, depending on whether n is even or odd. In the case were n is even, we are applying L_{AB} to a Mayer-Vietoris presentation that looks homotopically like

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & D \end{array}$$

to produce a Mayer-Vietoris presentation that looks homotopically like

$$\begin{array}{ccc} Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & D \end{array}$$

In case n is odd, we apply L_{AC} to convert a Mayer-Vietoris presentation of the latter type to one of the former type. Consequently, it suffices to discuss the passage from E^n_- to E^{n+1}_- where n is odd, for the case where n is even is just symmetric.

So, let n be odd, and assume we have a commuting diagram of Mayer-Vietoris presentations

$$\begin{array}{ccc} E^n_- & \longrightarrow & D_- \\ {\scriptstyle c_n}\downarrow & & \downarrow{\scriptstyle =} \\ F^n_- & \longrightarrow & D_- \end{array}$$

where the components of $c_n : E^n_- \longrightarrow F^n_-$ are weak homotopy equivalences and the horizontal arrows are the canonical maps. We need to produce a similar diagram for $E^{n+1}_- = L_{AC}(E^n_-)$ and F^{n+1}_-.

Let us consider first the commuting diagram

$$\begin{array}{ccccc} E^n_- & \longrightarrow & E^{n+1}_- = L_{AC}(E^n_-) & \longrightarrow & D_- \\ {\scriptstyle c_n}\downarrow & & \downarrow & & \downarrow{\scriptstyle =} \\ F^n_- & \longrightarrow & L_{AC}(F^n_-) & \longrightarrow & D_- \end{array}$$

obtained by applying L_{AC} to the first column of the diagram above. Since each component of c_n is a weak homotopy equivalence, each component of the mapping cone $C(c_n) = T(c_n)/E^n_-$ is weakly contractible. Commuting the construction of L_{AC} with that of $C(c_n)$, which is possible, because both are colimits, we find that $C(L_{AC}(c_n)) = L(C(c_n))$. Applying the (A, C) variant of lemma 5.5 to $C(c_n)$, we find that $C(c_n) \longrightarrow L(C(c_n))$ is a weak homotopy equivalence on each component. This means that each component of $C(L_{AC}(c_n))$ is weakly contractible. Thus, we conclude that each component of $L(c_n)$ is a weak homotopy equivalence.

So, to prove what we want, it suffices to prove that F^{n+1}_- is a Mayer-Vietoris presentation and to factor $L_{AC}(F^n_-) \longrightarrow D_-$ as $L_{AC}(F^n_-) \longrightarrow F^{n+1}_- \longrightarrow D_-$ where the components of the first arrow are weak homotopy equivalences. Our argument that F^{n+1}_- has the homotopy cocartesian property of a Mayer-Vietoris presentation is reminiscent of the inductive argument given in the proof of Theorem 2.7. Our inductive hypothesis is that F^n_- is a Mayer-Vietoris presentation. By looking at a cofibration sequence of squares, we see that F^{n+1}_- has the homotopy cocartesian property if the square

$$\begin{array}{ccc} (D_n \cdot C_A / D_{n-1} \cdot B_A) \wedge_A D & \longrightarrow & (D_{n+1} \cdot B / D_{n-1} \cdot B) \wedge_B D \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & 0 \end{array}$$

is weakly homotopy cocartesian. But this means we need to see that the top row is a weak homotopy equivalence.

Let $n = 2k - 1$. We need to refine the filtration on D by A-modules to interpolate something between $D_{2k-2} \cdot B_A$ and $D_{2k-1} \cdot C_A$. It helps to think of noncommutative polynomials

in variables b and c. The smaller term corresponds to polynomials of total degree less than or equal to $2k-1$ and, if the degree is $2k-1$, then the highest degree term ends with a b. The larger term corresponds to polynomials of degree less than or equal to $2k$, and, if the degree is $2k$, then the highest degree term ends with c. To get the larger object from the smaller one we have to add polynomials of degree $2k-1$ that end in c and the polynomials of degree $2k$ that end in c. Obviously, this should be done in two steps. We define an A-submodule of D, $D_{2k-(3/2)}$, by taking the image of $\mathbb{T}_{\leq 2k-1}(B \vee C)$ in D, as in section 2. Clearly

$$D_{2k-2} \cdot B_A \longrightarrow D_{2k-(3/2)} \longrightarrow D_{2k-1} \cdot C_A$$

and manipulations similar to those in the proof of Theorem 2.7 show that

$$D_{2k-(3/2)}/D_{2k-2} \cdot B_A \cong \widetilde{M}^C_{2k-1} \quad \text{and} \quad D_{2k-1} \cdot C_A/D_{3k-(3/2)} \cong \widetilde{M}^C_{2k}.$$

Now we also have formulas

$$D_{2k} \cdot B/D_{2k-1} \cdot B \cong \widetilde{M}^C_{2k} \wedge_A B \quad \text{and} \quad D_{2k-1} \cdot B/D_{2k-2} \cdot B \cong \widetilde{M}^C_{2k-1} \wedge_A B$$

Using a two step filtration on the source and the target of the arrow

$$(D_n \cdot C_A/D_{n-1} \cdot B_A) \wedge_A D \longrightarrow (D_{n+1} \cdot B/D_{n-1} \cdot B) \wedge_B D$$

shows that this arrow is indeed a weak homotopy equivalence.

It remains to produce a map

$$L(F^{2k-1}_-) \longrightarrow F^{2k}_-$$

which is a weak equivalence at each corner. By construction

$$L(F^{2k-1}_-)_A \longrightarrow F^{2k}_A = D_{2k-1} \cdot C_A$$

is a weak equivalence, and $L(F^{2k-1}_-)_B$ maps to the smallest right B-submodule of D containing $D_{2k-1} \cdot C$. By an argument similar to that given for the passage to E^1_- from E^0_-, this module is $D_{2k} \cdot B$. Still following the outline of the earlier argument, we can show that

$$L(F^{2k-1}_-)_B \longrightarrow F^{2k}_C = D_{2k} \cdot B$$

is a weak homotopy equivalence. The first step is to work relative to F^{2k-1}_-, then to relate these B corner quotients to the A-corner quotients. There is an extra step this time, due to the two step filtrations on these quotients. □

Now we take up the following theorem. This result describes the main property of a Mayer-Vietoris presentations localized in our sense. It is also the first result that requires an extra hypothesis on the generalized free product diagram. The proof is given after the statements and proofs of the two lemmas to follow.

Theorem 5.8. *Suppose that the S-algebras A, B, and C are connective, and suppose that*

$$\begin{array}{ccc} M_A & \xrightarrow{\phi_{BA}} & M_B \\ {\scriptstyle \phi_{CA}}\downarrow & & \downarrow{\scriptstyle \phi_{DB}} \\ M_C & \xrightarrow[\phi_{DC}]{} & M_D \end{array}$$

is a Mayer-Vietoris presentation of modules for which there is an integer N such that $q < N$ implies $\pi_q M_A = \pi_q M_B = \pi_q M_C = \pi_q M_D = 0$. We say M_- has homotopy that is bounded below. If the structure maps ϕ_{BA} and ϕ_{CA} are weak homotopy equivalences, then ϕ_{DB} and ϕ_{DC} are also weak homotopy equivalences.

