The Representation Type of the Full Transformation Semigroup T_4

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ABSTRACT. Let T_n be the semigroup of all transformations of a set of n elements and k a field of characteristic 0. According to Ponizovskii, the semigroup algebra kT_n is of finite representation type if $n \le 3$. According to Putcha, kT_n is of infinite representation type if $n \ge 5$. Here, we deal with the remaining case $n=4$ and show that kT_4 is also of finite representation type. Note that the quiver of kT_4 already has been exhibited by Putcha, here we determine the relations. It turns out that kT_4 is a string algebra and its global dimension is 3.

The semigroup algebras kT_n.

Given sets X, Y, Z and maps $f\colon X \to Y$ and $g\colon Y \to Z$, we denote the composition by $gf\colon X \to Z$. We denote by T_n the semigroup of all endo maps of the set $\{1, 2, \dots, n\}$ of all integers $1 \le i \le n$, and by T_{nm} the set of elements in T_n whose image is of cardinality at most m, where $0 \le m \le n$. Of course, we obtain in this way a chain of ideals of T_n

$$\emptyset = T_{n0} \subset T_{n1} \subset \cdots \subset T_{n,n-1} \subset T_{nn} = T_n,$$

and actually these are all the ideals of T_n. If we consider the corresponding semigroup algebras $I_m = kT_{nm}$ (here we suppress the index n), we obtain a chain of ideals

$$(*) \qquad 0 = I_0 \subset I_1 \subset \cdots \subset I_{n-1} \subset I_n = kT_n.$$

It has been shown by Hewitt and Zuckerman [HZ] that the simple kT_n-modules correspond bijectively to the partitions of m with $1 \le m \le n$. More precisely, the simple kT_n-modules annihilated by I_{m-1}, but not by I_m, correspond to the partitions of m. Given a partition $\lambda = (\lambda_1, \dots, \lambda_t)$ of m, let $L(\lambda)$ be the corresponding simple left kT_n-module, and $(\lambda)L$ its dual (thus this is a simple right kT_n-module). For a partition $\lambda = (\lambda_1, \dots, \lambda_t)$, we usually will write $L(\lambda_1, \dots, \lambda_t)$ instead of $L((\lambda_1, \dots, \lambda_t))$, and we will use such a convention also in similar cases.

As Putcha [Pu] has shown, the ideal chain $(*)$ is a heredity chain of the algebra kT_n (this means that the following three conditions are satisfied for all $1 \le m \le n$:

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first, the ideal I_m is idempotent, second, $I_m R I_m \subseteq I_{m-1}$, where R is the radical of kT_n, and third, I_m/I_{m-1} is a projective left kT_n/I_{m-1}-module). In particular kT_n is a quasi-hereditary algebra in the sense of Cline-Parshall-Scott [CPS]. It follows that the category of all left kT_n-modules is a highest weight category with respect to the following weight set Λ_n: Let $\Lambda(t)$ be the set of all partitions of t. Let Λ_n be the disjoint union of the sets $\Lambda(t)$, where $1 \le t \le n$, and consider Λ_n as a poset with $\lambda < \lambda'$ iff $\lambda \in \Lambda(t)$, $\lambda' \in \Lambda(t')$ and $t > t'$.

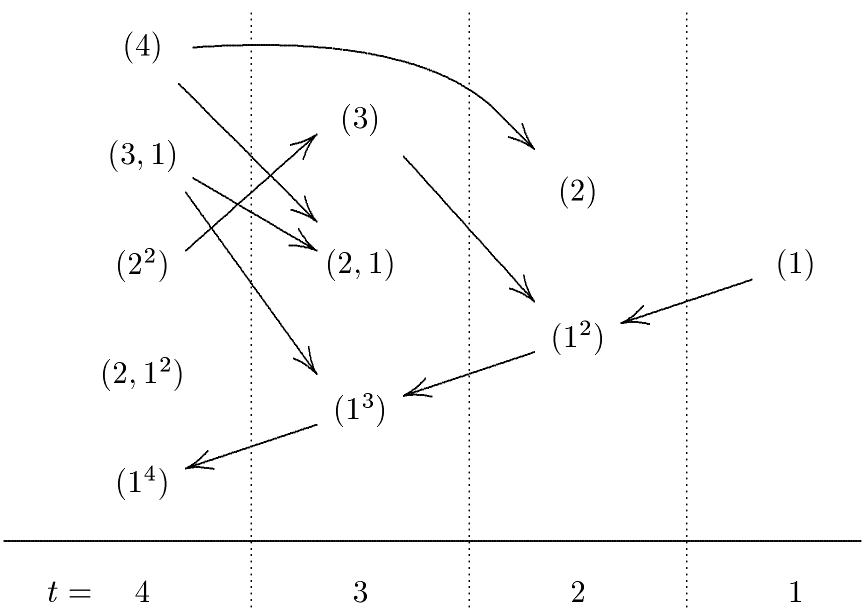
Let $\Delta(\lambda)$ be the standard left kT_n-module with top $L(\lambda)$; similarly, we denote by $(\lambda)\Delta$ the standard right kT_n-module with top $(\lambda)L$; of course, the dual of $(\lambda)\Delta$ is just the costandard left kT_n-module $\nabla(\lambda)$ with socle $L(\lambda)$.