The following corollary describes the first property of the localization functor that makes it useful for our study of the K-theory of Mayer-Vietoris presentations with respect to coarse equivalences. The result also gives precise form to the intuitive idea that after localization with respect to a generalized homology theory what remains is exactly what that homology theory sees; in the case of our Mayer-Vietoris presentations, that is just the weak homotopy type of the D-module.

Corollary 5.9. *Suppose that the S-algebras A, B, and C are connective, and suppose that M_- is a Mayer-Vietoris presentation of finite or homotopy finite modules, and let $M'_- = L(M_-)$ be its localization in the sense defined above. Then all four structure maps ϕ'_{BA}, ϕ'_{CA}, ϕ'_{DB}, and ϕ'_{DC}, of M'_- are weak homotopy equivalences.*

Proof. Since M_- is a Mayer-Vietoris presentation of finite or homotopy finite modules, it has homotopy that is bounded below. Clearly, since the S-algebras involved are connective, the localization construction described above preserves the property of having homotopy that is bounded below. Also, it guarantees that the structure maps ϕ'_{BA} and ϕ'_{CA} are weak homotopy equivalences. By theorem 5.8 the remaining two structure maps ϕ'_{DB} and ϕ'_{DC} are weak homotopy equivalences. □

There is one case where the theorem is obvious, and that is when the module M_A is weakly contractible. Under this condition, the hypothesis implies that M_B and M_C are also weakly contractible. Applying the generalization of Theorem III.3.8 of [6, page 60] in which a cell R-module is replaced by a q-cofibrant R module, $M_A \wedge_A D$, $M_B \wedge_B D$, $M_C \wedge_C D$ are also weakly contractible. (For readers not familiar with the terminology, a q-fibrant R-module is a retract of a cell R-module [6, Theorem VII.4.14, page 143]. Consequently, the q-cofibrant case follows formally from the cell case.) Finally, by the homotopy property of a Mayer-Vietoris presentation, M_D is also weakly contractible. Since we already know M_B and M_C are weakly contractible, the result is clearly true. In the next few paragraphs, we reduce the general situation to this simple case.

Recall that E_- denotes the elementary Mayer-Vietoris presentation

$$E_- = \begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

Suppose $M_-^n = (M_A^n, M_B^n, M_C^n, M_D^n)$ is a Mayer-Vietoris presentation in which $\pi_q M_A^n = 0$ for $q < n$, and suppose also that ϕ_{BA} and ϕ_{CA} are weak equivalences. Let W^n be a sufficiently large wedge of n-spheres and choose a map

$$W^n \wedge A \longrightarrow M_A^n$$

so that $\pi_n(W^n \wedge A) \longrightarrow \pi_n M_A^n$ is surjective. Then the induced map $\pi_n(W^n \wedge B) \longrightarrow \pi_n M_B^n$ will also be surjective, since ϕ_{BA} induces an isomorphism $\pi_n M_A \longrightarrow \pi_n M_B$, by hypothesis. Similar assertions are true about the (A, C)-leg of the diagram. Define M_-^{n+1} by means of the following pushout diagram of Mayer-Vietoris presentations.

$$\begin{array}{ccc} W^n \wedge E & \rightarrowtail & W^n \wedge I \wedge E_- \\ \downarrow & & \downarrow \\ M_-^n & \rightarrowtail & M_-^{n+1} \end{array}$$

The following lemma records the essential properties of M_-^{n+1}.

Lemma 5.10. *With M_-^{n+1} constructed as above, we have*

$$\pi_q M_A^{n+1} = \pi_q(M_B^{n+1}) = \pi_q(M_C^{n+1}) = 0 \quad \text{for } q < n+1,$$

and there is a cofibration sequence of Mayer-Vietoris presentations

$$M_-^n \rightarrowtail M_-^{n+1} \twoheadrightarrow \Sigma W^n \wedge E_-$$

where ΣW^n is the wedge of $(n+1)$-spheres obtained by suspending W^n.

□

If we combine this construction with the localization functor, we obtain the following result.

Lemma 5.11. *Let $M_-^n = (M_A^n, M_B^n, M_C^n, M_D^n)$ be a Mayer-Vietoris presentation in which $\pi_q M_A^n = 0$ for $q < n$, and suppose also that ϕ_{BA}^n and ϕ_{CA}^n are weak homotopy equivalences. Construct M_-^{n+1} as above, and let*

$$(M_-^{n+1})' = L(M_-^{n+1}).$$

Then the Mayer-Vietoris presentation $(M_-^{n+1})'$ has the following properties.

1. *The morphisms $(\phi_{BA}^{n+1})'$ and $(\phi_{CA}^{n+1})'$ are weak homotopy equivalences.*
2. *For the module $(M_-^{n+1})'_A$ we have $\pi_q(M_-^{n+1})'_A = 0$ for $q < n+1$.*
3. *The relative homotopy groups along the B-D and C-D edges remain the same. That is,*

   $$\pi_*((M_-^{n+1})'_D, (M_-^{n+1})'_B) \cong \pi_*(M_D^n, M_B^n) \quad \text{and} \quad \pi_*((M_-^{n+1})'_D, (M_-^{n+1})'_C) \cong \pi_*(M_D^n, M_C^n).$$

Proof. According to Theorem 5.4 the functor L preserves cofibration sequences. The following diagram, in which the rows are cofibration sequences of Mayer-Vietoris presentations, displays the situation.

$$\begin{array}{ccccc} M_-^n & \rightarrowtail & M_-^{n+1} & \twoheadrightarrow & \Sigma W^n \wedge E_- \\ \downarrow & & \downarrow & & \downarrow \\ L(M_-^n) & \rightarrowtail & L(M_-^{n+1}) & \twoheadrightarrow & \Sigma W^n \wedge L(E_-) \end{array}$$

The first statement in the proposition simply recalls an essential property of the functor L; the second statement follows by noting that, in view of the vanishing of the homotopy groups of M_A^{n+1}, M_B^{n+1}, and M_C^{n+1} in dimensions strictly smaller than $n{+}1$ proved in lemma 5.10, the L construction really starts to modify M_A^{n+1} in dimensions $n{+}1$ and higher.

The third statement is proven in two steps. First the facts that

$$\pi_*((M_-^n)'_D, (M_-^n)'_B) \cong \pi_*(M_D^n, M_B^n) \quad \text{and} \quad \pi_*((M_D^n)', (M_C^n)') \cong \pi_*(M_D^n, M_C^n)$$

follow from lemma 5.6 for the strongest possible reason. Namely, all the component arrows of the lefthand vertical map of Mayer-Vietoris presentations are homotopy equivalences, in view of the hypothesis on M^n and the lemma. For the second step, observe that the lower row is a cofibration sequence of Mayer-Vietoris presentations. If we extract some of the data then we have a ladder diagram

$$\begin{array}{ccccc} (M_-^n)'_B & \rightarrowtail & (M_-^{n+1})'_B & \twoheadrightarrow & \Sigma W^n \wedge L(E_-)_B \\ \downarrow & & \downarrow & & \downarrow \\ (M_-^n)'_D & \rightarrowtail & (M_-^{n+1})'_D & \twoheadrightarrow & \Sigma W^n \wedge L(E_-)_D \end{array}$$

where the righthand vertical arrow is a weak homotopy equivalence, by our sample calculation in proposition 5.7. Thus, the iterated homotopy quotient of the lefthand square is weakly contractible. Calculating this quotient by doing the columns first shows that

$$\pi_*((M_-^{n+1})'_D, (M_-^{n+1})'_B) \cong \pi_*((M_-^n)'_D, (M_-^n)'_B).$$

We similarly argue that

$$\pi_*((M_-^{n+1})'_D, (M_-^{n+1})'_C) \cong \pi_*((M_-^n)'_D, (M_-^n)'_C)$$

to complete the proof. □

Now we are ready to prove Theorem 5.8.

Proof of Theorem 5.8. Since A is connective, and the Mayer-Vietoris presentation M_- is homotopy finite, there is an integer n_0 such that $\pi_q M_A = 0$ for $q < n_0$. Then, by hypothesis we also have the groups $\pi_q M_B$ and $\pi_q M_C$ vanishing in the same range. Clearly the proof of

Theorem III.3.8 in [6, page 60] also shows that smash product by q-cofibrant module also preserves weak N-equivalences, for any N. So we also have

$$\pi_q(M_A \wedge_A D) = \pi_q(M_B \wedge_B D) = \pi_q(M_C \wedge_C D) = 0$$

for $q < n_0$. By the homotopy cocartesian property of a Mayer-Vietoris square, $\pi_q M_D = 0$ for $q < n_0$. By exact sequences we conclude that $\pi_q(M_D, M_B) = \pi_q(M_D, M_C) = 0$ for $q < n_0$. These remarks get us started.

Now let $n \geq n_0 + 1$. We show $\pi_q(M_D, M_B) = \pi_q(M_D, M_C) = 0$ for $q < n$ as follows. Inductively make a sequence of cofibrations

$$M_- = \widetilde{M}_-^{n_0} \rightarrowtail \widetilde{M}_-^{n_0+1} \rightarrowtail \cdots \rightarrowtail \widetilde{M}_-^{n}$$

as follows. Given $\widetilde{M}_-^k$, which has no homotopy in M_A below dimension k, make $\widetilde{M}_-^{k+1}$ in two steps by first killing the k-th homotopy group using the procedure of lemma 5.10 and then applying the localization functor L, as in lemma 5.11. As we read from lemma 5.11, we are making the A-module in the Mayer-Vietoris presentations more and more highly connected, and we restore the BA- and CA-structure maps to being weak homotopy equivalences by applying the L-functor. Lemma 5.11 applies at each stage so that

$$\pi_q(M_D, M_B) \cong \pi_q(\widetilde{M}_D^n, \widetilde{M}_B^n) \qquad \text{and} \qquad \pi_q(M_D, M_C) \cong \pi_q(\widetilde{M}_D^n, \widetilde{M}_C^n).$$

But with $\widetilde{M}_-^n$ and for $q < n$ we are in the stiuation discussed in the first paragraph of the proof. Repeating that argument, we find that

$$0 = \pi_q(\widetilde{M}_D^n, \widetilde{M}_B^n) \cong \pi_q(M_D, M_B)$$

and, similarly, $0 = \pi_q(\widetilde{M}_D^n, \widetilde{M}_C^n) = \pi_q(M_D, M_C)$. Since n was arbitrary, we have the conclusion. □

Now we will see that, for any Mayer-Vietoris presentation $M_- \in \mathcal{MV}$ without homotopy in arbitrarily low dimensions, the localization $L(M_-)$ is approximated by relatively finite Mayer-Vietoris presentations in a sense made precise by the next theorem. As mentioned in the introduction to the section, the property we are proving here is an analogue of the convergence property of possessed by Bousfield localization functors of finite type.

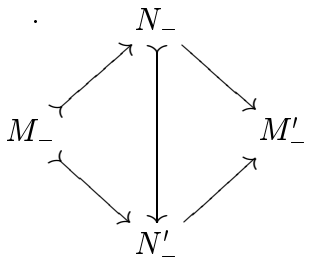
Fix a Mayer-Vietoris presentation $M_- \in \mathcal{MV}$ and assume that there is an N such that $\pi_q(M_A) = \pi_q(M_B) = \pi_q(M_C) = \pi_q(M_D) = 0$ for all $q < N$. Define category of diagrams $\mathcal{D}(M_-)$ as follows. An object of $\mathcal{D}(M_-)$ is a diagram

$$M_- \rightarrowtail N_- \longrightarrow M'_-$$

where each of the modules N_A, N_B, N_C has vanishing homotopy below dimension N and is a finite cell extension of the corresponding component of M_-, and where the D-component of the diagram reduces to

$$M_D \xrightarrow{=} N_D \xrightarrow{=} M'_D.$$

A morphism of $\mathcal{D}(M_-)$ is a commutative diagram



Define a diagram

$$\begin{array}{ccc} M''_A & \longrightarrow & M''_B \\ \downarrow & & \downarrow \\ M''_C & \longrightarrow & M''_D \end{array}$$

by setting

$$M''_A = \operatorname*{colim}_{\mathcal{D}(M)} N_A,\ M''_B = \operatorname*{colim}_{\mathcal{D}(M)} N_B,\ M''_C = \operatorname*{colim}_{\mathcal{D}(M)} N_C$$

and $M''_D = M_D$. Obtain the structure maps of the diagram by the universal property of the colimit. Also note that the universal property of the colimit yields a canonical map of diagrams

$$M''_- \longrightarrow M'_-.$$

In the process of defining the components of M''_- we are taking colimits of relative cellular inclusions over a filtering category, so each component of M''_- is a cell module relative to the corresponding component of M_-, by a slight generalization of [6, Proposition III.2.6, page 56]

Proposition 5.12. *The preceding diagram M''_- is a Mayer-Vietoris presentation with vanishing homotopy in dimensions below N.*

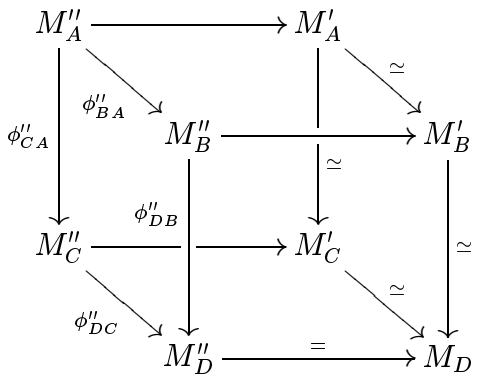
Proof. We check the homotopy cocartesian property by commuting colimits; the vanishing homotopy condition is preserved by the colimit construction. □

Theorem 5.13. *Let M_- be a Mayer-Vietoris presentation of homotopy finite modules, and let $M'_- = L(M_-)$ be its localization as constructed in Proposition 5.4. Then the components of the canonical map*

$$M''_- \longrightarrow M'_-$$

are all weak homotopy equivalences.