The semigroup algebra kT_4.

The dimension $d(\lambda)$ of the simple kT_4-module $L(\lambda)$ is known already from [HZ]:

λ	(4)	$(3,1)$	(2^2)	$(2,1^2)$	(1^4)	(3)	$(2,1)$	(1^3)	(2)	(1^2)	(1)
$d(\lambda)$	1	3	2	3	1	4	8	3	6	3	1

The quiver Q of kT_4 has been determined by Putcha [Pu]:



Here, an arrow $\lambda \to \lambda'$ indicates that $\mathrm{Ext}^1(L(\lambda), L(\lambda'))$ is 1-dimensional and otherwise $\mathrm{Ext}^1(L(\lambda), L(\lambda')) = 0$. For our further considerations, we have added the information on the weight set: the subsets $\Lambda(t)$ are separated by dotted lines; note that on the left, there are the small weights, on the right the large ones.

Proposition. *The Loewy length of the standard and of the costandard kT_4-modules is bounded by 2.*

Proof. Since the composition factors of the standard module $\Delta(\lambda)$ are of the form $L(\lambda')$ with $\lambda' \le \lambda$, the only standard modules to be considered are $\Delta(1)$ and $\Delta(1^2)$. Similarly, since the composition factors of the costandard module $\nabla(\lambda)$

are of the form $L(\lambda')$ with $\lambda' \leq \lambda$, the only costandard module to be considered is $\nabla(1^2)$. Of course, instead of looking at $\nabla(1^2)$, we also may consider its dual module $(1^2)\Delta$.

We claim that all three modules $\Delta(1), \Delta(1^2)$ and $(1^2)\Delta$ are of length 2. This assertion is implicitly contained in [Pu]. Actually, it can easily be retrieved from the knowledge of the quiver and the dimension of the simple modules, as we are going to show. Note that if e is an idempotent of $A = kT_n$ with $AeA + I_{t-1} = I_t$, then $\bigoplus_{\lambda\in\Lambda(t)} \Delta(\lambda)$ is a direct summand of $Ae/I_{t-1}e$, and similarly $\bigoplus_{\lambda\in\Lambda(t)} (\lambda)\Delta$ is a direct summand of eA/eI_{t-1}. Let $I = I_1$.

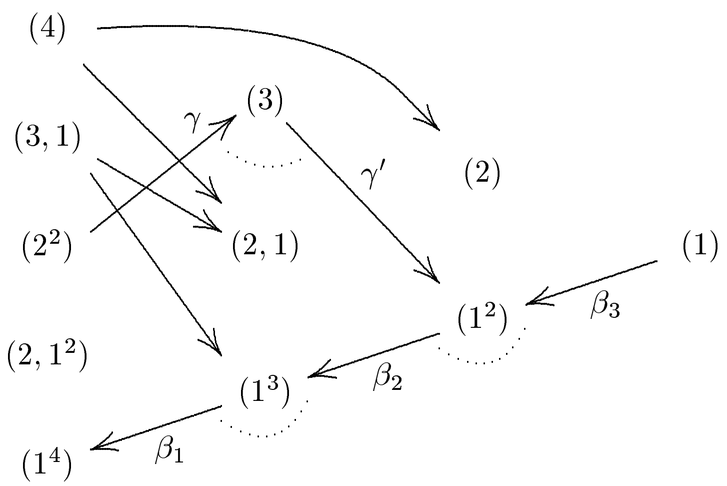
First, consider $\Delta(1)$. The constant map $e_1 \in T_4$ with $e_1(i) = 1$ for all $1 \leq i \leq 4$ is an idempotent and $Ae_1A = I$. Of course, I is 4-dimensional, a k-basis of I is given by the four constant maps. On the other hand, the quiver Q shows that $\Delta(1)$ has at least the composition factors $L(1)$ and $L(1^2)$; the module $L(1)$ is 1-dimensional, the module $L(1^2)$ is 3-dimensional, this shows that $I = \Delta(1)$ and that $\Delta(1)$ has just the two composition factors mentioned.