Proof. Display the components of this canonical map in the following diagram.



The arrows in the right half of the diagram are all weak homotopy equivalences by Theorem 5.4 and Theorem 5.8, and the lower horizontal arrow is an equality by construction. We will prove our claim by showing the arrows ϕ''_{BA} and ϕ''_{CA} are weak homotopy equivalences. Then we apply Theorem 5.8 to the lefthand Mayer-Vietoris presentation, obtaining that ϕ''_{DB} and ϕ''_{DC} are also weak homotopy equivalences. Since the diagram above commutes, the components $M''_A \longrightarrow M''_A$, and so on, must also be weak homotopy equivalences, which is the claim.

Let us now prove that ϕ''_{BA} is a weak homotopy equivalence. Suppose that the diagram

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ M''_A & \xrightarrow{\phi''_{BA}} & M''_B \end{array}$$

represent an element x of $\pi_n(\phi''_{BA})$. Our goal is to show that $x = 0$. We accomplish this task in essentially two steps. First we find an object of the diagram category $\mathcal{D}(M_-)$ through which the map of pairs (f, g) factors, and then we create another object of $\mathcal{D}(M_-)$ in which the map of pairs is visibly trivial.

Since S^{n-1} is compact and M''_A is a cell A-module relative to M_A, f factors through a finite relative cell submodule of M''_A by [6, Lemma III.2.3, page 55]. Consequently, there is an object N^0_- of $\mathcal{D}(M_-)$ and a factorization of f of the form

$$S^{n-1} \longrightarrow N^0_A \xrightarrow{\gamma^0_A} M''_A$$

where $\gamma^0_A : N^0_A \longrightarrow M''_A$ is the canonical map to the colimit. Note that the canonical map is an isomorphism

$$\operatorname*{colim}_{N'_- \in \mathcal{D}(M_-), N'_- \supset N^0_-} N'_B \longrightarrow \operatorname*{colim}_{N_- \in \mathcal{D}(M_-)} N_B$$

with an inverse induced by creating pushout diagrams

$$\begin{array}{ccc} M_- & \rightarrowtail & N_-^0 \\ \downarrow & & \downarrow \\ N_- & \rightarrowtail & N_-' \end{array}$$

Using this alternative description of M_B'' and the compactness of D^n, we conclude that there is a Mayer-Vietoris presentation N_-^1 in $\mathcal{D}(M_-)$ containing N_-^0 and such that $g : D^n \longrightarrow M_B''$ factors as

$$D^n \longrightarrow N_B^1 \xrightarrow{\gamma_B^1} M_B''$$

Thus, we have found a Mayer-Vietoris presentation N_-^1 in $\mathcal{D}(M_-)$ and a factorization

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ {\scriptstyle f^1}\downarrow & & \downarrow{\scriptstyle g^1} \\ N_A^1 & \xrightarrow{\phi_{BA}^1} & N_B^1 \\ {\scriptstyle \gamma_A^1}\downarrow & & \downarrow{\scriptstyle \gamma_B^1} \\ M_A'' & \xrightarrow{\phi_{BA}''} & M_B'' \end{array}$$

of the original diagram. We may write that we have found a Mayer-Vietoris presentation N_-^1 in $\mathcal{D}(M_-)$ and $x^1 \in \pi_n(\phi_{BA}^1)$ such that $\gamma_*^1(x^1) = x$, where $\gamma^1 : (N_B^1, N_A^1) \longrightarrow (M_B'', M_A'')$ is the canonical map.

Now we will create a Mayer-Vietoris presentation N_-^2 in $\mathcal{D}(M_-)$ together with a morphism $\gamma_1^2 : N_-^1 \longrightarrow N_-^2$ such that

$$0 = x^2 = (\gamma_1^2)_*(x^1) \in \pi_n(\phi_{BA}^2).$$

Then it follows that $x = (\gamma_1'')_*(x^1) = (\gamma_2'')_*(\gamma_1^2)_*(x^1) = 0$, as needed. Define N_A^2 and N_C^2 so that the following diagrams are pushouts.

$$\begin{array}{ccc} S^{n-1} \wedge A & \xrightarrow{f_1} & N_A^1 \\ \downarrow & & \downarrow \\ D^n \wedge A & \longrightarrow & N_A^2 \end{array} \qquad \begin{array}{ccc} S^{n-1} \wedge C & \xrightarrow{f_1} & N_C^1 \\ \downarrow & & \downarrow \\ D^n \wedge C & \longrightarrow & N_C^2 \end{array}$$

Let the structure map $\phi_{CA}^2 : N_A^2 \longrightarrow N_C^2$ be induced from ϕ_{CA}^1 by the pushout construction. To obtain N_B^2, a structure map $\phi_{BA}^2 : N_A^2 \longrightarrow N_B^2$, and a map $N_- \longrightarrow M_-'$, consider the

following diagram.

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ {\scriptstyle f^1}\downarrow & & \downarrow{\scriptstyle g^1} \\ N^1_A & \xrightarrow{\phi^1_{BA}} & N^1_B \\ {\scriptstyle \gamma'_{1,A}}\downarrow & & \downarrow{\scriptstyle \gamma'_{1,B}} \\ M'_A & \xrightarrow{\phi'_{BA}} & M'_B \end{array}$$

Applying the mapping cylinder construction to the top and bottom rows of the preceding diagram, we can represent the data by a map

$$G : (D^n \times \{0\} \cup S^{n-1} \times I, S^{n-1} \times \{1\}) \longrightarrow (T(\phi'_{BA}), M'_A)$$

such that $G|D^n \times \{0\} = j_2 \circ \gamma'_{1,B} \circ g^1$ and $G|S^{n-1} \times \{1\} = j_1 \circ \gamma'_{1,A} \circ f^1$. Here $j_2 : M'_B \longrightarrow T(\phi'_{BA})$ is the back inclusion of the mapping cylinder, and $j_1 : M'_A \longrightarrow T(\phi'_{BA})$ is the front inclusion.

By the characterization of $L(M_-) = M'_-$, $\pi_n(\phi'_{BA}) = 0$ so there is an extension of G to

$$H : (D^n, S^{n-1}) \times I \longrightarrow (T(\phi'_{BA}), M'_A)$$

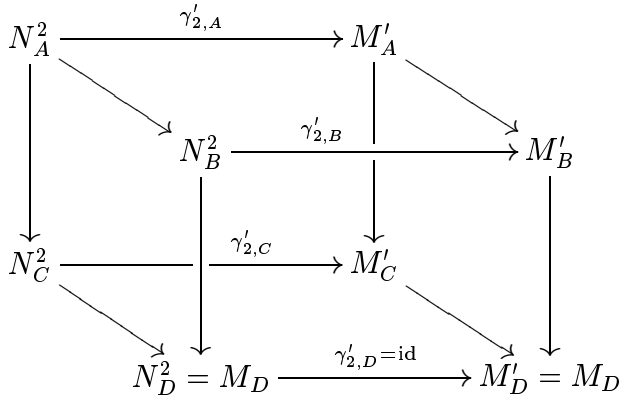
such that $H|D^n \times \{1\}$ factors through the front inclusion $j_1 : M'_A \longrightarrow T(\phi'_{BA})$.

Define N^2_B to make the following diagram a pushout.

$$\begin{array}{ccc} (D^n \times \{0\} \cup S^{n-1} \times I) \wedge B & \rightarrowtail & (D^n \times I) \wedge B \\ \downarrow & & \downarrow \\ N^1_B & \rightarrowtail & N^2_B \end{array}$$

where the lefthand vertical map is g^1 on $D^n \times \{0\}$ and $\phi^1_{BA} \circ f^1 \circ pr_1$ on $S^{n-1} \times I$. Creating this pushout is the analogue of attaching a cancelling cell pair in simple homotopy theory. Using H and the back projection $p : T(\phi'_{BA}) \longrightarrow M'_B$ we define $\gamma'_{2,B} : N^2_B \longrightarrow M'_B$. Let $\phi^2_{BA} : N^2_A \longrightarrow N^2_B$ be induced from ϕ^1_{BA} and from the inclusion $i_1 : D^n \longrightarrow D^n \times I$ that sends the disc to the $t = 1$ level of the cylinder. Thus, the n-cell added to N^1_A is mapped to the free face of the $(n+1, n)$-cell pair attached to N^1_B. Assembling the properties of the various definitions, the composite $\gamma'_{2,B} \circ \phi^2_{BA}$ factors through ϕ'_{BA}, so that $\gamma'_{2,A} : N^2_A \longrightarrow M'_A$ satisfying $\phi'_{BA} \circ \gamma'_{2,A} = \gamma'_{2,B} \circ \phi^2_{BA}$ exists. Finally, define $\phi^2_{DC} : N^2_C \longrightarrow N^2_D = M_D$ to be the map induced by $\phi^1_{DC} : N^1_C \longrightarrow M_D$ and by $\phi'_{DB} \circ (p \circ H \circ i_1)$ on the added disc. This finishes

the construction of the following commutative diagram.



In the diagram

$$\begin{array}{ccc} \pi_*(N^1_C \wedge D, N^1_A \wedge D) & \xrightarrow{\cong} & \pi_*(N^2_C \wedge D, N^2_A \wedge D) \\ \cong \downarrow & & \downarrow \\ \pi_*(N^1_D, N^1_B \wedge D) & \xrightarrow{\cong} & \pi_*(N^2_D, N^2_B \wedge D) \end{array}$$

the upper horizontal arrow is an isomorphism by excision and a homotopy, the lower horizontal arrow is an isomorphism by the five lemma, because $N^1_B \simeq N^2_B$, and the lefthand vertical arrow is an isomorphism because N^1_- is a Mayer-Vietoris presentation. Consequently, the righthand vertical arrow is an isomorphism as well, and it follows that the diagram N^2_- just defined satisfies the homotopy property of a Mayer-Vietoris presentation. By the definitions it is also clear that $(\gamma^2_1)_*(x^1) = 0$ in $\pi_n(\phi^2_{BA})$, so that $\pi_n(\phi''_{BA}) = 0$ follows as sketched above.

A similar argument proves $\pi_n(\phi''_{CA}) = 0$, which finishes the proof of the theorem. □

6 Mayer-Vietoris presentations and coarse equivalences

Consider a generalized free product diagram

$$\begin{array}{ccc} A & \xrightarrow{\beta} & B \\ {\scriptstyle\gamma}\downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in which the S-algebras A, B, and C are q-cofibrant, the maps β and γ are q-cofibrations, and A, B, and C are connective. In the case that the structure maps β and γ make B and C into relative left cell A-modules, examination of the filtration on D defined in section 2 shows that D will be connective if A, B, and C are. The S-algebra D will also be connective under the hypotheses given above. We need the hypotheses on A, B, and C so that Theorem 5.13 applies; we need D to be connective so that we can apply the Hurewicz theorem in the argument for the following theorem.

Theorem 6.1. *Suppose we are in a generalized free product situation where the S-algebras A, B, and C are q-cofibrant, the maps to B and C are q-cofibrations, and A, B, C, and D are connective. Then the forgetful functors*

$$u_D : \mathcal{MV}_f \longrightarrow \mathcal{M}(D)_f \quad \textit{and} \quad u_D : \mathcal{MV}_{hf} \longrightarrow \mathcal{M}(D)_{hf}$$

induce homotopy equivalences

$$wS_\bullet\mathcal{MV}_f \longrightarrow hS_\bullet\mathcal{M}(D)_f \quad \textit{and} \quad wS_\bullet\mathcal{MV}_{hf} \longrightarrow hS_\bullet\mathcal{M}(D)_{hf}.$$

Given this result, it is easy to prove the following statement which, when joined with theorems 3.21 and 4.2, says all the arrows in the diagrams displayed in the statement of theorem 3.21 are homotopy equivalences as long as the original generalized free product diagram satisfies some connectivity conditions.