Next, we deal with the weight (1^2). Let $e_2 \in T_4$ be defined by $e_2(1) = 1$, and $e_2(2) = e_2(3) = e_2(4) = 2$, this is an idempotent of A with $Ae_2A = I_2$. Note hat Ae_2/Ie_2 has a basis consisting of (the residue classes of) all the endo maps f with $f(1) \neq f(2) = f(3) = f(4)$, thus Ae_2/Ie_2 is 12-dimensional. On the other hand, Ae_2/Ie_2 maps onto $\Delta(2) \oplus \Delta(1^2)$. The module $\Delta(2) = L(2)$ is 6-dimensional, the module $\Delta(1^2)$ has as composition factors at least the simple modules $L(1^2)$ and $L(1^3)$, both being 3-dimensional. It follows that $Ae_2/Ie_2 = \Delta(2) \oplus \Delta(1^2)$ and that $\Delta(1^2)$ has just the two composition factors mentioned already.

Finally, consider e_2A/e_2I, it has a basis consisting of (the residue classes of) all the endo maps g with image equal to $\{1, 2\}$, thus e_2A/e_2I is 14-dimensional. The module e_2A/e_2I maps onto $(2)\Delta \oplus (1^2)\Delta$. The module $(2)\Delta$ has as composition factors at least the 6-dimensional module $(2)L$ and the 1-dimensional module $(4)L$, the module $(1^2)\Delta$ has as composition factors at least the 3-dimensional module $L(1^2)$ and the 4-dimensional module $L(3)$. It follows that $e_2A/e_2I = (2)\Delta \oplus (1^2)\Delta$ and that these two standard modules $(2)\Delta$ and $(1^2)\Delta$ both are of length 2.

Remark: It is not difficult to write down explicitly the direct decompositions $Ae_2/Ie_2 = \Delta(2) \oplus \Delta(1^2)$ and $e_2A/e_2I = (2)\Delta \oplus (1^2)\Delta$, as well as the submodules $L(1^2) \subset \Delta(1)$, $L(1^3) \subset \Delta(1^2)$, $(4)L \subset (2)\Delta$ and $(3)L \subset (1^2)\Delta$ which have been used in the proof.

The proposition yields three monomial relations: $\gamma'\gamma$ (this is the only composition of two arrows from left to right), $\beta_1\beta_2$, and $\beta_2\beta_3$ (these are the two possibilities of compositions of two arrows from right to left:



It remains to be seen that these are all the relations. Consider the path algebra kQ of the quiver, and denote by J the ideal generated by the three monomial relations $\beta_1\beta_2$, $\beta_2\beta_3$ and $\gamma'\gamma$. Then the basic algebra A^0 of A is a factor algebra of kQ/J, since the endomorphism ring of any simple A-module is equal to k. Take now an algebra $B = M(kQ/J; d)$ which is Morita equivalent to kQ/J such that the simple left B-module corresponding to the vertex $\lambda \in \Lambda = \Lambda_4$ has dimension $d(\lambda)$ (such an algebra is unique up to algebra isomorphism). Then A is a factor algebra of B. We show that B and A have the same dimension, thus A is isomorphic to B.

For $\lambda \in \Lambda$, let $P(\lambda)$ be the indecomposable projective left B-module corresponding to the vertex λ. Let us indicate the structure of the modules $P(\lambda)$ by writing down the corresponding composition factors in a suggestive way. All but two of the modules are serial (four are even simple), in these cases, the composition factors are written as a column, the upper entry is the top, the lowest entry the socle. For the two indecomposable projective modules $P(4)$ and $P(3,1)$ which are not serial, the radical is a direct sum of two serial modules and we use a similar convention for the serial summands.

$$
\begin{array}{ccccc}
P(4) & P(3,1) & P(2^2) & P(2,1^2) & P(1^4) \\
 & & & & \\
\begin{array}{c} L(4) \\ L(2)\oplus L(2,1) \end{array} &
\begin{array}{c} L(3,1) \\ L(2,1)\oplus L(1^3) \\ \phantom{L(2,1)\oplus{}} L(1^4) \end{array} &
\begin{array}{c} L(2^2) \\ L(3) \end{array} &
L(2,1^2) & L(1^4)
\end{array}
$$

and

$$
\begin{array}{cccccc}
P(3) & P(2,1) & P(1^3) & P(2) & P(1^2) & P(1) \\
 & & & & & \\
\begin{array}{c} L(3) \\ L(1^2) \\ L(1^3) \end{array} &
L(2,1) &
\begin{array}{c} L(1^3) \\ L(1^4) \end{array} &
L(2) &
\begin{array}{c} L(1^2) \\ L(1^3) \end{array} &
\begin{array}{c} L(1) \\ L(1^2) \end{array}
\end{array}
$$