Theorem 6.2. *Under the same hypotheses as above, the inclusion functors*

$$j_1 : \mathcal{MV}_{f,e} \longrightarrow \mathcal{MV}_{hf,e} \qquad \textit{and} \qquad j_2 : \mathcal{MV}_f \longrightarrow \mathcal{MV}_{hf}$$

induce homotopy equivalences

$$wS_\bullet(j_1) : wS_\bullet\mathcal{MV}_{f,e} \longrightarrow wS_\bullet\mathcal{MV}_{hf,e} \quad \textit{and} \quad wS_\bullet(j_2) : wS_\bullet\mathcal{MV}_f \longrightarrow wS_\bullet\mathcal{MV}_{hf}.$$

Proof. We have a commutative diagram

$$\begin{array}{ccc} wS_\bullet\mathcal{MV}_f & \longrightarrow & wS_\bullet\mathcal{MV}_{hf} \\ {\scriptstyle (u_D)_*}\downarrow & & \downarrow{\scriptstyle (u_D)_*} \\ hS_\bullet\mathcal{M}(D)_f & \longrightarrow & hS_\bullet\mathcal{M}(D)_{hf} \end{array}$$

in which the vertical arrows are homotopy equivalences by the preceding theorem. The lower arrow is already a homotopy equivalence, by the wellknown theorem 3.1. Therefore, the upper arrow is a homotopy equivalence as well. We also have a commutative diagram

$$\begin{array}{ccc} wS_\bullet\mathcal{MV}_{f,e} & \longrightarrow & wS_\bullet\mathcal{MV}_{hf,e} \\ \downarrow & & \downarrow \\ wS_\bullet\mathcal{MV}_f & \longrightarrow & wS_\bullet\mathcal{MV}_{hf} \end{array}$$

in which we proved the vertical arrows are homotopy equivalences in Theorem 3.21 and in which we have just proved the lower arrow is a homotopy equivalence. Therefore, the upper arrow is also a homotopy equivalence. This completes the proof of Theorem 6.2 and, thus, the constellation of results related to the diagrams in theorem 3.21 is complete. □

Returning to the main result, we obtain theorem 6.1 by an application of the approximation theorem [18, Theorem 1.6.7, page 354]. We have verified the basic hypotheses of the approximation theorem in section 3, so we will be checking only the two conditions *App 1* and *App 2* in this section. In the context of Theorem 6.1 we need to know

App 1. An arrow $f = (f_A, f_B, f_C, f_D)$ in $\mathcal{MV}$ is a w-map if and only if $u_D(f) = f_D$ is a homotopy equivalence.

App 2. If $M = (M_A, M_B, M_C, M_D)$ is a Mayer-Vietoris presentation and $f : u_D(M) = M_D \longrightarrow X$ is a map of D-modules, then there is a Mayer-Vietoris presentation $N = (N_A, N_B, N_C, N_D)$, a cofibration of Mayer-Vietoris presentations $h : M \rightarrowtail N$, and a map $g : u_D(N) = N_D \longrightarrow X$ such that $f = g \circ h_D$.

Condition *App 1* is satisfied by definition of the w-maps in $\mathcal{MV}_f$. We will now check condition *App 2.*

Before we launch into the details, let us sketch the main ideas of the proof. In general, checking condition *App 2* involves a step-by-step construction of N_- from M_-. It is easy to factor $f : M_D \longrightarrow X$ into a cofibration followed by a homotopy equivalence, and somewhat harder to accomplish this while keeping the intermediate module finite. In our situation we need to improve the whole Mayer-Vietoris presentation $M_- = (M_A, M_B, M_C, M_D)$. we use the localization functor L and its property of converting all structure maps in a Mayer-Vietoris presentation into weak homotopy equivalences to "spread" an obstruction to improving $f : M_D \longrightarrow X$ to a weak homotopy equivalence over $L(M_-)$. Then the finite approximation property of L proved in Theorem 5.13 shows that the "spread" obstruction to improving f and M_D is actually supported on a Mayer-Vietoris presentation that is corner by corner a finite cell extension of M_- We may then kill the obstruction in a standard way and iterate the procedure.

Proof of Theorem 6.1. We make the following preliminary considerations to start the verification of *App 2.* Let $M_- = (M_A, M_B, M_C, M_D)$ be a Mayer Vietoris presentation and let

$$f : M_D \longrightarrow X$$

be given. Since M_D and X are both finite and D is connective the groups

$$\pi_q(f) = \pi_q(T(f), M_D)$$

are zero for q sufficiently small. If $\pi_q(f) = 0$ for all q, then, of course, we may take $M = N$ and $h = \mathrm{id}$ to satisfy *App 2.*

Let us assume there is a first dimension n_0 where $\pi_{n_0}(f) \neq 0$. We frame an inductive procedure, as follows. Let n be an integer and let us assume that we have constructed a cofibration of Mayer Vietoris presentations

$$h^n : M_- \rightarrowtail M_-^n$$

and $g^n : M_D^n \longrightarrow X$ such that $f = g^n \circ h_D^n$ and $\pi_q(g^n) = 0$ for $q < n$. We want to improve the Mayer Vietoris presentation M^n and extend the map g^n to one with higher connectivity.

The following result is the first step in the procedure.

Lemma 6.3. *Let $\delta = \pi_0 D$. The δ-module $\pi_n(g^n)$ is finitely generated.*

Proof. By the Hurewicz theorem [6, Theorem IV.3.6, page 91]

$$\pi_n(g^n) \cong H_n^D(g^n; \delta)$$

and according to theorem IV.3.5 of [6, page 91] $H_*^D(g^n; \delta)$ may be calculated from a chain complex $C_*^D(g^n)$ of free δ-modules determined by the cell structure of $T(f)/M_D^n$. Since M_D^n, X, and $T(g^n)$ are all finite D-modules, this complex is finitely generated in each dimension. By Milnor's lemma 4.1 [10, page 369] the δ-module of cycles $Z_n^D(g^n)$ is finitely generated and stably free, so its quotient $H_n^D(g; \delta) \cong \pi_n(g^n)$ is also finitely generated. □

In view of the lemma, we may choose a finite wedge of $(n-1)$-spheres W, a map $\alpha : W \longrightarrow M_D^n$, and an extension $\bar{\alpha} : W \wedge I \longrightarrow X$ such that the diagram

$$\begin{array}{ccc} W & \longrightarrow & I \wedge W \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\bar{\alpha}} \\ M_D^n & \xrightarrow{g^n} & X \end{array}$$

commutes and such that the restrictions to the summands of W and $W \wedge I$ taken together represent a set of generators of $\pi_n(g^n)$.

Let us start to develop this situation into an improvement of the Mayer Vietoris presentation M_-^n. For the rest of the discussion let us write $M_- = M_-^n$, and drop the superscripts, for new ones will occur as we go. If we could lift $\alpha : W \longrightarrow M_D$ to $\tilde{\alpha} : W \longrightarrow M_A$ so that $\phi_{DA} \circ \tilde{\alpha} = \alpha$, there would be no difficulty in making the desired improvement to M_-. Granting the lifting, there is an extension to a map of Mayer-Vietoris presentations

$$W \wedge E_- \longrightarrow M_-$$

where E_- is the elementary Mayer-Vietoris presentation, and we could define N'_- by forming the following pushout diagram.

$$\begin{array}{ccc} W \wedge E_- & \rightarrowtail & W \wedge I \wedge E_- \\ \downarrow & & \downarrow \\ M_- & \rightarrowtail & N'_- \end{array}$$

The $(n-1)$-connected map $f : M_D \longrightarrow X$ would extend without difficulty to an $(n+1)$-connected map $f'_D : N'_D \longrightarrow X$, and we could continue. Postponing the explanation why the construction stops, we would seem to be in business.

However, in general, there is no way to get such a lifting. Here is where the L construction comes into play. We proceed as follows. Let $\eta : M_- \longrightarrow L(M_-) = M'_-$ be the natural transformation of Proposition 5.4 from the Mayer-Vietoris presentation M_- to its localization

$L(M_-) = M'_-$. Recall that $M'_D = M_D$ and that $\eta_D : M_D \longrightarrow M'_D = M_D$ is the identity. Recall also that all the structure maps of M'_- are weak homotopy equivalences. In particular,

$$\phi'_{DA} = \phi'_{DB} \circ \phi'_{BA} : M'_A \longrightarrow M'_D = M_D$$

is a weak homotopy equivalence. Since W is a wedge of spheres, there is map

$$\alpha' : W \longrightarrow M'_A$$

such that $\alpha \simeq \phi'_{DA} \circ \alpha'$. We may assume $\alpha = \phi'_{DA} \circ \alpha'$, if we compensate by changing $\bar{\alpha}$, so we assume this has been done.

Now we use the finite approximation property of M'_- proved in Theorem 5.13. Recall that we defined a category of diagrams $\mathcal{D}(M_-)$ in which an object of $\mathcal{D}(M_-)$ is a diagram

$$M_- \rightarrowtail N_- \longrightarrow M'_-$$

where the components of $M_- \rightarrowtail N_-$ are all finite cell extensions of M_A, M_B, and M_C and where the D-component of the diagram reduces to

$$M_D \xrightarrow{=} N_D \xrightarrow{=} M'_D.$$

(In particular, in $\mathcal{MV}_f$ or in $\mathcal{MV}_{hf}$ such an arrow $M_- \longrightarrow N_-$ is a cofibration.) Morphisms between such objects were defined to be cellular inclusions commuting with the reference maps from M_- and to M'_-. We defined a diagram

$$\begin{array}{ccc} M''_A & \longrightarrow & M''_B \\ \downarrow & & \downarrow \\ M''_C & \longrightarrow & M''_D \end{array}$$

by setting

$$M''_A = \operatorname*{colim}_{\mathcal{D}(M)} N_A,\ M''_B = \operatorname*{colim}_{\mathcal{D}(M)} N_B,\ M''_C = \operatorname*{colim}_{\mathcal{D}(M)} N_C$$

and $M''_D = M_D$, applying the universal property of the colimit to obtain the maps in the square above as well as a map of diagrams

$$\eta' : M''_- \longrightarrow M'_-.$$

We showed that M''_- is a Mayer-Vietoris presentation and in Theorem 5.13 that each component of η' is a weak homotopy equivalence.

Therefore, since the component $\eta'_A; M''_A \longrightarrow M'_A$ of η' is a weak homotopy equivalence, there is a lifting-up-to-homotopy of $\alpha' : W \longrightarrow M'_A$ to a map

$$\alpha'' : W \longrightarrow M''_A.$$

We may make adjustments of α' and α in their homotopy classes and a compensating adjustment to $\bar{\alpha}$, so we can assume $\alpha' = \eta'_A \circ \alpha''$. Since M''_A is a cell A-module (relative to M_A in the homotopy finite case), and W is compact, $W \wedge A \longrightarrow M''_A$ factors through a (relatively) finite cellular submodule by [6, Lemma III.2.3, page 55]. Consequently, there is a Mayer-Vietoris presentation M^0_- in $\mathcal{D}(M_-)$ such that $\alpha'' : W \longrightarrow M''_A$ factors through the canonical map $\gamma^0_A : M^0_A \longrightarrow M''_A$. Thus, starting from $\alpha : W \longrightarrow M_D$ we have obtained a cofibration of Mayer-Vietoris presentations

$$h^0 : M_- \rightarrowtail M^0_-$$

with $M^0_D = M_D$ and a map

$$\alpha^0 : W \longrightarrow M^0_A$$

such that $\phi^0_{DA} \circ \alpha^0 = \alpha$. We remark that M^0_A, M^0_B, and M^0_C are even finite cellular extensions of M_A, M_B, and M_C, respectively.

If E_- denotes the elementary Mayer-Vietoris presentation

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

we may extend α^0 to a map of Mayer-Vietoris presentations

$$\alpha^e : W \wedge E_- \longrightarrow M^0_-.$$

Now define M^1_- so that the righthand square in the following diagram is a pushout.

$$\begin{array}{ccccc} & & W \wedge E_- & \rightarrowtail & I \wedge W \wedge E_- \\ & & {\scriptstyle \alpha^e}\downarrow & & \downarrow \\ M_- & \rightarrowtail & M^0_- & \rightarrowtail & M^1_- \end{array}$$

In particular, we extract from the diagram above its D-component, obtaining the following diagram in which the square is also a pushout diagram of D-modules.

$$\begin{array}{ccccc} & & W \wedge D & \rightarrowtail & I \wedge W \wedge D \\ & & {\scriptstyle \alpha}\downarrow & & \downarrow \\ M_D & \overset{=}{\rightarrowtail} & M^0_D & \overset{h^1}{\rightarrowtail} & M^1_D \end{array}$$

Because M^1_D is defined as a pushout, there is also a unique map

$$g^1 : M^1_D \longrightarrow X, \text{ such that } g^1 \circ h^1 = g$$

induced from the diagram

$$\begin{array}{ccc} W \wedge D & \longrightarrow & I \wedge W \wedge D \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\bar{\alpha}} \\ M_D^n & \xrightarrow{\ g\ } & X \end{array}$$

which is just our earlier representation of generators of $\pi_n(g)$ canonically extended so that every object is now a D-module.

For $q \leq n$ the standard argument using the segment

$$H_q^D(M_D^1, M_D; \delta) \longrightarrow H_q^D(X, M_D, \delta) \longrightarrow H_q^D(X, M_D^1; \delta) \longrightarrow 0$$

of the long exact sequence associated to the triple (X, M_D^1, M_D) and the Hurewicz theorem shows that

$$\pi_q(g^1) \cong H_q^D(X, M_D^1; \delta) = 0$$

for $q \leq n$. To sum up, starting from the Mayer-Vietoris presentation M_- and an $(n-1)$-connected map $g : M_D \longrightarrow X$, we have constructed a cofibration $h^1 : M_- \rightarrowtail M_-^1$ and an n-connected map $g^1 : M_D^1 \longrightarrow X$ such that $g = g^1 \circ h^1$.