It is now easy to calculate the dimension $p(\lambda)$ of $P(\lambda)$:

λ	(4)	$(3,1)$	(2^2)	$(2,1^2)$	(1^4)	(3)	$(2,1)$	(1^3)	(2)	(1^2)	(1)
$p(\lambda)$	15	15	6	3	1	10	8	4	6	6	4

The dimension of B therefore is $\sum_{\lambda\in\Lambda} d(\lambda)p(\lambda) = 256$. Of course, the dimension of A is also $4^4 = 256$. This shows that B is isomorphic to A.

Theorem. *The algebra kT_4 is Morita equivalent to the path algebra of the quiver Q modulo the relations $\beta_1\beta_2$, $\beta_2\beta_3$ and $\gamma'\gamma$.*

Having determined a presentation by quivers and relations, it now is easy to derive many properties of kT_4.

Corollary 1. *The algebra kT_4 is a string algebra. It is of finite representation type, with 36 isomorphism classes of indecomposable left modules, all being thin.*

Recall from [BR] that a finite dimensional algebra is said to be *string algebra* provided it is given by a quiver with monomial relations, such that the following properties are satisfied: any vertex of the quiver is endpoint of at most two arrows and starting point of at most two arrows, and for any arrow β, there is at most one arrow α and at most one arrow γ such that $\alpha\beta$ and $\beta\gamma$ are not relations. It is well-known (and goes back to Gelfand and Ponomarev [GP]) how to construct all the indecomposable modules for a string algebra. In particular, a string algebra whose quiver is a tree has only finitely many isomorphism classes of indecomposables, and all the indecomposable modules are *thin* (i.e.: the Jordan-Hölder multiplicities are 0 or 1).

Proof: Our presentation shows that A is a string algebra. Since Q is a tree, it is of finite representation type and all the A-modules are thin: they are given by walks in the quivers which avoid the relations (see for example [BR]). Let us write down, for any natural number ℓ the number $i(\ell)$ of isomorphism classes of indecomposable A-modules of length ℓ

ℓ	1	2	3	4	5	6
$i(\ell)$	11	9	7	4	3	2

thus altogether there are 36 isomorphism classes.

Remark 1. After having determined the quiver Q of kT_4, Putcha already has noted that there are only finitely many isomorphism classes of indecomposable kT_4-modules of Loewy length at most 2. Our considerations show that there are only five additional isomorphism classes.

Remark 2. In order to see that kT_4 is of finite representation type, it is sufficient to verify the relations $\beta_1\beta_2$ and $\beta_2\beta_3$, since the factor algebra of kQ modulo the ideal generated by $\beta_1\beta_2$ and $\beta_2\beta_3$ already is a string algebra. Note that also the factor algebra of kQ modulo the ideal generated by $\beta_1\beta_2$ and $\gamma'\gamma$ is of finite representation type.

Corollary 2. *The global dimension of kT_4 is 3.*

Proof: The simple modules $L(2,1^2), L(1^4), L(2,1)$ and $L(2)$ are projective (since the corresponding vertices are sinks of the quiver). Here are the minimal projective presentations of the remaining simple A-modules $L(\lambda)$. There are four simple modules of projective dimension 1:

$$\begin{array}{ccccccccc}
0 & \to & L(2)\oplus L(2,1) & \to & P(4) & \to & L(4) & \to & 0\\
0 & \to & L(2,1)\oplus P(1^3) & \to & P(3,1) & \to & L(3,1) & \to & 0\\
0 & \to & P(1^2) & \to & P(3) & \to & L(3) & \to & 0\\
0 & \to & L(1^4) & \to & P(1^3) & \to & L(1^3) & \to & 0
\end{array}$$

two of projective dimension 2 and one of projective dimension 3:

$$\begin{array}{ccccccccccc}
0 & \to & P(1^2) & \to & P(3) & \to & P(2^2) & \to & L(2^2) & \to & 0\\
0 & \to & L(1^4) & \to & P(1^3) & \to & P(1^2) & \to & L(1^2) & \to & 0
\end{array}$$

$$0 \to L(1^4) \to P(1^3) \to P(1^2) \to P(1) \to L(1) \to 0$$

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