Because M_D and X are finite, or at least homotopy finite, if we repeat the procedure just described enough times, we eventually reach the situation where $g : M_D \longrightarrow X$ is such that $H_N^D(g; \delta) \neq 0$ for precisely one dimension N. In this case, the Hurewicz theorem and Milnor's lemma [10, page 369] tell us that $\pi_N(g) \cong H_N^D(g; \delta)$ is stably free over $\delta = \pi_0 D$. Make $\pi_N(g)$ free as follows. Choose a finite wedge of $(N-1)$-spheres W^0 of appropriate rank, define a cofibration of Mayer-Vietoris presentations

$$h^0 : M_- \rightarrowtail M_-^0 = M_- \cup_0 (W^0 \wedge E_-)$$

and extend $g : M_D \longrightarrow X$ to $g^0 : M_D^0 \longrightarrow X$ by mapping $W^0 \wedge D$ trivially to X. The result is that $\pi_N(g^0)$ is a free $\delta = \pi_0(D)$-module. In this case, if we choose a wedge of spheres W^1 so that the summands correspond precisely to a basis of $\pi_N(g^0)$, then the preceding argument yields a cofibration

$$h^1 : M_-^0 \rightarrowtail M_-^1$$

and an extension $g^1 : M_D^1 \longrightarrow X$ such that $\pi_*(g^1) = 0$. In the homotopy finite case, this is precisely what we need to complete the verification of *App 2*; in the finite case, the Whitehead theorem for D-modules tells us that this weak homotopy equivalence of cellular D-modules is a homotopy equivalence. □

Proof of theorem 1.1. This proof is mainly interpreting what we have proved in theorems 4.1 and 6.1 in the light of some well known facts about homotopy cartesian squares. Recall that a diagram

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle G} \\ Y & \xrightarrow[F]{} & Z \end{array}$$

commuting up to a homotopy $H : F \circ g \simeq G \circ f$ is homotopy cartesian if f, g, and H induce a homotopy equivalence from W to the homotopy pullback of X and Y over Z, which is the limit of the diagram

$$\begin{array}{ccccc} Y & & Z^I & & X \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle e_0} & & {\scriptstyle e_1}\searrow \quad \swarrow{\scriptstyle G} & \\ & Z & & Z & \end{array}$$

In case the diagram commutes on the nose, then we use the constant homotopy implicitly. Elementary arguments may be used to show that, if $\zeta : Z \longrightarrow Z'$ is a homotopy equivalence, then

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \zeta \circ G} \\ Y & \xrightarrow[\zeta \circ F]{} & Z' \end{array}$$

is also homotopy cartesian. One chooses a homotopy inverse ζ' to ζ and homotopies of the two compositions $\zeta' \circ \zeta$ and $\zeta \circ \zeta'$ to the appropriate identities and constructs from them a homotopy equivalence of homotopy pullbacks compatible with the maps from W. This explains how we may obtain the commuting homotopy cartesian squares

$$\begin{array}{ccc} vS_\bullet \mathcal{MV}_f^w & \longrightarrow & wS_\bullet \mathcal{MV}_f^w \\ \downarrow & & \downarrow \\ vS_\bullet \mathcal{MV}_f & \longrightarrow & hS_\bullet \mathcal{M}(D)_f \end{array} \quad \text{and} \quad \begin{array}{ccc} vS_\bullet \mathcal{MV}_{hf}^w & \longrightarrow & wS_\bullet \mathcal{MV}_{hf}^w \\ \downarrow & & \downarrow \\ vS_\bullet \mathcal{MV}_{hf} & \longrightarrow & wS_\bullet \mathcal{M}(D)_{hf} \end{array}$$

starting from the homotopy cartesian squares 3.1 supplied by the fibration theorem and applying the result of theorem 6.1.

Returning to the model homotopy cartesian square, it is more interesting to study what happens when we have a homotopy equivalence $\eta : Y \longrightarrow Y'$ with homotopy inverse $\eta' : Y' \longrightarrow Y$ and we consider the diagram

$$\begin{array}{ccc} W & \xrightarrow{f} & X \\ {\scriptstyle \eta \circ g}\downarrow & & \downarrow{\scriptstyle G} \\ Y' & \xrightarrow[F \circ \eta']{} & Z \end{array}$$

Even if the original diagram commutes, this diagram probably only commutes up to homotopy. In any case, one can use the homotopy associated with the original diagram together with the homotopies associated with $\eta' \circ \eta$ and $\eta \circ \eta'$ to produce homotopy equivalences between the homotopy pullback of the original diagram and the modified diagram, compatible with the canonical maps from W. Thus, making this change preserves homotopy cartesianness as well.

Now we apply our second remark to the homotopy equivalence of theorem 4.1 and its inverse as described in the remarks at the end of section 4. Thus, we obtain homotopy cartesian squares with the lower horizontal arrows as described in theorem 1.1 □

References

[1] M. Barr and C. Wells. *Toposes, triples, and theories.* Springer-Verlag, 1985.

[2] A.K. Bousfield, *The Boolean algebra of spectra,* Comment. Math. Helvetici **54**(1979), 368–377.

[3] A.K. Bousfield, *The localization of spectra with respect to homology,* Topology, **18**(1979), 257–281.

[4] B. Dundas, *Relative K-theory and topological cyclic homology,* Acta Math., **197**(1997), 223–242.

[5] W. Dwyer and J. Spalinski, *Homology theories and model categories*, in *A Handbook of algebraic topology*, edited by I.M. James. Elsevier Science, 1995.

[6] A.D. Elmendorf, I. Kriz, J.P. May, and M. Mandell, *Rings, modules, and algebras in stable homotopy theory,* Mathematical surveys and monographs; no. 47, American Mathematical Society, 1997.

[7] F.E.J. Linton, *Coequalizers in categories of algebras,* Lecture Notes in Math. 80, 1969, 75–90.

[8] S. MacLane, *Homology,* Third Corrected Printing. Springer Verlag, New York, 1975.

[9] *Finite Localizations,* Bol. Soc. Mat. Mex., **37**(1992), 383–389.

[10] J. Milnor, *Whitehead torsion,* Bull. A.M.S., **72**(1966), 358–426.

[11] D. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. Math., **106**(1984), 351–414.

[12] R. Schwänzl and R.E. Staffeldt, *The approximation theorem and the K-theory of generalized free products,* Trans. Amer. Math. Soc., **347**(1995), 3319–3345.

[13] R. Schwänzl and R.E. Staffeldt, *The Seifert-van Kampen theorem and generalized free products of S-algebras,* In preparation.

[14] R. Schwänzl, R. Staffeldt, and F. Waldhausen, *K-theory and generalized free products of S-algebras: the* Nil*-term,* In preparation.

[15] R. Schwänzl, R. Staffeldt, and F. Waldhausen, *Stable K-theory and topological Hochschild homology of A_∞-rings*, Contemporary Mathematics, **199**(1996), 161–174.

[16] P. Vogel, *On the vanishing of* Nil*-groups,* Preprint, 1990.

[17] F. Waldhausen, *Algebraic K-theory of generalized free products,* Ann. Math., **108** (1978), 135–256.

[18] F. Waldhausen, *Algebraic K-theory of spaces,* Lecture Notes in Math. 1126 (1985), 318–419.

[19] F. Waldhausen, *Algebraic K-theory of spaces, localization, and the chromatic filtration of stable homotopy,* Lecture Notes in Math. 1051 (1984), 173–196.